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Research article

A note on explicit conditions for diagonal stability

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Abstract: In this short note, we presented a number of alternative explicit necessary and sufficient conditions for diagonal stability along with a new proof of a well-known result in this regard.

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1. Introduction

Throughout this note, we shall deal with real square matrices only. A matrix A is called (Lyapunov) diagonally stable if there exists a positive diagonal matrix D such that

$$DA + A^T D$$

is positive definite. We comment that in practice, the setting of $DA + A^T D$ being negative definite is commonly used. We switch this to being positive definite so the notion of P-matrices can be handled more conveniently. The existence of such matrix D means that

$$V(x) = x^T Dx$$

is a diagonal quadratic Lyapunov function for a linear system in the form

$$\dot{x}(t) = Ax(t).$$

For detailed background material and existing results on diagonal stability, as well as its connection to other types of matrix stability, we refer the reader to the surveys in [1, 2].

This type of matrix stability has various applications in networks [3], dynamical systems [4], and computations [5]. The study of diagonal stability is of considerable importance, addressing both theoretical and practical aspects.

Despite significant research efforts, determining the diagonal stability of a matrix remains a difficult problem. Two most notable characterizations for the diagonal stability of a matrix $A \in \mathbb{R}^{n\times n}$ can be found in the literature. First, it was shown in [6] that a necessary and sufficient condition for A to be diagonally stable is the existence of a positive diagonal entry in AH, where H is any nonzero positive semi-definite matrix. The other, see [7], depends on the P- matrix property of the Hadamard multiplication A with any positive semi-definite matrix S with $s_{ii} \neq 0$ for all i. Recall that a matrix $A \in \mathbb{R}^{n\times n}$ is said to be a P-matrix if all its principal minors are positive.

A well-known necessary condition [8] for A to be diagonally stable is that A must be a P-matrix. This P-matrix condition is also sufficient when A is 2×2; see [6]. Such a condition is said to be explicit because it can be checked directly using the entries of A.

For diagonal stability of a 3×3 matrix, two explicit necessary and sufficient conditions can be found in the literature. The first one is as follows.

Theorem 1.1. [7, Theorem 4.1] *A 3×3 matrix $A = [a_{i,j}]$ is diagonally stable if, and only if,*

(i) A is a P-matrix, and

(ii) $\max\{1, \omega_1, \omega_2, \omega_3\} < \frac{1}{2}(1 + \omega_1 + \omega_2 + \omega_3)$, *where* $\omega_i = \sqrt{a_{i,i}\,(A^{-1})_{i,i}}$, $i = 1, 2, 3$.

The second one [8, Theorem 4] is not as explicit as the above since it is formulated as a pair of quadratic inequalities. An alternative and more transparent proof of this result can be found in [9]. The results in [9] are further explored in [10, 11].

This note shall concern Theorem 1.1 only. It should be pointed out here that no explicit necessary and sufficient condition is known for cases beyond 3×3. The main difficulty for this situation is due to the much larger number of parameters being involved in cases of higher dimensions.

The proof of Theorem 1.1 in [7] is done by establishing a positive lower bound for a multivariable objective function through some rather sophisticated parametrization and algebraic manipulations. This has been the main motivation for this note. Our goals are to present a more transparent proof of Theorem 1.1 by first establishing a number of elementary algebraic inequalities and, along the way, to formulate several useful alternative explicit necessary and sufficient conditions for diagonal stability.

Before introducing some preliminary existing results, let us set the stage first. Given two matrices A and B of the same size, they are said to share a common diagonal solution if there exists a positive diagonal matrix D such that

$$DA + A^T D \text{ and } DB + B^T D$$

are both positive definite. In addition, we denote by $M = [m_{i,j}]$ the matrix of minors of A, i.e. $m_{i,j}$ is the determinant of the sub-matrix obtained from A by deleting its ith row and jth column.

We now quote the following two results from [10], which will serve as our starting point.

Lemma 1.1. [10, Theorem 2.1] *A 3×3 matrix $A = [a_{i,j}]$ is diagonally stable if, and only if,*

(i) $a_{3,3} > 0$, and

(ii) $\hat{A} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ *and* $S = \frac{1}{a_{3,3}} \begin{bmatrix} m_{2,2} & m_{2,1} \\ m_{1,2} & m_{1,1} \end{bmatrix}$ *share a common diagonal solution.*

Note that the matrix S in Lemma 1.1 is the Schur complement of $a_{3,3}$ in A. For background material, see, for example, [12, 13].

