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Research article

High power sums of Fourier coefficients of holomorphic cusp forms and their applications

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Abstract: Let $\lambda_f(n)$ be the nth normalized Fourier coefficient of a holomorphic cusp form f for the full modular group. In this paper, we established asymptotic formulae for high power sums of Fourier coefficients of cusp forms and further improved previous results. Moreover, as an application, we studied the signs of the sequences $\{\lambda_f(n)\}$ and $\{\lambda_f(n)\lambda_g(n)\}$ in short intervals, and presented some quantitative results for the number of sign changes for $n \leq x$.

Keywords: cusp form; L-function; Fourier coefficient; sign change
Mathematics Subject Classification: 11F30, 11F66, 11N37

1. Introduction and main results

Let H_k^* denote the set of all primitive holomorphic cusp forms of even integral weight $k \geq 2$ for the full modular group $SL(2, \mathbb{Z})$. Every $f \in H_k^*$ has a Fourier expansion at the cusp ∞ of the type

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}.$$

The Fourier coefficient $\lambda_f(n)$ satisfies the multiplicative relation

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\Big(\frac{mn}{d^2}\Big).$$

In 1974, Deligne [1] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leq d(n) \ll n^{\epsilon}, \tag{1.1}$$

where $d(n)$ is the divisor function.

For $f \in H_k^*$, we define the i-th symmetric power L-function attached to f as

$$L(\mathrm{sym}^i f, s) = \prod_p \prod_{m=0}^{i} \left(1 - \frac{\alpha_f(p)^{i-m}\beta_f(p)^m}{p^s}\right)^{-1} \tag{1.2}$$

provided that $\Re s > 1$, where $\alpha_f(p)$ and $\beta_f(p)$ are two complex numbers satisfying

$$\alpha_f(p)\beta_f(p) = |\alpha_f(p)| = |\beta_f(p)| = 1, \quad \lambda_f(p) = \alpha_f(p) + \beta_f(p).$$

We can express it as a Dirichlet series:

$$L(\mathrm{sym}^i f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f}(n)}{n^s} = \prod_p \left(1 + \sum_{\nu=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f}(p^\nu)}{p^{\nu s}}\right), \tag{1.3}$$

where $\lambda_{\mathrm{sym}^i f}(n)$ is a real multiplicative function, and

$$\lambda_{\mathrm{sym}^i f}(p) = \sum_{m=0}^{i} \alpha_f(p)^{i-m}\beta_f(p)^m = \lambda_f(p^i).$$

It's easy to see that

$$\begin{cases} L(\mathrm{sym}^0 f, s) = \zeta(s), \\ L(\mathrm{sym}^1 f, s) = L(f, s). \end{cases}$$

Let $f, g \in H_k^*$ be two different cusp forms. The Rankin-Selberg L-function attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined by

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) = \prod_p \prod_{m=0}^{i} \prod_{n=0}^{j} \left(1 - \frac{\alpha_f(p)^{i-m}\beta_f(p)^m \alpha_g(p)^{j-n}\beta_g(p)^n}{p^s}\right)^{-1}$$

for $\Re s > 1$. Further, this can also be written as

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)}{n^s} = \prod_p \left(1 + \sum_{\nu=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p^\nu)}{p^{\nu s}}\right). \tag{1.4}$$

Then, we get

$$\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p) = \sum_{m=0}^{i} \sum_{n=0}^{j} \alpha_f(p)^{i-m}\beta_f(p)^m \alpha_f(p)^{j-n}\beta_f(p)^n = \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j g}(p),$$

where $i, j \geq 1$ are integers. In particular, we have

$$\begin{cases} L(\mathrm{sym}^1 f \times \mathrm{sym}^1 g, s) = L(f \times g, s), \\ L(\mathrm{sym}^1 f \times \mathrm{sym}^j g, s) = L(f \times \mathrm{sym}^j g, s). \end{cases}$$

For a more comprehensive investigation on basic properties of symmetric power L-functions and Rankin-Selberg L-functions, the interested readers can refer to [2, Chapter 13].

There are many hidden structures underlying the Fourier coefficients $\lambda_f(n)$. In analytic number theory, it is a classical problem to estimate the sums of the type

$$\sum_{n\le x}\lambda_f(n)^l\lambda_g(n)^m, \tag{1.5}$$

where $l, m \in \mathbb{N}$. The study of O-results on the sum (1.5) is of great significance and has attracted much attention of many number theorists. For $l = 1$ and $m = 0$, the best result to date was given by Wu [3]. Rankin [4] and Selberg [5] studied the average behavior of the power sum for the case $l = 2$ and $m = 0$. More recently, Huang [6] established the better result

$$\sum_{n\le x}\lambda_f(n)^2 = Cx + O(x^{\frac{3}{5}-\frac{1}{560}+\epsilon}).$$

Fomenko [7] solved the problem when $l = 3, 4$ and $m = 0$. Lü [8] improved Fomenko's result and successfully established the results with $l = 6, 8$ and $m = 0$ for the first time. Shortly afterward, Lau et al. [9] considered more general cases and obtained better results. Recently, Newton and Thorne [10] proved in a general setting that $\mathrm{sym}^i f$ is automorphic for $i \ge 1$. On the basis of the deep results of Newton and Thorne, by applying some techniques of analytic number theory, Xu [11] and Liu [12] investigated the average behavior of the power sums (1.5) with $l \in \mathbb{N}$ and $m = 0$. Hua [13] focused on the sum (1.4) with $l \ge 9, m = 0$ over indices that are sums of two squares.

For the power sum (1.5) with $m > 0$, Ogg [14] first established an asymptotic formula for $l, m = 1$. Subsequently, Fomenko [7] considered the sum of coefficient of the Rankin-Selberg L-function, and then successfully attained the O-results of the case of $l = 1, m = 2$ and $l = 2, m = 2$. In 2014, Lü [15] showed that

$$\sum_{n\le x}\lambda_f(n)\lambda_g(n) \ll x^{\frac{3}{5}}\log x^{-\frac{2}{3}(1-\frac{8}{3\pi})}$$

and

$$\sum_{n\le x}\lambda_f(n)^2\lambda_g(n)^2 = Cx + O(x^{\frac{7}{8}+\epsilon}), \tag{1.6}$$

which improved the results of Fomenko [7]. The current best known estimate for (1.6) is due to He [16], who showed that

$$\sum_{n\le x}\lambda_f(n)^2\lambda_g(n)^2 = Cx + O(x^{\frac{13}{15}+\epsilon}).$$

Lü [17] also investigated the cases of $l = m = 3, l = 4, m = 2$, and $l = 4, m = 4$.

The first purpose of this paper is to further improve the upper bounds on the error term of the sum (1.5) with $m = 0$ and $m \ge 2$, respectively. The result is formulated in the following theorem:

Theorem 1.1. *Let $f \in H_k^*$ and $g \in H_k^*$ be two different nonzero cusp forms:*

(i) *For $l = 2r \ge 6$, we have*

$$\sum_{n\le x}\lambda_f(n)^l = xP_l(\log x) + O(x^{\theta_l+\epsilon}),$$

where $P_l(y)$ denotes a polynomial in y of degree $\binom{l}{r} - \binom{l}{r-1} - 1$, and

$$\theta_l = \begin{cases} \frac{3271}{3391}, & l = 6, \\ 1 - \theta_{2,r}^{-1}, & l \ge 8. \end{cases}$$

Here,

$$\theta_{2,r} = \frac{13}{42r}\binom{l}{r-1} + \frac{18}{5(r-1)}\binom{l}{r-2} + \frac{1}{2}\Big(\sum_{n=1}^{r-2}\frac{(l-2n+1)^2}{n}\binom{l}{n-1} + l + 1\Big) - \frac{3}{14}.$$

