Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 25028–25048.
DOI: 10.3934/math.20241220
Received: 01 July 2024
Revised: 05 August 2024
Accepted: 16 August 2024
Published: 27 August 2024

Research article

Analysis of finite-time stability in genetic regulatory networks with interval time-varying delays and leakage delay effects

Nayika Samorn[1], Kanit Mukdasai[2] and Issaraporn Khonchaiyaphum[3,*]

[1] Faculty of Agriculture and Technology, Nakhon Phanom University, Nakhon Phanom 48000, Thailand

[2] Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand

[3] Department of Mathematics, Faculty of Engineering, Rajamangala University of Technology Isan, Khon Kaen Campus, Khon Kaen 40000, Thailand

* **Correspondence:** Email: Issaraporn.kh@rmuti.ac.th.

Abstract: We primarily examined the effect of leakage delays on finite-time stability problems for genetic regulatory networks with interval time-varying delays. Since leakage delays can occur within the negative feedback components of networks and significantly impact their dynamics, they may potentially cause instability or suboptimal performance. The derived criteria encompass both leakage delays and discrete interval time-varying delays through the construction of a Lyapunov-Krasovskii function. We employed the estimation of various integral inequalities and a reciprocally convex technique. Additionally, these models consider lower bounds on delays, which may be either positive or zero, and allow for the derivatives of delays to be either positive or negative. Consequently, new criteria for genetic regulatory networks with interval time-varying delays under the effect of leakage delays are expressed in the form of linear matrix inequalities. Ultimately, a numerical example is presented to show the effect of leakage delays and to emphasize the significance of our theoretical findings.

Keywords: finite-time stability; genetic regulatory networks; leakage delays; lyapunov-Krasovskii functional; interval time-varying delay
Mathematics Subject Classification: 34K20, 93D05, 93D40, 34D20

1. Introduction

Through their mRNAs, protein expression products, and other components, DNA segments in a cell interact with each other passively, forming dynamic genetic regulatory networks (GRNs) that work as a sophisticated dynamic system to govern biological processes, where the systems of regulatory linkages between DNA, RNA, and proteins comprise GRNs. Modeling genetic networks with dynamical system models, which are a useful tool for analyzing gene regulation mechanisms in living organisms, comes naturally. To examine genetic regulatory developments in life forms, the mathematical models of GRNs comprise formidable tools, which can be approximately separated into two categories, including discrete models [1, 2, 6, 16, 22] and continuous models [4, 5, 9, 10, 12]. The components in continuous models characterize the concentrations of mRNAs and proteins as continuous data, allowing for a more complete interpretation of GRNs' nonlinear dynamical behavior.

A differential equation is often used to describe a continuous model. Thus, erroneous estimates could result from theoretical models that fail to consider delays. Time delays frequently occur in various applications, and it is widely acknowledged that they can significantly impact system performance, leading to degradation and instability. This realization has sparked significant research interest in the field of stability analysis for systems with time delays over recent years, such as differential equations [35], neural networks [11, 13, 24], bidirectional associative memory fuzzy neural networks [36], fuzzy competitive neural networks [37], and neutral-type neural networks [14, 25, 27]. More accurate descriptions of GRNs are possible using differential equation models with delayed states, also called delayed GRNs, which offer a better display of the nature of life. To indicate the time needed for transcription, translation, phosphorylation, protein degradation, translocation, and posttranslational modification, time delays are often established in cellular models. Particularly for eukaryotes, delays may affect the firmness and dynamics of the whole system considerably. While there are limited quantitative assessments of the delays, developments in quantifying RNA splicing delays should be taken into account. Because the time delays in biochemical reactions align with or surpass other important time scales defining the cellular system, and the feedback loops related to these delays are robust, and these delays become pivotal for explaining transient processes. This suggests that in cases where delay durations are substantial, both analytical and numerical modeling need to consider the impact of time delays, which has led to the study of various stability forms of the system, for example, in [5, 10, 31, 33, 38]. A common type of time delay, referred to as leakage delay (i.e., the delay associated with leaking or forgetting items) might arise within the negative feedback components of networks and has a substantial impact on the dynamics, potentially resulting in instability or suboptimal performance in GRNs. In fact, various beneficial algorithms and computational tools have been devised to address this phenomenon, with numerous notable findings published in previous literature [7, 17, 28, 29, 32].

When a built system has been implemented in the field, there is typically some unpredictability; be it a windy sky or a rough road, we may encounter uncertainty in the operational area. Uncertainty can also be found in the system's defining variables, such as torque constants and physical dimensions. As a consequence of the dynamic nature of a system under investigation, one of the most critical attributes of that system is stability. In general, when a system is chaotic, or simply lacks stability, it is not especially useful. It is preferable to work with systems that are stable or offer periodic behavior, with the caveat that chaotic behavior can be readily understood in some systems, and there can also sometimes be good reasons for people to actively seek chaos in a system for various applications. Therefore, we found research that demonstrated various stability criteria such as asymptotic stability analysis, exponential stability analysis [3, 34], H infinity performance [25], finite-time stability (FTS) analysis [15, 23], and so on. The concept of FTS differs from classical stability in two significant ways.

