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Research article

Statistical inference for the bathtub-shaped distribution using balanced and unbalanced sampling techniques

Nuran M. Hassan[1], **M. Nagy**[2,*] **and Subhankar Dutta**[3]

[1] Department of Basic Science, Faculty of Engineering, Modern Academy, Cairo, Egypt

[2] Department of Statistics and Operations Research, College of Science, King Saud University, Saudi Arabia

[3] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Chennai Campus, Tamil Nadu, India

* **Correspondence:** Email: mnaji@ksu.edu.sa.

Abstract: In order to reduce errors and enhance precision while estimating the unknown parameters of the distributions, it is crucial to choose a representative sample. The common estimation methods that estimate the parameters associated with the bathtub-shaped distribution include maximum likelihood (ML), maximum product of spacings estimation (MPSE), and Cramér-von Mises estimation (CME) methods. However, four modifications are used with the sample selection technique. They are simple random sampling (SRS), ranked set sampling (RSS), maximum ranked set sampling (MaxRSS), and double ranked set sampling (DBRSS), which is due to small sample sizes. Based on the estimation methods such as ML, MPSE, and CME, the ranked set sampling techniques do not have simple functions to manage them. The MaxRSS matrix has variable dimensions but requires fewer observations than RSS. DBRSS requires a greater number of observations than MaxRSS and RSS. According to simulation studies, the RSS, MaxRSS, and DBRSS estimators were more effective than the SRS estimator for different sample sizes. Additionally, MaxRSS was discovered to be the most efficient RSS-based technique. Other techniques, however, proved more effective than RSS for high mean squared errors. The CM method estimated the true values of the parameters more accurately and with smaller biases than ML and MPSE. The MPSE method was also found to have significant biases and to be less accurate in estimating the values of the parameters when compared to the other estimate methods. Finally, two datasets demonstrated how the bathtub-shaped distribution could be feasible based on different sampling techniques.

Keywords: bathtub-shaped distribution; maximum product of spacings estimation method; Cramér-von Mises estimator method; maximum ranked set sampling; double ranked set sampling
Mathematics Subject Classification: 62D05, 62F10, 62G30

1. Introduction

McIntyre [1] introduced the concept of ranked set sampling (RSS). He used the RSS technique to estimate forage and forage yields in agriculture. Its purpose is to preserve the characteristics of SRS estimators while using as little information as feasible about those estimators that may be obtained through visual examination or other low-cost techniques. To further improve inference precision, he minimized the number of measured observations required. Here is a rundown of how the RSS technique works: The target population is first used to choose an SRS consisting of n sets, each with size n. Next, in a cycle j, choose the i^{th} element from the i^{th} collection, where $\{i = 1, \cdots, n\}$ and $\{j = 1, \cdots, r\}$. An RSS sample of size n can be obtained by repeating the procedure r times. Double ranked set sampling (DBRSS) was introduced by Al-Saleh and Al-Kadiri [2] to estimate the population mean. Using a double ranked set, the researcher must locate n^3 units in order to select a sample of size n. This may be challenging when an epidemic breaks out in a region or when data arrives in packets of varying sizes and causes queuing issues.

In addition, there may be a lack of experimental units, or ranking may be difficult, time-consuming, and expensive. The DBRSS technique is characterized as follows: To obtain a sample of size n, n^3 units from the intended population are selected. Allocate these units across n collections of size n^2 at random. Use the RSS on n sets to generate n ranked set samples of size n each. Repeat the RSS procedure on these n ranked sets of equal size to generate an n-by-n sample of double ranked sets. When estimating the mean of symmetric distributions such as the normal and exponential, RSS is more accurate than SRS, according to Al-Saleh and Al-Hadrami [3]. MaxRSS was created by Eskandarzadeh et al. in [4]. By halving the sample of traditional RSS, this technique is successful and can produce an estimator that is more accurate than traditional RSS. The MaxRSS technique is explained as follows: To begin with, choose n sets from a simple random sample (SRS), where the size of the i^{th} set is i for $i = 1, \cdots, n$. The top observation in each set should then be determined. The maximum statistical measure for $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ units. Subsequently, execute the preceding steps r times to produce a MaxRSS of size nr. Wang et al. [5] used the ML method to estimate the inverse Gaussian distribution based on MaxRSS with unequal sample sizes. A few changes to RSS were made in order to estimate the gamma/Gompertz distribution's parameters by Hassan et al. [6]. Based on RSS modification, Hassan et al. [7] used ML to estimate the parameters of the exponentiated exponential distribution. The parameters of the inverted Kumaraswamy distribution based on RSS and SRS were estimated by Nagy et al. [8] using a variety of estimation techniques, such as maximum likelihood, maximum product of spacings, ordinary least squares, weighted least squares, Cramer–von Mises, and Anderson–Darling. Chen [9] defined a "bathtub-shaped" distribution, or an increasing hazard rate function (CBL) distribution, as a lifetime distribution with two parameters. This distribution is extensively used in practice since it may depict the lifetimes of different mechanical and electrical items. Based on Chen's bathtub-shaped distribution, Tahmasebi and Jafari [10] developed an expanded distribution. Estimating the unknown parameters of the CBL distribution was accomplished by a variety of researchers using estimation methods. Zhang et al. [11] estimated the unknown parameters of the CBL distribution based on type-I hybrid censoring and Bayesian and E-Bayesian methods. Sarhan et al. [12] estimated the CBL distribution parameters using SRS-based Bayesian and ML methods. For more references, one may see Sindhu et al. [13], Sindhu and Atangana [14], Dutta and Kayal [15], Dutta et al. [16], and Dutta et al. [17].

The probability density function (PDF) and cumulative distribution function (CDF) of a distribution with a bathtub shape are followed by

$$f(x; \alpha, \lambda) = \alpha\lambda x^{\lambda-1} e^{\alpha(1-e^{x^{\lambda}})+x^{\lambda}}; \quad x > 0,\ \alpha, \lambda > 0, \tag{1.1}$$

and

$$F(x;\alpha,\lambda) = 1 - e^{\alpha(1-e^{x^{\lambda}})} \quad x > 0,\ \alpha,\lambda > 0, \tag{1.2}$$

where α and λ are the scale and shape parameters, respectively (see Figure 1).

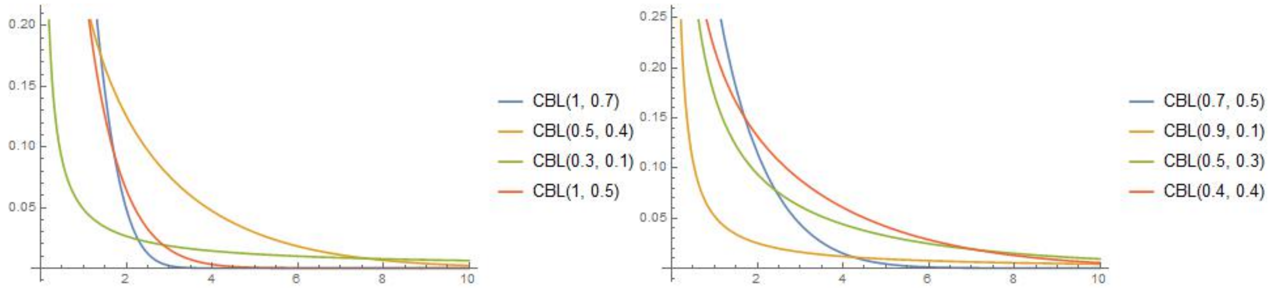

Figure 1. PDF of CBL distribution with different parameter values.

Numerous real-world studies, such as determining the strength with specific materials, the mortality rate, or the incubation time of a fatal illness, frequently result in the study of distributions with bathtub-shaped hazard functions. Yet, this distribution has not been applied in statistical literature with RSS modifications and different methods of estimation. Our aim of this work is to estimate the unknown parameters of this distribution with various method of estimation under RSS modifications.

This article presents a comparison between the MaxRSS technique and other sampling techniques, namely ML, MPSE, and CME estimation methods, in terms of estimating unknown parameters for CBL distribution using the SRS technique. The following sections of this article are arranged as follows: The PDF and CDF functions for the sampling technique are shown in Section 2. The three different estimation methods are discussed in Sections 3, 4, and 5 for the four sampling techniques discussed in this article. The effectiveness of the SRS-based estimators are compared to those of its counterparts, RSS, MaxRSS, and DBRSS, in Section 6 using the results of a simulation using the Monte Carlo method. The outcomes demonstrate that MaxRSS is more efficient than all other methods and minimizesthe mean squared error. The CBL distribution is applied to two real datasets in Section 7 to demonstrate the flexibility of the distribution, and Section 8 summarized the findings from Sections 6 and 7.

2. Some sampling techniques

Various sample techniques forthe selection of a unit, including RSS, MaxRSS, and DBRSS, are explained in this section along with the PDF and CDF functions for each. Assume that X is a continuous random variable with PDF and CDF as $f(x)$ and $F(x)$, respectively.

