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Research article

Geometric topology of CR-warped products in six-dimensional sphere

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Abstract: In this paper, we prove that a three-dimensional CR-warped product submanifold of a nearly Kaehler six-dimensional sphere, under some restrictions, contains finite fundamental groups. Using another approach from the theory of eigenvalues and also the gradient of Ricci curvature, we obtain similar results.

Keywords: CR-warped products; nearly Kaehler manifold; fundamental groups
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1. Introduction

The finite fundamental group (also known as the fundamental groupoid) of a space is a generalization of the fundamental group, which considers not just loops based at a single point but also paths and homotopies between loops that start and end at different points. The study of finite fundamental groups has many applications in several branches of mathematics. From an algebraic topological point of view, the finite fundamental groups are important objects of study in algebraic topology as they provide a way to distinguish between spaces that have different homotopic properties. In particular, the classification of finite fundamental groups of surfaces is a classical problem in algebraic topology. Given representation theory, the finite fundamental group of a space can be used to construct representations of the group in terms of linear transformations on vector spaces. This leads to the study of the representation theory of finite groups, which has applications in many areas of mathematics and physics (see, for instance, [1, 20, 22, 28, 34, 35]).

The six-dimensional nearly Kähler manifolds are special classes of six-dimensional Riemannian manifolds that possess a nearly Kähler structure. A nearly Kähler structure is a Riemannian structure that satisfies all the axioms of a Kähler structure except the integrability condition of the complex structure [38]. This means that the almost complex structure of a nearly Kähler manifold is not necessarily integrable, but it is "as close as possible" to being integrable. A nearly Kähler structure is equivalent to a Hermitian structure with a closed, non-degenerate 3-form in the six-dimensional manifold. Using the warped product theory and also the notion of CR-submanifolds [3] and warped product manifolds [4]. Chen defined the CR-warped product submanifold of almost Hermitian manifolds in his series of papers [7–9]. Later, such submanifolds have been studied by many authors for example [10, 11, 17, 33, 36]. From Hiepko's result, we know that if a manifold is a warped product manifold, it has two integrable distributions. Deshmukh and Ghazal [12] proved the non-existence of a 4-dimensional CR-submanifold with an integrable, totally real distribution $\mathcal{D}^{\perp}$ on $\mathbb{S}^6$. Therefore, one has to consider only three-dimensional CR-warped product submanifolds of nearly Kaehler six-spheres . We note that Sekigawa [32] obtained an example of a three-dimensional warped product, the CR-submanifold of $\mathbb{S}^6$. Later, this example was generalized by Hashimoto and Mashimo [16]. As a conclusion, there are many three-dimensional CR-warped product submanifolds in $\mathbb{S}^6$ [29].

On the other hand, Sahin and Sahin [31] have examined compact minimum contact CR-submanifolds of odd-dimensional unit spheres and derived topological sphere theorems. According to their results, CR-warped product submanifolds of odd-dimensional unit spheres are homeomorphic to the sphere if an inequality involving warping function and fiber scalar curvature is satisfied. As an example, it has been shown that for a 5-dimensional unit sphere, a 4-dimensional compact minimal contact CR-warped product submanifold is homeomorphic to the sphere if $\|\nabla \ln f\|^2 < 1$ is satisfied. Here, f represents a non-constant warping function. Another result obtained by the same authors when they applied Bonnet–Myers's theorem is about the fundamental group and they also obtained another result about homology groups by applying Leung's theorem. Inspired by the above results, we extend such a hypothesis in the setting of CR-warped products in nearly Kaehler's six-dimensional sphere.

2. Basic ideas

In this section, we recall some basic notions, formulas, and definitions of the nearly Kahler structure of $\mathbb{S}^6$ (see, for instance, [13–15] for more detail). Suppose $\{u_0, u_1, \cdots, u_7\}$ is the standard frame for $\mathbb{R}^8$. Thus, all points $r \in \mathbb{R}^8$ are uniquely introduced as $r = \lambda u_0 + x$, for some real number λ and $x \in \text{span}\{u_1, \cdots, u_7\}$. Hence, r can be regarded as a Cayley number. If $\lambda = 0$, r can be viewed as purely imaginary. For x_1, x_2, two purely imaginary Cayley numbers, the multiplication "$\cdot$" can be defined as:

$$x_1 \cdot x_2 =< x_1, x_2 > u_0 + x_1 \times x_2,$$

where "$\times$" is defined as follows:

$u_i \times u_j$	1	2	3	4	5	6	7
1	0	u_3	$-u_2$	u_5	$-u_4$	u_7	$-u_6$
2	$-u_3$	0	u_1	u_6	$-u_7$	$-u_4$	u_5
3	u_2	$-u_1$	0	$-u_7$	$-u_6$	u_5	u_4
4	$-u_5$	$-u_6$	u_7	0	u_1	u_2	$-u_3$
5	u_4	u_7	u_6	$-u_1$	0	$-u_3$	$-u_2$
6	$-u_7$	u_4	$-u_5$	$-u_2$	u_3	0	u_1
7	u_6	$-u_5$	$-u_4$	u_3	u_2	$-u_1$	0

We can check that the multiplication "×" is not commutative and not associative. Suppose C_+ is the set of all Cayley numbers that are pure and imaginary. Now, this set

$$\mathbb{S}^6(1) = \Big\{x \in C_+ | < x, x >= 1\Big\},$$

denotes the 6-dimensional unit sphere with the origin as a center. The perpendicular subspace of C_+ to $p \in \mathbb{S}^6$ forms the tangent space $T_p\mathbb{S}^6$ of $\mathbb{S}^6$. The almost complex structure $\mathbf{J}$ on $T_p\mathbb{S}^6$, which is an endomorphism that is $\mathbf{J}^2 = -\mathbb{I}\mathrm{d}$, can be defined as:

$$\mathbf{J}_p\mathcal{Z}_1 = p \times \mathcal{Z}_1,$$

where $p \in \mathbb{S}^6(1)$, $\mathcal{Z}_1 \in T_p\mathbb{S}^6$. The compact Lie group G_2 is the group of automorphisms of C. G_2 works normally at $\mathbb{S}^6(1)$ and maintains $\mathbf{J}$ and the metric at $\mathbb{S}^6(1)$. Now, to obtain the nearly Kahler structure on $(\mathbb{S}^6, < \cdot, \cdot >, \mathbf{J})$, let us define $G(\mathcal{Z}_1, \mathcal{Z}_2) = (\widetilde{\nabla}_{\mathcal{Z}_1}\mathbf{J})\mathcal{Z}_2$, for $\mathcal{Z}_1, \mathcal{Z}_2 \in \Gamma(\mathbb{S}^6)$ and $\widetilde{\nabla}$ Levi–Civita connection at $\mathbb{S}^6$. It is clear that if $G(\mathcal{Z}_1, \mathcal{Z}_1) = 0$, then $(\widetilde{\nabla}_{\mathcal{Z}_1}\mathbf{J})\mathcal{Z}_1 = 0$. The above almost complex structure $\mathbf{J}$, together with the induced Riemannian metric g on $\mathbb{S}^6$, gives the following nearly Kaehlerian structure [15]

$$\mathbf{J}^2 = -\mathbb{I}, \quad g(\mathbf{J}\mathcal{Z}_1, \mathbf{J}\mathcal{Z}_2) = g(\mathcal{Z}_1, \mathcal{Z}_2), \quad (\widetilde{\nabla}_{\mathcal{Z}_1}\mathbf{J})\mathcal{Z}_1 = 0. \tag{2.1}$$

Hence, the curvature tensor $\tilde{R}$ of $\mathbb{S}^6$ is defined by

$$\tilde{R}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4) = g(\mathcal{Z}_1, \mathcal{Z}_4)g(\mathcal{Z}_2, \mathcal{Z}_3) - g(\mathcal{Z}_1, \mathcal{Z}_3)g(\mathcal{Z}_2, \mathcal{Z}_4), \tag{2.2}$$

$\forall\ \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4 \in \Gamma(T\mathbb{S}^6)$.

Let us assume that $\mathcal{N}$ is a $2m$-dimensional Riemannian submanifold of a nearly Kähler manifold, $\widetilde{\mathcal{N}}^{2m}$. Let us denote by $\Gamma(T\mathcal{N})$ the section of the tangent bundle of $\mathcal{N}$ and by $\Gamma(T\mathcal{N}^\perp)$ the set of all normal vector fields of $\mathcal{N}$, respectively. ∇ represents here the Levi–Civita connection on the tangent bundle $T\mathcal{N}$, and $\nabla^\perp$ represents here the Levi–Civita connection on the normal bundle $T\mathcal{N}^\perp$, respectively. Thus, the Gauss and Weingarten formulas are given by:

$$\overline{\nabla}_{\mathcal{Z}_1}\mathcal{Z}_2 = \nabla_{\mathcal{Z}_1}\mathcal{Z}_2 + \boldsymbol{B}(\mathcal{Z}_1, \mathcal{Z}_2), \tag{2.3}$$

$$\overline{\nabla}_{\mathcal{Z}_1}\xi = -A_\xi\mathcal{Z}_1 + \nabla^\perp_{\mathcal{Z}_1}\xi, \tag{2.4}$$

for all $\mathcal{Z}_1, \mathcal{Z}_2 \in \Gamma(T\mathcal{N})$ and $\xi \in \Gamma(T\mathcal{N}^\perp)$, where A_ξ and $\boldsymbol{B}$ are the shape operators and respectively of the second fundamental form, given by:

$$g(A_\xi\mathcal{Z}_1, \mathcal{Z}_2) = g(\boldsymbol{B}(\mathcal{Z}_1, \mathcal{Z}_2), \xi). \tag{2.5}$$