(ii) *For $l \geq 2$, $m \geq 2$, we have*

$$\sum_{n\leq x}\lambda_f(n)^l\lambda_g(n)^m = xP_{l,m}(\log x) + O(x^{\theta_{l,m}+\epsilon}),$$

where $P_{l,m}(y)$ denotes a polynomial in y of degree $\Big(\binom{l}{[\frac{l}{2}]} - \binom{l}{[\frac{l}{2}]-1}\Big)\Big(\binom{m}{[\frac{m}{2}]} - \binom{l}{[\frac{m}{2}]-1}\Big) - 1$ for $2 \mid l$ and $2 \mid m$; otherwise, 0. In the O-term, we have

$\theta_{2,2} = \frac{773}{893} = 0.865621\cdots$	$\theta_{2,3} = \theta_{3,2} = \frac{206}{221} = 0.932126\cdots$	$\theta_{2,4} = \theta_{4,2} = \frac{349}{361} = 0.966759\cdots$
$\theta_{2,6} = \theta_{6,2} = \frac{14611}{14731} = 0.991853\cdots$	$\theta_{4,4} = \frac{3677}{3707} = 0.991907\cdots$	$\theta_{4,6} = \theta_{6,4} = \frac{29941}{30001} = 0.998000\cdots$

$$\theta_{l,m} = \begin{cases} 1-\theta_{2,2,r,\bar r}^{-1}, & l=2r, m=2\bar r, l=m=6, l\geq 8 \text{ or } m\geq 8,\\ 1-\theta_{1,1,t,\bar t}^{-1}, & l=2t+1, m=2\bar t+1, l,m\geq 3,\\ 1-\theta_{1,2,t,\bar r}^{-1}, & l=2t+1, m=2\bar r, l\neq 3 \text{ or } m\neq 2,\\ 1-\theta_{2,1,r,\bar t}^{-1}, & l=2r, m=2\bar t+1, l\neq 2 \text{ or } m\neq 3.\end{cases}$$

Here,

$$\begin{aligned}\theta_{2,2,r,\bar r} =& \frac{13\binom{l}{r-1}\binom{m}{\bar r-1}}{42r\bar r} + \frac{18\binom{l}{r-2}\binom{m}{\bar r-1}}{5\bar r(r-1)} + \frac{18\binom{l}{r-1}\binom{m}{\bar r-2}}{5r(\bar r-1)} + \frac{\sum_{n_1=1}^{r-2}\frac{(l-2n_1+1)^2}{\bar r n_1}\binom{l}{n_1-1}\binom{m}{\bar r-1} + (l+1)\frac{1}{\bar r}\binom{m}{\bar r-1}}{2}\\ &+ \frac{\sum_{n_2=1}^{\bar r-2}\frac{(m-2n_2+1)^2}{rn_2}\binom{l}{r-1}\binom{m}{n_2-1}(m+1)\frac{1}{r}\binom{l}{r-1}}{2} + \frac{(l+1)(m+1)}{2} + \frac{\sum_{n_1=1}^{r-1}\frac{(l-2n_1+1)^2(m+1)}{n_1}\binom{l}{n_1-1}}{2}\\ &+ \frac{\sum_{n_2=1}^{\bar r-1}\frac{(m-2n_2+1)^2(l+1)}{n_2}\binom{m}{n_2-1}}{2} + \frac{\sum_{n_1=1}^{r-1}\sum_{n_2=1}^{\bar r-1}\frac{(l-2n_1+1)^2(m-2n_2+1)^2}{n_1n_2}\binom{l}{n_1-1}\binom{m}{n_2-1}}{2} - \frac{3}{14},\\ \theta_{1,1,t,\bar t} =& \frac{(l+1)(m+1)}{2} + \frac{\sum_{n_1=1}^{t}\sum_{n_2=1}^{\bar t}\frac{(l-2n_1+1)^2(m-2n_2+1)^2}{n_1n_2}\binom{l}{n_1-1}\binom{m}{n_2-1}}{2} + \frac{\sum_{n_1=1}^{t}\frac{(l-2n_1+1)^2(m+1)}{n_1}\binom{l}{n_1-1}}{2}\\ &+ \frac{\sum_{n_2=1}^{\bar t}\frac{(m-2n_2+1)^2(l+1)}{n_2}\binom{m}{n_2-1}}{2},\\ \theta_{1,2,t,\bar r} =& \frac{4\binom{l}{t-1}\binom{m}{\bar r-1}}{3t\bar r} + \frac{81\binom{l}{t-1}\binom{m}{\bar r-2}}{5t(\bar r-1)} + \frac{\frac{4(m+1)}{t}\binom{l}{t-1} + \sum_{n_2=1}^{\bar r-2}\frac{4(m-2n_2+1)^2}{tn_2}\binom{l}{t-1}\binom{m}{n_2-1}}{2}\end{aligned}$$

$$+\frac{(l+1)(m+1)}{2}+\frac{\sum_{n_1=1}^{t-1}\frac{(l-2n_1+1)^2(m+1)}{n_1}\binom{l}{n_1-1}}{2}+\frac{\sum_{n_2=1}^{\bar{r}}\frac{(m-2n_2+1)^2(l+1)}{n_2}\binom{m}{n_2-1}}{2}$$
$$+\frac{\sum_{n_1=1}^{t-1}\sum_{n_2=1}^{\bar{r}}\frac{(l-2n_1+1)^2(m-2n_2+1)^2}{n_1n_2}\binom{l}{n_1-1}\binom{m}{n_2-1}}{2},$$

$$\theta_{2,1,r,\bar{t}}=\frac{4\binom{l}{r-1}\binom{m}{\bar{t}-1}}{3r\bar{t}}+\frac{81\binom{l}{r-2}\binom{m}{\bar{t}-1}}{5\bar{t}(r-1)}+\frac{\frac{4(l+1)}{\bar{t}}\binom{m}{\bar{t}-1}+\sum_{n_1=1}^{t-2}\frac{4(l-2n_1+1)^2}{\bar{t}n_1}\binom{l}{n_1-1}\binom{m}{\bar{t}-1}}{2}$$
$$+\frac{(l+1)(m+1)}{2}+\frac{\sum_{n_1=1}^{r}\frac{(l-2n_1+1)^2(m+1)}{n_1}\binom{l}{n_1-1}}{2}+\frac{\sum_{n_2=1}^{\bar{t}-1}\frac{(m-2n_2+1)^2(l+1)}{n_2}\binom{m}{n_2-1}}{2}$$
$$+\frac{\sum_{n_1=1}^{r}\sum_{n_2=1}^{\bar{t}-1}\frac{(l-2n_1+1)^2(m-2n_2+1)^2}{n_1n_2}\binom{l}{n_1-1}\binom{m}{n_2-1}}{2}.$$

Remark 1.2. *Compared with Liu [12, Theorem 1.1], He [16, Proposition 4.2], and [17, Theorems 1.4 and 1.5], we improve the previous results. In fact, we have*

Old	$\theta'_6 = 0.9647\cdots$	$\theta'_8 = 0.9914\cdots$	$\theta'_{2,2} = 0.8666\cdots$	$\theta'_{4,2} = 0.9687\cdots$	$\theta'_{4,4} = 0.9921\cdots$
New	$\theta_6 = 0.9646\cdots$	$\theta_8 = 0.9913\cdots$	$\theta_{2,2} = 0.8656\cdots$	$\theta_{4,2} = 0.9667\cdots$	$\theta_{4,4} = 0.9919\cdots$

As an application of Theorem 1.1, we then investigate quantitative results of the sign changes of $\lambda_f(n)$ and $\lambda_f(n)\lambda_g(n)$. The sign changes of the sequence of Fourier coefficients in short intervals was first investigated by Murty [18]. Later, Meher and Murty [19] established a lower bound for the number of sign changes of the sequence $\{\lambda_f(n)\}$. In addition, the analogous questions of simultaneous sign changes of $\lambda_f(n)\lambda_g(n)$ was considered by Kumari and Murty [20], where f and g are two different cusp forms. In 2019, He [16] improved the result of Kumari and Murty [20]. Here, we obtain the better quantitative results for sign changes of the sequences $\{\lambda_f(n)\}$ and $\{\lambda_f(n)\lambda_g(n)\}$.