Ranked set sampling: Suppose that $\{X^{(1)}_{(i;i)j}\}$ are random samples for n sets of size n. Each sample is independent and represents an SRS sample. The i^{th} order statistic unit is displayed, derived from the i^{th} sample of size n, where $i = \{1, \cdots, n\}$, and it contains an amount j cycles, for $j = \{1, \cdots, r\}$ see Figure 2. Then the PDF and CDF of $X^{(1)}_{(i:i)j}$ are expressed as

$$f_n(x^{(1)}_{(i:i)j};\theta) = \frac{n!}{(i-1)!(n-i)!} f(x_{(i:i)j};\theta)\ [F(x_{(i:i)j};\theta)]^{i-1} \times [1 - F(x_{(i:i)j};\theta)]^{n-i}, \tag{2.1}$$

and

$$F_n(x^{(1)}_{(i:i)j};\theta) = \sum_{t=i}^{n} \binom{n}{t} \, [F(x_{(i:i)j};\theta)]^t \; [1 - F(x_{(i:i)j};\theta)]^{n-t}\,. \tag{2.2}$$

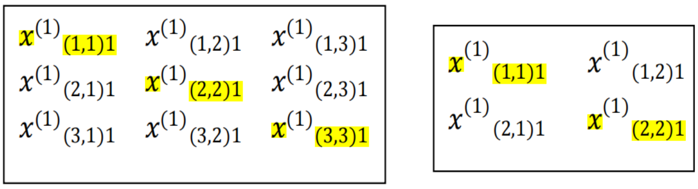

Figure 2. An illustrative examples for RSS with different sample sizes $n = 3$ and $n = 2$ for $r = 1$.

Double ranked set sampling: Let $\{X^{(2)}_{(i)j}\}$ be a DBRSS, that is, $X^{(2)}_{(i)j}$ is the i^{th} order statistic unit for $i = \{1, \cdots, n\}$ and number of cycles $j = \{1, \cdots, r\}$ of the RSS $\{X^{(1)}_{(i)j}\}$ and each of $X^{(2)}_{(i)j}$ are collected from independent ranked set samples of size n. It shows that $\{X^{(2)}_{(1)j}, X^{(2)}_{(2)j}, \cdots, X^{(2)}_{(n)j}\}$ are the order statistics units of the not identical independent random variables from $\{X^{(1)}_{(1)j}, X^{(1)}_{(2)j}, \cdots, X^{(1)}_{(n)j}\}$ (see Figure 3). Then, the CDF and PDF of $X^{(2)}_{(i)j}$ are given by

$$F_n(x^{(2)}_{(i)j};\theta) = \sum_{t=i}^{n} \sum_{S_t} \left[\prod_{k=1}^{t} F(x^{(1)}_{(i)_k j};\theta) \prod_{k=t+1}^{n} [1 - F(x^{(1)}_{(i)_k j};\theta)] \right] \tag{2.3}$$

and

$$f_n(x^{(2)}_{(i)j};\theta) = \sum_{t=i}^{n} \sum_{S_t} \left[\prod_{k=1}^{t} f(x^{(1)}_{(i)_k j};\theta) \prod_{k=t+1}^{n} (-f(x^{(1)}_{(i)_k j};\theta)) \right], \tag{2.4}$$

where S_t is the set of the entire permutations $(i_1, i_2, \cdots, i_n)$ of the integers $\{1, \cdots, n\}$ for which $i_1 < i_2 < \cdots < i_t$ and $i_{t+1} < i_{t+2} < \cdots < i_n$ (see David and Nagaraja [18]).

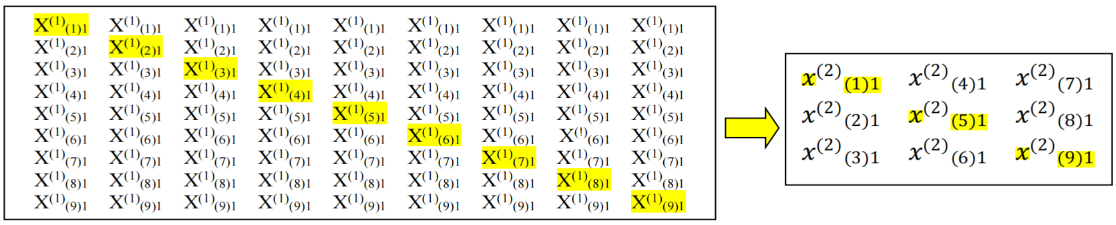

Figure 3. An illustrative example for DBRSS with a sample size $n = 3$ for $r = 1$.

Maximum ranked set sampling: Let $\{X_{i(1)}, \cdots, X_{i(n)}\}$ be drawn randomly from X for $\{i = 1, \cdots, n\}$. In that case, $X_{i:i}$ will be the $Max\{X_{(i:1)}, X_{(i:2)}, \cdots, X_{(i:i)}\}$ for $\{i = 1, \cdots, n\}$, see Figure 4. The PDF and CDF of $X_{(i:i)j}$ are written as

$$f_n(x_{(i:i)j};\theta) = i f(x_{(i:i)j};\theta)[F(x_{(i:i)j};\theta)]^{i-1}, \tag{2.5}$$

and

$$F_n(x_{(i:i)j};\theta) = [F(x_{(i:i)j};\theta)]^i. \tag{2.6}$$

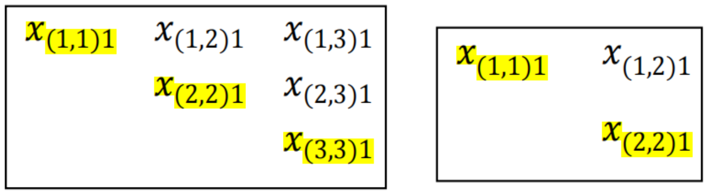

Figure 4. An illustrative examples for MaxRSS with different sample sizes $n = 3$ and $n = 2$ for $r = 1$.

3. Maximum likelihood estimation method

This section discusses parameter estimation using the maximum likelihood method of the bathtub-shaped distribution. Using the SRS, RSS, MaxRSS, and DBRSS techniques under one cycle $r = 1$, first derivative equations are discovered. In order to solve these equations to determine the typical estimators using sampling techniques, a numerical method for a complex mathematical procedure is required.

3.1. MLE using SRS

Let $X_1, X_2, \cdots, X_n$ be an independent random sample of size n from a population with a CBL distribution with a set of unknown parameter vector $\varphi = \begin{bmatrix} \alpha \\ \lambda \end{bmatrix}$. Then the likelihood function of SRS (ML_{SRS}) and log-likelihood function ($\log ML_{SRS}$) will be

$$ML_{SRS}(\varphi) = \prod_{i=1}^{n} \alpha\lambda x_i^{\lambda-1} e^{\alpha(1-e^{x_i^\lambda})+x_i^\lambda},$$

and

$$\log ML_{SRS}(\varphi) = n\ \log(\alpha) + n\ \log(\lambda) + (\lambda - 1)\sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n}(\alpha(1 - e^{x_i^\lambda}) + x_i^\lambda).$$

First derivatives of the $\log ML_{SRS}$ for α and λ are as follows:

$$\frac{\partial \log ML_{SRS}(\varphi)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}(1 - e^{x_i^\lambda}) = 0, \tag{3.1}$$

and

$$\frac{\partial \log ML_{SRS}(\varphi)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n}(x_i^\lambda \log x_i)(-\alpha e^{x_i^\lambda} + 1) = 0. \tag{3.2}$$

It is not possible to analytically determine $\hat{\alpha}_{SRS}$ and $\hat{\lambda}_{SRS}$ by solving the nonlinear equations in Eqs (3.1) and (3.2). Numerical methods exist for solving it. The simplest form of the estimators for the SRS look like this:

$$\hat{\alpha}_{SRS} = \frac{-n}{\sum_{i=1}^{n}(1 - e^{x_i^\lambda})},$$

and

$$\hat{\lambda}_{SRS} = \frac{-n}{\sum_{i=1}^{n} log(x_i) + \sum_{i=1}^{n}(x_i^\lambda ln x_i)(\frac{n e^{x_i^\lambda}}{\sum_{i=1}^{n}(1-e^{x_i^\lambda})} + 1)}.$$

3.2. MLE using RSS

We apply the the RSS technique to the CBL distribution by substituting Eqs (1.1) and (1.2) into Eq (2.1) to get ML_{RSS}. Suppose that $\{X_{(i;i)j}\}$ are random samples for n sets of size n. Each sample is independent and represents an SRS sample. It shows the i^{th} order statistic unit from the i^{th} sample of size n where $i = \{1, \cdots, n\}$ and has a number of cycles j where $j = \{1, \cdots, r\}$. Then the likelihood function of RSS (ML_{RSS}) and log-likelihood function ($logML_{RSS}$) for one cycle $j = 1$ are given by

$$ML_{RSS}(\varphi) = \prod_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!}[\alpha\lambda x_{(i,i)}^{\lambda-1} e^{\alpha(1-e^{x_{(i,i)}^{\lambda}})+x_{(i,i)}^{\lambda}}][1 - e^{\alpha(1-e^{x_{(i,i)}^{\lambda}})}]^{i-1} \times [e^{\alpha(1-e^{x_{(i,i)}^{\lambda}})}]^{n-i},$$

and

$$\begin{aligned} logML_{RSS}(\varphi) =& log(\prod_{i=1}^{n} \frac{n!}{(i-1)!(n-i)!}) + n\,log(\alpha) + n\,log(\lambda) + (\lambda - 1)\sum_{i=1}^{n} log(x_{(i,i)}) \\ &+ \sum_{i=1}^{n}(\alpha(1 + n - i)(1 - e^{x_{(i,i)}^{\lambda}}) + x_{(i,i)}^{\lambda}) + \sum_{i=1}^{n}(i-1)log(1 - e^{\alpha(1-e^{x_{(i,i)}^{\lambda}})}). \end{aligned}$$

