AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 25184–25204.
DOI: 10.3934/math.20241228
Received: 24 May 2024
Revised: 08 July 2024
Accepted: 26 July 2024
Published: 28 August 2024

Research article

Efficiency conditions in multiple-objective optimal control models under generalized hypotheses

Savin Treanţă[1,2,3,*], Cristina-Florentina Marghescu[1] and Laura-Gabriela Matei[4]

[1] Department of Applied Mathematics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania
[2] Academy of Romanian Scientists, 54 Splaiul Independentei, 050094 Bucharest, Romania
[3] Fundamental Sciences Applied in Engineering-Research Center (SFAI), National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania
[4] Department of Mathematics-Informatics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania

* **Correspondence:** Email: savin.treanta@upb.ro.

Abstract: Since not every problem in optimization theory involves convex functionals, in this study, we introduced new classes of generalized convex functionals. More precisely, under generalized hypotheses, we stated new efficiency conditions associated with a class of multiple-objective optimal control models. To this end, we first defined the G_θ-Fritz John problem and, by considering it, we established a link between the solutions of G_θ-Fritz John problem and efficient solutions of the considered model (P). In addition, we formulated the G_θ-necessary efficiency conditions for a feasible solution in (P). After that, we established a connection between the newly defined concept of $G_\theta - KT$ points to (P) and the efficient solutions of (P). Finally, we turned our attention to the G_θ-sufficient efficiency conditions for a feasible solution to (P). More precisely, we established that any feasible solution to (P) will be an efficient solution if the assumption of G_θ-convexity (and/or G_θ-quasiconvexity, G_θ-strictly quasiconvexity, G_θ-monotonic quasiconvexity) is imposed on the involved functionals.

Keywords: multiple-objective optimal control model; efficiency conditions; generalized hypotheses
Mathematics Subject Classification: 65K10, 26B25, 49K20, 90C30

1. Introduction

The importance of convexity in optimization theory has been well established over time. However, as we all know, the concept of convexity is no longer adequate for many mathematical models in engineering, economics, decision sciences, and mechanics. Thus, in this paper, we will consider a multiple-objective optimal control model determined by not necessary convex functionals. The crucial role of multi-objective optimization problems is well-known. Multi-objective programming problems are used to solve a wide range of real-world issues, such as those in engineering, finance, and production planning. These types of problems have become the focus of increasing amounts of research over the years in a variety of mathematical fields, including optimal control theory, game theory, statistics, and finance. Important duality theorems and necessary and sufficient optimality criteria for multi-objective variational problems have been thoroughly studied by a large number of scholars (see, for instance, Bector and Husain [8], Bhatia and Kumar [9], and Gulati and Mehndiratta [11], Arana-Jiménez et al. [6], Yu and Lu [31]). Moreover, necessary and sufficient efficiency criteria for multi-objective fractional control models involving multiple integrals were established by Mititelu and Treanţă [21].

Over time, many researchers have focused on optimization issues including uncertainty, because the empirical mechanisms are highly complex and often entail uncertainty in the original data. Robustness in optimization problems generated by curvilinear integrals with applications in mechanics was examined by Treanţă and Das [27]. Also, Treanţă [28] discussed robust saddle-point criterion in second-order partial differential equations and partial differential equation constrained-control problems. For a given multi-time control problem with data uncertainty, Baranwal et al. [7] constructed two significant dual models in the literature (namely, Mond-Weir and Wolfe-type duals), and established the corresponding robust duality theorems. The necessary and sufficient optimality hypotheses for a variational control problem embracing data uncertainty were recently stated in Treanţă [29].

Controlled optimization problems are fundamental in many fields of operations research, such as control of space structures, light control design, or production control. Several academics have contributed to formulating and investigating the optimality conditions for some classes of controlled optimization models with equality, inequality, or isoperimetric constraints, inspired by the practical viewpoints of the controlled optimization problem (see, for example, Jacobson et al. [15], Urziceanu [30], Jayswal et al. [16], Malanowski [19]). Treanţă [26], under the premise of KT-invexity, established that every Kuhn-Tucker point must be an optimal solution to the considered control model (see de Oliveira et al. [22], Arana-Jiménez et al. [5]). This technique sparked numerous efforts to investigate optimality criteria in the context of optimization problems. For more information in this direction, the reader can consult Mititelu [20], Prusinska and Tretyakov [23], Soolaki et al. [24], Bhushan et al. [10], Almetwally et al. [1], Anchitaalagammai et al. [2], Li and Guo [12, 13, 17], and references therein.

Since not every problem in optimization theory involves convex functionals, in this study, we introduce new classes of generalized convex functionals. More precisely, in accordance to Antczak [3, 4], Linh and Penot [18], and Gupta and Srivastava [14], we formulate the concepts of G_θ-convexity, G_θ-quasiconvexity, G_θ-strictly quasiconvexity, and G_θ-monotonic quasiconvexity associated with multiple integral type functionals. By considering these new theoretical elements, we study through various techniques (by using the newly defined concepts of G_θ-Fritz John problem and $G_\theta - KT$ points) the efficiency criteria for a multi-dimensional first-order PDE-constrained optimal control problem. In addition to the novelty elements mentioned above, the authors formulate an illustrative example of a real-valued double integral functional that is G_θ-convex but not convex at a given point.

As a consequence, it is evident that every convex functional is G_θ-convex, and in this case, G_θ is taken to be the identity map.

The structure of the article is as follows. Section 2 introduces notations, definitions, problem formulation, and preliminary results. Section 3 contains the G_θ-Fritz John problem and, by considering it, we establish a link between the solutions of G_θ-Fritz John problem and efficient solutions of the considered model (P). In addition, we formulate the G_θ-necessary efficiency conditions for a feasible solution in (P). After that, in Section 4, we establish a connection between the newly defined concept of $G_\theta - KT$ points to (P) and the efficient solutions of (P). Finally, in Section 5, we turn our attention to the G_θ-sufficient efficiency conditions for a feasible solution to (P). More precisely, we establish that any feasible solution to (P) will be an efficient solution if the assumption of G_θ-convexity (and/or G_θ-quasiconvexity, G_θ-strictly quasiconvexity, G_θ-monotonic quasiconvexity) is imposed on the involved functionals. Finally, we wrap up the paper in Section 6.

2. Problem formulation and preliminary results

This section consists of some basic notations and notions that will be helpful in the formulation of the problem and set up the main results:

- Consider the three finite dimensional Euclidean spaces $\mathbb{R}^m$, $\mathbb{R}^n$, and $\mathbb{R}^k$; also, let $x = (x^\alpha), \alpha = \overline{1,m}$, $g = \left(g^i\right), i = \overline{1,n}$, and $f = (f^j), j = \overline{1,k}$ are the local coordinates of $\mathbb{R}^m$, $\mathbb{R}^n$, and $\mathbb{R}^k$, respectively.
- Let $\mathcal{K} = \mathcal{K}_{x_0,x_1} \subset \mathbb{R}^m$ be a compact subset [for instance, a hyperparallelepiped fixed by the diagonally opposite points $x_0 = \left(x_0^\alpha\right)$ and $x_1 = \left(x_1^\alpha\right)$] in $\mathbb{R}^m$; also, let $dx = dx^1 dx^2 \cdots dx^m$ be the volume element in $\mathbb{R}^m \supset \mathcal{K}$.
- Let $\mathcal{G}$ be the space of piecewise smooth state functions $g : \mathcal{K} \mapsto \mathbb{R}^n$ endowed with norm

$$\|g\| = \|g\|_\infty + \sum_{\alpha=1}^{m} \|g_\alpha\|_\infty ,$$

  where $\|g\|_\infty = \max\left(\left|g^1\right|, \left|g^2\right|, \ldots, |g^n|\right)$ and $g_\alpha = \frac{\partial g}{\partial x^\alpha}$; also, let $\mathcal{F}$ be the space of continuous control functions $f : \mathcal{K} \mapsto \mathbb{R}^k$, equipped with the uniform norm.
- For any two points $a = \left(a^\ell\right)$ and $b = \left(b^\ell\right)$ in $\mathbb{R}^l$, the following convention will be used in this paper:

$$a = b \Leftrightarrow a^\ell = b^\ell, \quad a \leq b \Leftrightarrow a^\ell \leq b^\ell, \quad a < b \Leftrightarrow a^\ell < b^\ell,$$

$$a \preceq b \Leftrightarrow a \leq b,\ a \neq b, \quad \ell = \overline{1,l}.$$

Definition 2.1. A function $h : \mathbb{R} \mapsto \mathbb{R}$ is said to be a *strictly increasing function* if the following implication is satisfied:

$$u < e \Rightarrow h(u) < h(e), \quad \forall u, e \in \mathbb{R}.$$

Considering the above-mentioned mathematical elements, we formulate the following multi-dimensional first-order PDE-constrained control model:

$$\text{(P)} \quad \min_{(g,f)} \Big\{ \int_{\mathcal{K}} \theta(x, g(x), f(x)) dx$$

$$= \left(\int_{\mathcal{K}} \theta^1(x, g(x), f(x))dx, \cdots, \int_{\mathcal{K}} \theta^w(x, g(x), f(x))dx \right) \Big\}, \tag{1}$$

subject to

$$\frac{\partial g^i}{\partial x^\alpha}(x) = \mathcal{U}^i_\alpha(x, g(x), f(x)), \quad \alpha = \overline{1, m},\ i = \overline{1, n}, \tag{2}$$

$$\mathcal{W}_\beta(x, g(x), f(x)) \leq 0, \quad \beta \in \mathcal{Q} = \overline{1, q}, \tag{3}$$

$$g(x_0) = g_0, \quad g(x_1) = g_1, \tag{4}$$

where $x \in \mathcal{K},\ \theta = (\theta^\kappa) : \mathcal{K} \times \mathcal{G} \times \mathcal{F} \mapsto \mathbb{R}^w, \kappa = \overline{1, w},\ \mathcal{W}_\beta : \mathcal{K} \times \mathcal{G} \times \mathcal{F} \mapsto \mathbb{R}, \beta \in \mathcal{Q} = \overline{1, q},\ \mathcal{U}_\alpha = \left(\mathcal{U}^i_\alpha\right) : \mathcal{K} \times \mathcal{G} \times \mathcal{F} \mapsto \mathbb{R}^n, \alpha = \overline{1, m}$, are continuously differentiable functionals. Also, we assume the constraints $\mathcal{U}_\alpha$ satisfy the complete integrability conditions (closeness conditions) $D_\nu \mathcal{U}_\alpha = D_\alpha \mathcal{U}_\nu,\ \alpha, \nu = \overline{1, m},\ \alpha \neq \nu$, where D_ν is the total derivative.