Theorem 1.3. *Let $\lambda_f(n)$ and $\lambda_g(n)$ be the coefficients of $L(f,s)$ and $L(g,s)$, respectively.*

(i) *Suppose $f \in H_k^*$. Then, for any r_1 with $\frac{67}{112} < r_1 < 1$, the sequence $\{\lambda_f(n)\}$ has at least one sign change for $n \in (x, x+x^{r_1}]$. Moreover, the number of sign changes for $n \le x$ is $\gg x^{1-r_1}$ for sufficiently large x.*

(ii) *Suppose $f, g \in H_k^*$. Then, for any r_2 with $\frac{773}{893} < r_2 < 1$, the sequence $\{\lambda_f(n)\lambda_g(n)\}$ has at least one sign change for $n \in (x, x+x^{r_2}]$. Moreover, the number of sign changes for $n \le x$ is $\gg x^{1-r_2}$ for sufficiently large x.*

Remark 1.4. *In view of*

$$\frac{3}{5}=0.6>0.5982\cdots=\frac{67}{112},\quad \frac{13}{15}=0.8666\cdots>0.8656\cdots=\frac{773}{893},$$

we improve the results of Meher and Murty [19, Theorem 1.2] and He [16, Theorem 1.5].

2. Preliminaries

In this section, we will recall and establish some preliminary results which are used to prove the main theorems in this paper.

We define

$$F_l(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)^l}{n^s} \quad \text{and} \quad F_{l,m}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)^l \lambda_g(n)^m}{n^s}. \tag{2.1}$$

Lemma 2.1. *Let $f \in H_k^*$, then we have*

$$F_l(s) = G_l(s)H_l(s),$$

where

$$G_l(s) = \prod_{n=0}^{[\frac{l}{2}]} L(\mathrm{sym}^{l-2n} f, s)^{(\binom{l}{n} - \binom{l}{n-1})}.$$

$\binom{l}{n}$ is the binomial coefficient with the convention that $\binom{l}{n} = 0$ if $n < 0$, and the function $H_l(s)$ admits a Dirichlet series convergent absolutely in $\Re s > 1/2$. Noting $H_l(s) \neq 0$ for $\Re s = 1$.

Proof. This can be found in Xu [11, Lemma 5]. □

Based on Ivić [21, Theorem 8.4], Bourgain [22, Theorem 5], and Ramachandra and Sankaranarayanan [23, Lemma 2], we give the following lemma:

Lemma 2.2. *For any $\epsilon > 0$, we have*

$$\int_0^T \left|\zeta(\frac{5}{7} + i\tau)\right|^{12} d\tau \ll_\epsilon T^{1+\epsilon} \tag{2.2}$$

uniformly for $T \geq 1$ and

$$\zeta(\sigma + i\tau) \ll (|\tau + 1|)^{\max\{\frac{13}{42}(1-\sigma),0\}+\epsilon} \tag{2.3}$$

uniformly for $1/2 \leq \sigma \leq 2$ and $|\tau| \geq 1$. Moreover, for $U > U_0$, where U_0 is sufficiently large, there exists a $T^* \in (U, 2U)$ such that

$$\max_{\sigma \geq \frac{1}{2}} |\zeta(\sigma + iT^*)| \ll_\epsilon \exp(C(\log\log U)^2). \tag{2.4}$$

Lemma 2.3. *Let $f \in H_k^*$, then we have*

$$\int_0^T \left|L(f, \frac{5}{8} + i\tau)\right|^4 d\tau \ll_{f,\epsilon} T^{1+\epsilon} \tag{2.5}$$

uniformly for $T \geq 1$ and

$$L(f, \sigma + i\tau) \ll_{f,\epsilon} (|\tau + 1|)^{\max\{\frac{2}{3}(1-\sigma),0\}+\epsilon}, \tag{2.6}$$

$$L(\mathrm{sym}^2 f, \sigma + i\tau) \ll_{f,\epsilon} (|\tau + 1|)^{\max\{\frac{6}{5}(1-\sigma),0\}+\epsilon}, \tag{2.7}$$

$$L\Big(f \times \mathrm{sym}^2 g,\ \sigma + i\tau\Big) \ll_{f,g,\epsilon} (|\tau + 1|)^{\frac{27}{10}(1-\sigma)+\epsilon} \tag{2.8}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$ and $|\tau| \geq 1$.

Proof. These are Ivić [24, Theorem 2], Good [25, Corollary 3], Lin et al. [26, Corollary 1.2], and Lin and Sun [27, Corollary 1.3], respectively. □

For general L-functions, we have the following averaged or individual convexity bounds (see [28]):

Lemma 2.4. *Suppose that $\mathfrak{L}(s)$ is a general L-function of degree m. Then, for any $\epsilon > 0$, we have*

$$\int_T^{2T} |\mathfrak{L}(\sigma + i\tau)|^2 d\tau \ll T^{m(1-\sigma)+\epsilon} \tag{2.9}$$

uniformly for $1/2 \le \sigma \le 2$ *and* $T \ge 1$*, and*

$$\mathfrak{L}(\sigma + i\tau) \ll (|\tau| + 1)^{max\{\frac{m}{2}(1-\sigma),0\}+\epsilon} \tag{2.10}$$

uniformly for $1/2 \le \sigma \le 1 + \epsilon$ *and* $|\tau| \ge 1$.

Remark 2.5. *According to the Euler product (1.2), the degree of* $L(\mathrm{sym}^j f, s)$ *is* $j + 1$*. In the proof of Theorem 1.1, we take* $m = j + 1$ *in Lemma 2.4 for* $L(\mathrm{sym}^j f, s)$, $j \ge 3$*. Similarly, take* $m = (i + 1)(j + 1)$ *for the Rankin-Selberg L-function* $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$.

3. Proof of Theorem 1.1

3.1. Proof of (i) in Theorem 1.1

By the Perron formula ([29, Proposition 5.54]) with (1.1), we obtain

$$\sum_{n\le x} \lambda_f(n)^l = \frac{1}{2\pi i}\int_{1+\epsilon-iT}^{1+\epsilon+iT} F_l(s)\frac{x^s}{s}ds + O_{f,\epsilon}\left(\frac{x^{1+\epsilon}}{T}\right)$$

uniformly for $2 \le T \le x$, where the implied constant depends only on f and ϵ. From Lemma 2.1, we can easily get that the point $s = 1$ is the only pole of the integrand in the region $\sigma_0 \le \sigma \le 1 + \epsilon$ and $|\tau| \le T$ for any $\sigma_0 \in [1/2 + \epsilon, 1)$. Using the Cauchy residue theorem, we get

$$\sum_{n\le x} \lambda_f(n)^l = \operatorname*{Res}_{s=1} F_l(s)\frac{x^s}{s} + \frac{1}{2\pi i}\left(\int_{\sigma_0+iT}^{1+\epsilon+iT} + \int_{\sigma_0-iT}^{1+\epsilon-iT} + \int_{\sigma_0-iT}^{\sigma_0+iT}\right) F_l(s)\frac{x^s}{s}ds + O_{f,\epsilon}\left(\frac{x^{1+\epsilon}}{T}\right).$$

The factorization expression of $F_l(s)$ in Lemma 2.1 contains $\zeta(s)^{\binom{l}{n}-\binom{l}{n-1}}$, which means $s = 1$ is a pole of order $\binom{l}{n} - \binom{l}{n-1}$ of $F_l(s)$ in the half-plane $\Re s > 1/2$. Thus, by standard argument in complex analysis, we know the residue at $s = 1$ is equal to $xP_l(\log x)$, where $P_l(\log x)$ is a polynomial of degree $\binom{l}{r} - \binom{l}{r-1} - 1$ for $l = 2r$; otherwise, 0. Thus, we get

$$\sum_{n\le x} \lambda_f(n)^l = xP_l(\log x) + \frac{1}{2\pi i}\left(\int_{\sigma_0+iT}^{1+\epsilon+iT} + \int_{\sigma_0-iT}^{1+\epsilon-iT} + \int_{\sigma_0-iT}^{\sigma_0+iT}\right) F_l(s)\frac{x^s}{s}ds + O_{f,\epsilon}\Big(\frac{x^{1+\epsilon}}{T}\Big). \tag{3.1}$$