First derivatives of the $logML_{RSS}$ for α and λ are as follows:

$$\frac{\partial logML_{RSS}(\varphi)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}(1 + n - i)(1 - e^{x_{(i,i)}^{\lambda}}) + \sum_{i=1}^{n}(i-1)\frac{-e^{\alpha(1-e^{x_{(i,i)}^{\lambda}})}(1 - e^{x_{(i,i)}^{\lambda}})}{1 - e^{\alpha(1-e^{x_{(i,i)}^{\lambda}})}} = 0, \tag{3.3}$$

and

$$\begin{aligned} \frac{\partial logML_{RSS}(\varphi)}{\partial\lambda} =& \frac{n}{\lambda} + \sum_{i=1}^{n} log(x_{(i,i)}) + \sum_{i=1}^{n}(x_{(i,i)}^{\lambda} lnx_{(i,i)})(-\alpha(1 + n - i)e^{x_{(i,i)}^{\lambda}} + 1) \\ &+ \sum_{i=1}^{n}(i-1)\frac{-e^{\alpha(1-e^{x_{(i,i)}^{\lambda}})}(-\alpha e^{x_{(i,i)}^{\lambda}} x_i^{\lambda} lnx_{(i,i)})}{1 - e^{\alpha(1-e^{x_{(i,i)}^{\lambda}})}} = 0. \end{aligned} \tag{3.4}$$

Nonlinear equations such as Eqs (3.3) and (3.4) cannot be solved analytically. They have a numerical solution.

3.3. MLE using DBRSS

To obtain $\hat{\alpha}_{DBRSS}$ and $\hat{\lambda}_{DBRSS}$ of the CBL distribution using the DBRSS technique, Eqs (1.1) and (1.2) have been used in Eq (2.4) in this subsection. Let $\{X_1^{(2)}, \cdots, X_n^{(2)}\}$ be a DBRSS; that is, $X_i^{(2)}$ is the i^{th} order statistic unit for $i = \{1, \cdots, n\}$ of the RSS $\{X_1^{(1)}, \cdots, X_n^{(1)}\}$ and each of $X_i^{(2)}$ are collected from independent ranked set samples of size n. It appears that $\{X_1^{(2)}, \cdots, X_n^{(2)}\}$ are the order statistics units of the not identical independent random variables from $\{X_1^{(1)}, \cdots, X_n^{(1)}\}$. Then the likelihood function of DBRSS (ML_{DBRSS}) and log likelihood function ($logML_{DBRSS}$) will be

$$ML_{DBRSS}(\varphi) = \prod_{i=1}^{l}[\alpha\lambda x_i^{(1)[\lambda-1]} e^{\alpha(1-e^{x_i^{(1)\lambda}})+x_i^{(1)\lambda}}] \times \prod_{i=l+1}^{n}[-\alpha\lambda x_i^{(1)[\lambda-1]} e^{\alpha(1-e^{x_i^{(1)\lambda}})+x_i^{(1)\lambda}}],$$

and

$$\log ML_{DBRSS}(\varphi) = l\,\log(\alpha) + l\,\log(\lambda) + (\lambda - 1)\sum_{i=1}^{l} \log(x_i^{(1)})$$

$$+\sum_{i=1}^{l}(\alpha(1-e^{x_i^{(1)\lambda}})+x_i^{(1)\lambda})-(n-l)\ \log(\alpha)$$
$$-(n-l)\ \log(\lambda)-(\lambda-1)\sum_{i=l+1}^{n}\log(x_i^{(1)})$$
$$-\sum_{i=l+1}^{n}(\alpha(1-e^{x_i^{(1)\lambda}})+x_i^{(1)\lambda}).$$

First derivatives of the $logML_{DBRSS}$ are given by:

$$\frac{\partial \log ML_{DBRSS}(\varphi)}{\partial\alpha}=\frac{l}{\alpha}+\sum_{i=1}^{l}(1-e^{x_i^{(1)\lambda}})-\frac{n-l}{\alpha}-\sum_{i=l+1}^{n}(1-e^{x_i^{(1)\lambda}})=0, \tag{3.5}$$

and

$$\begin{aligned}\frac{\partial \log ML_{DBRSS}(\varphi)}{\partial\lambda}=&\frac{l}{\lambda}+\sum_{i=1}^{l}\log(x_i^{(1)})+\sum_{i=1}^{l}(x_i^{(1)\lambda}lnx_i^{(1)})(-\alpha e^{x_i^{(1)\lambda}}+1)\\&-\frac{n-l}{\lambda}-\sum_{i=l+1}^{n}\log(x_i^{(1)})-\sum_{i=l+1}^{n}(x_i^{(1)\lambda}lnx_i^{(1)})(-\alpha e^{x_i^{(1)\lambda}}+1)=0.\end{aligned} \tag{3.6}$$

It is not possible to determine $\hat{\alpha}_{DBRSS}$ and $\hat{\lambda}_{DBRSS}$ analytically by solving Eqs (3.5) and (3.6). There are numerical methods for solving them.

3.4. MLE using MaxRSS

We use the MaxRSS technique to derive ML_MaxBSS estimators for the CBL distribution by substituting Eqs (1.1) and (1.2) into Eq (2.5). Let $\{X_{i(1)},\cdots,X_{i(i)}\}$ be n sets drawn at random from X for $\{i=1,\cdots,n\}$, where $X_{i:i}$= max$\{X_{i:1},\cdots,X_{i:i}\}$ for $i=\{1,\cdots,n\}$, and it will represent the sample from MaxRSS,

$$ML_{MaxBSS}(\varphi)=\prod_{i=1}^{n}i[\alpha\lambda x_{i:i}^{\lambda-1}e^{\alpha(1-e^{x_{i:i}^{\lambda}})+x_{i:i}^{\lambda}}][1-e^{\alpha(1-e^{x_{i:i}^{\lambda}})}]^{i-1},$$

and

$$\begin{aligned}logML_{MaxRSS}(\varphi)=&log(\prod_{i=1}^{n}i)+n\,log(\alpha)+n\,log(\lambda)+(\lambda-1)\sum_{i=1}^{n}log(x_{i:i})+\sum_{i=1}^{n}(\alpha(1-e^{x_{i:i}^{\lambda}})+x_{i:i}^{\lambda})\\&+\sum_{i=1}^{n}(i-1)log(1-e^{\alpha(1-e^{x_{i:i}^{\lambda}})}).\end{aligned}$$

First derivatives of the $logML_{MaxRSS}$ are given by:

$$\frac{\partial logML_{MaxRSS}(\varphi)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}(1-e^{x_{i:i}^{\lambda}})+\sum_{i=1}^{n}(i-1)\frac{-e^{\alpha(1-e^{x_{i:i}^{\lambda}})}(1-e^{x_{i:i}^{\lambda}})}{1-e^{\alpha(1-e^{x_{i:i}^{\lambda}})}}=0, \tag{3.7}$$

and

$$\frac{\partial logML_{MaxRSS}(\varphi)}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}log(x_{i:i})+\sum_{i=1}^{n}(x_{i:i}^{\lambda}lnx_{i:i})(-\alpha e^{x_{i:i}^{\lambda}}+1)$$

$$+\sum_{i=1}^{n}(i-1)\frac{-e^{\alpha(1-e^{x_{i:i}^{\lambda}})}(-\alpha e^{x_{i:i}^{\lambda}}x_{i:i}^{\lambda}lnx_{i:i})}{1-e^{\alpha(1-e^{x_{i:i}^{\lambda}})}}=0. \tag{3.8}$$

Equations (3.7) and (3.8) cannot be solved through analytical methods, but they can be effectively solved using numerical techniques.

4. Maximum product of spacings estimation method

In continuous univariate distributions, Cheng and Amin [19] proposed the maximum product of spacings estimation method (MPSE), which was endorsed by Ranneby [20] as an alternative to the ML method. For further details on the MPS estimation method, refer to El Sherpieny et al. [21] and Ahmad and Almetwally [22]. It is possible to define uniform spaces for a sample of size n randomly chosen from the bathtub-shaped distribution as $D_i(\varphi) = F(x_i;\varphi) - F(x_{i-1};\varphi)$ for $x_1 < ... < x_n$ where $i = \{1, 2, \cdots, n+1\}$, $x_0 \to -\infty$ and $x_{n+1} \to \infty$. MPSE estimators of unknown parameters are the value that maximizes the given MPSE function for unknown parameter vector $\varphi = \begin{bmatrix} \alpha \\ \lambda \end{bmatrix}$,

$$MPSE(\varphi) = \frac{1}{n+1}\sum_{i=1}^{n+1} log(F(x_i;\varphi) - F(x_{i-1};\varphi)).$$

This section describes the MPSE method for estimating parameters for the bathtub-shaped distribution. Using SRS, RSS, MaxRSS, and DBRSS techniques under one cycle $j = 1$, the first derivative equations are discovered. Solving these equations and determining the estimators using sampling techniques requires a numerical method.

