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Research article

Properties and applications of generalized 1-parameter 3-variable Hermite-based Appell polynomials

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Abstract: We present a novel framework for introducing generalized 3-variable 1-parameter Hermite-based Appell polynomials. These polynomials are characterized by generating function, series definition, and determinant definition, elucidating their fundamental properties. Moreover, utilizing a factorization method, we established recurrence relations, shift operators, and various differential equations, including differential, integrodifferential, and partial differential equations. Special attention is given to exploring the specific cases of 3-variable 1-parameter generalized Hermite-based Bernoulli, Euler, and Genocchi polynomials, offering insights into their unique features and applications.

Keywords: 1-parameter generalized Hermite polynomials; Appell polynomials; determinant definition; recurrence relations; differential equations
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1. Introduction and preliminaries

In recent years, notable progress has been in developing various generalizations of special functions within mathematical physics. These advancements provide a robust analytical framework for solving a wide array of mathematical physics problems and have extensive practical applications across diverse domains. Particularly, the significance of generalized Hermite polynomials has been underscored, as noted in previous studies [1, 2]. These polynomials find utility in addressing challenges in quantum mechanics, optical beam transport, and a spectrum of problems spanning partial differential equations to abstract group theory.

The "2-variable Hermite Kampé de Feriet polynomials (2VHKdFP)", denoted as $\mathcal{D}_n(v_1, v_2)$ [3], are expressed through the following generating function:

$$e^{\nu_1\xi+\nu_2\xi^2} = \sum_{n=0}^{\infty} \mathcal{D}_n(\nu_1, \nu_2)\frac{\xi^n}{n!}. \tag{1.1}$$

Similarly, the "2-variable 1-parameter Hermite polynomials (2V1PHP)", represented as $\mathcal{D}_n(\nu_1, \nu_2, C)$, are defined using the subsequent generating function [4]:

$$C^{\nu_1\xi+\nu_2\xi^2} = \sum_{n=0}^{\infty} \mathcal{D}_n(\nu_1, \nu_2, C)\frac{\xi^n}{n!}, \quad C > 1. \tag{1.2}$$

The "3-variable Hermite polynomials (3VHP)", denoted as $\mathcal{D}_n(\nu_1, \nu_2, \nu_3)$ [5], are characterized by the following generating function:

$$e^{\nu_1\xi+\nu_2\xi^2+\nu_3\xi^3} = \sum_{n=0}^{\infty} \mathcal{D}_n(\nu_1, \nu_2, \nu_3)\frac{\xi^n}{n!}. \tag{1.3}$$

Further motivated by expressions (1.2) and (1.3), we give following representation to the "generalized 3-variable 1-parameter Hermite polynomials (g1P3VHP)", represented as $\mathcal{D}_n(\nu_1, \nu_2, \nu_3, C)$ defined by:

$$C^{\nu_1\xi+\nu_2\xi^2+\nu_3\xi^3} = \sum_{n=0}^{\infty} \mathcal{D}_n(\nu_1, \nu_2, \nu_3, C)\frac{\xi^n}{n!}, \quad C > 1. \tag{1.4}$$

The first few values of $\mathcal{D}_n(\nu_1, \nu_2, \nu_3, C)$ are:

$$\mathcal{D}_0(\nu_1, \nu_2, \nu_3, C) = 1,$$

$$\mathcal{D}_1(\nu_1, \nu_2, \nu_3, C) = \nu_1 \ln C,$$

$$\mathcal{D}_2(\nu_1, \nu_2, \nu_3, C) = \frac{(\nu_1 \ln C)^2}{2} + \nu_2 \ln C,$$

$$\mathcal{D}_3(\nu_1, \nu_2, \nu_3, C) = \frac{(\nu_1 \ln C)^3}{6} + \frac{(\nu_1 \ln C)(\nu_2 \ln C)}{2} + \nu_3 \ln C.$$

In their 3-variable formulation, these polynomials find widespread application across numerous fields in both pure and applied mathematics and physics. They serve as fundamental tools in addressing problems ranging from Laplace's equation in parabolic coordinates to various quantum mechanics and probability theory scenarios. Notably, for any integral value of n, these polynomials represent specific solutions to the heat or generalized heat problem facilitated by the corresponding existence of Gauss-Weierstrass transforms.

Appell polynomials, named after the French mathematician Paul Appell [6], constitute a significant class of special functions in mathematical analysis. They are typically defined as solutions to certain differential equations, often arising in studying various physical and mathematical phenomena. Appell polynomials possess distinctive properties, including recurrence relations, generating function, and explicit formulas, which render them invaluable in diverse areas such as probability theory, mathematical physics, and combinatorics. These polynomials exhibit remarkable versatility as fundamental tools for formulating and solving differential equations, integral transforms, and other mathematical problems. Their rich structure and wide-ranging applications make Appell polynomials an indispensable subject of study in modern mathematics. The generating relation gives the Appell polynomials:

$$\mathcal{R}(\xi)e^{\nu_1\xi} = \sum_{r=0}^{\infty} \mathcal{R}_r(\nu_1)\frac{\xi^r}{r!}, \tag{1.5}$$

where,

$$\mathcal{R}(\xi) = \sum_{r=0}^{\infty} \mathcal{R}_r\frac{\xi^r}{r!}; \quad \mathcal{R}_0 \neq 0. \tag{1.6}$$

Within the context of contemporary classical umbral calculus pioneered by Roman [7], the intrinsic characteristics of Appell sequences are seamlessly managed. Notably, the entire collection of Appell sequences exhibits closure properties when subjected to the operation of the umbral composition of polynomial sequences. This phenomenon extends to form an abelian group, accentuating the structural coherence and algebraic integrity inherent within Appell sequences. Further, Non-separable polynomials like multivariate Appell polynomials have several potential application benefits. In many real-world problems, variables are interdependent rather than independent. Non-separable polynomials can naturally capture these interactions without forcing an artificial separation of variables, (see for instance [8,9]). This can be particularly useful in fields like physics, economics, and machine learning, where interactions between variables are critical. Non-separable polynomials provide a richer set of functional forms, allowing for more flexible modeling of complex relationships. This can lead to better fitting models in regression analysis and more accurate representations of multidimensional phenomena. These polynomials can naturally represent higher-order interactions between variables, which can be difficult to capture with separable polynomials. This feature is valuable in scenarios where the combined effect of multiple variables is non-trivial.

Consider $\{\Psi_n(\nu_1)\}_{n=0}^{\infty}$, which signifies a series of polynomials, we can observe that

$$\deg(\Psi_n(\nu_1)) = n, \ (n \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}.$$

The differential operators $\mathcal{Z}_n^-$ and $\mathcal{Z}_n^+$ meeting the criteria

$$\mathcal{Z}_n^-\{\Psi_n(\nu_1)\} = \Psi_{n-1}(\nu_1), \tag{1.7}$$

$$\mathcal{Z}_n^+\{\Psi_n(\nu_1)\} = \Psi_{n+1}(\nu_1) \tag{1.8}$$

are referred to as multiplicative and derivative operators. $\{\Psi_n(\nu_1)\}_{n=0}^{\infty}$ is a series of polynomials that is considered quasi-monomial if and only if Eqs (1.7) and (1.8) hold, [10–14]. A differential equation like this can be found by finding the derivative and multiplicative operators for a given polynomial family as

$$(\mathcal{Z}_{n+1}^-\mathcal{Z}_n^+)\{\Psi_n(\nu_1)\} = n\Psi_n(\nu_1). \tag{1.9}$$

The factorization technique is the name given to this process. Determining the multiplicative operator $\mathcal{Z}_n^+$ and the derivative operator $\mathcal{Z}_n^-$ forms the basis of the factorization approach [15–18]. The monomiality principle is another way to think about this method. When the factorization approach is applied to the domain of multivariable special functions, new analytical techniques are presented to solve a wide variety of partial differential equations frequently encountered in practical situations.

Differential equations cover a wide range of topics in “physics, engineering, and pure and applied mathematics”. Problems from various scientific and technical fields typically take the form of differential equations, solved using specialized functions. Differential equation theory has attracted renewed attention in the last thirty years due to developments in nonlinear analysis, dynamical systems, and their useful applications in science and engineering.

Several studies employing different generating function approaches and analytical procedures have been conducted to present and analyze hybrid families of special polynomials methodically [19–21]. The “recurrence relations, explicit relations, functional and differential equations, summation formulae, symmetric and convolution identities, and determinant techniques” are just a few of the fundamental characteristics of multi-variable hybrid special polynomials that make them important. “Number theory, combinatorics, classical and numerical analysis, theoretical physics, approximation theory, and other fields of pure and practical mathematics” are just a few of the fields in which these polynomials can be useful to researchers. Various scientific areas can use the qualities of hybrid special polynomials to address new problems.

The article is organized as follows: In Section 2, we provide an overview for the 3-variable 1-parameter generalized Hermite-based Appell polynomials using the determinant formulation, series definition, and generating function. We also describe how to derive the related differential, integrodifferential, and partial differential equations. In order to demonstrate the usefulness of the major conclusions, we examine instances from this polynomial family in Section 3. In Section 4, we investigate specific cases of the 1-parameter, 3-variable, generalized Hermite-based Appell polynomials. Finally, the last part contains closing thoughts.