The absolute convergence of $H_l(s)$ for $\Re s > 1/2 + \epsilon$ yields $H_l(s) \ll 1$. Hence, (3.1) can be written as

$$\sum_{n\le x} \lambda_f(n)^l = xP_l(\log x) + O_{f,\epsilon}\Big(\frac{x^{1+\epsilon}}{T} + \Re_l^h + \Re_l^v\Big), \tag{3.2}$$

where

$$\Re_l^h = \frac{1}{T}\int_{\sigma_0}^{1+\epsilon} |G_l(\sigma + iT)| x^{\sigma} d\sigma$$

and

$$\Re_l^v = x^{\sigma_0}\int_1^T |G_l(\sigma_0 + i\tau)|\frac{d\tau}{\tau} \ll x^{\sigma_0+\epsilon} \sup_{1\leq T_1\leq T}\frac{1}{T_1}\int_{T_1}^{2T_1} |G_l(\sigma_0 + i\tau)| d\tau.$$

Our goal is to test for constraints on $\Re_l^h$ and $\Re_l^v$, which can certify $\Re_l^h \ll x^{1+\epsilon}/T$ and $\Re_l^v \ll x^{1+\epsilon}/T$.

Let us consider $2 \mid l, l \geq 6$. When the power of $\zeta(s)$ is less than 12, (2.2) cannot be used directly. In order to get better results, we consider it separately.

Case 1. For $l = 6$, according to Lemma 2.1, we have

$$G_6(s) = \zeta(s)^5 L(\mathrm{sym}^2 f, s)^9 L(\mathrm{sym}^4 f, s)^5 L(\mathrm{sym}^6 f, s).$$

Taking $U = x^{\frac{120}{3391}}$ in (2.4), there must exist a $T^* \in (U, 2U)$ such that

$$\zeta(\sigma + iT^*) \ll_{\epsilon} \exp(C(\log\log U)^2) \ll U^{\epsilon}.$$

Suppose that $T = \delta U$ with $1 < \delta < 2$. Now, we choose

$$\sigma_0 = \frac{5}{7},\ \ T = T^* = \delta U = \delta x^{\frac{120}{3391}}$$

in (3.2). Then, we obtain

$$\begin{aligned}\Re_6^h &\ll \frac{1}{T}\int_{\frac{5}{7}}^{1+\epsilon}\Big(\frac{T}{\delta}\Big)^{5\epsilon} T^{*(9\times\frac{6}{5}+5\times\frac{5}{2}+\frac{7}{2})(1-\sigma)+\epsilon} x^{\sigma} d\sigma \\ &\ll T^{\frac{129}{5}+\epsilon}\int_{\frac{5}{7}}^{1+\epsilon}\left(\frac{x}{T^{\frac{134}{5}}}\right)^{\sigma} d\sigma \ll \frac{x^{1+\epsilon}}{T} + x^{\frac{5}{7}+\epsilon}T^{\frac{233}{35}+\epsilon}.\end{aligned} \tag{3.3}$$

Applying the Hölder's inequality, we obtain

$$\begin{aligned}\Re_6^v \ll & x^{\frac{5}{7}+\epsilon}\sup_{1\leq T_1\leq T}\frac{1}{T_1}\int_{T_1}^{2T_1}\left|G_6(\frac{5}{7}+i\tau)\right| d\tau \\ \ll & x^{\frac{5}{7}+\epsilon}\sup_{1\leq T_1\leq T}\frac{1}{T_1}\left(\int_{T_1}^{2T_1}|\zeta(\frac{5}{7}+i\tau)|^{12}d\tau\right)^{\frac{5}{12}}\left(\int_{T_1}^{2T_1}|L(\mathrm{sym}^4 f, \frac{5}{7}+i\tau)|^{\frac{60}{7}}d\tau\right)^{\frac{7}{12}} \\ & \times L(\mathrm{sym}^2 f, \frac{5}{7}+iT_1)^9 L(\mathrm{sym}^6 f, \frac{5}{7}+iT_1).\end{aligned}$$

By (2.2), (2.7), and Lemma 2.4, we have

$$\Re_6^v \ll x^{\frac{5}{7}+\epsilon}T^{-1+\frac{5}{12}(1+\epsilon)+(5\times\frac{7}{12}+\frac{5}{2}\times\frac{46}{7}\times\frac{7}{12}+9\times\frac{6}{5}+\frac{7}{2})(1-\frac{5}{7})+\epsilon} \ll x^{\frac{5}{7}+\epsilon}T^{\frac{2971}{420}+\epsilon}. \tag{3.4}$$

Combining (3.2)–(3.4), we obtain

$$\sum_{n\leq x}\lambda_f(n)^6 = xP_6(\log x) + O\Big(\frac{x^{1+\epsilon}}{T} + x^{\frac{5}{7}+\epsilon}T^{\frac{2971}{420}+\epsilon}\Big).$$

Recall that

$$T=\delta x^{\frac{120}{3391}}$$

with $1<\delta<2$. Thus, we get the required result.

Case 2. When $l=2r\geq 8$, according to Lemma 2.1, we obtain

$$G_l(s)=\prod_{n=0}^{r}L(\mathrm{sym}^{l-2n}f,s)^{\left(\binom{l}{n}-\binom{l}{n-1}\right)}.$$

Take $U=x^{\theta_{2,r}^{-1}}$ in (2.4), where

$$\theta_{2,r}=\frac{13}{42r}\binom{l}{r-1}+\frac{18}{5(r-1)}\binom{l}{r-2}+\frac{\sum_{n=1}^{r-2}\frac{(l-2n+1)^2}{n}\binom{l}{n-1}+l+1}{2}-\frac{3}{14}.$$

Then, there must exist a $T^*\in(U,2U)$ such that

$$\zeta(\sigma+iT^*)\ll_\epsilon \exp(C(\log\log U)^2)\ll U^\epsilon.$$

Suppose that $T=\delta U$ with $1<\delta<2$. Now, we choose

$$\sigma_0=\frac{5}{7},\ \ T=T^*=\delta U=\delta x^{\theta_{2,r}^{-1}}$$

in (3.2). Then, we obtain

$$\begin{aligned}\Re_l^h&\ll\frac{1}{T}\int_{\frac{5}{7}}^{1+\epsilon}\left(\frac{T}{\delta}\right)^{\frac{1}{r}\binom{l}{r-1}\epsilon}T^{\left(\frac{18}{5(r-1)}\binom{l}{r-2}+\frac{\sum_{n=1}^{r-2}\frac{(l-2n+1)^2}{n}\binom{l}{n-1}+l+1}{2}\right)(1-\sigma)+\epsilon}x^\sigma d\sigma\\&\ll T^{\theta_{2,r}-\frac{13}{42r}\binom{l}{r-1}+\frac{3}{14}-1+\epsilon}\int_{\frac{5}{7}}^{1+\epsilon}\left(\frac{x}{T^{\theta_{2,r}-\frac{13}{42r}\binom{l}{r-1}+\frac{3}{14}}}\right)^\sigma d\sigma\\&\ll\frac{x^{1+\epsilon}}{T}+x^{\frac{5}{7}+\epsilon}T^{\frac{2}{7}\left(\theta_{2,r}-\frac{13}{42r}\binom{l}{r-1}+\frac{3}{14}\right)-1+\epsilon}.\end{aligned}\tag{3.5}$$