Let $\mathcal{D} = \{(g, f) \in \mathcal{G} \times \mathcal{F} \mid (g, f)$ satisfying the conditions (2)–(4)$\}$ be the set of all feasible solutions to (P). Also, let $\mathcal{Q}(\bar{g}, \bar{f})$ denote the set of indices of active constraints at $(\bar{g}, \bar{f})$, that is,

$$\mathcal{Q}(\bar{g}, \bar{f}) = \left\{ \beta \in \mathcal{Q} : \mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x)) = 0 \right\}.$$

Definition 2.2. A point $(\bar{g}, \bar{f}) \in \mathcal{D}$ is said to be an *efficient solution* to the multidimensional first-order PDE-constrained control problem (P) if, for all $(g, f) \in \mathcal{D}$, we have

$$\int_{\mathcal{K}} \theta(x, \bar{g}(x), \bar{f}(x))dx \leq \int_{\mathcal{K}} \theta(x, g(x), f(x))dx.$$

Definition 2.3. A point $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$ is said to be a *stationary point* of the vector functional θ if

$$\frac{\partial \theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x)) + \frac{\partial \theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x)) = 0,$$

for all $(g, f) \in \mathcal{G} \times \mathcal{F}$.

Definition 2.4. Let $\mathcal{G} \times \mathcal{F}$ be a convex set and let $\theta : \mathcal{K} \times \mathcal{G} \times \mathcal{F} \mapsto \mathbb{R}^w$ be a continuously differentiable functional. Then, the vector-valued multiple integral functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *convex at* $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$ if the following inequality

$$\int_{\mathcal{K}} \theta(x, g(x), f(x))dx - \int_{\mathcal{K}} \theta(x, \bar{g}(x), \bar{f}(x))dx$$

$$\geq \int_{\mathcal{K}} \left[\frac{\partial \theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x)) + \frac{\partial \theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x)) \right] dx,$$

is satisfied for all $(g, f) \in \mathcal{G} \times \mathcal{F}$. If the above inequality is satisfied for all $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$, then the functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *convex on* $\mathcal{G} \times \mathcal{F}$.

Definition 2.5. Let $\mathcal{G} \times \mathcal{F}$ be a convex set and let $\theta : \mathcal{K} \times \mathcal{G} \times \mathcal{F} \mapsto \mathbb{R}^w$ be a continuously differentiable functional. Then, the vector-valued multiple integral functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *quasiconvex at* $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$ if the following inequality

$$\int_{\mathcal{K}} \theta(x, g(x), f(x))dx \leq \int_{\mathcal{K}} \theta(x, \bar{g}(x), \bar{f}(x))dx,$$

implies

$$\int_{\mathcal{K}} \left[\frac{\partial \theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x)) + \frac{\partial \theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x)) \right] dx \leq 0,$$

for all $(g, f) \in \mathcal{G} \times \mathcal{F}$. If the above implication is satisfied for all $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$, then the functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *quasiconvex on* $\mathcal{G} \times \mathcal{F}$.

Definition 2.6. Let $\mathcal{G} \times \mathcal{F}$ be a convex set and let $\theta : \mathcal{K} \times \mathcal{G} \times \mathcal{F} \mapsto \mathbb{R}^w$ be a continuously differentiable functional. Then, the vector-valued multiple integral functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *strictly quasiconvex at* $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$ if the following inequality

$$\int_{\mathcal{K}} \theta(x, g(x), f(x))dx \leq \int_{\mathcal{K}} \theta(x, \bar{g}(x), \bar{f}(x))dx,$$

implies

$$\int_{\mathcal{K}} \left[\frac{\partial \theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x)) + \frac{\partial \theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x)) \right] dx < 0,$$

for all $(g, f) \in \mathcal{G} \times \mathcal{F}$. If the above implication is satisfied for all $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$, then the functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *strictly quasiconvex on* $\mathcal{G} \times \mathcal{F}$.

Definition 2.7. Let $\mathcal{G} \times \mathcal{F}$ be a convex set and let $\theta : \mathcal{K} \times \mathcal{G} \times \mathcal{F} \mapsto \mathbb{R}^w$ be a continuously differentiable functional. Then, the vector-valued multiple integral functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *monotonic quasiconvex at* $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$ if the following equality

$$\int_{\mathcal{K}} \theta(x, g(x), f(x))dx = \int_{\mathcal{K}} \theta(x, \bar{g}(x), \bar{f}(x))dx,$$

implies

$$\int_{\mathcal{K}} \left[\frac{\partial \theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x)) + \frac{\partial \theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x)) \right] dx = 0,$$

for all $(g, f) \in \mathcal{G} \times \mathcal{F}$. If the above implication is satisfied for all $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$, then the functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *monotonic quasiconvex on* $\mathcal{G} \times \mathcal{F}$.

Now, on the lines of Antczak [3, 4], Linh and Penot [18], and Gupta and Srivastava [14], we introduce the concept of G_θ-convexity for a vector-valued multiple integral functional. In a similar way, we can introduce the concepts of G_θ-quasiconvexity, G_θ-strictly quasiconvexity, or G_θ-monotonic quasiconvexity.

Definition 2.8. Let $\mathcal{G} \times \mathcal{F}$ be a convex set, $\theta : \mathcal{K} \times \mathcal{G} \times \mathcal{F} \mapsto \mathbb{R}^w$ a continuously differentiable functional, and $G_\theta : I_\theta \mapsto \mathbb{R}^w$ a strictly increasing vector-valued differentiable function, where $I_\theta \subset \mathbb{R}^w$ denote the range of θ. Then, the vector-valued multiple integral functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *G_θ-convex at* $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$ if the following inequality

$$\int_{\mathcal{K}} G_\theta(\theta(x, g(x), f(x)))dx - \int_{\mathcal{K}} G_\theta(\theta(x, \bar{g}(x), \bar{f}(x)))dx \tag{5}$$

$$\geq \int_{\mathcal{K}} \Bigg[G'_\theta(\theta(x, \bar{g}(x), \bar{f}(x))) \bigg(\frac{\partial \theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x))$$

$$+ \frac{\partial \theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x)) \bigg) \Bigg] dx, \quad \forall (g, f) \in \mathcal{G} \times \mathcal{F},$$

is satisfied, with $G_\theta = (G_\theta^\kappa)$, $G'_\theta = \frac{\partial G_\theta^\kappa}{\partial \theta^\kappa}$, $\theta = (\theta^\kappa)$, $\kappa = \overline{1, w}$. If inequality (5) is satisfied for any $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$, then the vector-valued multiple integral functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *G_θ-convex on* $\mathcal{G} \times \mathcal{F}$.

In the example given below, we consider a real-valued double integral functional and we show that it is G_θ-convex but not convex at a given point. As a consequence, it is evident that every convex functional is G_θ-convex, and in this case, G_θ is taken to be the identity map.

Example 2.1. Let $w = 1$, $\mathcal{G} = \{g : \mathcal{K}_{x_0, x_1} \to [-2, 2] \subset \mathbb{R}\}$, $\mathcal{F} = \{f : \mathcal{K}_{x_0, x_1} \to [-1, 1] \subset \mathbb{R}\}$, where $\mathcal{K}_{x_0, x_1}$ is a square fixed with the diagonally opposite points $x_0 = \left(x_0^1, x_0^2\right) = (-2, -2)$ and $x_1 = \left(x_1^1, x_1^2\right) = (2, 2)$ in $\mathbb{R}^2$. Now, we consider the real-valued functional $\theta : \mathcal{K}_{x_0, x_1} \times \mathcal{G} \times \mathcal{F} \mapsto \mathbb{R}$, defined by $\theta(x, g(x), f(x)) = \ln[g(x) + f(x) + 5]$, that generates the real-valued double integral functional

$$\Theta : \mathcal{G} \times \mathcal{F} \to \mathbb{R},$$

$$\Theta(g, f) = \int_{\mathcal{K}_{x_0, x_1}} \theta(x, g(x), f(x))dx^1 dx^2 = \int_{\mathcal{K}_{x_0, x_1}} \ln[g(x) + f(x) + 5]dx^1 dx^2. \tag{6}$$

Also, let us define

$$\bar{g}(x) = \frac{x^1 + x^2}{2}, \ \bar{f}(x) = \frac{2x^1 + x^2}{6}, \quad x = (x^1, x^2) \in \mathcal{K}_{x_0, x_1},$$

and consider $x^1 = x^2 = 1$. We have

$$\int_{\mathcal{K}_{x_0, x_1}} \theta(x, g(x), f(x))dx^1 dx^2 - \int_{\mathcal{K}_{x_0, x_1}} \theta(x, \bar{g}(x), \bar{f}(x))dx^1 dx^2$$

$$- \int_{\mathcal{K}_{x_0, x_1}} \Big[\frac{\partial \theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x))$$

$$+ \frac{\partial \theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x)) \Big] dx^1 dx^2$$

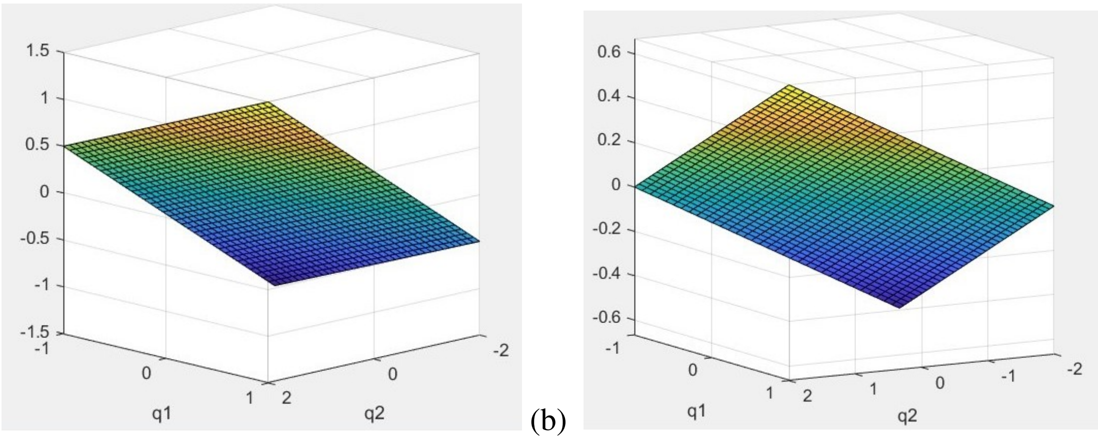
$$= \int_{\mathcal{K}_{x_0, x_1}} \left(\ln[g(x) + f(x) + 5] - \ln \frac{13}{2} - \frac{2}{13} \left[g(x) + f(x) - \frac{3}{2} \right] \right) dx^1 dx^2$$

$$\ngeq 0, \quad \forall (g, f) \in \mathcal{G} \times \mathcal{F} \text{ (see Figure1(a))}.$$

Thus, the above-mentioned real-valued double integral functional is not convex at $\left(1, \frac{1}{2}\right) \in \mathcal{G} \times \mathcal{F}$.