First, it addresses systems whose functionality is limited to a fixed finite time interval. Second, FTS necessitates predefined constraints on system variables. This is particularly relevant for systems that are recognized to operate exclusively within a finite time span and in cases where practical considerations dictate that system variables must adhere to specific bounds [8].

After thorough investigation, FTS for GRNs with delays has been studied in references [21,24,26], as well as [39]. For example, the researchers in reference [21] explored the FTS for GRNs that have demonstrated impulsive effects, employing the Lyapunov function method. In [24], the topic of FTS for switched GRNs with time-varying delays was examined through the use of Wirtinger's integral inequality. Furthermore, the effect of leakage delay has been studied in various works, including references [14, 18, 27]. For instance, in [14], the researchers proposed sufficient conditions that guarantee the asymptotic stability of GRNs, which have a neutral delay and are affected by leakage delay through the application of the Lyapunov functional. Additionally, in reference [18], the authors focused on global asymptotic stability analysis aimed at stabilizing switched stochastic GRNs that exhibit both leakage and impulsive effects. This stabilization has depended on both time-varying delay and distributed time-varying delay terms, utilizing contemporary Lyapunov-Krasovskii functional and integral inequality techniques. However, it has been determined that there are no studies focusing on analyzing the FTS while considering the effect of leakage delay for GRNs. The successful completion of such research could contribute to a deeper comprehension of leakage delay and potentially offer opportunities to improve the stability criteria of GRNs. This research concerns the effect of the leakage delays on FTS criteria for GRNs with interval time-varying delays. The criteria aim to consider the effect of leakage delays. Consequently, we employ the construction of a Lyapunov-Krasovskii (LK) function and estimate various integral inequalities as well as reciprocally convex techniques to establish them. These improvements enable us to specify the stability criteria concerning the effect of leakage delays on FTS. This refinement simplifies the representation of stability criteria in the form of linear matrix inequalities (LMIs). Ultimately, we offer a numerical example to demonstrate the effect of leakage delays and emphasize the significance of our theoretical findings.

The principal contributions of this research can be encapsulated as follows:

(i) Targeted focus on leakage delays: We specifically address the impact of leakage delays on the finite-time stability of GRNs, filling a gap in existing research and providing new data on the stability dynamics influenced by these delays.

(ii) Consideration of variable delay limits: We take into account the lower limits on delays, which can vary between positive values and zero, and accommodate the derivatives of delays ranging from negative to positive. This comprehensive approach allows for the one that offers a better representation of the nature of organism and modeling of real biological scenarios.

(iii) Empirical validation: A numerical example is provided to demonstrate the implications of leakage delays on the stability criteria. This example underscores the impact of leaked delays on GNRs.

(iv) Contribution to broader fields: The insights gained from this study have the potential to advance multiple fields, including systems biology, biotechnology, and medicine.

By addressing these aspects, this research significantly enhances our understanding of the interplay between leakage delays and finite-time stability in genetic regulatory networks, paving the way for future studies and applications in various scientific and medical domains.

2. Problem formulation and preliminaries

The first step is to introduce the following symbols or notations: $\mathbb{R}^n$ and $\mathbb{R}^{n\times r}$ represent the n-dimensional Euclidean space and the set of all $n \times r$ real matrices, respectively; $G > 0$ ($G \geq 0$) signifies that the symmetric matrix G is positive (semi-positive) definite; $G < 0$ ($G \leq 0$) signifies that the symmetric matrix G is negative (semi-negative) definite; The quadratic from $||\varphi(t)||_N^2$ is defined as: $||\varphi(t)||_N^2 = \varphi^T(t)N\varphi(t)$ or $||\varphi(t)||_N = \sqrt{\varphi^T(t)N\varphi(t)}$ for any state vector $\varphi(t)$, and $N \geq 0$.