Applying the Hölder inequality and (2.2), we obtain

$$\begin{aligned}\Re_l^v\ll&x^{\frac{5}{7}+\epsilon}\sup_{1\leq T_1\leq T}\frac{1}{T_1}\int_{T_1}^{2T_1}\left|G_l(\frac{5}{7}+i\tau)\right|d\tau\\\ll&x^{\frac{5}{7}+\epsilon}\sup_{1\leq T_1\leq T}\frac{1}{T_1}\left(\int_{T_1}^{2T_1}|\zeta(\frac{5}{7}+i\tau)|^{12}d\tau\right)\zeta(\frac{5}{7}+iT_1)^{\frac{13}{41r}\binom{l}{r-1}-12}\\&\times\prod_{n=0}^{r-2}L(\mathrm{sym}^{l-2n}f,\frac{5}{7}+iT_1)^{(\binom{l}{n}-\binom{l}{n-1})}\\\ll&x^{\frac{5}{7}+\epsilon}T^{\frac{2}{7}\theta_{2,r}-1+\epsilon}.\end{aligned}\tag{3.6}$$

Combining (3.2), (3.5), and (3.6), we obtain

$$\sum_{n\leq x}\lambda_f(n)^l=xP_l(\log x)+O\left(\frac{x^{1+\epsilon}}{T}+x^{\frac{5}{7}+\epsilon}T^{\frac{2}{7}\theta_{2,r}-1+\epsilon}\right).$$

Recall that $T=\delta x^{\theta_{2,r}^{-1}}$ with $1<\delta<2$. Thus, we get the required result.

3.2. Proof of (ii) in Theorem 1.1

To prove these results, we will use the following proposition:

Proposition 3.1. *Let $f, g \in H_k^*$, then we have*

$$F_{l,m}(s) = G_{l,m}(s)H_{l,m}(s),$$

where

$$G_{l,m}(s) = \prod_{n_1=0}^{[\frac{l}{2}]}\prod_{n_2=0}^{[\frac{m}{2}]} L(sym^{l-2n_1}f \times sym^{m-2n_2}g, s)^{\left(\binom{l}{n_1}-\binom{l}{n_1-1}\right)\left(\binom{m}{n_2}-\binom{m}{n_2-1}\right)}, \tag{3.7}$$

$\binom{l}{n_1}$ and $\binom{m}{n_2}$ are the binomial coefficients with the convention that $\binom{l}{n_1} = 0$ and $\binom{m}{n_2} = 0$ if $n_1, n_2 < 0$, and the function $H_{l,m}(s)$ admits a Dirichlet series convergent absolutely in $\Re s > 1/2$. Note $H_{l,m}(s) \neq 0$ for $\Re s = 1$.

Proof. For $f \in H_k^*$, we know

$$F_{l,m}(s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)^l \lambda_g(n)^m}{n^s} = \prod_p \Big(1 + \sum_{\nu \geq 1} \frac{\lambda_f(p^\nu)^l \lambda_g(p^\nu)^m}{p^{\nu s}}\Big).$$

By [11, Lemma 5], we get that

$$\lambda_f(p)^l = \sum_{n_1=0}^{[\frac{l}{2}]} \left(\binom{l}{n_1} - \binom{l}{n_1-1}\right) \lambda_{\mathrm{sym}^{l-2n_1}f}(p),$$

$$\lambda_g(p)^m = \sum_{n_2=0}^{[\frac{m}{2}]} \left(\binom{m}{n_2} - \binom{m}{n_2-1}\right) \lambda_{\mathrm{sym}^{m-2n_2}g}(p).$$

Hence, the coefficient of p^{-s} is

$$\begin{aligned} \lambda_f(p)^l \lambda_g(p)^m &= \left(\sum_{n_1=0}^{[\frac{l}{2}]} \left(\binom{l}{n_1} - \binom{l}{n_1-1}\right) \lambda_{\mathrm{sym}^{l-2n_1}f}(p)\right) \\ &\quad \times \left(\sum_{n_2=0}^{[\frac{m}{2}]} \left(\binom{m}{n_2} - \binom{m}{n_2-1}\right) \lambda_{\mathrm{sym}^{m-2n_2}g}(p)\right) \\ &= \sum_{n_1=0}^{[\frac{l}{2}]}\sum_{n_2=0}^{[\frac{m}{2}]} \left(\binom{l}{n_1} - \binom{l}{n_1-1}\right)\left(\binom{m}{n_2} - \binom{m}{n_2-1}\right) \lambda_{\mathrm{sym}^{l-2n_1}f\times\mathrm{sym}^{m-2n_2}g}(p). \end{aligned}$$

We define

$$H_{l,m}(s) = F_{l,m}(s)/G_{l,m}(s),$$

and its p-local factor is of the form $1 + O(p^{-2s})$. So, the Euler product (hence, the Dirichlet series) of $H_{l,m}(s)$ converges absolutely in $\Re s > 1/2$. □

Utilizing the similar method in Section 3.1 and the decomposition in Proposition 3.1, we can get the claim easily.

3.2.1. Even l and m

Case 1. When $l = m = 2$, according to Proposition 3.1, we have

$$G_{2,2}(s) = \zeta(s)L(\mathrm{sym}^2 f, s)L(\mathrm{sym}^2 g, s)L(\mathrm{sym}^2 f \times \mathrm{sym}^2 g, s).$$

Taking $U = x^{120/893}$ in (2.4), there must exist a $T^* \in (U, 2U)$ such that

$$\zeta(\sigma + iT^*) \ll_\epsilon \exp(C(\log\log U)^2) \ll U^\epsilon.$$

Suppose that $T = \delta U$ with $1 < \delta < 2$. Now, we choose

$$\sigma_0 = \frac{5}{7}, \ \ T = T^* = \delta U = \delta x^{\frac{120}{893}}$$

in (3.2). Then,

$$\begin{aligned}\Re^h_{2,2} &\ll \frac{1}{T}\int_{\frac{5}{7}}^{1+\epsilon}\Big(\frac{T}{\delta}\Big)^{\epsilon} T^{(\frac{6}{5}+\frac{6}{5}+\frac{9}{2})(1-\sigma)+\epsilon} x^\sigma d\sigma \\ &\ll T^{\frac{59}{10}+\epsilon}\int_{\frac{5}{7}}^{1+\epsilon}\left(\frac{x}{T^{\frac{69}{10}}}\right)^{\sigma} d\sigma \ll \frac{x^{1+\epsilon}}{T} + x^{\frac{5}{7}+\epsilon}T^{\frac{34}{35}+\epsilon}.\end{aligned} \tag{3.8}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}\Re^v_{2,2} \ll& x^{\frac{5}{7}+\epsilon}\sup_{1\le T_1\le T}\frac{1}{T_1}\int_{T_1}^{2T_1}\Big|G_{2,2}(\frac{5}{7}+i\tau)\Big|d\tau \\ \ll& x^{\frac{5}{7}+\epsilon}\sup_{1\le T_1\le T}\frac{1}{T_1}\left(\int_{T_1}^{2T_1}|\zeta(\frac{5}{7}+i\tau)|^{12}d\tau\right)^{\frac{1}{12}}\left(\int_{T_1}^{2T_1}|L(\mathrm{sym}^2 f\times\mathrm{sym}^2 g, \frac{5}{7}+i\tau)|^2 d\tau\right)^{\frac{1}{2}} \\ &\times\left(\int_{T_1}^{2T_1}|L(\mathrm{sym}^2 f, \frac{5}{7}+i\tau)|^{\frac{12}{5}}d\tau\right)^{\frac{5}{12}} L(\mathrm{sym}^2 g, \frac{5}{7}+iT_1).\end{aligned}$$

By (2.2), (2.7), and Lemma 2.4, we have

$$\Re^v_{2,2} \ll x^{\frac{5}{7}+\epsilon}T^{-1+\frac{1}{12}(1+\epsilon)+(\frac{9}{2}+\frac{2}{5}\times\frac{5}{12}\times\frac{6}{5}+\frac{5}{12}\times 3+\frac{6}{5})(1-\frac{5}{7})+\epsilon} \ll x^{\frac{5}{7}+\epsilon}T^{\frac{473}{420}+\epsilon}. \tag{3.9}$$

Combining (3.2), (3.8), and (3.9), we obtain

$$\sum_{n\le x}\lambda_f(n)^2\lambda_g(n)^2 = xP_{2,2}(\log x) + O\Big(\frac{x^{1+\epsilon}}{T} + x^{\frac{5}{7}+\epsilon}T^{\frac{473}{420}+\epsilon}\Big).$$

Recall that $T = \delta x^{\frac{120}{893}}$ with $1 < \delta < 2$, Thus, we get the required result.