2. Polynomials based on generalized Hermite polynomials with three variables and one parameter

In this section, we introduce a hybrid family known as the generalized 3-variable 1-parameter Hermite-based Appell polynomials (g3V1PHAP). Additionally, various properties of these polynomials are established. To obtain the generating function for the g3V1PHAP, a key result is demonstrated as follows:

Theorem 2.1. *For the generalized 3-variable 1-parameter Hermite-based Appell polynomials* ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3; C)$*, the succeeding generating relation is demonstrated:*

$$\mathcal{R}(\xi)C^{\nu_1\xi+\nu_2\xi^2+\nu_3\xi^3} = \sum_{n=0}^{\infty} {}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3; C)\frac{\xi^n}{n!}, \quad C > 1, \quad |\xi| < 1, \tag{2.1}$$

or, equivalently

$$\mathcal{R}(\xi)e^{\ln C(\nu_1\xi+\nu_2\xi^2+\nu_3\xi^3)} = \sum_{n=0}^{\infty} {}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3; C)\frac{\xi^n}{n!}, \quad C > 1. \tag{2.2}$$

Proof. Substituting the exponents of ξ, *i.e.,* ν_1^0, ν_1^1, $\nu_1^2, \cdots, \nu_1^n$ in the expansion of $e^{\nu_1\xi}$ by the polynomials ${}_{\mathcal{D}}\mathcal{R}_0(\nu_1, \nu_2, \nu_3; C)$, ${}_{\mathcal{D}}\mathcal{R}_2(\nu_1, \nu_2, \nu_3; C)$, $\cdots, {}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3; C)$ in the left-hand part and ν_1 by ${}_{\mathcal{D}}\mathcal{R}_1(\nu_1, \nu_2, \nu_3; C)$ in right-hand part of the expression (1.5), further adding up the expressions in left-hand part of the resultant expression, we have

$$\mathcal{R}(\xi)\sum_{n=0}^{\infty}\mathcal{D}_n(v_1,v_2,v_3;C)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\mathcal{R}_n(\mathcal{D}_1(v_1,v_2,v_3;C))\frac{\xi^n}{n!},\tag{2.3}$$

which indicates the resulting g3V1PHAP in the r.h.s. by applying Eq (1.3) in the l.h.s. ${}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2,v_3;C)$ that is

$${}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2,v_3;C):=\mathcal{R}_n\{\mathcal{D}_1(v_1,v_2,v_3;C)\},$$

leading to (2.1). The generating function (2.2) is obtained by simplifying the l.h.s. of Eq (2.1). □

The following theorem gives the series definition for the g3V1PHAP ${}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2,v_3;C)$:

Theorem 2.2. *For the g3V1PHAP ${}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2,v_3;C)$, the succeding series representation is demonstrated*

$${}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2,v_3;C)=n!\sum_{k=0}^{[n/3]}\frac{{}_{\mathcal{D}}\mathcal{R}_{n-3k}(v_1,v_2)v_3^k}{(n-3k)!\,k!}(\ln C)^{n-2k},\tag{2.4}$$

where

$$\mathcal{R}(\xi)\,C^{v_1\xi+v_2\xi^2}=\sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2)\frac{\ln C\xi^n}{n!}.$$

Proof. Inserting the expressions (1.2) and expansion of $e^{\ln C(v_3\xi^3)}$ in left hand part of the expression (2.1) or (2.2), it follows that

$$\sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2,v_3;C)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2)\frac{(\ln C\,\xi)^n}{n!}\sum_{k=0}^{\infty}v_3^k\frac{(\ln C\,\xi^3)^k}{k!},\tag{2.5}$$

thus, operating the Cauchy-product rule yields the expression:

$$\sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2,v_3;C)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]}n!\frac{{}_{\mathcal{D}}\mathcal{R}_{n-3k}(v_1,v_2)v_3^k}{(n-3k)!\,k!}(\ln C)^{n-2k}\frac{\xi^n}{n!}.\tag{2.6}$$

Assertion (2.4) is obtained by comparing the coefficients of the identical powers of ξ on both sides of the above expression. □

In order to give the determinant definition for the g3V1PHAP ${}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2,v_3;C)$, we demonstrate the succeeding result.

Theorem 2.3. *The g3V1PHAP ${}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2,v_3;C)$ of degree n give rise to the succeeding determinant representation:*

$$\begin{aligned}&{}_{\mathcal{D}}\mathcal{R}_0(v_1,v_2,v_3;C)=\frac{1}{\delta_0},\\&{}_{\mathcal{D}}\mathcal{R}_n(v_1,v_2,v_3;C)=\frac{(-1)^n}{(\delta_0)^{n+1}}\begin{vmatrix}1 & \mathcal{D}_1(v_1,v_2,v_3;C) & \mathcal{D}_2(v_1,v_2,v_3;C) & \cdots & \mathcal{D}_{n-1}(v_1,v_2,v_3;C) & \mathcal{D}_n(v_1,v_2,v_3;C)\\ \delta_0 & \delta_1 & \delta_2 & \cdots & \delta_{n-1} & \delta_n\\ 0 & \delta_0 & \binom{2}{1}\delta_1 & \cdots & \binom{n-1}{1}\delta_{n-2} & \binom{n}{1}\delta_{n-1}\\ 0 & 0 & \delta_0 & \cdots & \binom{n-1}{2}\delta_{n-3} & \binom{n}{2}\delta_{n-2}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & \delta_0 & \binom{n}{n-1}\delta_1\end{vmatrix},\end{aligned}\tag{2.7}$$

where $n = 1, 2, \cdots$ *;* $\delta_0, \delta_1, \cdots, \delta_n \in \mathbb{R}$*;* $\delta_0 \neq 0$ *and*

$$\begin{aligned} \delta_0 &= \tfrac{1}{\mathcal{R}_0}, \\ \delta_n &= -\tfrac{1}{\mathcal{R}_0}\Big(\sum_{k=1}^{n}\tbinom{n}{k}\gamma_k\,\delta_{n-k}\Big),\ n = 1, 2, \cdots, \end{aligned} \tag{2.8}$$

and $\mathcal{D}_n(\nu_1, \nu_2, \nu_3; C)$ *is defined in* (1.4) *and* $\mathcal{R}(\xi)$ *in* (1.6).

Proof. Consider γ_n and δ_n two numerical sequences with

$$\mathcal{R}(\xi) = \gamma_0 + \frac{\xi}{1!}\gamma_1 + \frac{\xi^2}{2!}\gamma_2 + \cdots + \frac{\xi^n}{n!}\gamma_n + \cdots,\ n = 0, 1, \cdots;\ \gamma_0 \neq 0, \tag{2.9}$$

$$\hat{\mathcal{R}}(\xi) = \delta_0 + \frac{\xi}{1!}\delta_1 + \frac{\xi^2}{2!}\delta_2 + \cdots + \frac{\xi^n}{n!}\delta_n + \cdots,\ n = 0, 1, \cdots;\ \delta_0 \neq 0, \tag{2.10}$$

satisfying

$$\mathcal{R}(\xi)\hat{\mathcal{R}}(\xi) = 1. \tag{2.11}$$

Consideration of the Cauchy-product rule yields:

$$\mathcal{R}(\xi)\hat{\mathcal{R}}(\xi) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\gamma_k\,\delta_{n-k}\frac{\xi^n}{n!}, \tag{2.12}$$

thus producing

$$\sum_{k=0}^{n}\binom{n}{k}\gamma_k\,\delta_{n-k} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n > 0. \end{cases} \tag{2.13}$$

Therefore

$$\begin{cases} \delta_0 = \tfrac{1}{\gamma_0}, \\ \delta_n = -\tfrac{1}{\gamma_0}\Big(\sum_{k=1}^{n}\tbinom{n}{k}\gamma_k\,\delta_{n-k}\Big),\ \ n = 1, 2, \cdots. \end{cases} \tag{2.14}$$

By multiplying expression (2.1) by $\hat{\mathcal{R}}(\xi)$ on both sides, it follows that

$$\mathcal{R}(\xi)\hat{\mathcal{R}}(\xi)C^{\nu_1\xi+\nu_2\xi^2+\nu_3\xi^3} = \hat{\mathcal{R}}(\xi)\sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3; C)\frac{\xi^n}{n!}. \tag{2.15}$$

Thus, in consideration of the expressions (1.4), (2.10) and (2.11), it follows that

$$\sum_{n=0}^{\infty}H_n^{(m)}(\nu_1, \nu_2, \nu_3, C)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C)\frac{\xi^n}{n!}\sum_{k=0}^{\infty}\delta_k\frac{\xi^k}{k!}. \tag{2.16}$$