The submanifold $\mathcal{N}$ is totally umbilical [3, 4, 7–9] if H satisfies $\boldsymbol{B}(\mathcal{Z}_1, \mathcal{Z}_2) = g(\mathcal{Z}_1, \mathcal{Z}_2)H$. The submanifold $\mathcal{N}$ is totally geodesic if $\boldsymbol{B} \equiv 0$ and minimal if $H = 0$, where H represents the mean curvature vector described by $H = \frac{1}{m}trace(h)$. The covariant derivative of σ is computed by the following relation:

$$(\nabla_{\mathcal{Z}_3}\boldsymbol{B})(\mathcal{Z}_1, \mathcal{Z}_2) = \nabla^{\perp}_{\mathcal{Z}_3}\boldsymbol{B}(\mathcal{Z}_1, \mathcal{Z}_2) - \boldsymbol{B}(\nabla_{\mathcal{Z}_3}\mathcal{Z}_1, \mathcal{Z}_2) - \boldsymbol{B}(\mathcal{Z}_1, \nabla_{\mathcal{Z}_3}\mathcal{Z}_2). \tag{2.6}$$

The Gauss and Codazzi equations are characterized by the following relation:

$$\begin{aligned}\overline{R}(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4) =& R(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4) + g(\boldsymbol{B}(\mathcal{Z}_1, \mathcal{Z}_3), \boldsymbol{B}(\mathcal{Z}_2, \mathcal{Z}_4)) \\ &- g(\boldsymbol{B}(\mathcal{Z}_1, \mathcal{Z}_4), \boldsymbol{B}(\mathcal{Z}_2, \mathcal{Z}_3)), \end{aligned} \tag{2.7}$$

$$(\overline{R}(\mathcal{Z}_1, \mathcal{Z}_2)\mathcal{Z}_3)^{\perp} = (\nabla_{\mathcal{Z}_1}\boldsymbol{B})(\mathcal{Z}_2, \mathcal{Z}_3) - (\nabla_{\mathcal{Z}_2}\boldsymbol{B})(\mathcal{Z}_1, \mathcal{Z}_3), \tag{2.8}$$

for every $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4 \in \Gamma(T\mathcal{N})$.

Definition 2.1. *Let $\mathcal{N}$ be a Riemannian submanifold of a nearly Kähler manifold, $\widetilde{\mathcal{N}}^{2m}$. Then $\mathcal{N}$ is a real submanifold if $\mathbf{J}(T\mathcal{N}) \subset T\mathcal{N}$, and $\mathcal{N}$ is a complex submanifold if $\mathbf{J}(T\mathcal{N}) \subset T\mathcal{N}^{\perp}$.*

Definition 2.2. *A $\mathbb{CR}$-submanifold of nearly Kähler manifold $\widetilde{\mathcal{N}}^{2m}$ whose tangent bundle can be decomposed as $T\mathcal{N} = \mathfrak{D} \oplus \mathfrak{D}_{\perp}$. Here, $\mathfrak{D}$ represents a complex distribution, and $\mathfrak{D}_{\perp}$ can be represented as a real distribution. Moreover, if there is a Riemannian metric on $\mathcal{N}$ of the form $g = g_{\mathcal{N}_T} + f^2 g_{\mathcal{N}_{\perp}}$, then $\mathcal{N}$ is a $\mathbb{CR}$-warped product of the form $\mathcal{N} = \mathcal{N}_T \times_f \mathcal{N}_{\perp}$.*

As we know, f represents a warping function for the warped product $\mathcal{N}_T \times_f \mathcal{N}_{\perp}$. For the general warped product manifold $\mathcal{N}_1 \times_f \mathcal{N}_2$, we have the following useful formulas [4]

$$\nabla_{\mathcal{Z}_1}\mathcal{Z}_2 = \frac{\mathcal{Z}_1 f}{f}\mathcal{Z}_2, \tag{2.9}$$

$$\mathcal{R}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{Z}_1 = \mathcal{R}^{\mathcal{N}_2}(\mathcal{Z}_2, \mathcal{Z}_3)\mathcal{Z}_1 + \left(\frac{||f||^2}{f}\right)\Big\{g(\mathcal{Z}_2, \mathcal{Z}_1)\mathcal{Z}_3 - g(\mathcal{Z}_3, \mathcal{Z}_1)\mathcal{Z}_1\Big\}, \tag{2.10}$$

where $\mathcal{R}^{\mathcal{N}_2}$ is the curvature tensor of fiber $\mathcal{N}_2$ and ∇f is the gradient of f.