On the other hand, if we consider the strictly increasing function $G_\theta : I_\theta \mapsto \mathbb{R}$ defined by $G_\theta(\theta(x, g(x), f(x))) = e^{2\theta(x,g(x),f(x))}$, then we obtain

$$\int_{\mathcal{K}_{x_0,x_1}} G_\theta(\theta(x, g(x), f(x)))dx^1dx^2 - \int_{\mathcal{K}_{x_0,x_1}} G_\theta\left(\theta(x, \bar{g}(x), \bar{f}(x))dx^1dx^2\right.$$
$$- \int_{\mathcal{K}_{x_0,x_1}} \left[G'_\theta(\theta(x, \overline{g}(x), \overline{f}(x)))\left(\frac{\partial\theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x))\right.\right.$$
$$\left.\left.+\frac{\partial\theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x))\right)\right]dx^1dx^2$$
$$= \int_{\mathcal{K}_{x_0,x_1}} \left((g(x) + f(x) + 5)^2 - \left(\frac{13}{2}\right)^2 - 13\left[g(x) + f(x) - \frac{3}{2}\right]\right)dx^1dx^2$$
$$\geq 0, \quad \forall(g, f) \in \mathcal{G} \times \mathcal{F} \text{ (see Figure 1(b)).}$$



(a) (b)

Figure 1. Convexity and G_θ-convexity associated with $\Theta(g, f)$.

Thus, the real-valued double integral functional given in (6) is G_θ-convex at $\left(1, \frac{1}{2}\right) \in \mathcal{G}\times\mathcal{F}$. Hence, we have shown that the real-valued double integral functional $\Theta(g, f) = \int_{\mathcal{K}_{x_0,x_1}} \theta(x, g(x), f(x))dx^1dx^2$ is G_θ-convex at $\left(1, \frac{1}{2}\right) \in \mathcal{G} \times \mathcal{F}$, but not convex at $\left(1, \frac{1}{2}\right) \in \mathcal{G} \times \mathcal{F}$.

Definition 2.9. Let $\mathcal{G}\times\mathcal{F}$ be a convex set, $\theta : \mathcal{K}\times\mathcal{G}\times\mathcal{F} \mapsto \mathbb{R}^w$ a continuously differentiable functional, and $G_\theta : I_\theta \mapsto \mathbb{R}^w$ a strictly increasing vector-valued differentiable function, where $I_\theta \subset \mathbb{R}^w$ denote the range of θ. Then, the vector-valued multiple integral functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is said to be *G_θ-quasiconvex at* $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$ if the following inequality

$$\int_{\mathcal{K}} G_\theta(\theta(x, g(x), f(x)))dx \leq \int_{\mathcal{K}} G_\theta(\theta(x, \bar{g}(x), \bar{f}(x)))dx,$$

implies

$$\int_{\mathcal{K}} \left[G'_\theta(\theta(x, \overline{g}(x), \overline{f}(x)))\left(\frac{\partial\theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x))\right.\right.$$

$$+\frac{\partial\theta}{\partial f(x)}(x,\bar{g}(x),\bar{f}(x))(f(x)-\bar{f}(x))\Bigg)\Bigg]dx \leq 0, \quad \forall (g,f)\in\mathcal{G}\times\mathcal{F},$$

with $G_\theta = (G_\theta^\kappa),\ G'_\theta = \frac{\partial G_\theta^\kappa}{\partial\theta^\kappa},\ \theta = (\theta^\kappa),\ \kappa = \overline{1,w}$. If the above inequality is satisfied for any $(\bar{g},\bar{f})\in\mathcal{G}\times\mathcal{F}$, then the vector-valued multiple integral functional $\Theta(g,f) = \int_{\mathcal{K}}\theta(x,g(x),f(x))dx$ is said to be *G_θ-quasiconvex on $\mathcal{G}\times\mathcal{F}$*.

Definition 2.10. Let $\mathcal{G}\times\mathcal{F}$ be a convex set, $\theta : \mathcal{K}\times\mathcal{G}\times\mathcal{F} \mapsto \mathbb{R}^w$ a continuously differentiable functional, and $G_\theta : I_\theta \mapsto \mathbb{R}^w$ a strictly increasing vector-valued differentiable function, where $I_\theta \subset \mathbb{R}^w$ denote the range of θ. Then, the vector-valued multiple integral functional $\Theta(g,f) = \int_{\mathcal{K}}\theta(x,g(x),f(x))dx$ is said to be *G_θ-strictly quasiconvex at* $(\bar{g},\bar{f})\in\mathcal{G}\times\mathcal{F}$ if the following inequality

$$\int_{\mathcal{K}} G_\theta(\theta(x,g(x),f(x)))dx \leq \int_{\mathcal{K}} G_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))dx,$$

implies

$$\int_{\mathcal{K}}\Bigg[G'_\theta(\theta(x,\overline{g}(x),\overline{f}(x)))\Bigg(\frac{\partial\theta}{\partial g(x)}(x,\bar{g}(x),\bar{f}(x))(g(x)-\bar{g}(x))$$
$$+\frac{\partial\theta}{\partial f(x)}(x,\bar{g}(x),\bar{f}(x))(f(x)-\bar{f}(x))\Bigg)\Bigg]dx < 0, \quad \forall (g,f)\in\mathcal{G}\times\mathcal{F},$$

with $G_\theta = (G_\theta^\kappa),\ G'_\theta = \frac{\partial G_\theta^\kappa}{\partial\theta^\kappa},\ \theta = (\theta^\kappa),\ \kappa = \overline{1,w}$. If the above inequality is satisfied for any $(\bar{g},\bar{f})\in\mathcal{G}\times\mathcal{F}$, then the vector-valued multiple integral functional $\Theta(g,f) = \int_{\mathcal{K}}\theta(x,g(x),f(x))dx$ is said to be *G_θ-strictly quasiconvex on $\mathcal{G}\times\mathcal{F}$*.

Definition 2.11. Let $\mathcal{G}\times\mathcal{F}$ be a convex set, $\theta : \mathcal{K}\times\mathcal{G}\times\mathcal{F} \mapsto \mathbb{R}^w$ a continuously differentiable functional, and $G_\theta : I_\theta \mapsto \mathbb{R}^w$ a strictly increasing vector-valued differentiable function, where $I_\theta \subset \mathbb{R}^w$ denote the range of θ. Then, the vector-valued multiple integral functional $\Theta(g,f) = \int_{\mathcal{K}}\theta(x,g(x),f(x))dx$ is said to be *G_θ-monotonic quasiconvex at* $(\bar{g},\bar{f})\in\mathcal{G}\times\mathcal{F}$ if the following inequality

$$\int_{\mathcal{K}} G_\theta(\theta(x,g(x),f(x)))dx = \int_{\mathcal{K}} G_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))dx,$$

implies

$$\int_{\mathcal{K}}\Bigg[G'_\theta(\theta(x,\overline{g}(x),\overline{f}(x)))\Bigg(\frac{\partial\theta}{\partial g(x)}(x,\bar{g}(x),\bar{f}(x))(g(x)-\bar{g}(x))$$
$$+\frac{\partial\theta}{\partial f(x)}(x,\bar{g}(x),\bar{f}(x))(f(x)-\bar{f}(x))\Bigg)\Bigg]dx = 0, \quad \forall (g,f)\in\mathcal{G}\times\mathcal{F},$$

with $G_\theta = (G_\theta^\kappa),\ G'_\theta = \frac{\partial G_\theta^\kappa}{\partial\theta^\kappa},\ \theta = (\theta^\kappa),\ \kappa = \overline{1,w}$. If the above inequality is satisfied for any $(\bar{g},\bar{f})\in\mathcal{G}\times\mathcal{F}$, then the vector-valued multiple integral functional $\Theta(g,f) = \int_{\mathcal{K}}\theta(x,g(x),f(x))dx$ is said to be *G_θ-monotonic quasiconvex on $\mathcal{G}\times\mathcal{F}$*.

In the next theorem, we establish a connection between stationary points and minimum points associated with a G_θ-convex vector-valued multiple integral functional.

Theorem 2.1. *If the vector-valued multiple integral type functional*

$$\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx,$$

is G_θ-convex on $\mathcal{G} \times \mathcal{F}$, then every stationary point of θ is global minimum in $\mathcal{G} \times \mathcal{F}$ for Θ.