4.1. MPSE using SRS

Let $\{x_1, x_2, \cdots, x_n\}$ be a random sample of the MPSE function of of size n having a CBL distribution using CDF, which is given by Eq (1.2),

$$MPSE_{SRS}(\varphi) = \frac{1}{n+1}\sum_{i=1}^{n+1} log(e^{\alpha(1-e^{x_{i-1}^{\lambda}})} - e^{\alpha(1-e^{x_i^{\lambda}})}).$$

First derivatives of the MPS_{SRS} for α and λ are shown in Eqs (4.1) and (4.2). These nonlinear equations cannot be analytically resolved. They are solvable numerically.

$$\frac{\partial MPSE_{SRS}(\varphi)}{\partial\alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{e^{\alpha(1-e^{x_{i-1}^{\lambda}})}(1-e^{x_{i-1}^{\lambda}}) - e^{\alpha(1-e^{x_i^{\lambda}})}(1-e^{x_i^{\lambda}})}{e^{\alpha(1-e^{x_{i-1}^{\lambda}})} - e^{\alpha(1-e^{x_i^{\lambda}})}} = 0, \tag{4.1}$$

and

$$\begin{aligned}\frac{\partial MPSE_{SRS}(\varphi)}{\partial\lambda} =& \frac{1}{n+1}\sum_{i=1}^{n+1}[\frac{e^{\alpha(1-e^{x_{i-1}^{\lambda}})}(-\alpha e^{x_{i-1}^{\lambda}}x_{i-1}^{\lambda}lnx_{i-1})}{e^{\alpha(1-e^{x_{i-1}^{\lambda}})} - e^{\alpha(1-e^{x_i^{\lambda}})}} \\ &- \frac{e^{\alpha(1-e^{x_i^{\lambda}})}(-\alpha e^{x_i^{\lambda}}x_i^{\lambda}lnx_i)}{e^{\alpha(1-e^{x_{i-1}^{\lambda}})} - e^{\alpha(1-e^{x_i^{\lambda}})}}] = 0.\end{aligned} \tag{4.2}$$

4.2. MPSE using RSS

The MPSE function of $\{X_{(1;1)1}, \cdots, X_{(1;1)n_1}; \cdots, X_{(n;n)1}, X_{(n;n)2}, \cdots, X_{(n;n)n_n}\}$ of n sets of size n for each set having a CBL distribution using the CDF of the RSS technique, which is given by Eq (2.2), can be given as

$$MPSE_{RSS}(\varphi) = \frac{1}{n+1}\sum_{i=1}^{n+1} log[\sum_{t=l}^{n}\binom{n}{t}[(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t} - (1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}]].$$

First derivatives of the $MPSE_{RSS}$ for the parameters of the CBL distribution are given by Eqs (4.3) and (4.4). Analytical solutions are not possible for these equations. Numerical solutions are a viable option for solving them.

$$\begin{aligned}\frac{\partial MPSE_{RSS}(\varphi)}{\partial\alpha} =& \sum_{i=1}^{n+1}\sum_{t=l}^{n}\binom{n}{t}[[-t(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})}(1-e^{x^{\lambda}_{(i,i)}})(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}\\
&\times\frac{1}{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}-(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}}]\\
&+\frac{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(n-t)(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})}(1-e^{x^{\lambda}_{(i,i)}})}{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}-(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}}\\
&+\frac{t(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})}(1-e^{x^{\lambda}_{(i,i)-1}})(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}}{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}-(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}}\\
&-\frac{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(n-t)(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})}(1-e^{x^{\lambda}_{(i,i)-1}})}{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}-(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}}\\
=&0,\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\frac{\partial MPSE_{RSS}(\varphi)}{\partial\lambda} =& \sum_{i=1}^{n+1}\sum_{t=l}^{n}\binom{n}{t}[[t(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})}(\alpha e^{x^{\lambda}_{(i,i)}}x^{\lambda}_{(i,i)}lnx_{(i,i)})(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}\\
&\times\frac{1}{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}-(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}}]\\
&+\frac{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(n-t)(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})}(-\alpha e^{x^{\lambda}_{(i,i)}}x^{\lambda}_{(i,i)}lnx_{(i,i)})}{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}-(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}}\\
&-\frac{t(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})}(\alpha e^{x^{\lambda}_{(i,i)-1}}x^{\lambda}_{(i,i)-1}lnx_{(i,i)-1})(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}}{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}-(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}}\\
&+[(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(n-t)(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})}\\
&\times\frac{(-\alpha e^{x^{\lambda}_{(i,i)-1}}x^{\lambda}_{(i,i)-1}lnx_{(i,i)-1})}{(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}-(1-e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{t}(e^{\alpha(1-e^{x^{\lambda}_{(i,i)-1}})})^{n-t}}]\\
=&0.\end{aligned}\tag{4.4}$$

4.3. MPSE using DBRSS

The MPS function of $\{X_1^{(2)}, X_2^{(2)}, \cdots, X_i^{(2)}\}$ of DBRSS comes from RSS of $\{X_1^{(1)}, X_2^{(1)}, \cdots, X_i^{(1)}\}$ sets having a CBL distribution. By using the CDF of the DBRSS technique, which is given by Eq (2.3), we have

$$MPSE_{DBRSS}(\varphi) = \frac{1}{n+1}[\sum_{i=1}^{l} log((1 - e^{\alpha(1-e^{x_i^{(1)\lambda}})}) - (1 - e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})})) + \sum_{i=l+1}^{n+1} log(e^{\alpha(1-e^{x_i^{(1)\lambda}})} - e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})}).$$

First derivatives of the $MPSE_{DBRSS}$ for α and λ are in Eqs (4.5) and (4.6) given below:

$$\frac{\partial MPSE_{DBRSS}(\varphi)}{\partial \alpha} = \frac{1}{n+1}[\sum_{i=1}^{l} \frac{-e^{\alpha(1-e^{x_i^{(1)\lambda}})}(1 - e^{x_i^{(1)\lambda}}) + e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})}(1 - e^{x_{i-1}^{(1)\lambda}})}{(1 - e^{\alpha(1-e^{x_i^{(1)\lambda}})}) - (1 - e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})})} + \sum_{i=l+1}^{n+1} \frac{e^{\alpha(1-e^{x_i^{(1)\lambda}})}(1 - e^{x_i^{(1)\lambda}}) - e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})}(1 - e^{x_{i-1}^{(1)\lambda}})}{e^{\alpha(1-e^{x_i^{(1)\lambda}})} - e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})}}] = 0, \quad (4.5)$$

and

$$\frac{\partial MPSE_{DBRSS}(\varphi)}{\partial \lambda} = \frac{1}{n+1}[\sum_{i=1}^{l}[\frac{e^{\alpha(1-e^{x_i^{(1)\lambda}})}(\alpha e^{x_i^{(1)\lambda}} x_i^{(1)\lambda} ln x_i^{(1)})}{(1 - e^{\alpha(1-e^{x_i^{(1)\lambda}})}) - (1 - e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})})} - \frac{e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})}(-\alpha e^{x_{i-1}^{(1)\lambda}} x_{i-1}^{(1)\lambda} ln x_{i-1}^{(1)})}{(1 - e^{\alpha(1-e^{x_i^{(1)\lambda}})}) - (1 - e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})})}] + \sum_{i=l+1}^{n+1}[\frac{e^{\alpha(1-e^{x_i^{(1)\lambda}})}(-\alpha e^{x_i^{(1)\lambda}} x_i^{(1)\lambda} ln x_i^{(1)})}{e^{\alpha(1-e^{x_i^{(1)\lambda}})} - e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})}} - \frac{e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})}(-\alpha e^{x_{i-1}^{(1)\lambda}} x_{i-1}^{(1)\lambda} ln x_{i-1}^{(1)})}{e^{\alpha(1-e^{x_i^{(1)\lambda}})} - e^{\alpha(1-e^{x_{i-1}^{(1)\lambda}})}}] = 0. \quad (4.6)$$

These equations cannot be analytically resolved. They are solvable numerically.

4.4. MPSE using MaxRSS

The MPSE function of $\{X_{i(1)}, X_{i(1)}, \cdots, X_{i(i)}\}$ of n sets of size i where $i = \{1, \cdots, n\}$ having a CBL distribution using the CDF of the MaxRSS technique, which is given by Eq (2.6), is given by

$$MPSE_{MaxRSS}(\varphi) = \frac{1}{n+1}\sum_{i=1}^{n+1} log((e^{\alpha(1-e^{x_{i:i}^{\lambda}})})^i - (e^{\alpha(1-e^{x_{i:i-1}^{\lambda}})})^i).$$