Applying the Cauchy-product approach to the series on the r.h.s. of Eq (2.16), the preceding equality yields the following system of infinite equations in the unknowns ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C),\ n=0,1,\cdots$:

$$\begin{cases} {}_{\mathcal{D}}\mathcal{R}_0(\nu_1,\nu_2,\nu_3,C)\delta_0 = 1, \\ {}_{\mathcal{D}}\mathcal{R}_0(\nu_1,\nu_2,\nu_3,C)\delta_1 + {}_{\mathcal{D}}\mathcal{R}_1(\nu_1,\nu_2,\nu_3,C)\delta_0 = \mathcal{D}_1(\nu_1,\nu_2,\nu_3,C), \\ {}_{\mathcal{D}}\mathcal{R}_0(\nu_1,\nu_2,\nu_3,C)\delta_2 + \binom{2}{1}{}_{\mathcal{D}}\mathcal{R}_1(\nu_1,\nu_2,\nu_3,C)\delta_1 + {}_{\mathcal{D}}\mathcal{R}_2(\nu_1,\nu_2,\nu_3,C)\delta_0 = \mathcal{D}_2(\nu_1,\nu_2,\nu_3,C), \\ \vdots \\ {}_{\mathcal{D}}\mathcal{R}_0(\nu_1,\nu_2,\nu_3,C)\delta_n + \binom{n}{1}{}_{\mathcal{D}}\mathcal{R}_1(\nu_1,\nu_2,\nu_3,C)\delta_{n-1} + \cdots + {}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\delta_0 = \mathcal{D}_n(\nu_1,\nu_2,\nu_3,C), \\ \vdots \end{cases} \tag{2.17}$$

From the initial equation in system (2.17), we derive the first segment of assertion (2.7). Furthermore, the particular configuration of system (2.17) (lower triangular form) allows us to ascertain the unknowns ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)$. Using Cramer's rule to solve the first $n+1$ expressions, we arrive at:

$${}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C) = \frac{1}{\delta_0^{n+1}} \begin{vmatrix} \delta_0 & 0 & 0 & \cdots & 0 & 1 \\ \delta_1 & \delta_0 & 0 & \cdots & 0 & \mathcal{D}_1(\nu_1,\nu_2,\nu_3,C) \\ \delta_2 & \binom{2}{1}\delta_1 & \delta_0 & \cdots & 0 & \mathcal{D}_2(\nu_1,\nu_2,\nu_3,C) \\ . & . & . & \cdots & . & . \\ \delta_{n-1} & \binom{n-1}{1}\delta_{n-2} & \binom{n-1}{2}\delta_{n-3} & \cdots & \delta_0 & \mathcal{D}_{n-1}(\nu_1,\nu_2,\nu_3,C) \\ \delta_n & \binom{n}{1}\delta_{n-1} & \binom{n}{2}\delta_{n-2} & \cdots & \binom{n}{n-1}\delta_1 & \mathcal{D}_n(\nu_1,\nu_2,\nu_3,C) \end{vmatrix}. \tag{2.18}$$

We now acquire the second part of statement (2.7) by transferring the $(n+1)$-th column to the first position by n adjacent column transpositions and realising that the determinant of a square matrix equals that of its transpose. □

3. Recurrence relations and shift operators

Theorem 3.1. *The g3V1PHAP ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)$ adhere to the succeeding recurrence relation:*

$$\begin{aligned} {}_{\mathcal{D}}\mathcal{R}_{n+1}(\nu_1,\nu_2,\nu_3,C) &= (\nu_1 \ln C + \gamma_0)\, {}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C) + \sum_{k=1}^{n}\binom{n}{k}\gamma_k\, {}_{\mathcal{D}}\mathcal{R}_{n-k}(\nu_1,\nu_2,\nu_3,C) \\ &\quad + 2n\nu_2 \ln C\, {}_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3,C) + 3n(n-1)\nu_3 \ln C\, {}_{\mathcal{D}}\mathcal{R}_{n-2}(\nu_1,\nu_2,\nu_3,C), \end{aligned} \tag{3.1}$$

where

$${}_{\mathcal{D}}\mathcal{R}_{-k}(\nu_1,\nu_2,\nu_3,C) := 0, \quad k = 1,2,\cdots \tag{3.2}$$

and the following expansion yields the coefficients $\{\gamma_k\}_{k\in\mathbb{N}_0}$:

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)} = \sum_{n=0}^{\infty}\gamma_n\frac{\xi^n}{n!}. \tag{3.3}$$

Proof. After taking ξ into account and differentiating both sides of the generating function (2.1), we arrive at:

$$\frac{\partial}{\partial\xi}\left\{\mathcal{R}(\xi)C^{\nu_1\xi+\nu_2\xi^2+\nu_3\xi^3}\right\} = \frac{\partial}{\partial\xi}\left\{\sum_{n=0}^{\infty} {}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3;C)\frac{\xi^n}{n!}\right\} \tag{3.4}$$

which can be simplified as

$$\left\{\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)} + \nu_1\ln(C) + 2\nu_2\ln(C)\xi + 3\nu_3\ln(C)\xi^2\right\}\sum_{n=0}^{\infty} {}_{\mathcal{D}}\mathcal{R}_{n+1}(\nu_1,\nu_2,\nu_3;C)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty} n\,{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3;C)\frac{\xi^{n-1}}{n!}.$$

Further, the preceding expression in consideration of the Cauchy-Product formula can be expressed as

$$\sum_{n=0}^{\infty}\Bigg[\sum_{k=0}^{n}\binom{n}{k}\gamma_k\,{}_{\mathcal{D}}\mathcal{R}_{n-k}(\nu_1,\nu_2,\nu_3;C) + \nu_1\ln(C)_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3;C) + 2n\nu_2\ln(C)_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3;C)$$
$$+3n(n-1)\nu_3\ln(C)_{\mathcal{D}}\mathcal{R}_{n-2}(\nu_1,\nu_2,\nu_3;C)\Bigg]_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3;C)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty} {}_{\mathcal{D}}\mathcal{R}_{n+1}(\nu_1,\nu_2,\nu_3;C)\frac{\xi^n}{n!}. \tag{3.5}$$

Assertion (3.1) is obtained by comparing the coefficients of the identical powers of ξ on both sides of the preceding statement. □

Theorem 3.2. *The g3V1PHAP ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)$ adhere to the succeeding shift operators:*

$${}_{\nu_1}\mathcal{L}_n^- := \frac{1}{n(\ln C)}D_{\nu_1}, \tag{3.6}$$

$${}_{\nu_2}\mathcal{L}_n^- := \frac{1}{n(\ln C)}D_{\nu_1}^{-1}D_{\nu_2}, \tag{3.7}$$

$${}_{\nu_3}\mathcal{L}_n^- := \frac{1}{n(\ln C)}D_{\nu_1}^{-2}D_{\nu_3}, \tag{3.8}$$

$${}_{\nu_1}\mathcal{L}_n^+ := (\nu_1\ln C + \gamma_0) + \sum_{k=1}^{n}\frac{\gamma_k}{k!}(\ln C)^{-k}D_{\nu_1}^k + 2\nu_2 D_{\nu_1} + 3\nu_3(\ln C)^{-1}D_{\nu_1}^2, \tag{3.9}$$

$${}_{\nu_2}\mathcal{L}_n^+ := (\nu_1\ln C + \gamma_0) + \sum_{k=1}^{n}\frac{\gamma_k}{k!}(\ln C)^{-k}D_{\nu_1}^{-k}D_{\nu_2}^k + 2\nu_2 D_{\nu_1}^{-1}D_{\nu_2} + 3\nu_3(\ln C)^{-1}D_{\nu_1}^{-2}D_{\nu_2}^2, \tag{3.10}$$

and

$${}_{\nu_3}\mathcal{L}_n^+ := (\nu_1\ln C + \gamma_0) + \sum_{k=1}^{n}\frac{\gamma_k}{k!}(\ln C)^{-k}D_{\nu_1}^{-k}D_{\nu_3}^k + 2\nu_2 D_{\nu_1}^{-1}D_{\nu_3} + 3\nu_3(\ln C)^{-1}D_{\nu_1}^{-4}D_{\nu_3}^2 \tag{3.11}$$

respectively, where

$$D_{\nu_1} := \frac{\partial}{\partial_{\nu_1}},\ \ D_{\nu_2} := \frac{\partial}{\partial_{\nu_2}};\ \ D_{\nu_3} := \frac{\partial}{\partial_{\nu_3}}\ \ D_{\nu_1}^{-1} := \int_0^{\nu_1} g(\xi)d\xi. \tag{3.12}$$

Proof. After rearranging the powers and differentiating both sides of Eq (2.1) concerning ν_1, we equate the coefficients of the identical powers of ξ in both sides of the resulting equation as follows:

$$D_{\nu_1}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\} = n(\ln C)\,{}_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3,C), \tag{3.13}$$

as a result, the operator provided by Eq (3.6) satisfies equation

$${}_{\nu_1}\mathcal{L}_n^{-}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\} = {}_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3,C). \tag{3.14}$$

Subsequently, we differentiate both sides of Eq (2.1) concerning ν_2, rearrange the powers, and then calculate the coefficients of the identical powers of ξ on both sides of the resulting equation gives:

$$D_{\nu_2}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\} = (\ln C)n(n-1)\,{}_{\mathcal{D}}\mathcal{R}_{n-2}(\nu_1,\nu_2,\nu_3,C), \tag{3.15}$$

which further can be stated as

$$D_{\nu_2}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\} = n(\ln C)D_{\nu_1}\,{}_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3,C), \tag{3.16}$$

thus, it follows that

$$\frac{1}{n(\ln C)}D_{\nu_2}D_{\nu_1}^{-1}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\} = {}_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3,C). \tag{3.17}$$

Thus, the above equation is satisfied by the operator provided by Eq (3.7).