3. Main results

In this section, we prove that the compact CR-warped products on a minimal submanifold are homeomorphic to a sphere. First, let us recall some fundamental theorems:

Theorem 3.1. *[21] Let $\mathcal{N}^n$ be an oriented compact minimal n-dimensional submanifold of the unit sphere $\mathbb{S}^{n+p}$ with $n = 3$. If $||\mathbf{B}||^2 < n$, then $\mathcal{N}^n$ has finite fundamental groups.*

Theorem 3.2. *[21] Let $\mathcal{N}^n$ be an oriented compact minimal n-dimensional submanifold of the unit sphere $\mathbb{S}^N$. Let f be an eigenfunction on $\mathcal{N}^n$ corresponding to a non-zero eigenvalue λ, then*

$$\int_{\mathcal{N}} \left(\lambda + ||\mathbf{B}||^2 - n\right)||df||^2 dV \geq 0, \tag{3.1}$$

where dV represents the volume element on $\mathcal{N}^n$. The equality holds if and only if either $\mathcal{N}^n$ is totally geodesic, and λ is the first non-zero eigenvalue, or $n = 2$ and $N = 2m$, and $\mathcal{N}$ is isometric to $\mathbb{S}^2\sqrt{\frac{m(m+1)}{2}}$ and λ is the first non-zero eigenvalue.

Theorem 3.3. *[21] If v is a unit vector on the minimal n-dimensional submanifold $\mathcal{N}^n$ of the sphere $\mathbb{S}^{n+p}$, then the following inequality holds:*

$$\mathcal{R}ic(v,v) \geq \frac{n-1}{n}\Big(n - \|\mathbf{B}\|^2\Big), \tag{3.2}$$

where $\mathcal{R}ic(v,v)$ is the Ricci curvature in the direction of v in $\mathcal{N}^n$.

Theorem 3.4. *[23] Let $\mathcal{N}^n$ be an oriented compact minimal n-dimensional submanifold of the unit sphere $\mathbb{S}^{n+p}$ for $n \geq 2$. If the following inequality is satisfied:*

$$\|\mathbf{B}\|^2 \leq \frac{n(3n-2)}{5n-4}, \tag{3.3}$$

then, $\mathcal{N}^n$ is either a totally geodesic submanifold or a Veronese surface in $\mathbb{S}^4$.

Similarly, we have the following result:

Theorem 3.5. *[23] Let $\mathcal{N}^n$ be an oriented compact minimal n-dimensional submanifold of the unit sphere $\mathbb{S}^{n+p}$, and let n be odd. If the following inequality is satisfied:*

$$\|\mathbf{B}\|^2 \leq \frac{n(3n-5)}{5n-9}, \tag{3.4}$$

then $\mathcal{N}^n$ is either a totally geodesic submanifold or $n = 3$ and $\|\mathbf{B}\|^2 = 2$ on $\mathcal{N}^3$, the second fundamental form is given by:

$$\Big(\mathcal{B}^4_{ij}\Big) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Big(\mathcal{B}^5_{ij}\Big) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Big(\mathcal{B}^5_{ij}\Big) = 0, \quad \alpha \geq 6.$$

The following results are obtained for a CR-warped submanifold of a nearly Kaehler manifold.

Lemma 3.1. *[29] Let $\mathcal{N} = \mathcal{N}_T \times_f \mathcal{N}_\perp$ be a $\mathbb{CR}$-warped product submanifold in a near Kaehler manifold, then*

$$g(\mathbf{B}(\mathcal{Z}_1,\mathcal{Z}_2),\mathbf{J}\mathcal{Z}_3) = -\mathbf{J}\mathcal{Z}_1(\ln f)g(\mathcal{Z}_2,\mathcal{Z}_3), \tag{3.5}$$

$$g(\mathbf{B}(\mathcal{Z}_1,\mathbf{J}\mathcal{Z}_4),\mathbf{J}\mathcal{Z}_2) = 0, \tag{3.6}$$

for any $\mathcal{Z}_1,\mathcal{Z}_4 \in \Gamma(\mathfrak{D}^T)$ and $\mathcal{Z}_2,\mathcal{Z}_3 \in \Gamma(\mathfrak{D}^\perp)$.