First derivatives of the $MPSE_{MaxRSS}$ for the parameters of the CBL distribution are given by Eqs (4.7) and (4.8). It is not possible to find a solution to these equations using analytical methods. It can be resolved through numerical methods,

$$\frac{\partial MPSE_{MaxRSS}(\varphi)}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}(\frac{i(e^{\alpha(1-e^{x_{i:i}^{\lambda}})})^{i-1} e^{\alpha(1-e^{x_{i:i}^{\lambda}})}(1 - e^{x_{i:i}^{\lambda}})}{(e^{\alpha(1-e^{x_{i:i}^{\lambda}})})^i - (e^{\alpha(1-e^{x_{i:i-1}^{\lambda}})})^i} - \frac{i(e^{\alpha(1-e^{x_{i:i-1}^{\lambda}})})^{i-1}(e^{\alpha(1-e^{x_{i:i-1}^{\lambda}})}(1 - e^{x_{i:i-1}^{\lambda}})}{(e^{\alpha(1-e^{x_{i:i}^{\lambda}})})^i - (e^{\alpha(1-e^{x_{i:i-1}^{\lambda}})})^i}) = 0, \quad (4.7)$$

and

$$\frac{\partial MPSE_{MaxRSS}(\varphi)}{\partial \lambda} = \frac{1}{n+1}\sum_{i=1}^{n+1}\Big(\frac{i(e^{\alpha(1-e^{x^{\lambda}_{i:i}})})^{i-1}e^{\alpha(1-e^{x^{\lambda}_{i:i}})}(-\alpha e^{x^{\lambda}_{i:i-1}}x^{\lambda}_{i:i-1}lnx_{i:i-1})}{e^{\alpha(1-e^{x^{\lambda}_{i}})}-e^{\alpha(1-e^{x^{\lambda}_{i-1}})}}$$
$$-\frac{i(e^{\alpha(1-e^{x^{\lambda}_{i:i-1}})})^{i-1}e^{\alpha(1-e^{x^{\lambda}_{i:i-1}})}(-\alpha e^{x^{\lambda}_{i:i-1}}x^{\lambda}_{i:i-1}lnx_{i:i-1})}{e^{\alpha(1-e^{x^{\lambda}_{i:i}})}-e^{\alpha(1-e^{x^{\lambda}_{i:i-1}})}} = 0. \quad (4.8)$$

5. Cramér-von Mises estimation method

The Cramér-von Mises method, also known as CME, is a form of minimal distance estimators, specifically maximum goodness of fit estimators. This method is based on comparing the estimated CDF with the empirical distribution function. Choi and Bulgren [23] supported the choice of minimum distance estimators of the CME type by presenting empirical data showing that the bias of these estimators is lower compared to other minimum distance estimators. The CME function for the minimum distance of the bathtub-shaped distribution with the unknown parameter vector $\varphi = \begin{bmatrix} \alpha \\ \lambda \end{bmatrix}$ is defined by

$$CME(\varphi) = \frac{1}{12n} + \sum_{i=1}^{n}(F(x_i;\varphi) - \frac{2i-1}{2n})^2.$$

In this part of the article, we apply the CME estimation (CME) method to estimate the unknown parameters of the bathtub-shaped distribution. The first derivative is obtained based on the four sampling techniques (SRS, RSS, MaxRSS, and DBRSS) under one cycle $j = 1$. Solving these equations required numerical methods.

5.1. CME using SRS

Let $\{x_1, x_2, \cdots, x_n\}$ be a random sample of the CM function of size n having a CBL distribution using the CDF, which is given by Eq (1.2),

$$CME_{SRS}(\varphi) = \frac{1}{12n} + \sum_{i=1}^{n}((1 - e^{\alpha(1-e^{x^{\lambda}_{i}})}) - \frac{2i-1}{2n})^2.$$

First derivatives of the CME_{SRS} for the unknown parameters are given in Eqs (5.1) and (5.2). Analytical solutions are not possible for these equations. Numerical solutions are a viable option for solving them,

$$\frac{\partial CME_{SRS}(\varphi)}{\partial \alpha} = \sum_{i=1}^{n}2((1 - e^{\alpha(1-e^{x^{\lambda}_{i}})}) - \frac{2i-1}{2n})(-e^{\alpha(1-e^{x^{\lambda}_{i}})}(1 - e^{x^{\lambda}_{i}})) = 0, \quad (5.1)$$

and

$$\frac{\partial CME_{SRS}(\varphi)}{\partial \lambda} = \sum_{i=1}^{n}2((1 - e^{\alpha(1-e^{x^{\lambda}_{i}})}) - \frac{2i-1}{2n})(e^{\alpha(1-e^{x^{\lambda}_{i}})}(\alpha e^{x^{\lambda}_{i}}x^{\lambda}_{i}lnx_i)) = 0. \quad (5.2)$$

5.2. CME using RSS

The CME function of $\{X_{(1;1)1}, \cdots, X_{(1;1)n_1}, \cdots, X_{(n;n)1}, X_{(n;n)2}, \cdots, X_{(n;n)n_n}\}$ of n sets of size n for each set having a CBL using the CDF of the RSS technique, which is given by Eq (2.2), is given by

$$CME_{RSS}(\varphi) = \frac{1}{12n} + \sum_{i=1}^{n}(\sum_{t=i}^{n}\binom{n}{t}\ (1 - e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^t(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t} - \frac{2i-1}{2n})^2.$$

First derivatives of the CME_{RSS} for the unknown parameters of the CBL distribution are given by Eqs (5.3) and (5.4). It is not possible to find a solution to these equations using analytical methods. They can be resolved through numerical methods,

$$\begin{aligned}\frac{\partial CME_{RSS}(\varphi)}{\partial\alpha} =& \sum_{i=1}^{n} 2(\sum_{t=i}^{n}\binom{n}{t}\ (1 - e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^t(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t} - \frac{2i-1}{2n})\\ &\times \sum_{t=i}^{n}\binom{n}{t}(-t(1 - e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})}(1 - e^{x^{\lambda}_{(i,i)}})(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}\\ &+ (1 - e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^t(n-t)(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})}(1 - e^{x^{\lambda}_{(i,i)}})) = 0, \qquad (5.3)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial CME_{RSS}(\varphi)}{\partial\lambda} =& \sum_{i=1}^{n} 2(\sum_{t=i}^{n}\binom{n}{t}\ (1 - e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^t(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t} - \frac{2i-1}{2n})\\ &\times \sum_{t=i}^{n}\binom{n}{t}(t(1 - e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})}(\alpha e^{x^{\lambda}_{(i,i)}}x^{\lambda}_{(i,i)}lnx_{(i,i)})(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t}\\ &+ (1 - e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^t(n-t)(e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})})^{n-t-1}e^{\alpha(1-e^{x^{\lambda}_{(i,i)}})}\\ &\times(-\alpha e^{x^{\lambda}_{(i,i)}}x^{\lambda}_{(i,i)}lnx_{(i,i)}) = 0. \qquad (5.4)\end{aligned}$$

5.3. CME using DBRSS

The CME function of $\{X^{(2)}_1, X^{(2)}_2, \cdots, X^{(2)}_i\}$ of DBRSS comes from the RSS of $\{X^{(1)}_1, X^{(1)}_2, \cdots, X^{(1)}_i\}$ sets having a CBL distribution. By using the CDF of the DBRSS technique, which is given by Eq (2.3), we have

$$CME_{DBRSS}(\varphi) = \frac{1}{12n} + \sum_{i=1}^{l}((1 - e^{\alpha(1-e^{x^{(1)\lambda}_i})}) - \frac{2i-1}{2n})^2 + \sum_{i=l+1}^{n}((e^{\alpha(1-e^{x^{(1)\lambda}_i})}) - \frac{2i-1}{2n})^2.$$

First derivatives of the CME_{DBRSS} for α and λ are in Eqs (5.5) and (5.6). These nonlinear equations cannot be analytically resolved. They are solvable numerically,

$$\begin{aligned}\frac{\partial CME_{DBRSS}(\varphi)}{\partial\alpha} =& \sum_{i=1}^{l} 2((1 - e^{\alpha(1-e^{x^{(1)\lambda}_i})}) - \frac{2i-1}{2n})(-e^{\alpha(1-e^{x^{(1)\lambda}_i})})(1 - e^{x^{(1)\lambda}_i}))\\ &+ \sum_{i=l+1}^{n} 2((e^{\alpha(1-e^{x^{(1)\lambda}_i})}) - \frac{2i-1}{2n})(e^{\alpha(1-e^{x^{(1)\lambda}_i})}(1 - e^{x^{(1)\lambda}_i})) = 0, \qquad (5.5)\end{aligned}$$

and

$$\frac{\partial CME_{DBRSS}(\varphi)}{\partial\lambda} = \sum_{i=1}^{l} 2((1 - e^{\alpha(1-e^{x^{(1)\lambda}_i})}) - \frac{2i-1}{2n})(e^{\alpha(1-e^{x^{(1)\lambda}_i})}$$

$$\times (\alpha e^{x_i^{(1)\lambda}} x_i^{(1)\lambda} ln x_i^{(1)})) + \sum_{i=l+1}^{n} 2((e^{\alpha(1-e^{x_i^{(1)\lambda}})}) - \frac{2i-1}{2n})$$
$$\times (e^{\alpha(1-e^{x_i^{(1)\lambda}})}(-\alpha e^{x_i^{(1)\lambda}} x_i^{(1)\lambda} ln x_i^{(1)})) = 0. \quad (5.6)$$

5.4. CME using MaxRSS

The CME function of $\{X_{i(1)}, X_{i(1)}, \cdots, X_{i(i)}\}$ of n sets of size i where $i = \{1, \cdots, n\}$ having a CBL distribution using the CDF of the MaxRSS technique, which is given by Eq (2.6), is

$$CME_{MaxRSS}(\varphi) = \frac{1}{12n} + \sum_{i=1}^{n}((1 - e^{\alpha(1-e^{x_{i:i}^{\lambda}})})^i - \frac{2i-1}{2n})^2.$$

First derivatives of the CME_{MaxRSS} for α and λ are in Eqs (5.7) and (5.8), which are given below:

$$\frac{\partial CME_{MaxRSS}(\varphi)}{\partial \alpha} = \sum_{i=1}^{n} 2((1 - e^{\alpha(1-e^{x_{i:i}^{\lambda}})})^i - \frac{2i-1}{2n}) i (1 - e^{\alpha(1-e^{x_{i:i}^{\lambda}})})^{i-1}$$
$$\times (-e^{\alpha(1-e^{x_{i:i}^{\lambda}})})(1 - e^{x_{i:i}^{\lambda}}) = 0, \quad (5.7)$$

and

$$\frac{\partial CME_{MaxRSS}(\varphi)}{\partial \lambda} = \sum_{i=1}^{n} 2((1 - e^{\alpha(1-e^{x_{i:i}^{\lambda}})})^i - \frac{2i-1}{2n}) i (1 - e^{\alpha(1-e^{x_{i:i}^{\lambda}})})^{i-1}$$
$$\times e^{\alpha(1-e^{x_{i:i}^{\lambda}})}(\alpha e^{x_{i:i}^{\lambda}} x_{i:i}^{\lambda} ln x_{i:i})) = 0. \quad (5.8)$$

These equations are not able to be solved using analytical methods, but they can be solved numerically.