Again differentiating both sides of Eq (2.1) with respect to ν_3, we have

$$D_{\nu_3}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\} = (\ln C)n(n-1)(n-2)\,{}_{\mathcal{D}}\mathcal{R}_{n-3}(\nu_1,\nu_2,\nu_3,C) \tag{3.18}$$

and further stated as

$$D_{\nu_3}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\} = n(\ln C)D_{\nu_1}^2\,{}_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3,C), \tag{3.19}$$

thus, it follows that

$$\frac{1}{n(\ln C)}D_{\nu_3}D_{\nu_1}^{-2}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\} = {}_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3,C). \tag{3.20}$$

Thus, the above equation is satisfied by the operator provided by Eq (3.8).

The raising operator (3.9) may be found using the following relation:

$${}_{\mathcal{D}}\mathcal{R}_{n-k}(\nu_1,\nu_2,\nu_3,C) = \left({}_{\nu_1}\mathcal{L}_{n-k+1}^{-}\ {}_{\nu_1}\mathcal{L}_{n-k+2}^{-}\cdots{}_{\nu_1}\mathcal{L}_{n-1}^{-}\ {}_{\nu_1}\mathcal{L}_{n}^{-}\right)\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\}. \tag{3.21}$$

Using Eq (3.6) in conjunction with Eq (3.21), we obtain

$$\begin{aligned}&{}_{\mathcal{D}}\mathcal{R}_{n-k}(\nu_1,\nu_2,\nu_3,C)\\ =&\left(\frac{1}{(n-k+1)(\ln C)}D_{\nu_1}\cdots\frac{1}{(n-1)(\ln C)}D_{\nu_1}\ \frac{1}{n(\ln C)}D_{\nu_1}\right)\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\}\end{aligned} \tag{3.22}$$

and further casted as

$$ {}_{\mathcal{D}}\mathcal{R}_{n-k}(\nu_1,\nu_2,\nu_3,C) = \frac{(n-k)!}{n!}(\ln C)^{-k} D_{\nu_1}^{k}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\}. \tag{3.23} $$

Further, we have

$$ {}_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3,C) = \frac{1}{n}(\ln C)^{-1} D_{\nu_1}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\}. \tag{3.24} $$

Thus inserting expressions (3.21) and (3.23) in Eq (3.1), we find

$$ \begin{aligned} {}_{\mathcal{D}}\mathcal{R}_{n+1}(\nu_1,\nu_2,\nu_3,C) = &\left((\nu_1 \ln C+\gamma_0)+\sum_{k=1}^{n}\frac{\gamma_k}{k!}(\ln C)^{-k}D_{\nu_1}^{-k}D_{\nu_2}^{k}+2\nu_2 D_{\nu_1}^{-1}D_{\nu_2}+3\nu_3(\ln C)^{-1}D_{\nu_1}^{-2}D_{\nu_2}^{2}\right)\\ &\times \{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\}, \end{aligned} \tag{3.25} $$

thus yielding the expression (3.9) of the raising operator ${}_{\nu_1}\mathcal{L}_n^{+}$.

We employ the relation below to determine the raising operator (3.10):

$$ {}_{\mathcal{D}}\mathcal{R}_{n-k}(\nu_1,\nu_2,\nu_3,C) = \left({}_{\nu_2}\mathcal{L}_{n-k+1}^{-}\ {}_{\nu_2}\mathcal{L}_{n-k+2}^{-}\cdots {}_{\nu_2}\mathcal{L}_{n-1}^{-}\ {}_{\nu_2}\mathcal{L}_{n}^{-}\right)\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\}. \tag{3.26} $$

Using Eq (3.7) in Eq (3.26) and simplifying, we find

$$ {}_{\mathcal{D}}\mathcal{R}_{n-k}(\nu_1,\nu_2,\nu_3,C) = \frac{(n-k)!}{n!}(\ln C)^{-k} D_{\nu_1}^{-k}D_{\nu_2}^{k}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\}. \tag{3.27} $$

Also, we have

$$ {}_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3,C) = \frac{1}{n}(\ln C)^{-1} D_{\nu_1}^{-1}D_{\nu_2}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\}. \tag{3.28} $$

Using Eqs (3.27) and (3.28) in Eq (3.1), we find

$$ \begin{aligned} {}_{\mathcal{D}}\mathcal{R}_{n+1}(\nu_1,\nu_2,\nu_3,C) = &\left((\nu_1 \ln C+\gamma_0)+\sum_{k=1}^{n}\tfrac{\gamma_k}{k!}(\ln C)^{-k}D_{\nu_1}^{-k}D_{\nu_2}^{k}+2\nu_2 D_{\nu_1}^{-1}D_{\nu_2}+3\nu_3(\ln C)^{-1}D_{\nu_1}^{-2}D_{\nu_2}^{2}\right)\\ &\times {}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C), \end{aligned} \tag{3.29} $$

thus yielding the expression (3.10) of the raising operator ${}_{\nu_2}\mathcal{L}_n^{+}$.

Last, we employ the relation below to determine the raising operator (3.11):

$$ {}_{\mathcal{D}}\mathcal{R}_{n-k}(\nu_1,\nu_2,\nu_3,C) = \left({}_{\nu_3}\mathcal{L}_{n-k+1}^{-}\ {}_{\nu_3}\mathcal{L}_{n-k+2}^{-}\cdots {}_{\nu_3}\mathcal{L}_{n-1}^{-}\ {}_{\nu_3}\mathcal{L}_{n}^{-}\right)\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\} \tag{3.30} $$

Using Eq (3.8) in Eq (3.30) and simplifying, we find

$$ {}_{\mathcal{D}}\mathcal{R}_{n-k}(\nu_1,\nu_2,\nu_3,C) = \frac{(n-k)!}{n!}(\ln C)^{-k} D_{\nu_1}^{-2k}D_{\nu_2}^{k}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\}. \tag{3.31} $$

Also, we have

$$ {}_{\mathcal{D}}\mathcal{R}_{n-1}(\nu_1,\nu_2,\nu_3,C) = \frac{1}{n}(\ln C)^{-1} D_{\nu_1}^{-2}D_{\nu_3}\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)\}. \tag{3.32} $$

Using Eqs (3.31) and (3.32) in Eq (3.1), we find

$$ \begin{aligned} {}_{\mathcal{D}}\mathcal{R}_{n+1}(\nu_1,\nu_2,\nu_3,C) = &\left((\nu_1 \ln C+\gamma_0)+\sum_{k=1}^{n}\tfrac{\gamma_k}{k!}(\ln C)^{-k}D_{\nu_1}^{-k}D_{\nu_3}^{k}+2\nu_2 D_{\nu_1}^{-1}D_{\nu_3}+3\nu_3(\ln C)^{-1}D_{\nu_1}^{-4}D_{\nu_3}^{2}\right)\\ &\times {}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C), \end{aligned} \tag{3.33} $$

thus yielding the expression (3.11) of the raising operator ${}_{\nu_3}\mathcal{L}_n^{+}$.

□

Next, we find the "differential, integrodifferential and partial differential equation" for the 3V1PGHbAP ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C)$. For this, we consider the following results:

Theorem 3.3. *The generalized 3-variable 1-parameter Hermite-based Appell polynomials* ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C)$ *satisfy the following differential equation:*

$$\left(\left(\nu_1 + \frac{\gamma_0}{\ln C}\right)D_{\nu_1} + \sum_{k=1}^{n} \frac{\gamma_k}{k!}(\ln C)^{-k-1}D_{\nu_1}^{k+1} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^2 + 3\nu_3(\ln C)^{-2}D_{\nu_1}^3 - (n+1)\right)$$
$$\times\, {}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C) = 0. \tag{3.34}$$

Proof. Making use of expressions (3.6) and (3.8) of the shift operators ${}_{\nu_1}\mathcal{L}_n^-$ and ${}_{\nu_1}\mathcal{L}_n^+$ in the factorization equation

$${}_{\nu_1}\mathcal{L}_{n+1}^-\ {}_{\nu_1}\mathcal{L}_n^+\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C)\} = {}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C),$$

we adhere to the expression (3.34). □

Theorem 3.4. *The generalized 3-variable 1-parameter Hermite-based Appell polynomials* ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C)$ *satisfy the following integrodifferential equations:*

$$\left(\left(\nu_1 + \frac{\gamma_0}{\ln C}\right)D_{\nu_2} + \sum_{k=1}^{n} \frac{\gamma_k}{k!}(\ln C)^{-k-1}D_{\nu_1}^{-k}D_{\nu_2}^{k+1} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{-1}D_{\nu_2}^2 + 3\nu_3(\ln C)^{-2}D_{\nu_1}^{-2}D_{\nu_2}^3 - (n+1)D_{\nu_1}\right)$$
$$\times\, {}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C) = 0, \tag{3.35}$$

$$\left(\left(\nu_1 + \frac{\gamma_0}{\ln C}\right)D_{\nu_3} + \sum_{k=1}^{n} \frac{\gamma_k}{k!}(\ln C)^{-k-1}D_{\nu_1}^{-k}D_{\nu_3}^{k+1} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{-1}D_{\nu_3}^2 + 3\nu_3(\ln C)^{-2}D_{\nu_1}^{-4}D_{\nu_3}^3 - (n+1)D_{\nu_1}^2\right)$$
$$\times\, {}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C) = 0, \tag{3.36}$$

$$\left(\left(\nu_1 + \frac{\gamma_0}{\ln C}\right)D_{\nu_3} + \sum_{k=1}^{n} \frac{\gamma_k}{k!}(\ln C)^{-k-1}D_{\nu_1}^{-k}D_{\nu_2}^{k}D_{\nu_3} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{-1}D_{\nu_2}^2D_{\nu_3} + 3\nu_3(\ln C)^{-2}D_{\nu_1}^{-2}D_{\nu_2}^3D_{\nu_3}\right.$$
$$\left. -(n+1)D_{\nu_1}^2\right){}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C) = 0, \tag{3.37}$$

$$\left(\left(\nu_1 + \frac{\gamma_0}{\ln C}\right)D_{\nu_2} + \sum_{k=1}^{n} \frac{\gamma_k}{k!}(\ln C)^{-k-1}D_{\nu_1}^{-k}D_{\nu_2}D_{\nu_3}^{k} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{-1}D_{\nu_2}D_{\nu_3} + 3\nu_3(\ln C)^{-2}D_{\nu_1}^{-4}D_{\nu_2}D_{\nu_3}^2\right.$$
$$\left. -(n+1)D_{\nu_1}\right){}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C) = 0. \tag{3.38}$$