Lemma 1.2. [10, Corollary 2.4] *Two* 2×2 *matrices* $A_1 = [a^{(1)}_{i,j}]$ *and* $A_2 = [a^{(2)}_{i,j}]$ *share a common diagonal solution if, and only if,*

(i) A_1 *and* A_2 *are P-matrices, and*

(ii) $\max_k x^{(k)}_1 < \min_k x^{(k)}_2$, *where, for* $k = 1, 2$,

$$x^{(k)}_1 = \left(\frac{a^{(k)}_{1,2}}{\sqrt{a^{(k)}_{1,1} a^{(k)}_{2,2}} + \sqrt{\det A_k}} \right)^2 \quad \text{and} \quad x^{(k)}_2 = \left(\frac{\sqrt{a^{(k)}_{1,1} a^{(k)}_{2,2}} + \sqrt{\det A_k}}{a^{(k)}_{2,1}} \right)^2 .$$

In Lemma 1.2, by convention, $x^{(k)}_2 = +\infty$ if the respective $a^{(k)}_{2,1} = 0$.

2. Main results

Let us start with a few technical lemmas.

Lemma 2.1. *Suppose that* $0 < \alpha < \beta \le \gamma < \delta$ *such that* $\alpha\delta = \beta\gamma = \lambda$. *Then,*

$$\beta + \gamma < \alpha + \delta.$$

Proof. The conclusion is equivalent to

$$\frac{\lambda}{\gamma} + \gamma < \frac{\lambda}{\delta} + \delta. \tag{2.1}$$

Note that $\gamma \ge \sqrt{\lambda}$ and $\delta > \sqrt{\lambda}$. Define $\phi(t) = \frac{\lambda}{t} + t$, where $t \ge \sqrt{\lambda}$. Since $\phi'(t) = -\frac{\lambda}{t^2} + 1 > 0$ for $t > \sqrt{\lambda}$, $\phi(t)$ is a strictly increasing function on $t \ge \sqrt{\lambda}$. Thus, (2.1) follows. □

In a similar way, we can easily show the following lemma whose proof is thus omitted.

Lemma 2.2. *Suppose that* $0 < \alpha \le \beta < \gamma \le \delta$ *such that* $\alpha\delta = \beta\gamma = \lambda$. *Then,*

$$\beta + \gamma \le \alpha + \delta.$$

The next lemma is a slightly enhanced version of [14, Lemma 1].

Lemma 2.3. [10, Lemma 3.3] *For any positive numbers a, b, c, and d, the following are equivalent:*

(i) $(a + b)^2 - (c - d)^2 > 0$ *and* $(c + d)^2 - (a - b)^2 > 0$.

(ii) $\max\{a, b, c, d\} < \frac{1}{2}(a + b + c + d)$.

Lemmas 1.1 and 1.2 lead directly to the result below, which is an alternative explicit necessary and sufficient condition for diagonal stability.

Theorem 2.1. *A* 3×3 *matrix A is diagonally stable if, and only if,* *(i) A is a P-matrix,*

(ii) $\left(\dfrac{a_{1,2}}{\sqrt{a_{1,1}a_{2,2}}+\sqrt{m_{3,3}}}\right)^2 < \left(\dfrac{\sqrt{m_{1,1}m_{2,2}}+\sqrt{a_{3,3}\det A}}{m_{1,2}}\right)^2 = \mu$, *and*

(iii) $\left(\dfrac{m_{2,1}}{\sqrt{m_{1,1}m_{2,2}}+\sqrt{a_{3,3}\det A}}\right)^2 < \left(\dfrac{\sqrt{a_{1,1}a_{2,2}}+\sqrt{m_{3,3}}}{a_{2,1}}\right)^2 = \nu$.

Proof. According to Lemma 1.1, the diagonal stability of $A \in \mathbb{R}^{3\times 3}$ is equivalent to: $(i)'$ $a_{33} > 0$, and $(ii)'$ $A_1 = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$ and $A_2 = \dfrac{1}{a_{3,3}} \begin{bmatrix} m_{2,2} & m_{2,1} \\ m_{1,2} & m_{1,1} \end{bmatrix}$ share a common diagonal solution.

Now, by Lemma 1.2, condition $(ii)'$ is equivalent to: $(iii)'$ A_1 and A_2 being P-matrices, and $(iv)'$ $\max_k x_1^{(k)} < \min_k x_2^{(k)}$, where

$$x_1^{(k)} = \left(\frac{a_{1,2}^{(k)}}{\sqrt{a_{1,1}^{(k)}a_{2,2}^{(k)}}+\sqrt{\det A_k}}\right)^2 \quad \text{and} \quad x_2^{(k)} = \left(\frac{\sqrt{a_{1,1}^{(k)}a_{2,2}^{(k)}}+\sqrt{\det A_k}}{a_{2,1}^{(k)}}\right)^2.$$

From $(i)'$ and $(iii)'$, (i) follows easily. Additionally, by noting that $\det A = a_{3,3}\det S$, and the condition $\max_k x_1^{(k)} < \min_k x_2^{(k)}$ is equivalent to $x_1^{(2)} < x_2^{(1)}$ and $x_1^{(1)} < x_2^{(2)}$, we obtain (ii) and (iii). □

We comment that since A is a P-matrix, all the square roots in Theorem 2.1 are well-defined and positive. Similar to Lemma 1.2, μ and ν are interpreted as $+\infty$ when the respective $m_{1,2}$ and $a_{2,1}$ are zero.