Now, we prove the main result of this paper.

Theorem 3.6. *Let $\mathcal{N}^3 = \mathcal{N}^2_T \times_f \mathcal{N}^1_\perp$ be an oriented compact minimal 3-dimensional $\mathbb{CR}$-warped product submanifold in a nearly Kaehler $\mathbb{S}^6$ if the following inequality is satisfied:*

$$\|\nabla f\|^2 < \frac{3f^2}{2}, \tag{3.7}$$

then, $\mathcal{N}^3$ has finite fundamental groups.

Proof. Assume that $\{u_1, u_2, u_3\}$ is an orthonormal frame of the $\mathbb{CR}$-warped product submanifold $\mathcal{N}^3 = \mathcal{N}_T^2 \times_f \mathcal{N}_\perp^1$ such that $\{u_1, u_2\}$ is tangent to $\mathcal{N}_T$ and $\{u_3\}$ is tangent to $\mathcal{N}_\perp$. Then we have

$$\|\boldsymbol{B}\|^2 = \|\boldsymbol{B}(\mathfrak{D}^T, \mathfrak{D}^T)\|^2 + \|\boldsymbol{B}(\mathfrak{D}^\perp, \mathfrak{D}^\perp)\|^2 + 2\|\boldsymbol{B}(\mathfrak{D}^\perp, \mathfrak{D}^T)\|^2. \tag{3.8}$$

The above equation can be expressed as

$$\begin{aligned}\|\boldsymbol{B}\|^2 =& \sum_{i,j}\sum_{\alpha} g(\boldsymbol{B}(u_i, u_j), \mathbf{J}u_\alpha^*)^2 + \sum_{\alpha,\beta,\gamma} g(\boldsymbol{B}(u_\alpha^*, u_\beta^*), \mathbf{J}u_\gamma^*)^2 \\ &+ 2\sum_{i}\sum_{\alpha,\beta} g(\boldsymbol{B}(u_i, u_\alpha^*), \mathbf{J}u_\beta^*)^2, \end{aligned} \tag{3.9}$$

for $i, j = 1, \cdots n_1 = \dim \mathcal{N}_T$ and $\alpha, \beta = 1, \cdots n_2 = \dim \mathcal{N}_\perp$. As we assumed that $n_1 = 2$ and $n_2 = 1$, then using Eqs (3.5) and (3.6), we derive the following:

$$\|\boldsymbol{B}\|^2 = 2\|\nabla \ln f\|^2 + \sum_{\alpha,\beta,\gamma} g(\boldsymbol{B}(u_i^*, u_j^*), \mathbf{J}u_\alpha^*)^2.$$

This is equivalent to the following:

$$\|\boldsymbol{B}\|^2 = 2\|\nabla \ln f\|^2 + \sum_{\alpha,\beta} g(\boldsymbol{B}(u_\alpha^*, u_\beta^*), \boldsymbol{B}(u_\alpha^*, u_\beta^*)). \tag{3.10}$$

From (2.7), we have

$$g(\widetilde{\mathbf{R}}(x, y)x, y) = g(\mathbf{R}(x, y)x, y) - g(\boldsymbol{B}(y, y), \boldsymbol{B}(x, x)) + g(\boldsymbol{B}(x, y), \boldsymbol{B}(x, y)). \tag{3.11}$$

As we have chosen the ambient manifold as a six-dimensional near Kaehler $\mathbb{S}^6$, then from (2.2), we have

$$g(\widetilde{\mathbf{R}}(x, y)x, y) = g(x, x)g(y, y) - g(x, y)^2. \tag{3.12}$$

Now combining (3.11), (3.12), and (2.10), we obtain

$$\begin{aligned} g(\boldsymbol{B}(x, y), \boldsymbol{B}(x, y)) =& \frac{\|\nabla f\|^2}{f^2}\|x\|^2\|y\|^2 - g(x, y)^2 + g(\boldsymbol{B}(y, y), \boldsymbol{B}(x, x)) \\ &+ \|x\|^2\|y\|^2 - g(x, y)^2 - g(\mathbf{R}^{\mathcal{N}_T}(x, y)x, y). \end{aligned} \tag{3.13}$$