Utilizing the similar method of $\lambda_f(n)^2\lambda_g(n)^2$, we can get the results of $\lambda_f(n)^2\lambda_g(n)^4$, $\lambda_f(n)^4\lambda_g(n)^2$, $\lambda_f(n)^4\lambda_g(n)^4$, $\lambda_f(n)^4\lambda_g(n)^6$, and $\lambda_f(n)^6\lambda_g(n)^4$ easily.

Case 2. When $l = m = 6$, $l \ge 8$, or $m \ge 8$, we have

$$G_{l,m}(s) = \prod_{n_1=0}^{r}\prod_{n_2=0}^{\bar{r}} L(\mathrm{sym}^{l-2n_1} f \times \mathrm{sym}^{m-2n_2} g, s)^{\left(\binom{l}{n_1}-\binom{l}{n_1-1}\right)\left(\binom{m}{n_2}-\binom{m}{n_2-1}\right)}$$

$$
\begin{aligned}
=&\zeta(s)^{(\binom{l}{r}-\binom{l}{r-1})(\binom{m}{\bar r}-\binom{m}{\bar r-1})}L(\mathrm{sym}^2 f,s)^{(\binom{l}{r-1}-\binom{l}{r-2})(\binom{m}{\bar r}-\binom{m}{\bar r-1})}L(\mathrm{sym}^2 g,s)^{(\binom{l}{r}-\binom{l}{r-1})(\binom{m}{\bar r-1}-\binom{m}{\bar r-2})}\\
&\times\prod_{n_1=0}^{r-2}L(\mathrm{sym}^{l-2n_1}f,s)^{\left(\binom{l}{n_1}-\binom{l}{n_1-1}\right)(\binom{m}{\bar r}-\binom{m}{\bar r-1})}\prod_{n_2=0}^{\bar r-2}L(\mathrm{sym}^{m-2n_2}g,s)^{(\binom{l}{r}-\binom{l}{r-1})\left(\binom{m}{n_2}-\binom{m}{n_2-1}\right)}\\
&\times\prod_{n_1=0}^{r-1}\prod_{n_2=0}^{\bar r-1}L(\mathrm{sym}^{l-2n_1}f\times\mathrm{sym}^{m-2n_2}g,s)^{\left(\binom{l}{n_1}-\binom{l}{n_1-1}\right)\left(\binom{m}{n_2}-\binom{m}{n_2-1}\right)},
\end{aligned}
$$

where $l=2r,\ m=2\bar r$.

Taking $U=x^{\theta_{2,2,r,\bar r}^{-1}}$ in (2.4), there must exist a $T^*\in(U,2U)$ such that

$$\zeta(\sigma+iT^*)\ll_\epsilon\exp(C(\log\log U)^2)\ll U^\epsilon.$$

Suppose that $T=\delta U$ with $1<\delta<2$. Now, we choose

$$\sigma_0=\frac{5}{7},\quad T=T^*=\delta U=\delta x^{\theta_{2,2,r,\bar r}^{-1}},$$

where

$$
\begin{aligned}
\theta_{2,2,r,\bar r}=&\frac{13\binom{l}{r-1}\binom{m}{\bar r-1}}{42r\bar r}+\frac{18\binom{l}{r-2}\binom{m}{\bar r-1}}{5\bar r(r-1)}+\frac{18\binom{l}{r-1}\binom{m}{\bar r-2}}{5r(\bar r-1)}+\frac{\sum\limits_{n_1=1}^{r-2}\frac{(l-2n_1+1)^2}{\bar r n_1}\binom{l}{n_1-1}\binom{m}{\bar r-1}+(l+1)\frac{1}{\bar r}\binom{m}{\bar r-1}}{2}\\
&+\frac{\sum\limits_{n_2=1}^{\bar r-2}\frac{(m-2n_2+1)^2}{rn_2}\binom{l}{r-1}\binom{m}{n_2-1}(m+1)\frac{1}{r}\binom{l}{r-1}}{2}+\frac{(l+1)(m+1)}{2}+\frac{\sum\limits_{n_1=1}^{r-1}\frac{(l-2n_1+1)^2(m+1)}{n_1}\binom{l}{n_1-1}}{2}\\
&+\frac{\sum\limits_{n_2=1}^{\bar r-1}\frac{(m-2n_2+1)^2(l+1)}{n_2}\binom{m}{n_2-1}}{2}+\frac{\sum\limits_{n_1=1}^{r-1}\sum\limits_{n_2=1}^{\bar r-1}\frac{(l-2n_1+1)^2(m-2n_2+1)^2}{n_1n_2}\binom{l}{n_1-1}\binom{m}{n_2-1}}{2}-\frac{3}{14}
\end{aligned}
$$

in (3.2). Then,

$$
\begin{aligned}
\Re_{l,m}^h\ll&\frac{1}{T}\int_{\frac57}^{1+\epsilon}\left(\frac{T}{\delta}\right)^{\frac{1}{r\bar r}\binom{l}{r-1}\binom{m}{\bar r-1}\epsilon}T^{\left(\theta_{2,2,r,\bar r}-\frac{13}{42r\bar r}\binom{l}{r-1}\binom{m}{\bar r-1}+\frac{3}{14}\right)(1-\sigma)}x^\sigma d\sigma\\
\ll&T^{\left(\theta_{2,2,r,\bar r}-\frac{13}{42r\bar r}\binom{l}{r-1}\binom{m}{\bar r-1}+\frac{3}{14}\right)-1+\epsilon}\int_{\frac57}^{1+\epsilon}\left(\frac{x}{T^{\left(\theta_{2,2,r,\bar r}-\frac{13}{42r\bar r}\binom{l}{r-1}\binom{m}{\bar r-1}+\frac{3}{14}\right)}}\right)^\sigma d\sigma\\
\ll&\frac{x^{1+\epsilon}}{T}+x^{\frac57+\epsilon}T^{\frac27\left(\theta_{2,2,r,\bar r}-\frac{13}{42r\bar r}\binom{l}{r-1}\binom{m}{\bar r-1}+\frac{3}{14}\right)-1+\epsilon}.
\end{aligned}
\tag{3.10}
$$

Applying the Hölder's inequality, we obtain

$$
\begin{aligned}
\Re_{l,m}^v\ll&x^{\frac57+\epsilon}\sup_{1\le T_1\le T}\frac{1}{T_1}\int_{T_1}^{2T_1}\left|G_{l,m}(\frac57+i\tau)\right|d\tau\\
\ll&x^{\frac57+\epsilon}\sup_{1\le T_1\le T}\frac{1}{T_1}\left(\int_{T_1}^{2T_1}|\zeta(\frac57+i\tau)|^{12}d\tau\right)T_1^{(\theta_{2,2,r,\bar r}-\frac72)(1-\frac57)+\epsilon}\\
\ll&x^{\frac57+\epsilon}T^{\frac27\theta_{2,2,r,\bar r}-1+\epsilon}.
\end{aligned}
\tag{3.11}
$$

Combining (3.2), (3.10), and (3.11), we obtain

$$\sum_{n\le x}\lambda_f(n)^l\lambda_g(n)^m = xP_1(\log x) + O\left(\frac{x^{1+\epsilon}}{T} + x^{\frac{5}{7}+\epsilon}T^{\frac{2}{7}\theta_{2,2,r,\bar{r}}-1+\epsilon}\right).$$

Recall that $T = \delta x^{\theta^{-1}_{2,2,r,\bar{r}}}$ with $1 < \delta < 2$. Thus, we get the required result.