6. Simulation using the Monte Carlo method

We use Monte Carlo simulations to test how accurate the point estimation methods are. These methods are MLE, MPSE, and CME methods under different values of n, for a single cycle $r = 1$. The true values of the parameter that were chosen for the shape parameter α are $\{0.7, 0.5\}$ and the scale parameter λ are $\{0.5, 0.4\}$. The R 4.0.3 software was used to carry out the simulation with $l = 1000$ repetition for SRS, RSS, MaxRSS, and DBRSS. Some major R packages were utilized for this purpose, including VGAM [24], bbmle [25], stats4 [26] and Matrix [27]. In addition, the estimates' values were obtained by solving the nonlinear equations using the 'Optim' function in the R software. The performance of the estimates have been compared based on mean squared error (MSE) and bias.

The simulation results for the CBL distribution are shown inthe following tables. Tables 1, 2, and 3, respectively, compare the point estimation results of the ML, MPSE, and CME methods. Under the assumption that the parameters to be estimated remain constant with an increase in sample size n, we approach the true parameter values and the mean square errors decrease. The simulation's outcomes are detailed below. The most accurate estimates were produced by MaxRSS-based estimators, which were estimators with minimal bias.

Table 1. ML-based estimators and biases for four different sampling techniques.

α	λ	n	$\hat{\alpha}_{SRS}$	$\hat{\lambda}_{SRS}$	$\hat{\alpha}_{RSS}$	$\hat{\lambda}_{RSS}$	$\hat{\alpha}_{MaxRSS}$	$\hat{\lambda}_{MaxRSS}$	$\hat{\alpha}_{DBRSS}$	$\hat{\lambda}_{DBRSS}$
0.7	0.5	6	1.0826	0.7222	0.7475	0.5535	0.8045	0.6239	0.7329	0.4824
			(−0.3826)	(−0.2222)	(−0.0475)	(−0.0535)	(−0.1045)	(−0.1239)	(−0.0329)	(0.0176)
		7	0.9451	0.6649	0.7475	0.5535	0.7669	0.6014	0.7252	0.4251
			(−0.2451)	(−0.1649)	(−0.0475)	(−0.0535)	(−0.0669)	(−0.1014)	(−0.0252)	(0.0749)
		8	0.9329	0.6533	0.7321	0.4658	0.7535	0.5733	0.7231	0.3945
			(−0.2329)	(−0.1533)	(−0.0321)	(0.0342)	(−0.0535)	(−0.0733)	(−0.0231)	(0.1055)
		9	0.8574	0.6217	0.7237	0.4060	0.7366	0.5692	0.7110	0.3667
			(−0.1574)	(−0.1217)	(−0.0237)	(0.0940)	(−0.0366)	(−0.0692)	(−0.0110)	(0.1333)
0.5	0.4	6	0.6317	0.4842	0.5382	0.2172	0.5444	0.4411	0.5251	0.1894
			(−0.1317)	(−0.0842)	(−0.0382)	(0.1828)	(−0.0444)	(−0.0411)	(−0.0251)	(0.2106)
		7	0.6352	0.4802	0.5328	0.1918	0.5275	0.4278	0.5154	0.1708
			(−0.1352)	(−0.0802)	(−0.0328)	(0.2082)	(−0.0275)	(−0.0278)	(−0.0154)	(0.2292)
		8	0.5939	0.4490	0.5206	0.1698	0.5268	0.4181	0.5164	0.1623
			(−0.0939)	(−0.0490)	(−0.0206)	(0.2302)	(−0.0268)	(−0.0181)	(−0.0164)	(0.2377)
		9	0.5797	0.4413	0.5166	0.1608	0.5249	0.4070	0.5095	0.1551
			(−0.0797)	(−0.0413)	(−0.0166)	(0.2392)	(−0.0249)	(−0.0070)	(−0.0095)	(0.2449)

Table 2. MPSE-based estimators and biases for four different sampling techniques.

α	λ	n	$\hat{\alpha}_{SRS}$	$\hat{\lambda}_{SRS}$	$\hat{\alpha}_{RSS}$	$\hat{\lambda}_{RSS}$	$\hat{\alpha}_{MaxRSS}$	$\hat{\lambda}_{MaxRSS}$	$\hat{\alpha}_{DBRSS}$	$\hat{\lambda}_{DBRSS}$
0.7	0.5	6	1.3180	0.6570	1.1243	0.5620	1.1231	0.5614	1.0426	0.5234
			(−0.6180)	(−0.157)	(−0.4243)	(−0.0620)	(−0.4231)	(−0.0614)	(−0.3426)	(−0.0234)
		7	1.1792	0.6212	1.0230	0.5360	1.0219	0.5417	0.9766	0.5173
			(−0.4792)	(−0.1212)	(−0.3230)	(−0.0360)	(−0.3219)	(−0.0417)	(−0.2766)	(−0.0173)
		8	1.0799	0.5975	0.9688	0.5282	0.9553	0.5263	0.9301	0.5073
			(−0.3799)	(−0.0975)	(−0.2688)	(−0.0282)	(−0.2553)	(−0.0263)	(−0.2301)	(−0.0073)
		9	1.0445	0.5838	0.9333	0.5331	0.9290	0.5196	0.9106	0.5040
			(−0.3445)	(−0.0838)	(−0.2333)	(−0.0331)	(−0.2290)	(−0.0196)	(−0.2106)	(−0.0040)
0.5	0.4	6	0.8953	0.5133	0.7641	0.4547	0.7543	0.4553	0.7349	0.4164
			(−0.3953)	(−0.1133)	(−0.2641)	(−0.0547)	(−0.2543)	(−0.0553)	(−0.2349)	(−0.0164)
		7	0.8185	0.4883	0.7118	0.4256	0.7145	0.4288	0.6877	0.4174
			(−0.3185)	(−0.0883)	(−0.2118)	(−0.0256)	(−0.2145)	(−0.0288)	(−0.1877)	(−0.0174)
		8	0.7496	0.4760	0.6740	0.4234	0.6815	0.4283	0.6571	0.4106
			(−0.2496)	(−0.0760)	(−0.1740)	(−0.0234)	(−0.1815)	(−0.0283)	(−0.1571)	(−0.0106)
		9	0.7038	0.4670	0.6441	0.4167	0.7268	0.4178	0.6456	0.4068
			(−0.2038)	(−0.0670)	(−0.1441)	(−0.0167)	(−0.2268)	(−0.0178)	(−0.1456)	(−0.0068)

Table 3. CME-based estimators and biases for four different sampling techniques.

α	λ	n	$\hat{\alpha}_{SRS}$	$\hat{\lambda}_{SRS}$	$\hat{\alpha}_{RSS}$	$\hat{\lambda}_{RSS}$	$\hat{\alpha}_{MaxRSS}$	$\hat{\lambda}_{MaxRSS}$	$\hat{\alpha}_{DBRSS}$	$\hat{\lambda}_{DBRSS}$
0.7	0.5	6	0.8796	0.4719	0.8065	0.4925	0.8507	0.4668	0.8046	0.4963
			(−0.1796)	(0.0281)	(−0.1065)	(0.0075)	(−0.1507)	(0.0332)	(−0.1046)	(0.0037)
		7	0.8636	0.4854	0.7922	0.5061	0.8619	0.4892	0.7895	0.5060
			(−0.1636)	(0.0146)	(−0.0922)	(−0.0061)	(−0.1619)	(0.0108)	(−0.0895)	(−0.0060)
		8	0.8594	0.5085	0.7660	0.5124	0.8465	0.5071	0.7699	0.5117
			(−0.1594)	(−0.0085)	(−0.0660)	(−0.0124)	(−0.1465)	(−0.0071)	(−0.0699)	(−0.0117)
		9	0.8554	0.5218	0.7558	0.5161	0.8412	0.5147	0.7646	0.5189
			(−0.1554)	(−0.0218)	(−0.0558)	(−0.0161)	(−0.1412)	(−0.0147)	(−0.0646)	(−0.0189)
0.5	0.4	6	0.6296	0.4171	0.5525	0.4142	0.6174	0.4202	0.5518	0.4162
			(−0.1296)	(−0.0171)	(−0.0525)	(−0.0142)	(−0.1174)	(−0.0202)	(−0.0518)	(−0.0162)
		7	0.5953	0.4239	0.5406	0.4185	0.5931	0.4202	0.5330	0.4113
			(−0.0953)	(−0.0239)	(−0.0406)	(−0.0185)	(−0.0931)	(−0.0202)	(−0.0330)	(−0.0113)
		8	0.5850	0.4261	0.5333	0.4135	0.5999	0.4383	0.5303	0.4127
			(−0.0850)	(−0.0261)	(−0.0333)	(−0.0135)	(−0.0999)	(−0.0383)	(−0.0303)	(−0.0127)
		9	0.5768	0.4337	0.5229	0.4111	0.5751	0.4332	0.5248	0.4117
			(−0.0768)	(−0.0337)	(−0.0229)	(−0.0111)	(−0.0751)	(−0.0332)	(−0.0248)	(−0.0117)