Proof. Let $(\bar{g}, \bar{f}) \in \mathcal{G} \times \mathcal{F}$ be a stationary point of the vector functional θ. Then, we have

$$\int_{\mathcal{K}} \Big[G'_\theta(\theta(x, \bar{g}(x), \bar{f}(x)))\Big(\frac{\partial\theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x)) \tag{7}$$

$$+\frac{\partial\theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x))\Big)\Big]dx = 0, \quad \forall (g, f) \in \mathcal{G} \times \mathcal{F}.$$

From the assumption that the vector-valued multiple integral type functional $\Theta(g, f) = \int_{\mathcal{K}} \theta(x, g(x), f(x))dx$ is G_θ-convex on $\mathcal{G} \times \mathcal{F}$, we have

$$\begin{aligned}
&\int_{\mathcal{K}} G_\theta(\theta(x, g(x), f(x)))dx - \int_{\mathcal{K}} G_\theta(\theta(x, \bar{g}(x), \bar{f}(x)))dx \\
&\geq \int_{\mathcal{K}} \Bigg[G'_\theta(\theta(x, \bar{g}(x), \bar{f}(x)))\Bigg(\frac{\partial\theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x)) \\
&\quad +\frac{\partial\theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x))\Bigg)\Bigg] dx, \quad \forall (g, f) \in \mathcal{G} \times \mathcal{F}.
\end{aligned}$$

By using the Eq (7), we get

$$\int_{\mathcal{K}} G_\theta(\theta(x, g(x), f(x)))dx \geq \int_{\mathcal{K}} G_\theta(\theta(x, \bar{g}(x), \bar{f}(x)))dx, \quad \forall (g, f) \in \mathcal{G} \times \mathcal{F}.$$

Since G_θ is an increasing function, we obtain

$$\int_{\mathcal{K}} \theta(x, g(x), f(x))dx \geq \int_{\mathcal{K}} \theta(x, \bar{g}(x), \bar{f}(x))dx, \quad \forall (g, f) \in \mathcal{G} \times \mathcal{F},$$

which concludes that $(\bar{g}, \bar{f})$ is a global minimum of the functional $\int_{\mathcal{K}} \theta(x, g(x), f(x))dx$. This completes the proof. □

3. Efficiency in (P) via G_θ-Fritz John problem

In this section, we first define the G_θ-Fritz John problem and, by considering it, we establish a link between the solutions of G_θ-Fritz John problem and the efficient solutions of (P). In addition, we formulate the G_θ-necessary efficiency conditions for a feasible solution in (P).

*G_θ-**Fritz John problem.*** *If it exists, find the point* $(\bar{g}(x), \bar{f}(x), \sigma, \lambda(x), \mu(x)) \in \mathcal{D} \times R_+^w \times \mathbb{R}^{nm} \times R_+^q$ *such that (with summation over repeated indices):*

$$\sigma^\kappa G_\theta'(\theta(x, \bar{g}(x), \bar{f}(x)))\frac{\partial \theta}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x))$$

$$+\lambda_i^\alpha(x) G'_{\mathcal{U}_\alpha^i}\left(\mathcal{U}_\alpha^i(x, \bar{g}(x), \bar{f}(x))\right)\frac{\partial \mathcal{U}_\alpha^i}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x))$$

$$+\mu^\beta(x) G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x))\right)\frac{\partial \mathcal{W}_\beta}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x)) + \frac{\partial \lambda_i^\alpha}{\partial x^\alpha} = 0, \quad i = \overline{1, n}, \tag{8}$$

$$\sigma^\kappa G_\theta'(\theta(x, \bar{g}(x), \bar{f}(x)))\frac{\partial \theta}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x))$$

$$+\lambda_i^\alpha(x) G'_{\mathcal{U}_\alpha^i}\left(\mathcal{U}_\alpha^i(x, \bar{g}(x), \bar{f}(x))\right)\frac{\partial \mathcal{U}_\alpha^i}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x))$$

$$+\mu^\beta(x) G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x))\right)\frac{\partial \mathcal{W}_\beta}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x)) = 0, \quad j = \overline{1, k}, \tag{9}$$

$$\mu^\beta(x)\left[G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x, g(x), f(x))\right) - G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x))\right)\right] \leq 0, \tag{10}$$

$$\beta \in \mathcal{Q},\ (g, f) \in \mathcal{D},\ (\textit{G-complementary slackness condition}),$$

$$\sigma \geq 0,\ \mu(x) \geq 0,\ (\sigma, \mu(x)) \neq (0, 0), \tag{11}$$

are satisfied, for all $x \in \mathcal{K}$, except at discontinuity points, where $G_\theta, G_{\mathcal{U}_\alpha^i}$, and $G_{\mathcal{W}_\beta}$ are differentiable increasing functions defined on I_θ (the image set of $\theta(x, g(x), f(x))$), $I_{\mathcal{U}_\alpha^i}$ (the image set of $\mathcal{U}_\alpha^i(x, g(x), f(x))$), and $I_{\mathcal{W}_\beta}$ (the image set of $\mathcal{W}_\beta(x, g(x), f(x))$), respectively.

Definition 3.1. We say the point $(\bar{g}(x), \bar{f}(x), \sigma, \lambda(x), \mu(x)) \in \mathcal{D} \times R_+^w \times \mathbb{R}^{nm} \times R_+^q$ is a *solution of the G_θ-Fritz John problem* if it satisfies the conditions (8) to (11).

Lemma 3.1. *Let $(\bar{g}, \bar{f}) \in \mathcal{D}$ and $\mu(x) \in R_+^q$ satisfy the classical slackness condition $\mu^\beta(x)\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x)) = 0$. Then, $(\bar{g}, \bar{f}) \in \mathcal{D}$ and $\mu(x) \in R_+^q$ also fulfil the G-complementary slackness condition, for all $(g, f) \in \mathcal{D}$, $\beta \in \mathcal{Q}$.*

Proof. If $\beta \notin \mathcal{Q}(\bar{g}, \bar{f})$, then from classical slackness condition it follows that $\mu^\beta(x) = 0$. Thus, the G-complementary slackness condition holds. If $\mu^\beta(x) > 0$, then again from classical slackness condition, we have $\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x)) = 0$, and therefore

$$\mathcal{W}_\beta(x, g(x), f(x)) \leq \mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x)) = 0, \quad \forall (g, f) \in \mathcal{D}.$$

Since $G_{\mathcal{W}_\beta}$ with $\beta \in \mathcal{Q}$ is an increasing function on $I_{\mathcal{W}_\beta}$, from the above inequality we get

$$G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x, g(x), f(x))\right) \leq G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x))\right), \quad \forall (g, f) \in \mathcal{D}.$$

Further, since $\mu^\beta(x) \geq 0,\ \beta \in \mathcal{Q}$, we have

$$\mu^\beta(x) G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x, g(x), f(x))\right) \leq \mu^\beta(x) G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x))\right), \quad \forall (g, f) \in \mathcal{D},$$

or, equivalently,

$$\mu^{\beta}(x)\left[G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,g(x),f(x))\right)-G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\right]\leq 0,\quad \forall(g,f)\in\mathcal{D},$$

and this completes the proof. □

In the next theorem, we establish a connection between solutions of the G_θ-Fritz John problem and the efficient solutions of (P).

Theorem 3.1. *If $(\bar{g},\bar{f})\in\mathcal{D}$ is an efficient solution to (P), then there exists $\sigma\in R^w_+$ and piecewise smooth functions $\lambda(x)\in\mathbb{R}^{nm}$, $\mu(x)\in R^q_+$ such that the point*

$$(\bar{g}(x),\bar{f}(x),\sigma,\lambda(x),\mu(x)),$$

is a solution to the G_θ-Fritz John problem.

Proof. The proof given below follows the same line as in Treanţă [29]. Let $(g,f)\in\mathcal{D}$ and the vector differentiable functions $p(x)\in\mathbb{R}^n$ and $q(x)\in\mathbb{R}^k$, such that $p|_{\partial\mathcal{K}}=q|_{\partial\mathcal{K}}=0$ (where $\partial\mathcal{K}$ denotes the boundary of $\mathcal{K}$). For $\epsilon_1>0$, $\epsilon_2>0$ and for the efficient solution $(\bar{g},\bar{f})$, we consider the ϵ-neighborhood defined by

$$\mathcal{V}_\epsilon=\left\{(g,f)\mid g=\bar{g}+\epsilon_1 p,\ f=\bar{f}+\epsilon_2 q\right\}.$$

Now, from the assumption that $(\bar{g},\bar{f})$ is an efficient solution to (P), we obtain that $(\epsilon_1,\epsilon_2)=(0,0)$ is a minimizer to the following problem:

$$\text{(P1)}\quad \min s(\epsilon_1,\epsilon_2)=\int_{\mathcal{K}}G_\theta\left(\theta\left(x,\bar{g}+\epsilon_1 p,\bar{f}+\epsilon_2 q\right)\right)dx,$$

subject to

$$u^i_\alpha(\epsilon_1,\epsilon_2)=\int_{\mathcal{K}}\left[G_{\mathcal{U}^i_\alpha}\left(\mathcal{U}^i_\alpha\left(x,\bar{g}+\epsilon_1 p,\bar{f}+\epsilon_2 q\right)\right)-\frac{\partial\bar{g}^i}{\partial x^\alpha}-\epsilon_1\frac{\partial p}{\partial x^\alpha}\right]dx=0,$$
$$h_\beta(\epsilon_1,\epsilon_2)=\int_{\mathcal{K}}G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta\left(x,\bar{g}+\epsilon_1 p,\bar{f}+\epsilon_2 q\right)\right)dx\leq 0,$$
$$x\in\mathcal{K},\ p|_{\partial\mathcal{K}}=0,\ q|_{\partial\mathcal{K}}=0.$$