In this paper, we suggest the GRNs with interval time-varying delays and leakage delays in the form

$$\begin{array}{rcl} \dot{m}_i(t) &=& -a_i m_i(t-\rho_a) + \sum_{j=1}^n w_{ij} g_j(p_j(t-r(t))) + \epsilon_i, \\ \dot{p}_i(t) &=& -c_i p_i(t-\rho_c) + d_i m_i(t-h(t)), \qquad i = 1,2,\ldots,n, \end{array} \tag{2.1}$$

where $m_i(\cdot)$ and $p_i(\cdot)$ are the concentrations of mRNA and protein of the ith node at time t, respectively. $a_i> 0$ and $c_i> 0$ denotes the degradation or dilution rates of mRNAs and proteins, retrospectively. d_i represents the translation. w_{ij} is defined as follows:

$$w_{ij} = \begin{cases} \gamma_{ij} & \text{if transcription factor } j \text{ is an activator of gene } i; \\ 0 & \text{if there is no link from node } j \text{ to } i; \\ -\gamma_{ij} & \text{if transcription factor } j \text{ is an repressor of gene } i. \end{cases}$$

$r(\cdot)$ and $h(\cdot)$ are the feedback regulation and translation delays, respectively, which are retrospectively satisfied by

$$0 < r_m \leq r(t) \leq r_M, \quad r_{dm} \leq \dot{r}(t) \leq r_{dM}, \tag{2.2}$$

$$0 < h_m \leq h(t) \leq h_M \quad h_{dm} \leq \dot{h}(t) \leq h_{dM}, \tag{2.3}$$

where $\rho_a > 0$ and $\rho_c > 0$ denote the leakage delays. $g_j(p_j(s)) = (p_j(s)/B_j)^{H_j}/(1 + p_j/B_j)^{H_j}$ where H_j is the monotonic function in Hill form, $\epsilon_i = \sum_{j\in U_k} \gamma_{ij}$ with $U_k = \{j|$ the jth transcription factor being a repressor of the kth gene, $j = 1,\ldots,n\}$, $B_j > 0$ is a constant which the feedback regulation of the protein on the transcription. When we let $(m^*, p^*)^T$ be an equilibrium point of (2.1), the equilibrium point can be shifted to the origin by transformation: $x_i(t) = m_i - m_i^*,\ y_i(t) = p_i - p_i^*$, and system (2.1) can be rewritten in the vector form

$$\begin{array}{rcl} \dot{x}(t) &=& -Ax(t-\rho_a) + Wf(y(t-r(t))), \\ \dot{y}(t) &=& -Cy(t-\rho_c) + Dx(t-h(t)), \\ x(t) &=& \phi(t), \qquad y(t) = \xi(t), \qquad t \in [-\tau, 0], \qquad \tau = \max\{h_M, r_M, \rho_a, \rho_c\}, \end{array} \tag{2.4}$$

where $A = diag(a_1, a_2, ..., a_n)$, $C = diag(c_1, c_2, ..., c_n)$, $W = [w_{ij}]_{n\times n}$, $f(y(t)) = g(y(t)) - p^*$ with $f(0) = 0$, $\phi(t)$, $t \in [-\max\{\rho_a, h_M\}, 0]$ and $\xi(t)$, $t \in [-\max\{\rho_c, r_M\}, 0]$ are the initial conditions.

Assumption 1. The regulatory function $f(\zeta(t)) = [f_1(\zeta_1(t)), f_2(\zeta_2(t)), ..., f_n(\zeta_n(t))]^T \in \mathbb{R}^n$ is assumed to satisfy the following condition

$$\wp_i^- \leq \frac{f_i(\zeta_1) - f_i(\zeta_2)}{\zeta_1 - \zeta_2} \leq \wp_i^+, \qquad f(0) = 0, \qquad \zeta_1, \zeta_2 \in \mathbb{R}, \qquad \zeta_1 \neq \zeta_2, i = 1, 2, ..., n, \tag{2.5}$$

where $\wp_i^-, \wp_i^+$ are real constants, and let $\eth^- = diag(\wp_1^-, \wp_2^-, \ldots, \wp_n^-)$, $\eth^+ = diag(\wp_1^+, \wp_2^+, \ldots, \wp_n^+)$ and $\wp_i = diag(\max\{|\wp_j^-|, |\wp_j^+|\})$, $\eth = diag(\wp_1, \wp_2, \ldots, \wp_n)$.

Then, the definition presented along with the several lemmas serves as a methodologies utilized to prove our primary outcomes.

Definition 2.1. (see [21]) Let $G \geq 0$ be a matrix, if

$$||\Phi(t)||_G^2 + ||\Psi(t)||_G^2 \leq c_1 \rightarrow ||x(t)||_G^2 + ||y||_G^2 \leq c_2, \forall t \in [0, T],$$

where $||\Phi(t)||_G = \sup_{-\max\{\rho_a, h_M\} \leq t \leq 0}\{||\phi(t)||_G, ||\dot{\phi}(t)||_G\}$ and $||\Psi(t)||_G = \sup_{-\max\{\rho_c, r_M\} \leq t \leq 0}\{||\xi(t)||_G, ||\dot{\xi}(t)||_G\}$, then the GRNs with interval time-vary delays and leakage delays (2.4) exhibits FTS concerning positive real numbers (c_1, c_2, T).