As shown in Tables 1, 2, and 3, the biases of the RSS, MaxRSS, and DBRSS techniques are almost always much smaller than those of SRS for a variety of true parameter values. When compared to ML and MPSE, the CME method is found to be more accurate at estimating the values of the parameters, with small biases. Perhaps the reason why CME has the smallest bais is that it does not depend on units chosen randomly, as happens with other methods, but rather on the value of i, which makes it more satiable than the other methods. Furthermore, when compared to the other estimation methods, the MPSE method is found to be less accurate at estimating the values of the parameters with large biases. This is because it uses two different values, x_i and x_{i-1}, at the same time, which makes it better than ML, which depends on only one value, x_i. Since estimators depend on observations that are chosen randomly, samples must be chosen very carefully to make the ML and MPSE methods more effective than before. When the sample size n has values larger than 20, all of the estimation methods will give the same results with differences in the MSE based on sampling techniques. In the case of sample size n, there is no need to use RSS modifications because it will take a lot more time to select the ranked units than the usual SRS. There are also problems in selecting units based on the sampling techniques, which creates differences between them in the results. For example, RSS was selected once from the equal sample sizes, while DBRSS was selected twice from the equal sample sizes, Finally, MaxRSS was selected once from the unequal sample sizes, which makes it better for saving time and energy than the other techniques using small sample sizes. This is the benefit of using these kinds of sizes. Next, we determined that, for various ranges of sample sizes, MaxRSS estimators outperform all sampling techniques, while DBRSS estimators outperform SRS and RSS estimators based on estimation methods. In terms of (α, λ), RSS estimators outperform SRS estimators for all sample sizes and estimation methods. The results are depicted in Table 4. This table displays the RE for each sampling technique based on estimation techniques for the various sample sizes. In addition, MSEs of RSS estimators are consistently bigger than MSEs of MaxRSS and DBRSS estimators, whereas MSEs for (α, λ) using RSS, MaxRSS, and DBRSS estimators almost always differ from MSEs of SRS estimators. Table 4 demonstrates that for various sample sizes, the MSEs of MaxRSS estimators are consistently lower than the MSEs of the other RSS estimators for various values of (α, λ).

Table 4. MSEs of the various sampling techniques based on estimation methods.

	α	λ	n	$\hat{\alpha}_{SRS}$	$\hat{\lambda}_{SRS}$	$\hat{\alpha}_{RSS}$	$\hat{\lambda}_{RSS}$	$\hat{\alpha}_{MaxRSS}$	$\hat{\lambda}_{MaxRSS}$	$\hat{\alpha}_{DBRSS}$	$\hat{\lambda}_{DBRSS}$
ML	0.7	0.5	6	0.0858	0.0250	0.0134	0.0146	0.0069	0.0011	0.0123	0.0013
			7	0.4719	0.0722	0.1336	0.0146	0.0538	0.0016	0.0500	0.0112
			8	0.3488	0.0717	0.0536	0.0326	0.0398	0.0038	0.0181	0.0262
			9	0.0709	0.0142	0.0372	0.0083	0.0013	0.0030	0.0066	0.0061
	0.5	0.4	6	0.2748	0.6078	0.0274	0.0274	0.0150	0.0097	0.0904	0.0270
			7	0.3336	0.0736	0.1652	0.0476	0.0207	0.0083	0.0797	0.0440
			8	0.7936	0.0831	0.1132	0.0761	0.0265	0.0122	0.0377	0.0655
			9	0.2085	0.1446	0.0124	0.0746	0.0109	0.0577	0.0124	0.0625
$MPSE$	0.7	0.5	6	0.0480	0.0133	0.0092	0.0119	0.0012	0.0042	0.0023	0.0063
			7	0.0981	0.0272	0.0795	0.0268	l0.0216	0.0058	0.0372	0.0179
			8	0.1449	0.0247	0.0113	0.0055	0.0062	0.0021	0.0083	0.0021
			9	0.0868	0.0110	0.0198	0.0072	0.0087	0.0024	0.0122	0.0041
	0.5	0.4	6	0.0288	0.0768	0.0136	0.0276	0.0019	0.0013	0.0075	0.0096
			7	0.0173	0.0398	0.0086	0.0034	0.0084	0.0011	0.0077	0.0012
			8	0.0685	0.0239	0.0124	0.0085	0.0019	0.0073	0.0020	0.0080
			9	0.0851	0.0153	0.0360	0.0119	0.0064	0.0061	0.0224	0.0023
CME	0.7	0.5	6	0.0882	0.0207	0.0049	0.0170	0.0038	0.0020	0.0040	0.0116
			7	0.0468	0.0302	0.0050	0.0045	0.0010	0.0012	0.0011	0.0024
			8	0.0410	0.0410	0.0368	0.0207	0.0010	0.0015	0.0012	0.0016
			9	0.0234	0.0176	0.0027	0.0112	0.0007	0.0017	0.0014	0.0020
	0.5	0.4	6	0.0431	0.0201	0.0126	0.0097	0.0024	0.0008	0.0100	0.0084
			7	0.0194	0.0144	0.0066	0.0072	0.0010	0.0020	0.0058	0.0066
			8	0.0250	0.0186	0.0043	0.0059	0.0012	0.0023	0.0015	0.0028
			9	0.0876	0.0452	0.0128	0.0133	0.0106	0.0015	0.0112	0.0023

7. Illustrative example

In this section, we test the feasibility of the the bathtub-shaped distribution using maximum likelihood, maximum product of spacing, and Cramér-von Mises estimation methods on two different real datasets. Dataset I from Nichols and Padgett [28] was used for illustration. It shows the observed fracture stress of 100 carbon fibers. Dataset II from Kundu and Raqab [29] were tested 74 fibers under tension at gauge lengths of 20 mm. These datasets were chosen based on their suitability for distribution, as shown in the values below, and also because they fulfill the condition of constructing matrices with small sample sizes from which one can choose to achieve different sampling techniques. The Kolmogorov-Smirnov ($K-St$), Anderson-Darling ($A-Dt$), and Cramér-von Mises ($C-Mt$) statistical tests were conducted on these datasets to assess distribution fit. In general, the distribution provides an adequate fit to the data, as indicated by the p values for all three tests (see Table 5).

Table 5. The $K-St$, $A-Dt$, $C-Mt$ and the p values of these test for two datasets.

	$K-St$	p *value*	$A-Dt$	p *value*	$C-Mt$	p *value*
Data I	0.9915	0.6232	0.4686	0.2439	0.0584	0.3926
Data II	0.9392	0.8520	0.2458	0.7498	0.0353	0.7617

Figure 5 displays the quantile-quantile (Q-Q) plot for the CBL distribution taking into account Data I and the histogram plot with the fitted PDF. The CBL fits the available data well, as this graph demonstrates. There are no outliers in the data, as shown in the boxplot in Figure 5.

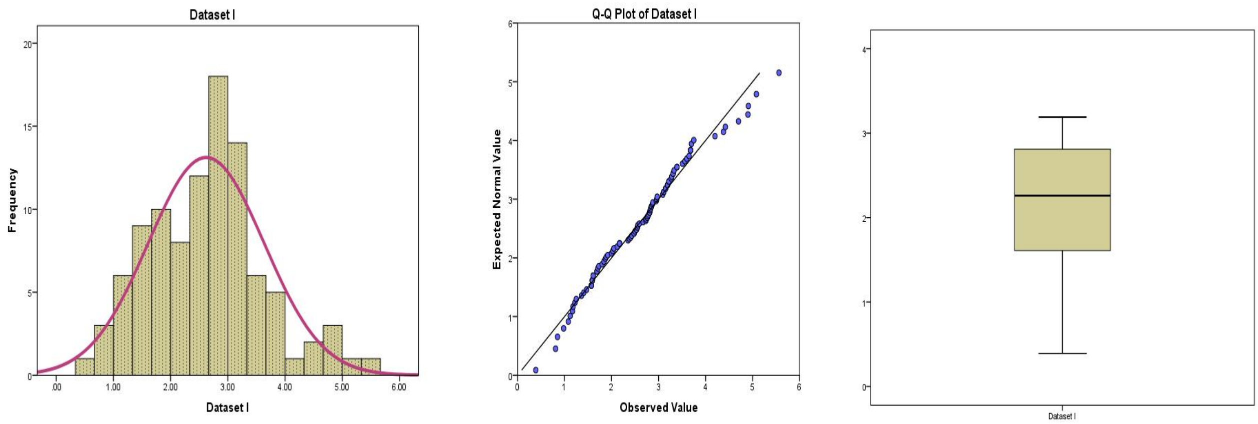

Figure 5. Histogram, Q-Q, and boxplot for the CBL distribution fit using Data I.