Proof. Making use of expressions (3.7), (3.10) and (3.8), (3.11) of the shift operators $\mathcal{L}_n^-$ and $\mathcal{L}_n^+$ in the factorization equation

$$\mathcal{L}_{n+1}^-\ \mathcal{L}_n^+\{{}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C)\} = {}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C),$$

we adhere to the expression (3.35) and (3.36).

Further, making use of expressions (3.7), (3.11) and (3.8), (3.10) of the shift operators $\mathcal{L}_n^-$ and $\mathcal{L}_n^+$ in above factorization relation, we adhere to the expression (3.37) and (3.38). □

Theorem 3.5. *The generalized 3-variable 1-parameter Hermite-based Appell polynomials* ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)$ *satisfy the following partial differential equations:*

$$\Bigg(\Big(\nu_1+\frac{\gamma_0}{\ln C}\Big)D_{\nu_1}^{n}D_{\nu_2}+nD_{\nu_1}^{n-1}D_{\nu_2}+\sum_{k=1}^{n}\frac{\gamma_k}{k!}(\ln C)^{-k-1}D_{\nu_1}^{n-k}D_{\nu_2}^{k+1}+2\nu_2(\ln C)^{-1}D_{\nu_1}^{n-1}D_{\nu_2}^{2}+3\nu_3(\ln C)^{-2}D_{\nu_1}^{n-2}D_{\nu_2}^{3}-(n+1)D_{\nu_1}^{n+1}\Bigg){}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)=0. \quad (3.39)$$

$$\Bigg(\Big(\nu_1+\frac{\gamma_0}{\ln C}\Big)D_{\nu_1}^{2n}D_{\nu_3}+2nD_{\nu_1}^{2n-1}D_{\nu_3}+\sum_{k=1}^{n}\frac{\gamma_k}{k!}(\ln C)^{-k-1}D_{\nu_1}^{2n-2k}D_{\nu_3}^{k+1}+2\nu_2(\ln C)^{-1}D_{\nu_1}^{2n-1}D_{\nu_3}^{2}+3\nu_3(\ln C)^{-2}\times D_{\nu_1}^{2n-4}D_{\nu_3}^{3}-(n+1)D_{\nu_1}^{2n+2}\Bigg){}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)=0. \quad (3.40)$$

$$\Bigg(\Big(\nu_1+\frac{\gamma_0}{\ln C}\Big)D_{\nu_1}^{n}D_{\nu_3}+nD_{\nu_1}^{n-1}D_{\nu_3}+\sum_{k=1}^{n}\frac{\gamma_k}{k!}(\ln C)^{-k-1}D_{\nu_1}^{n-k}D_{\nu_2}^{k}D_{\nu_3}+2\nu_2(\ln C)^{-1}D_{\nu_1}^{n-1}D_{\nu_2}^{2}D_{\nu_3}+3\nu_3(\ln C)^{-2}D_{\nu_1}^{n-2}D_{\nu_2}^{3}D_{\nu_3}-(n+1)D_{\nu_1}^{n+2}\Bigg){}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)=0. \quad (3.41)$$

$$\Bigg(\Big(\nu_1+\frac{\gamma_0}{\ln C}\Big)D_{\nu_1}^{2n}D_{\nu_2}+2nD_{\nu_1}^{2n-1}D_{\nu_2}+\sum_{k=1}^{n}\frac{\gamma_k}{k!}(\ln C)^{-k-1}D_{\nu_1}^{2n-k}D_{\nu_2}D_{\nu_3}^{k}+2\nu_2(\ln C)^{-1}\times D_{\nu_1}^{2n-1}D_{\nu_2}D_{\nu_3}+3\nu_3(\ln C)^{-2}D_{\nu_1}^{2n-4}D_{\nu_2}D_{\nu_3}^{2}-(n+1)D_{\nu_1}^{2n+1}\Bigg){}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)=0. \quad (3.42)$$

Proof. Differentiating the expressions (3.35) and (3.37) w.r.t. D_{ν_1} n times, we get the partial differential equations (3.39) and (3.41). Similarly, upon differentiating the expressions (3.36) and (3.38) w.r.t. D_{ν_1} $2n$ times, we get the partial differential equations (3.40) and (3.42). □

A few instances of how the above-derived results are applied are given in the next section.

4. Applications

By carefully choosing the function $\mathcal{R}(\xi)$, various distinct members within the extensive family of Appell polynomials can be derived. This flexibility allows for the customization of polynomial solutions tailored to specific mathematical or practical requirements, catering to diverse applications across numerous domains. This is because of its versatility, enabling researchers and practitioners to access a rich spectrum of polynomial functions, each suited to address different mathematical problems or modeling scenarios. The Bernoulli, Euler, and Genocchi polynomials are pivotal in numerous expansions and approximation formulas, serving as valuable assets in the analytical theory of numbers and classical and numerical analysis. Their significance extends to addressing a myriad of challenges in engineering and physics. These polynomials and numbers provide essential tools for mathematical research and problem-solving. The Appell polynomial family, combined with the data shown in Table 1, makes up a useful and strong instrument in the toolbox of scientists and mathematicians. They provide a modifiable problem-solving method by offering specialized answers to particular mathematical difficulties in various academic fields.

Table 1. Several members of the Appell polynomial family.

S. No.	**Name of the polynomials and related numbers**	$\mathcal{R}(\xi)$	**Generating expression**	**Series representation**
I.	Bernoulli	$\left(\frac{\xi}{e^{\xi}-1}\right)$	$\left(\frac{\xi}{e^{\xi}-1}\right)e^{\nu_1\xi} = \sum_{k=0}^{\infty}\mathcal{B}_k(\nu_1)\frac{\xi^k}{k!}$	$\mathcal{B}_k(\nu_1) = \sum_{m=0}^{k}\binom{k}{m}\mathcal{B}_m\xi^{k-m}$
	polynomials		$\left(\frac{\xi}{e^{\xi}-1}\right) = \sum_{k=0}^{\infty}\mathcal{B}_k\frac{\xi^k}{k!}$	
	and numbers [22]		$\mathcal{B}_k := \mathcal{B}_k(0)$	
II.	Euler	$\left(\frac{2}{e^{\xi}+1}\right)$	$\left(\frac{2}{e^{\xi}+1}\right)e^{\nu_1\xi} = \sum_{k=0}^{\infty}\mathcal{E}_k(\nu_1)\frac{\xi^k}{k!}$	$\mathcal{E}_k(\nu_1) = \sum_{m=0}^{k}\binom{k}{m}\frac{\mathcal{E}_m}{2^m}\left(\xi-\frac{1}{2}\right)^{k-m}$
	polynomials		$\frac{2\xi}{e^{2\xi}+1} = \sum_{k=0}^{\infty}\mathcal{E}_k\frac{\xi^k}{k!}$	
	and numbers [22]		$\mathcal{E}_k := 2^k\mathcal{E}_k\left(\frac{1}{2}\right)$	
III.	Genocchi	$\left(\frac{2\xi}{e^{\xi}+1}\right)$	$\left(\frac{2\xi}{e^{\xi}+1}\right)e^{\nu_1\xi} = \sum_{k=0}^{\infty}\mathcal{G}_k(\nu_1)\frac{\xi^k}{k!}$	$\mathcal{G}_k(\nu_1) = \sum_{m=0}^{k}\binom{k}{m}\mathcal{G}_m\xi^{k-m}$
	polynomials		$\frac{2\xi}{e^{\xi}+1} = \sum_{k=0}^{\infty}\mathcal{G}_k\frac{\xi^k}{k!}$	
	and numbers [23]		$\mathcal{G}_k := \mathcal{G}_k(0)$	

The "generating function, series definition, recurrence relation, shift operators, as well as the differential, integrodifferential, and partial differential equations" for certain members of the 3-variable, 1-parameter generalized Hermite-based Appell polynomials ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)$ is derived by examining the following examples:

Example 4.1. Consider

$$\left(\frac{\xi}{e^{\xi}-1}\right) = \mathcal{R}(\xi)$$

in generating function (2.1), the 3V1PGHbAP ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)$ reduces to the 3-variable 1-parameter generalized Hermite-based Bernoulli polynomials (3V1PGHbBP) ${}_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3,C)$ defined by the following generating function:

$$\left(\frac{\xi}{e^{\xi}-1}\right)C^{\nu_1\xi+\nu_2\xi^2+\nu_3\xi^3} = \sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3;C)\frac{\xi^n}{n!}, \quad C>1, \quad |\xi|<2\pi, \tag{4.1}$$

or, equivalently

$$\left(\frac{\xi}{e^{\xi}-1}\right)e^{\ln C(\nu_1\xi+\nu_2\xi^2+\nu_3\xi^3)}=\sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3;C)\frac{\xi^n}{n!},\quad C>1,\quad |\xi|<2\pi. \tag{4.2}$$

Further, the remaining corresponding results for the 3-variable 1-parameter generalized Hermite-based Bernoulli polynomials (3V1PGHbBP) ${}_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3,C)$ are established by setting similar substitutions in the following manner:

The 3-variable 1-parameter generalized Hermite-based Bernoulli polynomials ${}_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3;C)$ are defined by the following series definition:

$${}_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3;C)=n!\sum_{k=0}^{[n/3]}\frac{{}_{\mathcal{D}}\mathcal{B}_{n-3k}(\nu_1,\nu_2)\nu_3^k}{(n-3k)!\,k!}(\ln C)^{n-2k}, \tag{4.3}$$

where

$$\left(\frac{\xi}{e^{\xi}-1}\right)C^{\nu_1\xi+\nu_2\xi^2}=\sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2)\frac{\ln C\xi^n}{n!}.$$

Next, setting $\delta_0=1$ and $\delta_i=\frac{1}{i+1}$ $(i=1,2,\cdots,n)$ in expression (2.7), the 3-variable 1-parameter generalized Hermite-based Bernoulli polynomials ${}_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3;C)$ of degree n are defined by the following determinant definition:

$$\begin{aligned}&{}_{\mathcal{D}}\mathcal{B}_0(\nu_1,\nu_2,\nu_3;C)=\tfrac{1}{\delta_0},\\&{}_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3;C)=\tfrac{(-1)^n}{(\delta_0)^{n+1}}\begin{vmatrix}1&\mathcal{D}_1(\nu_1,\nu_2,\nu_3;C)&\mathcal{D}_2(\nu_1,\nu_2,\nu_3;C)&\cdots&\mathcal{D}_{n-1}(\nu_1,\nu_2,\nu_3;C)&\mathcal{D}_n(\nu_1,\nu_2,\nu_3;C)\\1&\frac{1}{2}&\frac{1}{3}&\cdots&\frac{1}{n}&\frac{1}{n+1}\\0&1&\binom{2}{1}\frac{1}{2}&\cdots&\binom{n-1}{1}\frac{1}{n-1}&\binom{n}{1}\frac{1}{n}\\0&0&1&\cdots&\binom{n-1}{2}\frac{1}{n-2}&\binom{n}{2}\frac{1}{n-1}\\\vdots&\vdots&\vdots&\cdots&\vdots&\vdots\\0&0&0&\cdots&1&\binom{n}{n-1}\frac{1}{2}\end{vmatrix}.\end{aligned} \tag{4.4}$$

Also, the 3-variable, 1-parameter generalized Hermite-based Bernoulli polynomials ${}_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3,C)$ adhere to the following recurrence relation and shift operators:

$$\begin{aligned}{}_{\mathcal{D}}\mathcal{B}_{n+1}(\nu_1,\nu_2,\nu_3,C)&=(\nu_1\ln C-\tfrac{1}{2})\,{}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)-\sum_{k=1}^{n}\binom{n}{k}\tfrac{\mathcal{B}_{k+1}(1)}{k+1}\,{}_{\mathcal{D}}\mathcal{B}_{n-k}(\nu_1,\nu_2,\nu_3,C)\\&+2n\nu_2\ln C{}_{\mathcal{D}}\mathcal{B}_{n-2}(\nu_1,\nu_2,\nu_3,C)+3n(n-1)\nu_3\ln C{}_{\mathcal{D}}\mathcal{B}_{n-2}(\nu_1,\nu_2,\nu_3,C),\end{aligned} \tag{4.5}$$

where

$$\frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}=-\sum_{n=0}^{\infty}\frac{\mathcal{B}_{n+1}(1)}{n+1}\frac{\xi^n}{n!},$$

thus using (3.3), we find

$$\sum_{n=0}^{\infty}\gamma_n\frac{\xi^n}{n!}=-\sum_{n=0}^{\infty}\frac{\mathcal{B}_{n+1}(1)}{n+1}\frac{\xi^n}{n!},$$

which implies$\gamma_n = -\frac{\mathcal{B}_{n+1}(1)}{n+1}$; $\gamma_0 = -\frac{1}{2}$.

$$ {}_{\nu_1}\mathcal{L}_n^- := \frac{1}{n(\ln C)} D_{\nu_1}, \tag{4.6} $$

$$ {}_{\nu_2}\mathcal{L}_n^- := \frac{1}{n(\ln C)} D_{\nu_1}^{-1} D_{\nu_2}, \tag{4.7} $$

$$ {}_{\nu_3}\mathcal{L}_n^- := \frac{1}{n(\ln C)} D_{\nu_1}^{-2} D_{\nu_3}, \tag{4.8} $$

$$ {}_{\nu_1}\mathcal{L}_n^+ := (\nu_1 \ln C - \frac{1}{2}) - \sum_{k=1}^{n} \binom{n}{k} \frac{\mathcal{B}_{k+1}(1)}{k+1} (\ln C)^{-k} D_{\nu_1}^k + 2\nu_2 D_{\nu_1} + 3\nu_3 (\ln C)^{-1} D_{\nu_1}^2, \tag{4.9} $$

$$ {}_{\nu_2}\mathcal{L}_n^+ := (\nu_1 \ln C - \frac{1}{2}) - \sum_{k=1}^{n} \binom{n}{k} \frac{\mathcal{B}_{k+1}(1)}{k+1} (\ln C)^{-k} D_{\nu_1}^{-k} D_{\nu_2}^k + 2\nu_2 D_{\nu_1}^{-1} D_{\nu_2} + 3\nu_3 (\ln C)^{-1} D_{\nu_1}^{-2} D_{\nu_2}^2, \tag{4.10} $$

and

$$ {}_{\nu_3}\mathcal{L}_n^+ := (\nu_1 \ln C - \frac{1}{2}) - \sum_{k=1}^{n} \binom{n}{k} \frac{\mathcal{B}_{k+1}(1)}{k+1} (\ln C)^{-k} D_{\nu_1}^{-k} D_{\nu_3}^k + 2\nu_2 D_{\nu_1}^{-1} D_{\nu_3} + 3\nu_3 (\ln C)^{-1} D_{\nu_1}^{-4} D_{\nu_3}^2. \tag{4.11} $$

Finally, the 3-variable 1-parameter generalized Hermite-based Bernoulli polynomials ${}_{\mathcal{D}}\mathcal{B}_n(\nu_1, \nu_2, \nu_3, C)$ satisfy the following differential, integrodifferential and partial differential equations:

$$ \Big(\big(\nu_1 - \frac{1}{2\ln C}\big) D_{\nu_1} - \sum_{k=1}^{n} \binom{n}{k} \frac{\mathcal{B}_{k+1}(1)}{k+1} (\ln C)^{-k-1} D_{\nu_1}^{k+1} + 2\nu_2 (\ln C)^{-1} D_{\nu_1}^2 + 3\nu_3 (\ln C)^{-2} D_{\nu_1}^3 - (n+1) \Big) {}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C) = 0, $$

$$ \Big(\big(\nu_1 - \frac{1}{2\ln C}\big) D_{\nu_2} - \sum_{k=1}^{n} \binom{n}{k} \frac{\mathcal{B}_{k+1}(1)}{k+1} (\ln C)^{-k-1} D_{\nu_1}^{-k} D_{\nu_2}^{k+1} + 2\nu_2 (\ln C)^{-1} D_{\nu_1}^{-1} D_{\nu_2}^2 + 3\nu_3 (\ln C)^{-2} D_{\nu_1}^{-2} D_{\nu_2}^3 - (n+1) D_{\nu_1} \Big) {}_{\mathcal{D}}\mathcal{B}_n(\nu_1, \nu_2, \nu_3, C) = 0, $$

$$ \Big(\big(\nu_1 - \frac{1}{2\ln C}\big) D_{\nu_3} - \sum_{k=1}^{n} \binom{n}{k} \frac{\mathcal{B}_{k+1}(1)}{k+1} (\ln C)^{-k-1} D_{\nu_1}^{-k} D_{\nu_3}^{k+1} + 2\nu_2 (\ln C)^{-1} D_{\nu_1}^{-1} D_{\nu_3}^2 + 3\nu_3 (\ln C)^{-2} D_{\nu_1}^{-4} D_{\nu_3}^3 - (n+1) D_{\nu_1}^2 \Big) {}_{\mathcal{D}}\mathcal{B}_n(\nu_1, \nu_2, \nu_3, C) = 0, $$