Since $(0,0)$ is a minimizer of (MCP1), then there are the Lagrange multipliers σ^κ, $\lambda^\alpha_i(\bar{g},\bar{f})$ and $\mu^\beta(\bar{g},\bar{f})$ such that the following Fritz John conditions hold at $(0,0)$:

$$\sigma^\kappa\nabla s(0,0)+\lambda^\alpha_i(\bar{g},\bar{f})\nabla u^i_\alpha(0,0)+\mu^\beta(\bar{g},\bar{f})\nabla h_\beta(0,0)=0, \tag{12}$$

$$\mu^\beta(\bar{g},\bar{f})h_\beta(0,0)=0, \tag{13}$$

$$\sigma\geq 0,\ \mu^\beta(\bar{g},\bar{f})\geq 0, \tag{14}$$

where

$$\nabla s(\epsilon_1,\epsilon_2)=\left(\frac{\partial s}{\partial\epsilon_1},\frac{\partial s}{\partial\epsilon_2}\right)=\left(\int_{\mathcal{K}}G'_\theta\frac{\partial\theta}{\partial g^i}p\,dx,\int_{\mathcal{K}}G'_\theta\frac{\partial\theta}{\partial f^j}q\,dx\right),$$

$$\nabla u^i_\alpha(\epsilon_1,\epsilon_2)=\left(\frac{\partial u^i_\alpha}{\partial\epsilon_1},\frac{\partial u^i_\alpha}{\partial\epsilon_2}\right)=\left(\int_{\mathcal{K}}\left(G'_{\mathcal{U}^i_\alpha}\frac{\partial\mathcal{U}^i_\alpha}{\partial g^i}p-\frac{\partial p}{\partial x^\alpha}\right)dx,\int_{\mathcal{K}}G'_{\mathcal{U}^i_\alpha}\frac{\partial\mathcal{U}^i_\alpha}{\partial f^j}q\,dx\right),$$

$$\nabla h_\beta(\epsilon_1,\epsilon_2)=\left(\frac{\partial h_\beta}{\partial\epsilon_1},\frac{\partial h_\beta}{\partial\epsilon_2}\right)=\left(\int_{\mathcal{K}}G'_{\mathcal{W}_\beta}\frac{\partial\mathcal{W}_\beta}{\partial g^i}pdx,\int_{\mathcal{K}}G'_{\mathcal{W}_\beta}\frac{\partial\mathcal{W}_\beta}{\partial f^j}qdx\right).$$

Then, the relation (12) can be written as follows

$$\begin{aligned}&\sigma^\kappa\int_{\mathcal{K}}G'_\theta\frac{\partial\theta}{\partial g^i}pdx+\lambda_i^\alpha(\bar g,\bar f)\int_{\mathcal{K}}\left(G'_{\mathcal{U}^i_\alpha}\frac{\partial\mathcal{U}^i_\alpha}{\partial g^i}p-\frac{\partial p}{\partial x^\alpha}\right)dx\\&+\mu^\beta(\bar g,\bar f)\int_{\mathcal{K}}G'_{\mathcal{W}_\beta}\frac{\partial\mathcal{W}_\beta}{\partial g^i}pdx=0,\\&\sigma^\kappa\int_{\mathcal{K}}G'_\theta\frac{\partial\theta}{\partial f^j}qdx+\lambda_i^\alpha(\bar g,\bar f)\int_{\mathcal{K}}G'_{\mathcal{U}^i_\alpha}\frac{\partial\mathcal{U}^i_\alpha}{\partial f^j}qdx\\&+\mu^\beta(\bar g,\bar f)\int_{\mathcal{K}}G'_{\mathcal{W}_\beta}\frac{\partial\mathcal{W}_\beta}{\partial f^j}qdx=0,\end{aligned}$$

or,

$$\int_{\mathcal{K}}\sigma^\kappa G'_\theta\frac{\partial\theta}{\partial g^i}pdx+\int_{\mathcal{K}}\lambda_i^\alpha\left(G'_{\mathcal{U}^i_\alpha}\frac{\partial\mathcal{U}^i_\alpha}{\partial g^i}p-\frac{\partial p}{\partial x^\alpha}\right)dx+\int_{\mathcal{K}}\mu^\beta G'_{\mathcal{W}_\beta}\frac{\partial\mathcal{W}_\beta}{\partial g^i}pdx=0, \tag{15}$$

$$\int_{\mathcal{K}}\sigma^\kappa G'_\theta\frac{\partial\theta}{\partial f^j}qdx+\int_{\mathcal{K}}\lambda_i^\alpha G'_{\mathcal{U}^i_\alpha}\frac{\partial\mathcal{U}^i_\alpha}{\partial f^j}qdx+\int_{\mathcal{K}}\mu^\beta G'_{\mathcal{W}_\beta}\frac{\partial\mathcal{W}_\beta}{\partial f^j}qdx=0. \tag{16}$$

Since λ_i^α is a differentiable function at $x\in\mathcal{K}$ except at discontinuities, we have

$$\frac{\partial\left(\lambda_i^\alpha p\right)}{\partial x^\alpha}=\frac{\partial\lambda_i^\alpha}{\partial x^\alpha}p+\frac{\partial p}{\partial x^\alpha}\lambda_i^\alpha,$$

involving that

$$\int_{\mathcal{K}}\lambda_i^\alpha\frac{\partial p}{\partial x^\alpha}dx=\int_{\mathcal{K}}\frac{\partial\left(\lambda_i^\alpha p\right)}{\partial x^\alpha}dx-\int_{\mathcal{K}}\frac{\partial\lambda_i^\alpha}{\partial x^\alpha}pdx.$$

We obtain, due to the Gauss-Ostrogradsky formula, the following relation

$$\int_{\mathcal{K}}\frac{\partial\left(\lambda_i^\alpha p\right)}{\partial x^\alpha}dx=\int_{\partial\mathcal{K}}(\lambda_i^\alpha p)\,\vec{v}dx=0,$$

where $\vec{v}$ is the normal unit vector to the boundary $\partial\mathcal{K}$ and $p|_{\partial\mathcal{K}}=0$, implying that

$$\int_{\mathcal{K}}\lambda_i^\alpha\frac{\partial p}{\partial x^\alpha}dx=-\int_{\mathcal{K}}\frac{\partial\lambda_i^\alpha}{\partial x^\alpha}pdx.$$

Using the above equality in Eq (15), we get

$$\int_{\mathcal{K}}\left[\sigma^\kappa G'_\theta\frac{\partial\theta}{\partial g^i}+\lambda_i^\alpha G'_{\mathcal{U}^i_\alpha}\frac{\partial\mathcal{U}^i_\alpha}{\partial g^i}+\mu^\beta G'_{\mathcal{W}_\beta}\frac{\partial\mathcal{W}_\beta}{\partial g^i}+\frac{\partial\lambda_i^\alpha}{\partial x^\alpha}\right]pdx=0.$$

Now, by using a fundamental Lemma of variational calculus, from the above equality it follows

$$\sigma^{\kappa} G'_{\theta} \frac{\partial \theta}{\partial g^i} + \lambda_i^{\alpha} G'_{\mathcal{U}_{\alpha}^i} \frac{\partial \mathcal{U}_{\alpha}^i}{\partial g^i} + \mu^{\beta} G'_{\mathcal{W}_{\beta}} \frac{\partial \mathcal{W}_{\beta}}{\partial g^i} + \frac{\partial \lambda_i^{\alpha}}{\partial x^{\alpha}} = 0.$$

Thus, the condition (8) is fulfilled. Proceeding as above, together with Eq (16), we get

$$\sigma^{\kappa} G'_{\theta} \frac{\partial \theta}{\partial f^j} + \lambda_i^{\alpha} G'_{\mathcal{U}_{\alpha}^i} \frac{\partial \mathcal{U}_{\alpha}^i}{\partial f^j} + \mu^{\beta} G'_{\mathcal{W}_{\beta}} \frac{\partial \mathcal{W}_{\beta}}{\partial f^j} = 0.$$

Therefore, the condition (9) is also fulfilled. From the relation (13), we get

$$\int_{\mathcal{K}} \mu^{\beta} G_{\mathcal{W}_{\beta}} \left(\mathcal{W}_{\beta}(x, \bar{g}, \bar{f}) \right) dx = 0,$$

and by taking into account Lemma 3.1, we get

$$\int_{\mathcal{K}} \mu^{\beta} \left[G_{\mathcal{W}_{\beta}} \left(\mathcal{W}_{\beta}(x, g, f) \right) - G_{\mathcal{W}_{\beta}} \left(\mathcal{W}_{\beta}(x, \bar{g}, \bar{f}) \right) \right] dx \leq 0.$$

Hence, we obtain condition (10) and this completes the proof. □

4. Necessary efficiency in (P) via G_{θ}-Kuhn-Tucker point

In accordance to Treanţă and Arana-Jiménez [25], we introduce the definition of G_{θ}-Kuhn-Tucker point (in short, $G_{\theta} - KT$ point) to the problem (P).

Definition 4.1. A point $(\bar{g}, \bar{f}) \in \mathcal{D}$ is said to be a $G_{\theta} - KT$ *point* to (P) if there are piecewise smooth functions $\lambda(x) \in \mathbb{R}^{nm}$ and $\mu(x) \in R_+^q$ such that the following conditions

$$\begin{aligned} & G'_{\theta}(\theta(x, \bar{g}(x), \bar{f}(x))) \frac{\partial \theta}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x)) \\ & + \lambda_i^{\alpha}(x) G'_{\mathcal{U}_{\alpha}^i} \left(\mathcal{U}_{\alpha}^i(x, \bar{g}(x), \bar{f}(x)) \right) \frac{\partial \mathcal{U}_{\alpha}^i}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x)) \\ & + \mu^{\beta}(x) G'_{\mathcal{W}_{\beta}} \left(\mathcal{W}_{\beta}(x, \bar{g}(x), \bar{f}(x)) \right) \frac{\partial \mathcal{W}_{\beta}}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x)) + \frac{\partial \lambda_i^{\alpha}}{\partial x^{\alpha}} = 0, \quad i = \overline{1, n}, \end{aligned} \tag{17}$$

$$\begin{aligned} & G'_{\theta}(\theta(x, \bar{g}(x), \bar{f}(x))) \frac{\partial \theta}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x)) \\ & + \lambda_i^{\alpha}(x) G'_{\mathcal{U}_{\alpha}^i} \left(\mathcal{U}_{\alpha}^i(x, \bar{g}(x), \bar{f}(x)) \right) \frac{\partial \mathcal{U}_{\alpha}^i}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x)) \\ & + \mu^{\beta}(x) G'_{\mathcal{W}_{\beta}} \left(\mathcal{W}_{\beta}(x, \bar{g}(x), \bar{f}(x)) \right) \frac{\partial \mathcal{W}_{\beta}}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x)) = 0, \quad j = \overline{1, k}, \end{aligned} \tag{18}$$

(G-complementary slackness condition)

$$\mu^{\beta}(x) \left[G_{\mathcal{W}_{\beta}} \left(\mathcal{W}_{\beta}(x, g(x), f(x)) \right) - G_{\mathcal{W}_{\beta}} \left(\mathcal{W}_{\beta}(x, \bar{g}(x), \bar{f}(x)) \right) \right] \leq 0, \quad \text{(no summation)}$$

$$\beta \in Q,\ (g,f)\in \mathcal{D},\ \mu(x)\geq 0, \tag{19}$$

hold for all $x\in\mathcal{K}$, except at discontinuity points.