Lemma 2.2. *Let $N \in \mathbb{R}^{n\times n}$, $N = N^T > 0$ and $G \in \mathbb{R}^{n\times n}, G = G^T$ be any constant matrices. Then*

$$\lambda_{\min}(N^{-1}G)\varphi^T N\varphi \leq \varphi^T G\varphi \leq \lambda_{\max}(N^{-1}G)\varphi^T N\varphi,$$

where the expressions "$\lambda_{\min}(N^{-1}G)$" and "$\lambda_{\max}(N^{-1}G)$" denote the minimum real part and the maximum real part of the eigenvalues of $N^{-1}G$ respectively.

Lemma 2.3 (*Jensen's inequality* [28])**.** *Let $G \in \mathbb{R}^{n\times n}$, $G = G^T > 0$ be any constant matrix, δ_M be positive real constant and $\varphi : [-\delta_M, 0] \rightarrow \mathbb{R}^n$ be vector-valued function. Then,*

$$-\delta_M \int_{t-\delta_M}^{t} \varphi^T(s)G\varphi(s)ds \leq -\int_{t-\delta_M}^{t} \varphi^T(s)dsG\int_{t-\delta_M}^{t} \varphi(s)ds.$$

Lemma 2.4 (see [30])**.** *Let $G \in \mathbb{R}^{n\times n}$, $G = G^T > 0$ be any constant matrix, $\delta_m > 0, \delta_M > 0$ are real constants and $\varphi : [-\delta_M, -\delta_m,] \rightarrow \mathbb{R}^n$ be vector-valued function. Then,*

$$-(\delta_M - \delta_m)\int_{t-\delta_M}^{t-\delta_m} \varphi^T(s)G\varphi(s)ds \leq -\int_{t-\delta_M}^{t-\delta_m} \varphi^T(s)dsG\int_{t-\delta_M}^{t-\delta_m} \varphi(s)ds.$$

Lemma 2.5 (see [20])**.** *Let $G \in \mathbb{R}^{n\times n}$, $G = G^T > 0$ be any constant matrix, and any continuously differentiable function $z : [\delta_m, \delta_M] \rightarrow \mathbb{R}^n$. Then,*

$$(\delta_M - \delta_m)\int_{\delta_m}^{\delta_M} \varphi^T(s)G\varphi(s)ds \geq \aleph_1^T G\aleph_1 + 3\aleph_2^T G\aleph_2,$$

$$(\delta_M - \delta_m)\int_{\delta_m}^{\delta_M} \dot{\varphi}^T(s)G\dot{\varphi}(s)ds \geq \aleph_3^T G\aleph_3 + 3\aleph_4^T G\aleph_4 + 5\aleph_5^T G\aleph_5,$$

where

$$\aleph_1 = \int_{\delta_m}^{\delta_M} \varphi(s)ds, \quad \aleph_2 = \int_{\delta_m}^{\delta_M} \varphi(s)ds - \frac{2}{\delta_M - \delta_m}\int_{\delta_m}^{\delta_M}\int_{\theta}^{\delta_M} \varphi(s)dsd\theta,$$

$$\aleph_3 = \varphi(\delta_M) - \varphi(\delta_m), \quad \aleph_4 = \varphi(\delta_M) + \varphi(\delta_m) - \frac{2}{\delta_M - \delta_m}\int_{\delta_m}^{\delta_M} \varphi(s)ds,$$

$$\aleph_5 = \varphi(\delta_M) - \varphi(\delta_m) + \frac{6}{\delta_M - \delta_m}\int_{\delta_m}^{\delta_M} \varphi(s)ds - \frac{12}{(\delta_M - \delta_m)^2}\int_{\delta_m}^{\delta_M}\int_{\theta}^{\delta_M} \varphi(s)dsd\theta.$$

Lemma 2.6 (see [19])**.** *Let $\kappa_1, \kappa_2, \ldots, \kappa_N : \mathbb{R}^n \rightarrow \mathbb{R}$ have positive values in an open subset D of $\mathbb{R}^n$. Then, the reciprocally convex combination of κ_i over D satisfies*

$$\min_{\{\alpha_i|\alpha_1>0, \sum_i \alpha_i = 1\}} \sum_i \frac{1}{\alpha_i}\kappa_i(t) = \sum_i \kappa_i(t) + \max_{\eta_{i,j}(t)} \sum_{i\neq j} \eta_{i,j}(t),$$

subject to

$$\left\{\eta_{i,j} : \mathbb{R}^n \Rightarrow \mathbb{R}, \eta_{i,j} = \eta_{j,i}, \begin{bmatrix} \kappa_i(t) & \eta_{i,j}(t) \\ \eta_{j,i}(t) & \kappa_j(t) \end{bmatrix} \geq 0\right\}.$$