We conclude, based on Figure 6, that the CBL distribution fits Data II. It shows the fitted PDF and Q-Q diagrams along with the histogram graph for Data II. The boxplot on the graph indicates that there are no outlier values.

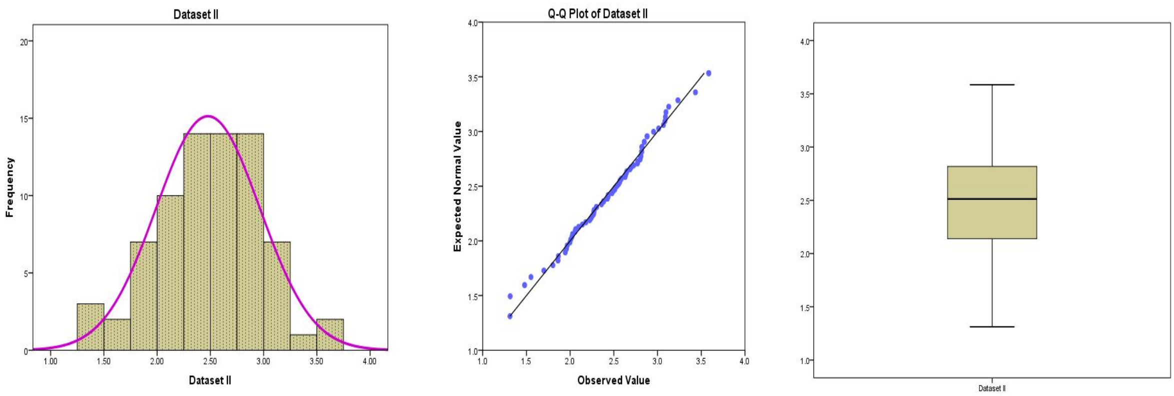

Figure 6. Histogram, Q-Q, and Boxplot for the CBL distribution fit using Data II.

In order to estimate model parameters, the ML, MPSE, and CME approaches were utilized. Standard error (SE), mean absolute error (MAE), mean bias error (MBE), and mean squared error (MSE) are some of the often-used error measures in this article. We have used these measures for comparison because they were different estimation methods that needed to unite the competition. There is no better error measure to standardize this comparison.

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(x_{obs} - x_{exp})^2, \qquad MAE = \frac{1}{n}\sum_{i=1}^{n}|x_{obs} - x_{exp}|,$$

$$MBE = \frac{1}{n}\sum_{i=1}^{n}(x_{obs} - x_{exp}), \quad and \quad SE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_{obs} - x_{exp})^2},$$

where x_{obs} are the values from the actual datasets and x_{exp} are the simulated values. Data I and Data II for sample size $n = 4$, and a one cycle $r = 1$ have their SE, MAE, MBE, and MSE values displayed in Tables 6–8. To use RSS, we created a symmetric matrix with a sample size of $n = 4$ from x_{obs} in Data I. We then chose units as usual and estimated the unknown CBL parameters. Following that, we simulated x_{exp} using the unknown parameters obtained from the estimate method. To use DBRSS, we created a matrix with sample size n^2 from x_{obs} of Data I and selected n units as described in the previous steps in Section 1. Finally, we followed the identical steps for RSS with Data I. To use MaxRSS, we created sets with variable sample size i from x_{obs} of Data I and selected n units as described in the previous steps in Section 1. Finally, we followed the identical steps for RSS with Data I. We repeated all of the steps they mentioned in selecting samples with Data II.

These tables use ML, MPSE, and CME to compare the CBL distribution to the SRS, RSS, MaxRSS, and DBRSS techniques. Using these measures, we can compare different sampling techniques for estimating the CBL distribution and make an informed decision on which one to choose. Using the numerical results from several datasets, the following is a summary of the conclusions: To begin, error measurement values calculated using the RSS, MaxRSS, and DBRSS techniques are all smaller than those calculated using the SRS technique. Second, the MaxRSS method works better than any other sampling strategy due to having the lowest measurement error. Third, it is clear that SRS is not as effective as alternative sampling techniques. For all methods of estimating, it has the greatest error measures. Fourth, the MPS method has the lowest sampling-based error measures when compared to other estimation methods. However, the largest sampling-based error measures are associated with the ML method. Lastly, for all estimation and sampling techniques, MSE and MBE have the same values.

Table 6. The estimators use sampling techniques with ML to select measures for datasets.

	n	Measures	$\hat{\alpha}$	$\hat{\lambda}$	MSE	MAE	MBE	SE
Data I	4	SRS	0.1443	1.5481	0.9864	0.8547	0.9864	0.9932
		RSS	0.2871	1.3695	0.8059	0.7828	0.8059	0.8977
		$MaxRSS$	0.7160	0.9597	0.1199	0.3021	0.1199	0.3462
		$DBRSS$	0.2500	0.66814	0.6365	0.6503	0.6365	0.7978
Data II		SRS	0.1439	0.7227	3.3270	1.5773	3.3270	1.8240
		RSS	0.1087	1.2753	2.4743	1.5592	2.4743	1.5729
		$MaxRSS$	0.4224	0.9323	1.2742	1.1158	1.2742	1.1288
		$DBRSS$	0.1760	0.7472	1.9156	1.3191	1.9156	1.3840

Table 7. The estimators use sampling techniques with MPSE to select measures for datasets.

	n	Measures	$\hat{\alpha}$	$\hat{\lambda}$	MSE	MAE	MBE	SE
Data I	4	SRS	0.6428	1.6551	0.37016	0.5764	0.3701	0.6084
		RSS	0.3981	1.4109	0.3571	0.5267	0.3571	0.5975
		$MaxRSS$	0.3981	1.4109	0.1736	0.2560	0.1736	0.4167
		$DBRSS$	0.2352	0.8497	0.2693	0.4298	0.2693	0.5189
Data II		SRS	0.0169	1.5487	0.3870	0.5293	0.3870	0.3870
		RSS	0.0400	1.9788	0.2984	0.4800	0.2984	0.5463
		$MaxRSS$	0.0466	1.9097	0.1029	0.1882	0.1029	0.3207
		$DBRSS$	0.0534	1.3046	0.1525	0.2883	0.1525	0.3905

Table 8. The estimators use sampling techniques with CME to select measures for datasets.

	n	Measures	$\hat{\alpha}$	$\hat{\lambda}$	MSE	MAE	MBE	SE
Data I	4	SRS	0.2909	1.0369	0.2949	0.4200	0.2949	0.5430
		RSS	0.4792	1.0830	0.2166	0.4039	0.2166	0.4654
		$MaxRSS$	0.9979	1.9972	0.1718	0.3910	0.1718	0.4145
		$DBRSS$	0.7994	0.9885	0.2079	0.3170	0.2079	0.4859
Data II		SRS	0.6285	0.5603	1.8033	1.2900	1.8033	1.3428
		RSS	0.6288	0.5256	1.0413	0.8661	1.0413	1.0204
		$MaxRSS$	0.6288	0.5256	0.9607	0.7463	0.9607	0.9801
		$DBRSS$	0.5017	0.4554	0.9455	0.9302	0.9455	0.9723

8. Summary and conclusions

The bathtub-shaped distribution's unknown parameters were calculated in this article utilizing ML, MPSE, and CME methods using SRS, RSS, MaxRSS, and DBRSS techniques. There are three sections dedicated to the conclusions. First, sampling techniques were used to obtain theoretical results through various estimation methods. The estimators were obtained by solving the first derivatives of these sampling-based estimation methods numerically. Second, numerical comparisons between SRS and various RSS techniques, based on the simulation results of the comparative study, showed that: Estimates produced with RSS, MaxRSS, and DBRSS approaches were typically more accurate than those produced with SRS estimators. These findings were based on the simulation results of the comparative study. Moreover, it has been demonstrated that DBRSS and MaxRSS techniques with high MSEs were more efficient than RSS. When it comes to varying sample sizes, SRS-based RSS techniques were less effective than MaxRSS. It was discovered that the CME method estimates the values of the parameter more precisely and with less bias than the ML and MPSE methods. The MPSE method was also found to have significant biases and to be less accurate in estimating the values of the parameters when compared to the other estimate methods. Finally, based on the outcomes of analyzing two real data sets, all error measurements based on the RSS, MaxRSS, and DBRSS techniques had lower values than those based on the SRS methodology. When compared to other MaxRSS and DBRSS techniques, the RSS accquired the highest values. Of all of the sampling techniques, the MaxRSS technique is the best. It had the lowest error measurement values, while DBRSS estimators outperformed SRS and RSS estimators based on estimation methods.

Author contributions

Nuran M. Hassan: Conceptualization, methodology, formal analysis, investigation, data curation, writing—original draft preparation, visualization; M. Nagy: Methodology, validation, resources, writing, review and editing, project administration; Subhankar Dutta: Conceptualization, methodology, validation, formal analysis, data curation, writing—original draft preparation, writing review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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