Inserting the above equation in (3.10), we arrive at

$$\begin{aligned} \|\boldsymbol{B}\|^2 =& 2\|\nabla f\|^2 + \Big(1 + \frac{\|\nabla f\|^2}{f^2}\Big)(1 - 1) + \sum_{\alpha,\beta} g(\boldsymbol{B}(u_\alpha^*, u_\alpha^*), \boldsymbol{B}(u_\beta^*, u_\beta^*)) \\ &- \sum_{\alpha,\beta} g(\mathbf{R}^{\mathcal{N}_\perp}(u_\alpha^*, u_\beta^*)u_\alpha^*, u_\beta^*), \end{aligned}$$

which implies that

$$\|\boldsymbol{B}\|^2 = 2\frac{\|\nabla f\|^2}{f^2} + \sum_{\alpha,\beta} g(\boldsymbol{B}(u_\alpha^*, u_\alpha^*), \boldsymbol{B}(u_\beta^*, u_\beta^*)) - 2\tau_{\mathcal{N}_\perp}.$$

As the leaf is 1-dimensional, then $\tau_{\mathcal{N}_\perp} = 0$. Substituting these values into the above equations above, one obtains:

$$\|\boldsymbol{B}\|^2 = \frac{2\|\nabla f\|^2}{f^2} + \sum_{\alpha,\beta} g(\boldsymbol{B}(u_\alpha^*, u_\alpha^*), \boldsymbol{B}(u_\beta^*, u_\beta^*)). \tag{3.14}$$

From (3.6), we easily obtain

$$g(\boldsymbol{B}(\mathbf{J}\mathcal{Z}_1, \mathbf{J}\mathcal{Z}_4), \mathbf{J}\mathcal{Z}_2) = 0. \tag{3.15}$$

Since $\mathcal{N}$ is minimal, then we define the mean curvature as:

$$\begin{aligned} 0 =& \frac{1}{3} trace(\boldsymbol{B}) = \frac{1}{3} \sum_{i=1}^{2} \boldsymbol{B}(u_i, u_i) + \boldsymbol{B}(u_3^*, u_3^*) \\ =& \sum_{i=1}^{2} g(\boldsymbol{B}(u_i, u_i), \mathbf{J}u_3^*)\mathbf{J}u_3^* + \sum_{i=1}^{2} g(\boldsymbol{B}(\mathbf{J}u_i, \mathbf{J}u_i), \mathbf{J}u_3^*)\mathbf{J}u_3^* + \boldsymbol{B}(u_3^*, u_3^*). \end{aligned}$$

From (3.6) and (3.15), we obtain

$$\boldsymbol{B}(u_3^*, u_3^*) = 0. \tag{3.16}$$

In conclusion, from (3.14) and (3.15), we have

$$\|\boldsymbol{B}\|^2 = \frac{2\|\nabla f\|^2}{f^2}. \tag{3.17}$$

If the inequity (3.7) is satisfied, then from the above inequality, we get $\|\boldsymbol{B}\|^2 < 3$. Hence, the proof is done using Theorem 3.1. □

We obtain another important result from the following:

Theorem 3.7. *[26] Let $\mathcal{N}^n$ be an n-dimensional compact minimal submanifold of $\mathbb{S}^{n+m}$, $m \geq 2$. If $\|\mathbf{B}\|^2 \leq \frac{2n}{3}$ everywhere in $\mathcal{N}^n$, then $\mathcal{N}^n$ is either a totally geodesic submanifold or a Veronese surface in $\mathbb{S}^{2+m}$.*

Using the above theorem, we prove the following:

Theorem 3.8. *If $\mathcal{N}^3 = \mathcal{N}_T^2 \times_f \mathcal{N}_\perp^1$ is an oriented compact minimal 3-dimensional $\mathbb{CR}$-warped product submanifold in a nearly Kaehler $\mathbb{S}^6$, and if the following inequality is satisfied*

$$\|\nabla f\|^2 \leq f^2, \tag{3.18}$$

then $\mathcal{N}^3$ is either a totally geodesic submanifold or a Veronese surface in $\mathbb{S}^5$ for $m = 3$.

Proof. From (3.17) and (3.18), we obtain $\|\mathbf{B}\|^2 \leq 2$ for $n = 3$ as submanifold dimension and complex dimension $m = 3$ for ambient manifold dimension. Now, applying Theorem 3.7, we obtain the desired result. □

One of the most important goals in Riemannian geometry is to find the bound of the Laplacian on a particular manifold [2, 18, 19, 24, 37]. We can reach this purpose by studying the eigenvalues of the Dirichlet boundary condition, which is denoted by $\upsilon_1(\Sigma) > 0$, on a complete noncompact Riemannian manifold $\mathcal{N}^n$ with the compact domain Σ in $\mathcal{N}^n$. In this respect, we have:

$$\Delta\sigma + \upsilon_1\sigma = 0, \quad \text{on } \Sigma \text{ and } \quad \sigma = 0 \text{ on } \partial\Sigma, \tag{3.19}$$

where Δ is the Laplacian on $\mathcal{N}^n$ and σ is a non-zero function defined on $\mathcal{N}^n$. Then, $\upsilon_1(\mathcal{N}^n)$ can be expressed as $inf_{\Sigma}\upsilon_1(\Sigma)$.