Lemma 2.7 (see [11,13]). *Suppose* **Assumption 1** *is valid, let diagonal matrices* $\Gamma_i > 0,\ i = 1, 2$. *Then*

$$\begin{bmatrix} \zeta(t) \\ f(\zeta(t)) \end{bmatrix}^T \begin{bmatrix} -\Gamma_1\Xi_1 & \Gamma_1\Xi_2 \\ \Gamma_1\Xi_2 & -\Gamma_1 \end{bmatrix} \begin{bmatrix} \zeta(t) \\ f(\zeta(t)) \end{bmatrix} \geq 0, \tag{2.6}$$

$$\begin{bmatrix} \zeta(t-r(t)) \\ f(\zeta(t-r(t))) \end{bmatrix}^T \begin{bmatrix} -\Gamma_2\Xi_1 & \Gamma_2\Xi_2 \\ \Gamma_2\Xi_2 & -\Gamma_2 \end{bmatrix} \begin{bmatrix} \zeta(t-r(t)) \\ f(\zeta(t-r(t))) \end{bmatrix} \geq 0, \tag{2.7}$$

where $\Xi_1 = diag(\wp_1^-\wp_1^+, \wp_2^-\wp_2^+, \dots, \wp_n^-\wp_n^+)$ *and* $\Xi_2 = diag(\frac{\wp_1^-+\wp_1^+}{2}, \frac{\wp_2^-+\wp_2^+}{2}, \dots, \frac{\wp_n^-+\wp_n^+}{2})$.

3. Main results

In this section, we present a theorem and a corollary related to GRNs. To begin with, the FTS result is formulated for the GRNs with interval time-varying delays, considering the effect of leakage delays.

Theorem 3.1. *Given that* **Assumption 1** *valid. For positive scalars* c_1, c_2, T, α, ρ_a, ρ_c, h_m, h_M, h_{dm}, h_{dM}, r_m, r_M, r_{dm} *and* r_{dM} *according conditions* (2.2)–(2.3), *if there exist matrices* $P_i > 0, i = 1, 2, 3$, $Q_i > 0, i = 1, 2, 3, 4$, $R_i > 0, i = 1, 2$, $S_i > 0, i = 1, 2, 3, 4$, $U_i > 0, i = 1, 2$, *any diagonal matrices* $\Gamma_i > 0, i = 1, 2$, *any appropriate dimensional matrices* $E_i, N_i, i = 1, 2, 3, 4$, *satisfying the following conditions:*

$$\begin{aligned} \begin{bmatrix} S_2 & X_{1i} \\ * & S_2 \end{bmatrix} &\geq 0, \quad i = 1, 2, 3, \\ \begin{bmatrix} S_4 & X_{2i} \\ * & S_4 \end{bmatrix} &\geq 0, \quad , i = 1, 2, 3, \\ \Theta &< 0, & (3.1) \\ \frac{\lambda_1}{\lambda_2} e^{\alpha T} c_1 &\leq c_2. & (3.2) \end{aligned}$$

Then, the system (2.4) *exhibits FTS concerning* $N > 0$ *and positive real numbers* (c_1, c_2, T), *where* $\Theta = \sum_{i=1}^{7} \Theta_i$ *is defined as*

$$\begin{aligned} \Theta_1 = {} & 2\begin{bmatrix} e_1 \\ e_5 \end{bmatrix}^T \begin{bmatrix} P_1 & E_1^T \\ 0 & E_2^T \end{bmatrix} \begin{bmatrix} e_5 \\ -e_5 - A(e_1 - e_{13}) + We_8 \end{bmatrix} + e_1^T P_2 e_1 - e^{\alpha h_m} e_2^T P_2 e_2 + e^{\alpha h_m} e_2^T P_3 e_2 \\ & + (h_{dM} e^{\alpha h_M} - e^{\alpha h_m}) e_3^T P_3 e_3 + (e^{\alpha h_M} - h_{dm} e^{\alpha h_m}) e_3^T P_3 e_3 - e^{\alpha h_M} e_4^T P_3 e_4 - \alpha e_1^T P_1 e_1, \\ \Theta_2 = {} & 2\begin{bmatrix} e_6 \\ e_{10} \end{bmatrix}^T \begin{bmatrix} P_3 & E_3^T \\ 0 & E_4^T \end{bmatrix} \begin{bmatrix} e_{10} \\ -e_{10} - C(e_6 - e_{14}) + De_3 \end{bmatrix} + e_6^T Q_2 e_6 - e^{\alpha r_m} e_7^T Q_2 e_7 + e^{\alpha h_m} e_7^T Q_3 e_7 \\ & + (r_{dM} e^{\alpha r_M} - e^{\alpha r_m}) e_8^T Q_3 e_8 + (e^{\alpha r_M} - r_{dm} e^{\alpha r_m}) e_9^T Q_3 e_9 - e^{\alpha r_m} e_9^T Q_3 e_9 + e_{11}^T Q_4 e_{11} \\ & + (r_{dM} e^{\alpha r_M} - e^{\alpha r_m}) e_{12}^T Q_4 e_{12} - \alpha e_6^T Q_1 e_6, \end{aligned}$$