In the next theorem, we establish a connection between $G_\theta - KT$ points to (P) and the efficient solutions of (P). Specifically, we show that conditions formulated in Definition 3.2 are necessary for the efficiency of a feasible point to (P).

Theorem 4.1. (G_θ-necessary efficiency conditions) *Let* $(\bar{g},\bar{f})\in\mathcal{D}$ *be a normal* ($\sigma>0$) *efficient solution to* (P)*, and the constraint conditions (for the existence of multipliers) hold. Then,* $(\bar{g},\bar{f})$ *is a* $G_\theta - KT$ *point to the problem* (P).

Proof. Supposing that the constraint conditions (for the existence of multipliers) hold, then proceeding on the lines of Treanţă and Arana-Jiménez [25], we can conclude from Theorem 3.1 that if $(\bar{g},\bar{f})\in\mathcal{D}$ is an efficient solution to the problem (P), then there are $\sigma\in R^w_+$ and piecewise smooth functions $\lambda(x)\in\mathbb{R}^{nm},\ \mu(x)\in R^q_+$ satisfying the following conditions (with summation over repeated indices):

$$\begin{aligned}
&\sigma^\kappa G'_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))\frac{\partial\theta}{\partial g^i(x)}(x,\bar{g}(x),\bar{f}(x))\\
&+\lambda^\alpha_i(x)G'_{\mathcal{U}^i_\alpha}\left(\mathcal{U}^i_\alpha(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{U}^i_\alpha}{\partial g^i(x)}(x,\bar{g}(x),\bar{f}(x))\\
&+\mu^\beta(x)G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{W}_\beta}{\partial g^i(x)}(x,\bar{g}(x),\bar{f}(x))+\frac{\partial\lambda^\alpha_i}{\partial x^\alpha}=0,\quad i=\overline{1,n},\\
&\sigma^\kappa G'_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))\frac{\partial\theta}{\partial f^j(x)}(x,\bar{g}(x),\bar{f}(x))\\
&+\lambda^\alpha_i(x)G'_{\mathcal{U}^i_\alpha}\left(\mathcal{U}^i_\alpha(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{U}^i_\alpha}{\partial f^j(x)}(x,\bar{g}(x),\bar{f}(x))\\
&+\mu^\beta(x)G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{W}_\beta}{\partial f^j(x)}(x,\bar{g}(x),\bar{f}(x))=0,\quad j=\overline{1,k},
\end{aligned}$$

(G-complementary slackness condition)

$$\begin{aligned}
&\mu^\beta(x)\left[G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,g(x),f(x))\right)-G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\right]\leq 0,\qquad \text{(no summation)}\\
&\beta\geq Q,\ (g,f)\in\mathcal{D},\\
&\sigma\geq 0,\ \mu(x)\geq 0,\ (\sigma,\mu(x))\neq(0,0),
\end{aligned}$$

hold for all $x\in\mathcal{K}$, except at discontinuities. Since a normal efficient solution to (P) is an efficient solution $(\bar{g},\bar{f})$ to (P) that satisfies the conditions (8)–(11) for all $\sigma>0$, then we can assume that $\sigma=\mathbf{1}=(1,\cdots,1)\in\mathbb{R}^w$ (without loss of generality) and the proof is complete. □

5. Sufficient efficiency in (P) via G_θ-convexity and/or G_θ-quasiconvexity

In this section, we turn our attention to the G_θ-sufficient efficiency conditions for a feasible solution to (P). More precisely, we will establish that any feasible solution to (P) will be an efficient solution if the assumption of G_θ-convexity (and/or G_θ-quasiconvexity, G_θ-strictly quasiconvexity, G_θ-monotonic quasiconvexity) is imposed on the involved functionals.

Theorem 5.1. *Let $(\bar{g},\bar{f})\in\mathcal{D}$ be a $G_\theta-KT$ point to (P) such that the G_θ-necessary efficiency conditions (17)–(19) are fulfilled. Also, we assume that the multiple integral functionals*

$$\int_{\mathcal{K}}\theta(x,g(x),f(x))dx,\ \int_{\mathcal{K}}\mu(x)\mathcal{W}(x,g(x),f(x))dx,$$

and

$$\int_{\mathcal{K}}\lambda(x)\left(\mathcal{U}(x,g(x),f(x))-\frac{\partial g(x)}{\partial x}\right)dx,$$

are G_θ-convex at $(\bar{g},\bar{f})$. Then, $(\bar{g},\bar{f})$ is an efficient solution to (P).

Proof. We proceed by contradiction and assume that $(\bar{g},\bar{f})$ is not an efficient solution to (P). Then there exists $\left(g^0,f^0\right)\in\mathcal{D}$ such that

$$\int_{\mathcal{K}}\theta\left(x,g^0(x),f^0(x)\right)dx\leq\int_{\mathcal{K}}\theta(x,\bar{g}(x),\bar{f}(x))dx,$$

Since $G_\theta:I_\theta\mapsto\mathbb{R}^w$ is an increasing function, from the above inequality it follows

$$\int_{\mathcal{K}}G_\theta\left(\theta\left(x,g^0(x),f^0(x)\right)\right)dx\leq\int_{\mathcal{K}}G_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))dx. \tag{20}$$

By hypotheses, the point $(\bar{g},\bar{f})$ satisfies the conditions (17)–(19), and by multiplying the relations (17) and (18) with $\left(g^0-\bar{g}\right)$ and $\left(f^0-\bar{f}\right)$, respectively, and then integrating and adding them, we obtain

$$\begin{aligned}
&\int_{\mathcal{K}}\left(g^0-\bar{g}\right)\Bigg\{G'_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))\frac{\partial\theta}{\partial g^i(x)}(x,\bar{g}(x),\bar{f}(x))\\
&+\lambda_i^\alpha(x)G'_{\mathcal{U}_\alpha^i}\left(\mathcal{U}_\alpha^i(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{U}_\alpha^i}{\partial g^i(x)}(x,\bar{g}(x),\bar{f}(x))\\
&+\mu^\beta(x)G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{W}_\beta}{\partial g^i(x)}(x,\bar{g}(x),\bar{f}(x))+\frac{\partial\lambda_i^\alpha}{\partial x^\alpha}\Bigg\}dx\\
&+\int_{\mathcal{K}}\left(f^0-\bar{f}\right)\Bigg\{G'_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))\frac{\partial\theta}{\partial f^j(x)}(x,\bar{g}(x),\bar{f}(x))\\
&+\lambda_i^\alpha(x)G'_{\mathcal{U}_\alpha^i}\left(\mathcal{U}_\alpha^i(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{U}_\alpha^i}{\partial f^j(x)}(x,\bar{g}(x),\bar{f}(x))\\
&+\mu^\beta(x)G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{W}_\beta}{\partial f^j(x)}(x,\bar{g}(x),\bar{f}(x))\Bigg\}dx=0,
\end{aligned} \tag{21}$$

$$i=\overline{1,n},\ j=\overline{1,k}.$$

Since the vector-valued multiple integral functional $\int_{\mathcal{K}}\theta(x,g(x),f(x))dx$ is G_θ-convex at $(\bar{g},\bar{f})$, we get

$$\begin{aligned}
&\int_{\mathcal{K}}\left\{G_\theta\left(\theta\left(x,g^0(x),f^0(x)\right)\right)-G_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))\right\}dx\\
&\geq\int_{\mathcal{K}}G'_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))\left(\frac{\partial\theta}{\partial g(x)}(x,\bar{g}(x),\bar{f}(x))\left(g^0(x)-\bar{g}(x)\right)\right.\\
&\left.+\frac{\partial\theta}{\partial f(x)}(x,\bar{g}(x),\bar{f}(x))\left(f^0(x)-\bar{f}(x)\right)\right)dx,
\end{aligned}$$

which in view of the condition (20), yields

$$\int_{\mathcal{K}} G'_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))\left(\frac{\partial\theta}{\partial g(x)}(x,\bar{g}(x),\bar{f}(x))\left(g^0(x)-\bar{g}(x)\right)\right.$$
$$\left.+\frac{\partial\theta}{\partial f(x)}(x,\bar{g}(x),\bar{f}(x))\left(f^0(x)-\bar{f}(x)\right)\right)dx \leq 0. \tag{22}$$

Again, from the assumption that $\int_{\mathcal{K}} \mu^\beta(x)\mathcal{W}_\beta(x,g(x),f(x))dx$ is G_θ-convex at $(\bar{g},\bar{f})$, we have

$$\int_{\mathcal{K}} \left\{\mu^\beta(x)G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta\left(x,g^0(x),f^0(x)\right)\right)-\mu^\beta(x)G_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\right\}dx$$
$$\geq \int_{\mathcal{K}} \mu^\beta(x)G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\left(\frac{\partial\mathcal{W}_\beta}{\partial g(x)}(x,\bar{g}(x),\bar{f}(x))\left(g^0(x)-\bar{g}(x)\right)\right.$$
$$\left.+\frac{\partial\mathcal{W}_\beta}{\partial f(x)}(x,\bar{g}(x),\bar{f}(x))\left(f^0(x)-\bar{f}(x)\right)\right)dx.$$

Since $\left(g^0,f^0\right)\in\mathcal{D}$, by using the condition (19) and the above inequality, it follows

$$\int_{\mathcal{K}} \mu^\beta(x)G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\left(\frac{\partial\mathcal{W}_\beta}{\partial g(x)}(x,\bar{g}(x),\bar{f}(x))\left(g^0(x)-\bar{g}(x)\right)\right.$$
$$\left.+\frac{\partial\mathcal{W}_\beta}{\partial f(x)}(x,\bar{g}(x),\bar{f}(x))\left(f^0(x)-\bar{f}(x)\right)\right)dx \leq 0, \tag{23}$$