3.2.2. Odd l and m

From (3.7), we have

$$\begin{aligned}G_{l,m}(s) =& \prod_{n_1=0}^{t}\prod_{n_2=0}^{\bar{t}} L(\mathrm{sym}^{l-2n_1}f\times \mathrm{sym}^{m-2n_2}g, s)^{\left(\binom{l}{n_1}-\binom{l}{n_1-1}\right)\left(\binom{m}{n_2}-\binom{m}{n_2-1}\right)}\\ =&\prod_{n_1=1}^{t} L(\mathrm{sym}^{l-2n_1}f\times \mathrm{sym}^{m}g, s)^{\left(\binom{l}{n_1}-\binom{l}{n_1-1}\right)}\prod_{n_2=1}^{\bar{t}} L(\mathrm{sym}^{l}f\times \mathrm{sym}^{m-2n_2}g, s)^{\left(\binom{m}{n_2}-\binom{m}{n_2-1}\right)}\\ &\times L(\mathrm{sym}^l f\times \mathrm{sym}^m g)\prod_{n_1=1}^{t}\prod_{n_2=1}^{\bar{t}} L(\mathrm{sym}^{l-2n_1}f\times \mathrm{sym}^{m-2n_2}g, s)^{\left(\binom{l}{n_1}-\binom{l}{n_1-1}\right)\left(\binom{m}{n_2}-\binom{m}{n_2-1}\right)},\end{aligned}$$

where $l = 2t + 1,\ m = 2\bar{t} + 1$. Then,

$$\begin{aligned}\Re^h_{l,m} &\ll \frac{1}{T}\int_{\frac{1}{2}}^{1+\epsilon} T^{\theta_{1,1,t,\bar{t}}(1-\sigma)}x^{\sigma}d\sigma \ll T^{\theta_{1,1,t,\bar{t}}-1}\int_{\frac{1}{2}}^{1+\epsilon}\left(\frac{x}{T^{\theta_{1,1,t,\bar{t}}}}\right)^{\sigma}d\sigma\\ &\ll \frac{x^{1+\epsilon}}{T} + x^{\frac{1}{2}+\epsilon}T^{\frac{1}{2}\theta_{1,1,t,\bar{t}}-1},\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}\theta_{1,1,t,\bar{t}} =& \frac{(l+1)(m+1)}{2} + \frac{\sum_{n_1=1}^{t}\frac{(l-2n_1+1)^2(m+1)}{n_1}\binom{l}{n_1-1}}{2} + \frac{\sum_{n_2=1}^{\bar{t}}\frac{(m-2n_2+1)^2(l+1)}{n_2}\binom{m}{n_2-1}}{2}\\ &+\frac{\sum_{n_1=1}^{t}\sum_{n_2=1}^{\bar{t}}\frac{(l-2n_1+1)^2(m-2n_2+1)^2}{n_1n_2}\binom{l}{n_1-1}\binom{m}{n_2-1}}{2}.\end{aligned}$$

Applying the Cauchy inequality and (2.9), we obtain

$$\begin{aligned}\Re^v_{l,m} \ll& x^{\frac{1}{2}+\epsilon}\sup_{1\le T_1\le T}\frac{1}{T_1}\int_{T_1}^{2T_1}\left|G_{l,m}(\frac{1}{2}+i\tau)\right|d\tau\\ \ll& x^{\frac{1}{2}+\epsilon}\sup_{1\le T_1\le T}\frac{1}{T_1}\int_{T_1}^{2T_1}\left|L(\mathrm{sym}^{l-2}f\times\mathrm{sym}^m g, s)\right|^2 d\tau\times T_1^{(\theta_{1,1,t,\bar{t}}-2\frac{(l-1)(m+1)}{2})(1-\frac{1}{2})+\epsilon}\\ \ll& x^{\frac{1}{2}+\epsilon}T^{\frac{1}{2}\theta_{1,1,t,\bar{t}}-1+\epsilon}.\end{aligned} \tag{3.13}$$

Combining (3.2), (3.12), and (3.13), we obtain

$$\sum_{n\le x}\lambda_f(n)^l\lambda_g(n)^m = xP_{l,m}(\log x) + O\left(\frac{x^{1+\epsilon}}{T} + x^{\frac{1}{2}+\epsilon}T^{\frac{1}{2}\theta_{1,1,t,\bar{t}}-1+\epsilon}\right).$$

By taking a fixed $T = x^{\theta^{-1}_{1,1,t,\bar{t}}}$, we obtain the result.

3.2.3. Odd l and even m

When $2 \nmid l, 2 \mid m$, because (2.5) cannot be used directly when the power of $L(f,s)$ is less than 4, we need to think about this case separately.

Case 1. When $l=3, m=2$, according to Proposition 3.1 and taking $\sigma_0 = \frac{5}{8}$, we have

$$G_{3,2}(s) = L(f,s)^2 L(f\times \mathrm{sym}^2 g, s)^2 L(\mathrm{sym}^3 f, s) L(\mathrm{sym}^3 f \times \mathrm{sym}^2 g, s).$$

Then, we get

$$\begin{aligned}
\Re^h_{3,2} &= \frac{1}{T}\int_{\frac{5}{8}}^{1+\epsilon} \left| L(f,s)^2 L(f\times \mathrm{sym}^2 g, s)^2 L(\mathrm{sym}^3 f, s) L(\mathrm{sym}^3 f \times \mathrm{sym}^2 g, s)\right| x^{\sigma} d\sigma \\
&\ll \frac{1}{T}\int_{\frac{5}{8}}^{1+\epsilon} T^{\left(2\times\frac{2}{3}+2\times\frac{27}{10}+\frac{4}{2}+\frac{12}{2}\right)(1-\sigma)+\epsilon} x^{\sigma} d\sigma \\
&\ll T^{\frac{206}{15}+\epsilon}\int_{\frac{5}{8}}^{1+\epsilon}\left(\frac{x}{T^{\frac{221}{15}}}\right)^{\sigma} d\sigma \\
&\ll \frac{x^{1+\epsilon}}{T} + x^{\frac{5}{8}+\epsilon} T^{\frac{181}{40}+\epsilon}.
\end{aligned} \tag{3.14}$$

In order to estimate $\Re^v_{3,2}$, we apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
\Re^v_{3,2} &\ll x^{\frac{5}{8}+\epsilon} \sup_{1\le T_1\le T} \frac{1}{T_1}\int_{T_1}^{2T_1} \left| G_{3,2}(\frac{5}{8}+i\tau)\right| d\tau \\
&\ll x^{\frac{5}{8}+\epsilon} \sup_{1\le T_1\le T} \frac{1}{T_1} T_1^{\left(2\times\frac{27}{10}+\frac{12}{2}\right)\left(1-\frac{5}{8}\right)+\epsilon}\left(\int_{T_1}^{2T_1}\left|L(\mathrm{sym} f, \frac{5}{8}+i\tau)\right|^4 d\tau\right)^{\frac{1}{2}} \\
&\quad \times\left(\int_{T_1}^{2T_1}\left|L(\mathrm{sym}^3 f, \frac{5}{8}+i\tau)\right|^2 d\tau\right)^{\frac{1}{2}} \\
&\ll x^{\frac{5}{8}+\epsilon} \sup_{1\le T_1\le T} \frac{1}{T_1} T_1^{\left(2\times\frac{27}{10}+\frac{12}{2}\right)\left(1-\frac{5}{8}\right)+\epsilon} T_1^{\frac{1}{2}+\epsilon} T_1^{\frac{1}{2}\times 4\times\left(1-\frac{5}{8}\right)+\epsilon} \\
&\ll x^{\frac{5}{8}+\epsilon} T^{\frac{181}{40}+\epsilon}.
\end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15) with (3.2) and $T = x^{\frac{15}{221}}$, we get the required result.