$$\Theta_3 = h_M^2 e_1^T R_1 e_1 + r_M^2 e_6^T R_2 e_6 - \aleph_{11}^T (h_m^2 R_1)\aleph_{11} - 3\aleph_{12}^T (h_m^2 R_1)\aleph_{12} - \aleph_{21}^T (r_m^2 R_2)\aleph_{21} - 3\aleph_{22}^T (r_m^2 R_2)\aleph_{22},$$

$$\begin{aligned}\Theta_4 = {} & e_5^T (h_m^2 S_1 + h_{Mm}^2 S_2) e_5 + e_{10}^T (r_m^2 S_3 + r_{Mm}^2 S_4) e_{10} - \aleph_{31}^T S_1 \aleph_{31} - 3\aleph_{41}^T S_1 \aleph_{41} - 5\aleph_{51}^T S_1 \aleph_{51} \\ & -\aleph_{32}^T S_3 \aleph_{32} - 3\aleph_{42}^T S_3 \aleph_{42} - 5\aleph_{52}^T S_3 \aleph_{52} + e^{\alpha h_m}(- \aleph_{61}^T Q \aleph_{61} - 3\aleph_{71}^T Q \aleph_{71} - 5\aleph_{81}^T Q \aleph_{81} - \aleph_{62}^T Q \aleph_{62} \\ & -3\aleph_{72}^T Q \aleph_{72} - 5\aleph_{82}^T Q \aleph_{82}) + e^{\alpha r_m}(- \aleph_{91}^T Q \aleph_{91} - 3\aleph_{101}^T Q \aleph_{101} - 5\aleph_{111}^T Q \aleph_{111} - \aleph_{92}^T Q \aleph_{92} \\ & -3\aleph_{102}^T Q \aleph_{102} - 5\aleph_{112}^T Q \aleph_{112}) + 2e^{\alpha h_m}(- \aleph_{61}^T X_{11} \aleph_{62} - 3\aleph_{71}^T X_{12} \aleph_{72} - 5\aleph_{81}^T X_{13} \aleph_{82}) \\ & +2e^{\alpha r_m}(- \aleph_{91}^T X_{21} \aleph_{92} - 3\aleph_{101}^T X_{22} \aleph_{102} - 5\aleph_{111}^T X_{23} \aleph_{112}),\end{aligned}$$

$$\Theta_5 = \rho_a^2 e_5^T U_1 e_5 + \rho_c^2 e_{10}^T U_2 e_{10} - e_{13}^T U_1 e_{13} - e_{14}^T U_2 e_{14},$$

$$\Theta_6 = 2\begin{bmatrix} e_6 \\ e_{11} \end{bmatrix}^T \begin{bmatrix} -\Gamma_1\Xi_1 & \Gamma_1\Xi_2 \\ \Gamma_1\Xi_2 & -\Gamma_1 \end{bmatrix} \begin{bmatrix} e_6 \\ e_{11} \end{bmatrix} + 2\begin{bmatrix} e_8 \\ e_{12} \end{bmatrix}^T \begin{bmatrix} -\Gamma_2\Xi_1 & \Gamma_2\Xi_2 \\ \Gamma_2\Xi_2 & -\Gamma_2 \end{bmatrix} \begin{bmatrix} e_8 \\ e_{12} \end{bmatrix},$$

$$\begin{aligned}\Theta_7 = {} & 2(e_1 - e_3 - e_{21})^T M_1 (- C(e_6 - e_{14}) + D e_3 - e_{10})^T + 2(e_1 - e_3 - e_{21})^T M_2 e_1 \\ & +2(e_1 - e_3 - e_{21})^T M_3 e_3 + 2(e_1 - e_3 - e_{21})^T M_4 e_{21},\end{aligned}$$

$$\aleph_{11} = \frac{1}{h_m} e_{15}, \quad \aleph_{12} = \frac{1}{h_m}(e_{15} - 2e_{22}), \quad \aleph_{21} = \frac{1}{r_m} e_{16}, \quad \aleph_{22} = \frac{1}{r_m}(e_{16} - 2e_{23}), \quad \aleph_{31} = e_1 - e_2,$$

$$\aleph_{41} = e_1 + e_2 - 2e_{15}, \quad \aleph_{51} = e_1 - e_2 + 6e_{15} - 12e_{22} \quad , \aleph_{32} = e_6 - e_7, \quad \aleph_{42} = e_6 + e_7 - 2e_{16},$$