The Dirichlet eigenvalues are the eigenvalues of the Laplace operator on a domain with Dirichlet boundary conditions. They have many important consequences in various areas of mathematics, including differential geometry, number theory, and mathematical physics. For example, for the Dirichlet eigenvalues, we can find the geometry of a domain. For example, the first Dirichlet eigenvalue of a domain is related to the diameter of the domain. The higher eigenvalues are related to the curvature of the domain and the way it is embedded in Euclidean space. In this sequel, the Dirichlet eigenvalues appear in the solution of the heat equation on a domain. The eigenvalues and the corresponding eigenfunctions determine the rate of decay of the solution. Assume that σ is the non-constant warping function on the compact warped product submanifold $\mathcal{N}^n$, then the minimum principle on υ_1 leads to (see, for instance, [5, 6])

$$\int_{\mathcal{N}^n} \|\nabla\sigma\|^2 dV \geq \upsilon_1 \int_{\mathcal{N}^n} (\sigma)^2 dV, \tag{3.20}$$

and the equality is satisfied if and only if

$$\Delta\sigma = \upsilon_1\sigma. \tag{3.21}$$

Based on the above classification, we obtain the following theorem:

Theorem 3.9. *Let $\mathcal{N}^3 = \mathcal{N}_T^2 \times_f \mathcal{N}_\perp^1$ be an oriented compact minimal 3-dimensional $\mathbb{CR}$-warped product submanifold on a near Kaehler $\mathbb{S}^6$, and if the following inequality is satisfied*

$$\|\nabla^2\sigma\|^2 + Ric(\nabla\sigma, \nabla\sigma) < \frac{3\upsilon_1 f^2}{2}, \tag{3.22}$$

then $\mathcal{N}^3$ has finite fundamental groups.

Proof. Let us consider Eq (3.20). With $\sigma = f$, one obtains:

$$\int_{\mathcal{N}^n} \|\nabla f\|^2 dV \geq \upsilon_1 \int_{\mathcal{N}^n} (f)^2 dV, \tag{3.23}$$

If f represents the first eigenfunction of the Laplacian, $\Delta f = div(\nabla f)$ for $\mathcal{B}^n$, connected to the first non-zero eigenvalue υ_1, such that $\Delta f = -\upsilon_1 f$, then recalling the Bochner formula (see [6]), this gives us the following relation for the differentiable function f:

$$\frac{1}{2}\Delta\|\nabla f\|^2 = \|\nabla^2 f\|^2 + Ric(\nabla f, \nabla f) + g(\nabla f, \nabla(\Delta f)).$$

By the integration of the previous equation, using the Stokes theorem, we have

$$\int_{\mathcal{N}_T\times\{1\}} ||\nabla^2 f||^2 dV + \int_{\mathcal{N}_T\times\{1\}} Ric(\nabla f, \nabla f) dV + \int_{\mathcal{N}_T\times\{1\}} g(\nabla f, \nabla(\Delta f)) dV = 0. \tag{3.24}$$

Now, using $\Delta f = -\upsilon_1 f$ and making some rearrangement in Eq (3.24), we derive

$$\int_{\mathcal{N}_T\times\{1\}} ||\nabla f||^2 dV = \frac{1}{\upsilon_1}\Big(\int_{\mathcal{N}_T\times\{1\}} ||\nabla^2\sigma||^2 dV + \int_{\mathcal{N}_T\times\{1\}} Ric(\nabla\sigma, \nabla\sigma) dV\Big). \tag{3.25}$$

Integrating with (3.22) and using the above equation, we arrive at

$$\int_{\mathcal{N}_T\times\{1\}} ||\nabla f||^2 dV < \frac{1}{\upsilon_1}\int_{\mathcal{N}_T\times\{1\}} \Big(\frac{3\upsilon_1 f^2}{2}\Big) dV = \int_{\mathcal{N}_T\times\{1\}} \Big(\frac{3f^2}{2}\Big) dV \tag{3.26}$$

which implies that

$$||\nabla f||^2 < \frac{3f^2}{2}. \tag{3.27}$$

Using Theorem 3.6, we get the desired proof. This completes the proof of the theorem. □

Using Theorem 3.2, we are ready to give the following result:

Theorem 3.10. *Let $\mathcal{N}^3 = \mathcal{N}_T^2 \times_f \mathcal{N}_\perp^1$ be a closed minimal 3-dimensional $\mathbb{CR}$-warped product submanifold in a near Kaehler $\mathbb{S}^6$. Then, we have*

$$\int_{\mathcal{N}} \Big(\upsilon_1 + \frac{2||\nabla f||^2}{f^2} - 3\Big) ||df||^2 dV \geq 0, \tag{3.28}$$

where dV is the volume element on $\mathcal{N}^n$. Equality holds if and only if either $\mathcal{N}^3$ is totally geodesic and υ_1 is the first non-zero eigenvalue.