$$ \Big(\big(\nu_1 - \frac{1}{2\ln C}\big) D_{\nu_3} - \sum_{k=1}^{n} \binom{n}{k} \frac{\mathcal{B}_{k+1}(1)}{k+1} (\ln C)^{-k-1} D_{\nu_1}^{-k} D_{\nu_2}^{k} D_{\nu_3} + 2\nu_2 (\ln C)^{-1} D_{\nu_1}^{-1} D_{\nu_2}^2 D_{\nu_3} + 3\nu_3 (\ln C)^{-2} D_{\nu_1}^{-2} D_{\nu_2}^3 D_{\nu_3} - (n+1) D_{\nu_1}^2 \Big) {}_{\mathcal{D}}\mathcal{B}_n(\nu_1, \nu_2, \nu_3, C) = 0, $$

$$ \Big(\big(\nu_1 - \frac{1}{2\ln C}\big) D_{\nu_2} - \sum_{k=1}^{n} \binom{n}{k} \frac{\mathcal{B}_{k+1}(1)}{k+1} (\ln C)^{-k-1} D_{\nu_1}^{-k} D_{\nu_2} D_{\nu_3}^{k} + 2\nu_2 (\ln C)^{-1} D_{\nu_1}^{-1} D_{\nu_2} D_{\nu_3} + 3\nu_3 (\ln C)^{-2} D_{\nu_1}^{-4} D_{\nu_2} D_{\nu_3}^2 $$

$$- (n+1)D_{\nu_1}\Big)_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3,C)=0,$$

$$\Big(\Big(\nu_1-\frac{1}{2\ln C}\Big)D_{\nu_1}^{n}D_{\nu_2}+nD_{\nu_1}^{n-1}D_{\nu_2}-\sum_{k=1}^{n}\binom{n}{k}\frac{\mathcal{B}_{k+1}(1)}{k+1}(\ln C)^{-k-1}D_{\nu_1}^{n-k}D_{\nu_2}^{k+1}+2\nu_2(\ln C)^{-1}D_{\nu_1}^{n-1}D_{\nu_2}^{2}$$

$$+3\nu_3(\ln C)^{-2}D_{\nu_1}^{n-2}D_{\nu_2}^{3}-(n+1)D_{\nu_1}^{n+1}\Big)_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3,C)=0,$$

$$\Big(\Big(\nu_1-\frac{1}{2\ln C}\Big)D_{\nu_1}^{2n}D_{\nu_3}+2nD_{\nu_1}^{2n-1}D_{\nu_3}-\sum_{k=1}^{n}\binom{n}{k}\frac{\mathcal{B}_{k+1}(1)}{k+1}(\ln C)^{-k-1}D_{\nu_1}^{2n-2k}D_{\nu_3}^{k+1}+2\nu_2(\ln C)^{-1}D_{\nu_1}^{2n-1}D_{\nu_3}^{2}$$

$$+3\nu_3(\ln C)^{-2}D_{\nu_1}^{2n-4}D_{\nu_3}^{3}-(n+1)D_{\nu_1}^{2n+2}\Big)_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3,C)=0,$$

$$\Big(\Big(\nu_1-\frac{1}{2\ln C}\Big)D_{\nu_1}^{n}D_{\nu_3}+nD_{\nu_1}^{n-1}D_{\nu_3}-\sum_{k=1}^{n}\binom{n}{k}\frac{\mathcal{B}_{k+1}(1)}{k+1}(\ln C)^{-k-1}D_{\nu_1}^{n-k}D_{\nu_2}^{k}D_{\nu_3}+2\nu_2(\ln C)^{-1}D_{\nu_1}^{n-1}D_{\nu_2}^{2}D_{\nu_3}$$

$$+3\nu_3(\ln C)^{-2}D_{\nu_1}^{n-2}D_{\nu_2}^{3}D_{\nu_3}-(n+1)D_{\nu_1}^{n+2}\Big)_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3,C)=0,$$

$$\Big(\Big(\nu_1-\frac{1}{2\ln C}\Big)D_{\nu_1}^{2n}D_{\nu_2}+2nD_{\nu_1}^{2n-1}D_{\nu_2}-\sum_{k=1}^{n}\binom{n}{k}\frac{\mathcal{B}_{k+1}(1)}{k+1}(\ln C)^{-k-1}D_{\nu_1}^{2n-k}D_{\nu_2}D_{\nu_3}^{k}$$

$$+2\nu_2(\ln C)^{-1}D_{\nu_1}^{2n-1}D_{\nu_2}D_{\nu_3}+3\nu_3(\ln C)^{-2}D_{\nu_1}^{2n-4}D_{\nu_2}D_{\nu_3}^{2}-(n+1)D_{\nu_1}^{2n+1}\Big)_{\mathcal{D}}\mathcal{B}_n(\nu_1,\nu_2,\nu_3,C)=0.$$

Example 4.2. Consider $\left(\frac{2}{e^{\xi}+1}\right)=\mathcal{R}(\xi)$ in generating function (2.1), the 3V1PGHbAP ${}_{\mathcal{D}}\mathcal{R}_n(\nu_1,\nu_2,\nu_3,C)$ reduces to the 3-variable 1-parameter generalized Hermite-based Euler polynomials (3V1PGHbEP) ${}_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C)$ defined by the following generating function:

$$\left(\frac{2}{e^{\xi}+1}\right)C^{\nu_1\xi+\nu_2\xi^2+\nu_3\xi^3}=\sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3;C)\frac{\xi^n}{n!},\quad C>1,\quad |\xi|<\pi, \tag{4.12}$$

or, equivalently

$$\left(\frac{2}{e^{\xi}+1}\right)e^{\ln C(\nu_1\xi+\nu_2\xi^2+\nu_3\xi^3)}=\sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3;C)\frac{\xi^n}{n!},\quad C>1,\quad |\xi|<\pi. \tag{4.13}$$

Further, the remaining corressponding results for the 3-variable 1-parameter generalized Hermite-based Euler polynomials (3V1PGHbEP) ${}_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C)$ are established by setting similar substitutions in the following manner:

The 3-variable 1-parameter generalized Hermite-based Euler polynomials ${}_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3;C)$ are defined by the following series definition:

$${}_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3;C)=n!\sum_{k=0}^{[n/3]}\frac{{}_{\mathcal{D}}\mathcal{E}_{n-3k}(\nu_1,\nu_2)\nu_3^k}{(n-3k)!\,k!}(\ln C)^{n-2k}, \tag{4.14}$$

where

$$\left(\frac{2}{e^{\xi}+1}\right)C^{\nu_1\xi+\nu_2\xi^2}=\sum_{n=0}^{\infty}{}_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2)\frac{\ln C\xi^n}{n!}.$$

Next, setting $\delta_0 = 1$ and $\delta_i = \frac{1}{2}$ $(i = 1, 2, \cdots, n)$ in expression (2.7), the 3-variable 1-parameter generalized Hermite-based Euler polynomials ${}_{\mathcal{D}}\mathcal{E}_n(\nu_1, \nu_2, \nu_3; C)$ of degree n are defined by the following determinant definition:

$$
\begin{aligned}
&{}_{\mathcal{D}}\mathcal{E}_0(\nu_1, \nu_2, \nu_3; C) = \frac{1}{\delta_0},\\
&{}_{\mathcal{D}}\mathcal{E}_n(\nu_1, \nu_2, \nu_3; C) = \frac{(-1)^n}{(\delta_0)^{n+1}}
\begin{vmatrix}
1 & \mathcal{D}_1(\nu_1, \nu_2, \nu_3; C) & \mathcal{D}_2(\nu_1, \nu_2, \nu_3; C) & \cdots & \mathcal{D}_{n-1}(\nu_1, \nu_2, \nu_3; C) & \mathcal{D}_n(\nu_1, \nu_2, \nu_3; C) \\
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\
0 & 1 & \frac{1}{2}\binom{2}{1} & \cdots & \frac{1}{2}\binom{n-1}{1} & \frac{1}{2}\binom{n}{1} \\
0 & 0 & 1 & \cdots & \frac{1}{2}\binom{n-1}{2} & \frac{1}{2}\binom{n}{2} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\binom{n}{n-1}
\end{vmatrix}.
\end{aligned}
\tag{4.15}
$$