Similarly, from the assumption that $\int_{\mathcal{K}} \lambda(x)\left(\mathcal{U}(x,g(x),f(x))-\frac{\partial g(x)}{\partial t}\right)dx$ is G_θ-convex at $(\bar{g},\bar{f})$ and feasibility of $\left(g^0,f^0\right)$ in (P), it results

$$\int_{\mathcal{K}} \left[\left(\lambda_i^\alpha(x)G'_{\mathcal{U}_\alpha^i}\left(\mathcal{U}_\alpha^i(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{U}_\alpha^i}{\partial g(x)}(x,\bar{g}(x),\bar{f}(x))+\frac{\partial\lambda_i^\alpha}{\partial x^\alpha}\right)\left(g^0(x)-\bar{g}(x)\right)\right.$$
$$\left.+\left(\lambda_i^\alpha(x)G'_{\mathcal{U}_\alpha^i}\left(\mathcal{U}_\alpha^i(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{U}_\alpha^i}{\partial f(x)}(x,\bar{g}(x),\bar{f}(x))\right)\left(f^0(x)-\bar{f}(x)\right)\right]dx \leq 0. \tag{24}$$

By adding the inequalities (22)–(24), we have

$$\int_{\mathcal{K}} (g^0-\bar{g})\left\{G'_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))\frac{\partial\theta}{\partial g^i(x)}(x,\bar{g}(x),\bar{f}(x))\right.$$
$$+\lambda_i^\alpha(x)G'_{\mathcal{U}_\alpha^i}\left(\mathcal{U}_\alpha^i(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{U}_\alpha^i}{\partial g^i(x)}(x,\bar{g}(x),\bar{f}(x))$$
$$\left.+\mu^\beta(x)G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{W}_\beta}{\partial g^i(x)}(x,\bar{g}(x),\bar{f}(x))+\frac{\partial\lambda_i^\alpha}{\partial x^\alpha}\right\}dx$$
$$+\int_{\mathcal{K}}\left(f^0-\bar{f}\right)\left\{G'_\theta(\theta(x,\bar{g}(x),\bar{f}(x)))\frac{\partial\theta}{\partial f^j(x)}(x,\bar{g}(x),\bar{f}(x))\right.$$
$$+\lambda_i^\alpha(x)G'_{\mathcal{U}_\alpha^i}\left(\mathcal{U}_\alpha^i(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{U}_\alpha^i}{\partial f^j(x)}(x,\bar{g}(x),\bar{f}(x))$$
$$\left.+\mu^\beta(x)G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x,\bar{g}(x),\bar{f}(x))\right)\frac{\partial\mathcal{W}_\beta}{\partial f^j(x)}(x,\bar{g}(x),\bar{f}(x))\right\}dx \leq 0,$$

which contradicts the relation (21). Hence, the proof is complete. □

The next theorems assert new G_θ-sufficient efficiency conditions under (strictly, monotonic) G_θ-quasiconvexity assumptions.

Theorem 5.2. *Let $(\bar{g}, \bar{f}) \in \mathcal{D}$ be a $G_\theta - KT$ point to (P) such that the G_θ-necessary efficiency conditions (17)–(19) are fulfilled. Also, we assume that the multiple integral functionals*

$$\Theta(g, f) := \int_{\mathcal{K}} \theta(x, g(x), f(x))dx, \;\; Y(g, f) := \int_{\mathcal{K}} \mu(x)\mathcal{W}(x, g(x), f(x))dx,$$

are G_θ-quasiconvex and G_θ-stricly quasiconvex, respectively, and

$$H(g, f) := \int_{\mathcal{K}} \lambda(x)\left(\mathcal{U}(x, g(x), f(x)) - \frac{\partial g(x)}{\partial x}\right)dx,$$

is G_θ-monotonic quasiconvex at $(\bar{g}, \bar{f})$. Then, $(\bar{g}, \bar{f})$ is an efficient solution to (P).

Proof. Let us assume that $(\bar{g}, \bar{f})$ is not an efficient solution to (P), and consider the following non-empty set

$$S = \left\{(g, f) \in \mathcal{D} \mid \Theta(g, f) \leq \Theta(\bar{g}, \bar{f}),\; H(g, f) = H(\bar{g}, \bar{f}),\; Y(g, f) \leq Y(\bar{g}, \bar{f})\right\}.$$

By hypothesis, for $(g, f) \in S$, we get

$$\Theta(g, f) \leq \Theta(\bar{g}, \bar{f}),$$

and by using the G_θ-quasiconvexity property, it follows

$$\int_{\mathcal{K}} \left[G'_\theta(\theta(x, \bar{g}(x), \bar{f}(x)))\left(\frac{\partial\theta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))(g(x) - \bar{g}(x)) + \frac{\partial\theta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))(f(x) - \bar{f}(x))\right)\right]dx \leq 0. \tag{25}$$

For $(g, f) \in S$, the equality $H(g, f) = H(\bar{g}, \bar{f})$ holds, and by using the G_θ-monotonic quasiconvexity property, it follows

$$\int_{\mathcal{K}} \left[\left(\lambda_i^\alpha(x)G'_{\mathcal{U}_\alpha^i}\left(\mathcal{U}_\alpha^i(x, \bar{g}(x), \bar{f}(x))\right)\frac{\partial\mathcal{U}_\alpha^i}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x)) + \frac{\partial\lambda_i^\alpha}{\partial x^\alpha}\right)\left(g^0(x) - \bar{g}(x)\right) + \left(\lambda_i^\alpha(x)G'_{\mathcal{U}_\alpha^i}\left(\mathcal{U}_\alpha^i(x, \bar{g}(x), \bar{f}(x))\right)\frac{\partial\mathcal{U}_\alpha^i}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))\right)\left(f^0(x) - \bar{f}(x)\right)\right]dx = 0. \tag{26}$$

Also, for $(g, f) \in S$, the inequality $Y(g, f) \leq Y(\bar{g}, \bar{f})$, and by using the G_θ-strictly quasiconvexity property, it follows

$$\int_{\mathcal{K}} \mu^\beta(x)G'_{\mathcal{W}_\beta}\left(\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x))\right)\left(\frac{\partial\mathcal{W}_\beta}{\partial g(x)}(x, \bar{g}(x), \bar{f}(x))\left(g^0(x) - \bar{g}(x)\right) + \frac{\partial\mathcal{W}_\beta}{\partial f(x)}(x, \bar{g}(x), \bar{f}(x))\left(f^0(x) - \bar{f}(x)\right)\right)dx < 0. \tag{27}$$

By hypotheses, the point $(\bar{g}, \bar{f})$ satisfies the conditions (17)–(19), and by multiplying the relations (17) and (18) with $\left(g^0 - \bar{g}\right)$ and $\left(f^0 - \bar{f}\right)$, respectively, and then integrating and adding them, we obtain

$$\int_{\mathcal{K}} \left(g^0 - \bar{g}\right) \left\{ G'_\theta(\theta(x, \bar{g}(x), \bar{f}(x))) \frac{\partial \theta}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x)) \right.$$

$$+\lambda_i^\alpha(x) G'_{\mathcal{U}_\alpha^i} \left(\mathcal{U}_\alpha^i(x, \bar{g}(x), \bar{f}(x))\right) \frac{\partial \mathcal{U}_\alpha^i}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x))$$

$$\left. +\mu^\beta(x) G'_{\mathcal{W}_\beta} \left(\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x))\right) \frac{\partial \mathcal{W}_\beta}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x)) + \frac{\partial \lambda_i^\alpha}{\partial x^\alpha} \right\} dx$$

$$+ \int_{\mathcal{K}} \left(f^0 - \bar{f}\right) \left\{ G'_\theta(\theta(x, \bar{g}(x), \bar{f}(x))) \frac{\partial \theta}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x)) \right.$$

$$+\lambda_i^\alpha(x) G'_{\mathcal{U}_\alpha^i} \left(\mathcal{U}_\alpha^i(x, \bar{g}(x), \bar{f}(x))\right) \frac{\partial \mathcal{U}_\alpha^i}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x))$$

$$\left. +\mu^\beta(x) G'_{\mathcal{W}_\beta} \left(\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x))\right) \frac{\partial \mathcal{W}_\beta}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x)) \right\} dx = 0, \tag{28}$$

$$i = \overline{1, n},\ j = \overline{1, k}.$$

By adding the inequalities (25)–(27), we have

$$\int_{\mathcal{K}} (g^0 - \bar{g}) \left\{ G'_\theta(\theta(x, \bar{g}(x), \bar{f}(x))) \frac{\partial \theta}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x)) \right.$$
$$+ \lambda_i^\alpha(x) G'_{\mathcal{U}_\alpha^i} \left(\mathcal{U}_\alpha^i(x, \bar{g}(x), \bar{f}(x))\right) \frac{\partial \mathcal{U}_\alpha^i}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x))$$
$$\left. +\mu^\beta(x) G'_{\mathcal{W}_\beta} \left(\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x))\right) \frac{\partial \mathcal{W}_\beta}{\partial g^i(x)}(x, \bar{g}(x), \bar{f}(x)) + \frac{\partial \lambda_i^\alpha}{\partial x^\alpha} \right\} dx$$
$$+ \int_{\mathcal{K}} \left(f^0 - \bar{f}\right) \left\{ G'_\theta(\theta(x, \bar{g}(x), \bar{f}(x))) \frac{\partial \theta}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x)) \right.$$
$$+ \lambda_i^\alpha(x) G'_{\mathcal{U}_\alpha^i} \left(\mathcal{U}_\alpha^i(x, \bar{g}(x), \bar{f}(x))\right) \frac{\partial \mathcal{U}_\alpha^i}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x))$$
$$\left. +\mu^\beta(x) G'_{\mathcal{W}_\beta} \left(\mathcal{W}_\beta(x, \bar{g}(x), \bar{f}(x))\right) \frac{\partial \mathcal{W}_\beta}{\partial f^j(x)}(x, \bar{g}(x), \bar{f}(x)) \right\} dx < 0,$$

which contradicts the relation (28). Hence, the proof is complete. □

Next, an immediate consequence of the previous theorem can be formulated as follows.