Proof. By replacing the value of $||\mathcal{B}||^2$ from (3.17) to (3.1), we obtain the desired result. The second part of Theorem 3.2 does not hold for us because $n = 3$. This completes the proof of the theorem. □

Using Theorem 3.3, we prove the following:

Theorem 3.11. *Let $\mathcal{N}^3 = \mathcal{N}_T^2 \times_f \mathcal{N}_\perp^1$ be a minimal 3-dimensional $\mathbb{CR}$-warped product submanifold in a near Kaehler $\mathbb{S}^6$, we have*

$$\mathcal{R}ic(v, v) \geq \frac{1}{f^2}\Big(2f^2 - \frac{4}{3}||\nabla f||^2\Big), \tag{3.29}$$

where $\mathcal{R}ic(v, v)$ is the Ricci curvature in the direction of v in $\mathcal{N}^3$.

Proof. By replacing the value of $||\mathcal{B}||^2$ from (3.17) in (3.6), we get the proof of the theorem. □

The following result is based on Theorem 3.4:

Theorem 3.12. *Let $\mathcal{N}^3 = \mathcal{N}_T^2 \times_f \mathcal{N}_\perp^1$ be a compact minimal 3-dimensional $\mathbb{CR}$-warped product submanifold in a near Kaehler $\mathbb{S}^6$. If the following inequality is satisfied*

$$||\nabla f||^2 \leq \Big(\frac{21}{22}\Big) f^2, \tag{3.30}$$

then $\mathcal{N}^3$ is either a totally geodesic submanifold or a Veronese surface in $\mathbb{S}^4$.

Proof. For $n = 3$, we derive the inequality from (3.18), that is

$$||\mathcal{B}||^2 \leq \Big(\frac{21}{11}\Big). \tag{3.31}$$

It means that Theorem 3.4 holds for $n = 3$. If the inequality (3.30) holds, then from (3.17), we obtain the required inequality (3.31). □

Immediately, as a consequence of Theorem 3.5, we have:

Theorem 3.13. *Let $\mathcal{N}^3 = \mathcal{N}_T^2 \times_f \mathcal{N}_\perp^1$ be a compact minimal 3-dimensional $\mathbb{CR}$-warped product submanifold in a near Kaehler $\mathbb{S}^6$ and 3 is an odd. If the following inequality is satisfied*

$$||\nabla f||^2 \leq f^2, \tag{3.32}$$

then $\mathcal{N}^3$ is either a totally geodesic submanifold or $||\nabla f||^2 = f^2$ on $\mathcal{N}^3$ and the second fundamental form is given by

$$\left(\mathcal{B}_{ij}^4\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \left(\mathcal{B}_{ij}^5\right) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \left(\mathcal{B}_{ij}^5\right) = 0, \quad \alpha \geq 6.$$

4. Conclusions

The present paper discusses topological sphere types theorems for oriented compact minimal $\mathbb{CR}$-warped product submanifolds in a nearly Kaehler manifold. We show that if an inequality involving the warping function and the scalar curvature of the fibers is satisfied, a compact minimal-dimensional $\mathbb{CR}$-warped product submanifolds in a nearly Kaehler manifold is homeomorphic to the sphere. In the particular case, of a 6-dimensional unit sphere, we show that a 3-dimensional compact minimal CR-warped product submanifold has finite fundamental groups homeomorphic to a sphere if $2||\nabla f||^2 < 3f^2$, is satisfied. By using Bonnet–Myers's theorem, we give a result about the fundamental group and by using Leung's theorem, we obtain a result about the topological properties of a CR-warped submanifold in the sense of [30, 39]

Authors' contributions

Conceptualization, A. A. and F. A. A.; methodology, A. A. and N. A.; software, F. A. A.; validation, A.A., F.A.A., and F.M.; formal analysis, A. A.; investigation, A. A.; resources, N. A.; data curation, A. A., F. M.; writing—original draft preparation, P.L. I.; A. A.; writing—review and editing, F. M.; P.L.I.; visualization, N. A.; P.L.; supervision, N. A.; project administration, F.A.A., and N.A; funding acquisition, N.A. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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