A consequence of Theorem 2.1 is the following, whose proof is straightforward and hence omitted.

Corollary 2.1. *Let A be a* 3×3 *matrix such that* $a_{1,2}$, $a_{2,1}$, $m_{1,2}$ *and* $m_{2,1}$ *are all nonzero. Then, A is diagonally stable if, and only if,*

(i) A is a P-matrix,

(ii) $\left(\dfrac{a_{1,2}}{\sqrt{a_{1,1}a_{2,2}}+\sqrt{m_{3,3}}}\right)^2 < \left(\dfrac{\sqrt{m_{1,1}m_{2,2}}+\sqrt{a_{3,3}\det A}}{m_{1,2}}\right)^2$,

(iii) $\left(\dfrac{\sqrt{a_{1,1}a_{2,2}}-\sqrt{m_{3,3}}}{a_{2,1}}\right)^2 < \left(\dfrac{m_{2,1}}{\sqrt{m_{1,1}m_{2,2}}-\sqrt{a_{3,3}\det A}}\right)^2$,

(iv) $\left(\dfrac{m_{2,1}}{\sqrt{m_{1,1}m_{2,2}}+\sqrt{a_{3,3}\det A}}\right)^2 < \left(\dfrac{\sqrt{a_{1,1}a_{2,2}}+\sqrt{m_{3,3}}}{a_{2,1}}\right)^2$, *and*

(v) $\left(\dfrac{\sqrt{m_{1,1}m_{2,2}}-\sqrt{a_{3,3}\det A}}{m_{1,2}}\right)^2 < \left(\dfrac{a_{1,2}}{\sqrt{a_{1,1}a_{2,2}}-\sqrt{m_{3,3}}}\right)^2$.

Next, using Theorem 2.1 and Corollary 2.1, we shall present another alternative explicit necessary and sufficient condition for diagonal stability.

For convenience, let us define

$$u_1 = \left(\sqrt{a_{1,1}a_{2,2}} - \sqrt{m_{3,3}}\right)\left(\sqrt{m_{1,1}m_{2,2}} - \sqrt{a_{3,3}\det A}\right),$$

$$u_2 = \left(\sqrt{a_{1,1}a_{2,2}} + \sqrt{m_{3,3}}\right)\left(\sqrt{m_{1,1}m_{2,2}} + \sqrt{a_{3,3}\det A}\right),$$

$$v_1 = a_{1,2}m_{1,2}, \text{ and } v_2 = a_{2,1}m_{2,1}.$$

Observe

$$u_1u_2 = v_1v_2. \tag{2.2}$$

Then, we can state the following.

Theorem 2.2. *A* 3×3 *matrix A is diagonally stable if, and only if,*

(i) A is a P-matrix, and

(ii) $v_1^2 + v_2^2 < u_1^2 + u_2^2$.

Proof. It suffices to show that condition (ii) here is equivalent to the corresponding conditions in Theorem 2.1 and Corollary 2.1.

When $a_{1,2} = 0$, conditions (ii) and (iii) in Theorem 2.1 reduce to $v_2^2 < u_2^2$ and, hence, are equivalent to $v_1^2 + v_2^2 < u_1^2 + u_2^2$ since $u_1 = v_1 = 0$. Similarly, when $a_{2,1} = 0$, conditions (ii) and (iii) in Theorem 2.1 become $v_1^2 < u_2^2$, i.e., $v_1^2 + v_2^2 < u_1^2 + u_2^2$ since $u_1 = v_2 = 0$. Clearly, such an argument applies also to the cases when either $m_{1,2} = 0$ or $m_{2,1} = 0$. Theorem 2.2, therefore, holds when any of the $a_{1,2}$, $a_{2,1}$, $m_{1,2}$, or $m_{2,1}$ is zero.

We assume next that $a_{1,2}$, $a_{2,1}$, $m_{1,2}$, and $m_{2,1}$ are nonzero. Consequently, $u_1^2, u_2^2, v_1^2, v_2^2$ are all positive. Besides, $u_1^2 < u_2^2$.

Suppose that conditions (ii) through (v) in Corollary 2.1 hold, i.e.,

$$u_1^2 < v_1^2, v_2^2 \text{ and } v_1^2, v_2^2 < u_2^2. \tag{2.3}$$

Without loss of generality, we assume $u_1^2 < v_1^2 \leq v_2^2 < u_2^2$. Besides, by (2.2), $u_1^2u_2^2 = v_1^2v_2^2$. These, accordingly to Lemma 2.1, imply condition (ii) here.