$$\aleph_{52} = e_6 - e_7 + 6e_{16} - 12e_{23}, \ \aleph_{61} = e_2 - e_3, \quad \aleph_{71} = e_2 + e_3 - 2e_{17}, \ \aleph_{81} = e_2 - e_3 + 6e_{17} - 12e_{24},$$

$$\aleph_{62} = e_3 - e_4, \quad \aleph_{72} = e_3 + e_4 - 2e_{18}, \quad \aleph_{82} = e_3 - e_4 + 6e_{18} - 12e_{25}, \quad \aleph_{91} = e_7 - e_8,$$

$$\aleph_{101} = e_7 + e_8 - 2e_{19}, \quad \aleph_{111} = e_7 - e_8 + 6e_{19} - 12e_{26}, \quad \aleph_{92} = e_8 - e_9, \quad \aleph_{102} = e_8 + e_9 - 2e_{20},$$

$$\aleph_{112} = e_8 - e_9 + 6e_{20} - 12e_{27}, \quad h_{Mm} = h_M - h_m, \quad r_{Mm} = r_M - r_m, \quad \beta_{1h} = \frac{e^{\alpha h_m} - 1}{\alpha},$$

$$\beta_{2h} = \frac{e^{\alpha h_M} - e^{\alpha h_m}}{\alpha}, \quad \beta_{3h} = \frac{e^{\alpha h_m} - \alpha h_m - 1}{\alpha^2} \quad \beta_{4h} = \frac{e^{\alpha h_M} - e^{\alpha h_m} + \alpha(h_m - h_M)}{\alpha^2}, \quad \beta_{1r} = \frac{e^{\alpha r_m} - 1}{\alpha},$$

$$\beta_{2r} = \frac{e^{\alpha r_M} - e^{\alpha r_m}}{\alpha}, \quad \beta_{3r} = \frac{e^{\alpha r_M} - 1}{\alpha}, \quad \beta_{4r} = \frac{e^{\alpha r_m} - \alpha r_m - 1}{\alpha^2}, \quad \beta_{5r} = \frac{e^{\alpha r_M} - e^{\alpha r_m} + \alpha(r_m - r_M)}{\alpha^2},$$

$$\beta_{1\rho} = \frac{e^{\alpha \rho_a} - 1}{\alpha}, \quad ,\beta_{2\rho} = \frac{e^{\alpha \rho_c} - 1}{\alpha}, \quad e_j = \begin{bmatrix} 0_{n\times(j-1)n} & I_n & 0_{n\times(j-14)n} \end{bmatrix}, j = 1, 2, \ldots, 27.$$

Proof. First, we employ the Newton-Leibniz formula to modify system (2.4), which is as follows

$$\begin{aligned} \dot{x}(t) &= -A\Big(x(t) - \int_{t-\rho_a}^{t} \dot{x}(s)ds\Big) + Wf(y(t - r(t))), \\ \dot{y}(t) &= -C\Big(y(t) - \int_{t-\rho_c}^{t} \dot{y}(s)ds\Big) + Dx(t - h(t)). \end{aligned} \tag{3.3}$$

Second, the LK functional is designed for the system (3.3) :

$$V(t) = \sum_{i=1}^{5} V_i(t), \tag{3.4}$$

where

$$\begin{aligned} V_1(t) &= x^T(t)P_1x(t) + \int_{t-h_m}^{t} e^{\alpha(t-s)} x^T(s)P_2x(s)ds + \int_{t-h_M}^{t-h_m} e^{\alpha(t-s)} x^T(s)P_3x(s)ds, \\ V_2(t) &= y^T(t)Q_1y(t) + \int_{t-r_m}^{t} e^{\alpha(t-s)} y^T(s)Q_2y(s)ds + \int_{t-r_M}^{t-r_m} e^{\alpha(t-s)} y^T(s)Q_3y(s)ds, \end{aligned}$$

$$+\int_{t-r(t)}^{t} e^{\alpha(t-s)} f^T(y(s))Q_4 f(y(s))ds$$

$$\begin{aligned} V_3(t) &= h_m \int_{-h_m}^{0}\int_{t+\theta}^{t} e^{\alpha(t-s)} x^T(s)R_1x(s)dsd\theta + r_m \int_{-r_m}^{0}\int_{t+\theta}^{t} e^{\alpha(t-s)} y^T(s)R_2y(s)dsd\theta, \\ V_4(t) &= h_m \int_{-h_m}^{0}\int_{t+\theta}^{t} e^{\alpha(t-s)} \dot{x}^T(s)S_1\dot{x}(s)dsd\theta + h_{Mm}\int_{-h_M}^{-h_m}\int_{t+\theta}^{t} e^{\alpha(t-s)} \dot{x}^T(s)S_2\dot{x}(s)dsd\theta \\ &\quad + r_m \int_{-r_m}^{0}\int_{t+\theta}^{t} e^{\alpha(t-s)} \dot{y}^T(s)S_3\dot{y}(s)dsd\theta + r_{Mm}\int_{-r_M}^{-r_m}\int_{t+\theta}^{t} e^{\alpha(t-s)} \dot{y}^T(s)S_4\dot{y}(s)dsd\theta, \\ V_5(t) &= \rho_a \int_{-\rho_a}^{0}\int_{t+\theta}^{t} e^{\alpha(t-s)} \dot{x}^T(s)U_1\dot{x}(s)dsd\theta + \rho_c \int_{-\rho_c}^{0}\int_{t+\theta}^{t} e^{\alpha(t-s)} \dot{y}^T(s)U_2\dot{y}(s)dsd\theta. \end{aligned}$$