Theorem 5.3. *Let $(\bar{g}, \bar{f}) \in \mathcal{D}$ be a $G_\theta-KT$ point to (P) such that the G_θ-necessary efficiency conditions (17)–(19) are fulfilled. Also, we assume that the multiple integral functionals*

$$\Theta(g, f) := \int_{\mathcal{K}} \theta(x, g(x), f(x)) dx,\ \ Y(g, f) := \int_{\mathcal{K}} \mu(x) \mathcal{W}_\beta(x, g(x), f(x)) dx,$$

are G_θ-strictly quasiconvex and G_θ-quasiconvex, respectively, and

$$H(g, f) := \int_{\mathcal{K}} \lambda(x) \left(\mathcal{U}(x, g(x), f(x)) - \frac{\partial g(x)}{\partial x} \right) dx,$$

is G_θ-monotonic quasiconvex at $(\bar{g}, \bar{f})$. Then, $(\bar{g}, \bar{f})$ is an efficient solution to (P).

Proof. The proof follows in the same manner as in Theorem 5.2, by replacing the sign "≤" in (25) with "<", and the sign "<" in (27) with "≤". □

6. Conclusions

In this study, we have formulated new conditions of efficiency for a class of multiple-objective optimal control models under generalized assumptions. In this regard, we first defined the G_θ-Fritz John problem and, by considering it, we established a link between the solutions of G_θ-Fritz John problem and efficient solutions of the considered model (P). In addition, we formulated the G_θ-necessary efficiency conditions for a feasible solution in (P). Also, a connection between the newly defined concept of $G_\theta - KT$ points to (P) and the efficient solutions of (P) was formulated. Finally, we turned our attention to the G_θ-sufficient efficiency conditions for a feasible solution to (P). In this regard, we established that any feasible solution to (P) is an efficient solution if the assumption of G_θ-convexity (and/or G_θ-quasiconvexity, G_θ-strictly quasiconvexity, G_θ-monotonic quasiconvexity) is imposed on the involved functionals.

As further developments associated with this paper, the authors mention the study of well-posedness and generalized well-posedness. Also, a duality theory related to this class of extremization problems could be another interesting topic.

Author contributions

Savin Treanţă, Cristina-Florentina Marghescu, Laura-Gabriela Matei: Conceptualization, Formal Analysis, Investigation, Methodology, Software, Visualisation, Writing- original draft; Savin Treanţă, Cristina-Florentina Marghescu, Laura-Gabriela Matei: Data Curation, Funding acquisition, Project administration, Supervision, Writing-review & editing; Savin Treanţă, Cristina-Florentina Marghescu, Laura-Gabriela Matei: Supervision, Writing- review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

References

1. E. M. Almetwally, T. M. Jawa, N. Sayed-Ahmed, C. Park, M. Zakarya, S. Dey, Analysis of unit-Weibull based on progressive type-II censored with optimal scheme, *Alexandria Engineer. J.*, **63** (2023), 321–338. https://doi.org/10.1016/j.aej.2022.07.064
2. J. V. Anchitaalagammai, T. Jayasankar, P. Selvaraj, M. Y. Sikkandar, M. Zakarya, M. Elhoseny, et al., Energy efficient cluster-based optimal resource management in IoT environment, *Comput. Mater. Contin.*, **70** (2022), 1247–1261. https://doi.org/10.32604/cmc.2022.017910
3. T. Antczak, New optimality conditions and duality results of G-type in differentiable mathematical programming, *Nonlinear Anal.*, **66** (2007), 1617–1632. https://doi.org/10.1016/j.na.2006.02.013
4. T. Antczak, On G-invex multiobjective programming. Part I. Optimality, *J. Global Optim.*, **43** (2009), 97–109. https://doi.org/10.1007/s10898-008-9299-5
5. M. Arana-Jiménez, R. Osuna-Gómez, A. Rufián-Lizana, G. Ruiz-Garzón, KT-invex control problem, *Appl. Math. Comput.*, **197** (2008), 489–496. https://doi.org/10.1016/j.amc.2007.07.064
6. M. Arana-Jiménez, G. Ruiz-Garzón, A. Rufián-Lizana, R. Osuna-Gómez, A necessary and sufficient condition for duality in multiobjective variational problems, *Eur. J. Oper. Res.*, **201** (2010), 672–681. https://doi.org/10.1016/j.ejor.2009.03.047
7. A. Baranwal, A. Jayaswal, Preeti, Robust duality for the uncertain multitime control optimization problems, *Int. J. Robust Nonlinear Control*, **32** (2022), 5837–5847. https://doi.org/10.1002/rnc.6113
8. C. R. Bector, I. Husain, Duality for multiobjective variational problems, *J. Math. Anal. Appl.*, **166** (1992), 214–229. https://doi.org/10.1016/0022-247X(92)90337-D
9. D. Bhatia, P. Kumar, Multiobjective control problem with generalized invexity, *J. Math. Anal. Appl.*, **189** (1995), 676–692. https://doi.org/10.1006/jmaa.1995.1045
10. S. Bhushan, A. Kumar, E. Hussam, M. S. Mustafa, M. Zakarya, W. R. Alharbi, On stratified ranked set sampling for the quest of an optimal class of estimators, *Alexandria Engineer. J.*, **86** (2024), 79–97. https://doi.org/10.1016/j.aej.2023.11.037
11. T. R. Gulati, G. Mehndiratta, Optimality and duality for second-order multiobjective variational problems, *Eur. J. Pure Appl. Math.*, **3** (2010), 786–805. Available from: `https://www.ejpam.com/index.php/ejpam/article/view/534`
12. Y. Guo, T. Li, Modeling the competitive transmission of the Omicron strain and Delta strain of COVID-19, *J. Math. Anal. Appl.*, **526** (2023), 127283. https://doi.org/10.1016/j.jmaa.2023.127283
13. Y. Guo, T. Li, Fractional-order modeling and optimal control of a new online game addiction model based on real data, *Commun. Nonlinear Sci. Numer. Simul.*, **121** (2023), 107221. https://doi.org/10.1016/j.cnsns.2023.107221
14. R. Gupta, M. Srivastava, Optimality and duality for nonsmooth multiobjective programming using G-type I functions, *Appl. Math. Comput.*, **240** (2014), 294–307. https://doi.org/10.1016/j.amc.2014.04.066
15. D. H. Jacobson, M. M. Lele, J. L. Speyer, New necessary conditions of optimality for control problems with state-variable inequality constraints, *J. Math. Anal. Appl.*, **35** (1971), 255–284. https://doi.org/10.1016/0022-247X(71)90219-8
16. A. Jayswal, Preeti, S. Treanţă, *Multi-dimensional control problems: Robust approach*, Springer Singapore, 2022. https://doi.org/10.1007/978-981-19-6561-6

17. T. Li, Y. Guo, Modeling and optimal control of mutated COVID-19 (Delta strain) with imperfect vaccination, *Chaos Soliton Fract.*, **156** (2022), 111825. https://doi.org/10.1016/j.chaos.2022.111825
18. N. T. H. Linh, J. P. Penot, Optimality conditions for quasiconvex programs, *SIAM J. Optim.*, **17** (2006), 500–510. https://doi.org/10.1137/040621843
19. K. Malanowski, Sufficient optimality conditions for optimal control subject to state constraints, *SIAM J. Control Optim.*, **35** (1997), 205–227. https://doi.org/10.1137/S0363012994267637
20. S. Mititelu, Optimality and duality for invex multi-time control problems with mixed constraints, *J. Adv. Math. Stud.*, **2** (2009), 25–35.
21. S. Mititelu, S. Treanţă, Efficiency conditions in vector control problems governed by multiple integrals, *J. Appl. Math. Comput.*, **57** (2018), 647–665. https://doi.org/10.1007/s12190-017-1126-z
22. V. A. de Oliveira, G. N. Silva, M. A. Rojas-Medar, KT-invexity in optimal control problems, *Nonlinear Anal.*, **71** (2009), 4790–4797. https://doi.org/10.1016/j.na.2009.03.055
23. A. Prusińska, A. Tret'yakov, Necessary p-th order optimality conditions for irregular Lagrange problem in calculus of variations, *Math. Commun.*, **19** (2014), 561–572. Available from: `https://hrcak.srce.hr/130116`
24. J. Soolaki, O. S. Fard, A. H. Borzabadi, Generalized Euler-Lagrange equations for fuzzy fractional variational calculus, *Math. Commun.*, **21** (2016), 199–218. Available from: `https://hrcak.srce.hr/170384`
25. S. Treanţă, M. Arana-Jiménez, KT-pseudoinvex multidimensional control problem, *Optim. Control Appl. Meth.*, **39** (2018), 1291–1300. https://doi.org/10.1002/oca.2410
26. S. Treanţă, A necessary and sufficient condition of optimality for a class of multidimensional control problems, *Optim. Control Appl. Methods*, **41** (2020), 2137–2148. https://doi.org/10.1002/oca.2645
27. S. Treanţă, K. Das, On robust saddle-point criterion in optimization problems with curvilinear integral functionals, *Mathematics*, **9** (2021), 1790. https://doi.org/10.3390/math9151790
28. S. Treanţă, Robust saddle-point criterion in second-order partial differential equation and partial differential inequation constrained control problems, *Int. J. Robust Nonlinear Control*, **31** (2021), 9282–9293. https://doi.org/10.1002/rnc.5767
29. S. Treanţă, Robust optimality in constrained optimization problems with application in mechanics, *J. Math. Anal. Appl.*, **515** (2022), 126440. https://doi.org/10.1016/j.jmaa.2022.126440
30. S. A. Urziceanu, Necessary optimality conditions in isoperimetric constrained optimal control problems, *Symmetry*, **11** (2019), 1380. https://doi.org/10.3390/sym11111380
31. G. Yu, Y. Lu, Multi-objective optimization problems and vector variational-like inequalities involving semi-strong E-convexity, *Fourth Int. Joint Conf. Comput. Sci. Optimiz.*, 476–479, 2011. https://doi.org/10.1109/CSO.2011.170

AIMS Press