Case 2. When l is odd, m is even, and $l\neq 3$ or $m \neq 2$. From (3.7), we have

$$\begin{aligned}
G_{l,m}(s) &= \prod_{n_1=0}^{t}\prod_{n_2=0}^{\bar r} L(\mathrm{sym}^{l-2n_1} f\times \mathrm{sym}^{m-2n_2} g, s)^{\left(\binom{l}{n_1}-\binom{l}{n_1-1}\right)\left(\binom{m}{n_2}-\binom{m}{n_2-1}\right)} \\
&= L(f,s)^{\left(\binom{l}{t}-\binom{l}{t-1}\right)\left(\binom{m}{\bar r}-\binom{m}{\bar r-1}\right)} \prod_{n_1=0}^{t-1}\prod_{n_2=0}^{\bar r} L(\mathrm{sym}^{l-2n_1} f\times \mathrm{sym}^{m-2n_2} g, s)^{\left(\binom{l}{n_1}-\binom{l}{n_1-1}\right)\left(\binom{m}{n_2}-\binom{m}{n_2-1}\right)} \\
&\quad \times L(f\times\mathrm{sym}^2 g, s)^{\left(\binom{l}{t}-\binom{l}{t-1}\right)\left(\binom{m}{\bar r-1}-\binom{m}{\bar r-2}\right)} \prod_{n_2=0}^{\bar r-2} L(f\times \mathrm{sym}^{m-2n_2} g, s)^{\left(\binom{l}{t}-\binom{l}{t-1}\right)\left(\binom{m}{n_2}-\binom{m}{n_2-1}\right)},
\end{aligned}$$

where $l = 2t + 1,\ m = 2\bar{r}$. By taking $\sigma_0 = \frac{5}{8}$, then

$$\begin{aligned}\Re^h_{l,m} &\ll \frac{1}{T}\int_{\frac{5}{8}}^{1+\epsilon} T^{\theta_{1,2,t,\bar{r}}(1-\sigma)} x^\sigma d\sigma \ll T^{\theta_{1,2,t,\bar{r}}-1}\int_{\frac{5}{8}}^{1+\epsilon}\left(\frac{x}{T^{\theta_{1,2,t,\bar{r}}}}\right)^\sigma d\sigma \\ &\ll \frac{x^{1+\epsilon}}{T} + x^{\frac{5}{8}+\epsilon}T^{\frac{3}{8}\theta_{1,2,t,\bar{r}}-1},\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\theta_{1,2,t,\bar{r}} =& \frac{4\binom{l}{t-1}\binom{m}{\bar{r}-1}}{3t\bar{r}} + \frac{81\binom{l}{t-1}\binom{m}{\bar{r}-2}}{5t(\bar{r}-1)} + \frac{\frac{4(m+1)}{t}\binom{l}{t-1} + \sum\limits_{n_2=1}^{\bar{r}-2}\frac{4(m-2n_2+1)^2}{tn_2}\binom{l}{t-1}\binom{m}{n_2-1}}{2} \\ &+ \frac{(l+1)(m+1)}{2} + \frac{\sum\limits_{n_1=1}^{t-1}\frac{(l-2n_1+1)^2(m+1)}{n_1}\binom{l}{n_1-1}}{2} + \frac{\sum\limits_{n_2=1}^{\bar{r}}\frac{(m-2n_2+1)^2(l+1)}{n_2}\binom{m}{n_2-1}}{2} \\ &+ \frac{\sum\limits_{n_1=1}^{t-1}\sum\limits_{n_2=1}^{\bar{r}}\frac{(l-2n_1+1)^2(m-2n_2+1)^2}{n_1n_2}\binom{l}{n_1-1}\binom{m}{n_2-1}}{2}.\end{aligned}$$

Applying the Hölder inequality, we obtain

$$\begin{aligned}\Re^v_{l,m} \ll& x^{\frac{5}{8}+\epsilon}\sup_{1\leqslant T_1\leqslant T}\frac{1}{T_1}\int_{T_1}^{2T_1}\left|G_{l,m}(\frac{5}{8}+i\tau)\right|d\tau \\ \ll& x^{\frac{5}{8}+\epsilon}\sup_{1\leqslant T_1\leqslant T}\frac{1}{T_1}\left(\int_{T_1}^{2T_1}\left|L(f,\frac{5}{8}+i\tau)\right|^4 d\tau\right)T_1^{(\theta_{1,2,t,\bar{r}}-4\times\frac{2}{3})(1-\frac{5}{8})+\epsilon} \\ \ll& x^{\frac{5}{8}+\epsilon}T^{\frac{5}{8}\theta_{1,2,t,\bar{r}}-1+\epsilon}.\end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17) with (3.2) and $T = x^{\theta^{-1}_{1,2,t,\bar{r}}}$, we get the required result.

3.2.4. Oven l and odd m

The proof of $2 \mid l, 2 \nmid m$ is similar to $2 \nmid l, 2 \mid m$, so it can be estimated in a similar way. In order to avoid repetition, we shall not prove it verbatim here.

4. Proof of Theorem 1.3

In order to prove Theorem 1.3, we state the following result of Meher and Murty [19, Theorem 1.1] to detect sign changes:

Lemma 4.1. *Suppose a sequence of real numbers* $\{a(n)\}$ *satisfies*

(1) $a(n) = O(x^{\alpha+\epsilon})$,

(2) $\sum\limits_{n\leq x} a(n) = O(x^{\beta+\epsilon})$,

(3) $\sum\limits_{n\leq x} a(n)^2 = cx + O(x^{\gamma+\epsilon})$,

where α, β, γ, and c are positive real constants. If $\alpha + \beta < 1$, then for any r with $\max\{\alpha + \beta, \gamma\} < r < 1$, the sequence $\{a(n)\}$ has at least one sign change for $n \in (x, x + x^r]$ for sufficiently large x. Moreover, the number of sign changes of $\{a(n)\}$ for $n \leq x$ is $\gg x^{1-r}$.

(i) Let $f \in H_k^*$. From Wu [3, Theorem 2] and Huang [6, Theorem 1], we have

$$\sum_{n\leq x} \lambda_f(n) \ll x^{\frac{1}{3}}(\log x)^{-0.118} \quad \text{and} \quad \sum_{n\leq x} \lambda_f(n)^2 = Cx + O(x^{\frac{3}{5}-\frac{1}{560}+\epsilon}). \tag{4.1}$$

Combining (1.1) and (4.1) in Lemma 4.1, we know that $\alpha = 0,\ \beta = 1/3,\ \gamma = 67/112$, which means

$$\max\{\alpha + \beta, \gamma\} = 67/112 < 1.$$

(ii) Let $f, g \in H_k^*$. In 2014, Lü [15] proved that

$$\sum_{n\leq x} \lambda_f(n)\lambda_g(n) \ll x^{\frac{3}{5}} \log x^{-\frac{2}{3}(1-\frac{8}{3\pi})}.$$

From Theorem 1.1, we improve the error term for the sharp-cut sum (1.5) with $l = m = 2$ from $O(x^{13/15+\epsilon})$ to $O(x^{773/893+\epsilon})$. With Lemma 4.1, we obtain $\alpha = 0,\ \beta = 3/5,\ \gamma = 773/893$, which means

$$\max\{\alpha + \beta, \gamma\} = 773/893 < 1.$$

Thus, we finish the proof of Theorem 1.3.

5. Conclusions

In this paper, we study the distribution of Fourier coefficients of holomorphic cusp forms. Let $\lambda_f(n)$ be the nth normalized Fourier coefficient of a holomorphic cusp form f for the full modular group. Combining the classical analytic method with property of some primitive automorphic L-functions, we establish asymptotic formulae for high power sums of Fourier coefficients of cusp forms. As an application, we also use a general criteria to detect the signs of $\lambda_f(n)$ and $\lambda_f(n)\lambda_g(n)$, and obtain some quantitative results for the number of sign changes for $n \leq x$. We are able to improve or extend previous results.

Author contributions

Guangwei Hu: writing-review and editing, supervision, validation, methodology, formal analysis, conceptualization, funding acquisition; Huixue Lao: writing-review and editing, resources, methodology, supervision, validation, formal analysis; Huimin Pan: writing-original draft, methodology, validation, formal analysis. All authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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