Conversely, assume that condition (ii) here holds but (2.3) is not satisfied. Let us consider the following possible scenarios:

- If $u_1^2 \geq v_1^2$ and $u_2^2 > v_2^2$, then $u_1^2u_2^2 > v_1^2v_2^2$, which contradicts (2.2).
- If $u_1^2 < v_1^2$ and $u_2^2 \leq v_2^2$, then $u_1^2u_2^2 < v_1^2v_2^2$, which again contradicts (2.2).
- If $u_1^2 \geq v_1^2$ and $u_2^2 \leq v_2^2$, then $v_1^2 \leq u_1^2 < u_2^2 \leq v_2^2$. By Lemma 2.2, we obtain $u_1^2 + u_2^2 \leq v_1^2 + v_2^2$, which contradicts condition (ii) here.

The above analysis shows that (2.3) must hold. The proof is now complete. □

We are now ready to present a new proof of Theorem 1.1.

Proof of Theorem 1.1. It suffices to show that condition (ii) in Theorem 1.1 and condition (ii) in Theorem 2.2 are equivalent.

It is easy to verify

$$u_1 + u_2 = 2\sqrt{a_{1,1}a_{2,2}m_{1,1}m_{2,2}} + 2\sqrt{a_{3,3}m_{3,3}\det A} > 0. \tag{2.4}$$

Moreover, we have

$$a_{1,1}m_{1,1} + a_{2,2}m_{2,2} - a_{3,3}m_{3,3} - \det A = a_{1,2}m_{1,2} + a_{2,1}m_{2,1}. \tag{2.5}$$

By Lemma 2.3 and $(A^{-1})_{i,i} = \dfrac{m_{i,i}}{\det A}$ for all i, condition (ii) in Theorem 1.1 is equivalent to

$$\left(\sqrt{a_{1,1}m_{1,1}} + \sqrt{a_{2,2}m_{2,2}}\right)^2 - \left(\sqrt{a_{3,3}m_{3,3}} - \sqrt{\det A}\right)^2 > 0 \tag{2.6}$$

and

$$\left(\sqrt{a_{3,3}m_{3,3}} + \sqrt{\det A}\right)^2 - \left(\sqrt{a_{1,1}m_{1,1}} - \sqrt{a_{2,2}m_{2,2}}\right)^2 > 0. \tag{2.7}$$

Expanding these expressions and using (2.4) and (2.5), we see that (2.6) and (2.7) are equivalent to

$$-u_1 - u_2 < v_1 + v_2 < u_1 + u_2,$$

i.e.

$$(v_1 + v_2)^2 < (u_1 + u_2)^2.$$

Next, by (2.2), the above is equivalent to

$$v_1^2 + v_2^2 < u_1^2 + u_2^2,$$

i.e. condition (ii) in Theorem 2.2. This completes the proof. □

Example 2.1. *Consider the matrix*

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 6 & 2 & -1 \\ -6 & -1 & 3 \end{bmatrix}.$$

It is clear that A is a P-matrix. Additionally, we have

$$A^{-1} = \frac{1}{53}\begin{bmatrix} 5 & 7 & -1 \\ -12 & 15 & 13 \\ 6 & 19 & 20 \end{bmatrix}.$$

Thus, we find that $\omega_1 = \sqrt{1 \cdot \frac{5}{53}} = 0.307$, $\omega_2 = \sqrt{2 \cdot \frac{15}{53}} = 0.752$, *and* $\omega_3 = \sqrt{3 \cdot \frac{20}{53}} = 1.064$. *Since* $\max\{1, \omega_1, \omega_2, \omega_3\} = 1.064 < \frac{1}{2}(1 + \omega_1 + \omega_2 + \omega_3) = 1.562$, *we conclude from Theorem 1.1 that A is indeed diagonally stable. In fact, the matrix* $D = \begin{bmatrix} 1 & & \\ & 0.9 & \\ & & 0.65 \end{bmatrix}$ *satisfies the condition that* $DA + A^T D$ *is positive definite.*

3. Conclusions

Explicit necessary and sufficient conditions for the diagonal stability are very interesting research problems due to their simplicity to verify. However, no such results exist for matrices beyond 3×3. In this short paper, we provide a shorter and more transparent proof of a well-known explicit characterization for the 3×3 case [7]. Our new proof relies on Redheffer's result [9, 11], which has been restated in [10] using the notion of Schur complements. Furthermore, we derive additional equivalent explicit conditions for the diagonal stability of a matrix $A \in \mathbb{R}^{3\times3}$. A natural direction for future work is developing similar results for higher dimension matrices.

Author contributions

Ali Algefary: Methodology, Formal analysis, Writing-original draft preparation,Writing-review and editing; Jianhong Xu: Formal analysis, Writing-original draft, Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors do not have any conflict of interest.

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