Also, the 3-variable, 1-parameter generalized Hermite-based Euler polynomials ${}_{\mathcal{D}}\mathcal{E}_n(\nu_1, \nu_2, \nu_3, C)$ adhere to the following recurrence relation and shift operators:

$$
\begin{aligned}
{}_{\mathcal{D}}\mathcal{E}_{n+1}(\nu_1, \nu_2, \nu_3, C) &= (\nu_1 \ln C - \tfrac{1}{2})\, {}_{\mathcal{D}}\mathcal{R}_n(\nu_1, \nu_2, \nu_3, C) - \textstyle\sum_{k=1}^{n} \binom{n}{k} \mathcal{E}_k \, {}_{\mathcal{D}}\mathcal{E}_{n-k}(\nu_1, \nu_2, \nu_3, C) \\
&\quad + 2n\nu_2 \ln C_{\mathcal{D}}\mathcal{E}_{n-2}(\nu_1, \nu_2, \nu_3, C) + 3n(n-1)\nu_3 \ln C_{\mathcal{D}}\mathcal{E}_{n-2}(\nu_1, \nu_2, \nu_3, C),
\end{aligned}
\tag{4.16}
$$

$$ {}_{\nu_1}\mathcal{L}_n^- := \frac{1}{n(\ln C)} D_{\nu_1}, \tag{4.17} $$

$$ {}_{\nu_2}\mathcal{L}_n^- := \frac{1}{n(\ln C)} D_{\nu_1}^{-1} D_{\nu_2}, \tag{4.18} $$

$$ {}_{\nu_3}\mathcal{L}_n^- := \frac{1}{n(\ln C)} D_{\nu_1}^{-2} D_{\nu_3}, \tag{4.19} $$

$$ {}_{\nu_1}\mathcal{L}_n^+ := (\nu_1 \ln C - \frac{1}{2}) + \sum_{k=1}^{n} \binom{n}{k} \mathcal{E}_k (\ln C)^{-k} D_{\nu_1}^k + 2\nu_2 D_{\nu_1} + 3\nu_3 (\ln C)^{-1} D_{\nu_1}^2, \tag{4.20} $$

$$ {}_{\nu_2}\mathcal{L}_n^+ := (\nu_1 \ln C - \frac{1}{2}) + \sum_{k=1}^{n} \binom{n}{k} \mathcal{E}_k (\ln C)^{-k} D_{\nu_1}^{-k} D_{\nu_2}^k + 2\nu_2 D_{\nu_1}^{-1} D_{\nu_2} + 3\nu_3 (\ln C)^{-1} D_{\nu_1}^{-2} D_{\nu_2}^2, \tag{4.21} $$

and

$$ {}_{\nu_3}\mathcal{L}_n^+ := (\nu_1 \ln C - \frac{1}{2}) + \sum_{k=1}^{n} \binom{n}{k} \mathcal{E}_k (\ln C)^{-k} D_{\nu_1}^{-k} D_{\nu_3}^k + 2\nu_2 D_{\nu_1}^{-1} D_{\nu_3} + 3\nu_3 (\ln C)^{-1} D_{\nu_1}^{-4} D_{\nu_3}^2. \tag{4.22} $$

Finally, the 3-variable 1-parameter generalized Hermite-based Euler polynomials ${}_{\mathcal{D}}\mathcal{E}_n(\nu_1, \nu_2, \nu_3, C)$ satisfy the following differential, integrodifferential and partial differential equations:

$$\Big(\big(\nu_1 - \frac{1}{2\ln C}\big)D_{\nu_1} + \sum_{k=1}^{n}\binom{n}{k}\mathcal{E}_k(\ln C)^{-k-1}D_{\nu_1}^{k+1} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{2}$$
$$+ 3\nu_3(\ln C)^{-2}D_{\nu_1}^{3} - (n+1)\Big)_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C) = 0,$$

$$\Big(\big(\nu_1 - \frac{1}{2\ln C}\big)D_{\nu_2} + \sum_{k=1}^{n}\binom{n}{k}\mathcal{E}_k(\ln C)^{-k-1}D_{\nu_1}^{-k}D_{\nu_2}^{k+1} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{-1}D_{\nu_2}^{2} + 3\nu_3(\ln C)^{-2}D_{\nu_1}^{-2}D_{\nu_2}^{3}$$
$$- (n+1)D_{\nu_1}\Big)_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C) = 0,$$

$$\Big(\big(\nu_1 - \frac{1}{2\ln C}\big)D_{\nu_3} + \sum_{k=1}^{n}\binom{n}{k}\mathcal{E}_k(\ln C)^{-k-1}D_{\nu_1}^{-k}D_{\nu_3}^{k+1} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{-1}D_{\nu_3}^{2} + 3\nu_3(\ln C)^{-2}D_{\nu_1}^{-4}D_{\nu_3}^{3}$$
$$- (n+1)D_{\nu_1}^{2}\Big)_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C) = 0,$$

$$\Big(\big(\nu_1 - \frac{1}{2\ln C}\big)D_{\nu_3} + \sum_{k=1}^{n}\binom{n}{k}\mathcal{E}_k(\ln C)^{-k-1}D_{\nu_1}^{-k}D_{\nu_2}^{k}D_{\nu_3} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{-1}D_{\nu_2}^{2}D_{\nu_3} + 3\nu_3(\ln C)^{-2}D_{\nu_1}^{-2}D_{\nu_2}^{3}D_{\nu_3}$$
$$- (n+1)D_{\nu_1}^{2}\Big)_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C) = 0,$$

$$\Big(\big(\nu_1 - \frac{1}{2\ln C}\big)D_{\nu_2} + \sum_{k=1}^{n}\binom{n}{k}\mathcal{E}_k(\ln C)^{-k-1}D_{\nu_1}^{-k}D_{\nu_2}D_{\nu_3}^{k} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{-1}D_{\nu_2}D_{\nu_3} + 3\nu_3(\ln C)^{-2}D_{\nu_1}^{-4}D_{\nu_2}D_{\nu_3}^{2}$$
$$- (n+1)D_{\nu_1}\Big)_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C) = 0,$$

$$\Big(\big(\nu_1 - \frac{1}{2\ln C}\big)D_{\nu_1}^{n}D_{\nu_2} + nD_{\nu_1}^{n-1}D_{\nu_2} + \sum_{k=1}^{n}\binom{n}{k}\mathcal{E}_k(\ln C)^{-k-1}D_{\nu_1}^{n-k}D_{\nu_2}^{k+1} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{n-1}D_{\nu_2}^{2}$$
$$+ 3\nu_3(\ln C)^{-2}D_{\nu_1}^{n-2}D_{\nu_2}^{3} - (n+1)D_{\nu_1}^{n+1}\Big)_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C) = 0,$$

$$\Big(\big(\nu_1 - \frac{1}{2\ln C}\big)D_{\nu_1}^{2n}D_{\nu_3} + 2nD_{\nu_1}^{2n-1}D_{\nu_3} + \sum_{k=1}^{n}\binom{n}{k}\mathcal{E}_k(\ln C)^{-k-1}D_{\nu_1}^{2n-2k}D_{\nu_3}^{k+1} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{2n-1}D_{\nu_3}^{2}$$
$$+ 3\nu_3(\ln C)^{-2}D_{\nu_1}^{2n-4}D_{\nu_3}^{3} - (n+1)D_{\nu_1}^{2n+2}\Big)_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C) = 0,$$

$$\Big(\big(\nu_1 - \frac{1}{2\ln C}\big)D_{\nu_1}^{n}D_{\nu_3} + nD_{\nu_1}^{n-1}D_{\nu_3} + \sum_{k=1}^{n}\binom{n}{k}\mathcal{E}_k(\ln C)^{-k-1}D_{\nu_1}^{n-k}D_{\nu_2}^{k}D_{\nu_3} + 2\nu_2(\ln C)^{-1}D_{\nu_1}^{n-1}D_{\nu_2}^{2}D_{\nu_3}$$
$$+ 3\nu_3(\ln C)^{-2}D_{\nu_1}^{n-2}D_{\nu_2}^{3}D_{\nu_3} - (n+1)D_{\nu_1}^{n+2}\Big)_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C) = 0,$$

$$\Big(\big(\nu_1 - \frac{1}{2\ln C}\big)D_{\nu_1}^{2n}D_{\nu_2} + 2nD_{\nu_1}^{2n-1}D_{\nu_2} + \sum_{k=1}^{n}\binom{n}{k}\mathcal{E}_k(\ln C)^{-k-1}D_{\nu_1}^{2n-k}D_{\nu_2}D_{\nu_3}^{k}$$
$$+ 2\nu_2(\ln C)^{-1}D_{\nu_1}^{2n-1}D_{\nu_2}D_{\nu_3} + 3\nu_3(\ln C)^{-2}D_{\nu_1}^{2n-4}D_{\nu_2}D_{\nu_3}^{2} - (n+1)D_{\nu_1}^{2n+1}\Big)_{\mathcal{D}}\mathcal{E}_n(\nu_1,\nu_2,\nu_3,C) = 0.$$

Similar results can be obtained for 3-variable 1-parameter generalized Hermite-based Genocchi polynomials ${}_{\mathcal{D}}\mathcal{G}_n(v_1, v_2, v_3, C)$.

5. Conclusions

We present a novel framework for introducing generalized 3-variable 1-parameter Hermite-based Appell polynomials. The essential characteristics of these polynomials are explained through generating functions, series definitions, and determinant definitions. This research uses a factorization technique to build recurrence relations, shift operators, and several differential equations, such as integrodifferential, partial, and differential. We focus on examining the special situations of 3-variable 1-parameter generalized Hermite-based Bernoulli, Euler, and Genocchi polynomials, providing an understanding of their special characteristics and use.

Future research could focus on extending the current framework to include more than three variables, exploring the associated complexities and new properties. Further examination of additional analytical properties, such as orthogonality, asymptotic behaviour, and zeros, is also warranted. Developing efficient computational algorithms to facilitate the practical application of these polynomials in various fields, such as numerical analysis, physics, and engineering, will be beneficial. Additionally, investigating their application in solving higher-order and more complex differential equations, particularly in modeling real-world phenomena, could yield significant insights. Interdisciplinary applications in finance, biology, and data science and enhanced graphical and numerical analyses could provide deeper insights and lead to new theoretical advancements.

Author contributions

The authors contribute equally to this article. Both authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in creating this article.

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Conflict of interest

The authors declare no conflicts of interest in this article.

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