Then, taking $\dot{V}_i(t)$ along the trajectory of system (3.3) with (2.2) and (2.3), we have

$$\begin{aligned} \dot{V}_1(t) &= 2x^T(t)P_1\dot{x}(t) + x^T(t)P_2x(t) - e^{\alpha h_m}x^T(t-h_m)P_2x(t-h_m) + e^{\alpha h_m}x(t-h_m)P_3x(t-h_m) \\ &\quad -(1-\dot{h}(t))e^{\alpha h(t)}x^T(t-h(t))P_3x(t-h(t)) + (1-\dot{h}(t))e^{\alpha h(t)}x^T(t-h(t))P_3x(t-h(t)) \\ &\quad -e^{\alpha h_M}x^T(t-h_M)P_3x(t-h_M) - \alpha x^T(t)P_1x(t) + \alpha V_1(t). \end{aligned}$$

Utilizing the zero equation $-\dot{x}(t) - A(x(t) - \int_{t-\rho_a}^{t} \dot{x}(s)ds) + Wf(y(t-r(t))) = 0$, we estimate the boundary $h_{dm} \le \dot{h}(t) \le h_{dM}$, and the exponential function term $1 = e^0 \le e^{\alpha h_m} \le e^{\alpha h(t)} \le e^{\alpha h_M}$, $-1 = -e^0 \ge -e^{\alpha h_m} \ge -e^{\alpha h(t)} \le -e^{\alpha h_M}$. From this, we derive the values of $\dot{V}_1(t)$ as follows.

$$\begin{aligned} \dot{V}_1(t) &\le 2\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \begin{bmatrix} P_1 & E_1^T \\ 0 & E_2^T \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ -\dot{x}(t) - A(x(t) - \int_{t-\rho_a}^{t} \dot{x}(s)ds) + Wf(y(t-r(t))) \end{bmatrix} + x^T(t)P_2x(t) \\ &\quad -e^{\alpha h_m}x^T(t-h_m)P_2x(t-h_m) + e^{\alpha h_m}x^T(t-h_m)P_3x(t-h_m) + (h_{dM}e^{\alpha h_M} - e^{\alpha h_m}) \\ &\quad \times x^T(t-h(t))P_3x(t-h(t)) + (e^{\alpha h_M} - h_{dm}e^{\alpha h_m})x^T(t-h(t))P_3x(t-h(t)) \\ &\quad -e^{\alpha h_M}x^T(t-h_M)P_3x(t-h_M) - \alpha x^T(t)P_1x(t) + \alpha V_1(t) \\ &\le \chi^T(t)\Theta_1\chi(t) + \alpha V_1(t). \end{aligned}$$

Using the same method, the zero equation as $-\dot{y}(t) - C(y(t) - \int_{t-\rho_c}^{t} \dot{y}(s)ds) + Dx(t-h(t))) = 0$ allows us to obtain the derivative of $V_2(t)$, which is

$$\begin{aligned} \dot{V}_2(t) &\le 2\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}^T \begin{bmatrix} Q_1 & E_3^T \\ 0 & E_4^T \end{bmatrix} \begin{bmatrix} \dot{y}(t) \\ -\dot{y}(t) - C(y(t) - \int_{t-\rho_c}^{t} \dot{y}(s)ds) + Dx(t-h(t))) \end{bmatrix}^T + y^T(t)Q_2y(t) \\ &\quad -e^{\alpha r_m}y^T(t-r_m)Q_2y(t-r_m) + e^{\alpha r_m}y^T(t-r_m)Q_3y(t-r_m) + (r_{dM}e^{\alpha r_M} - e^{\alpha r_m}) \\ &\quad \times y(t-r(t))Q_3y(t-r(t)) + (e^{\alpha r_M} - r_{dm}e^{\alpha r_m})y(t-r(t))Q_3y(t-r(t))) \\ &\quad -e^{\alpha r_M}y^T(t-r_M)Q_3y(t-r_M) + f^T(y(t))Q_4f(y(t)) + (r_{dM}e^{\alpha r_M} - e^{\alpha r_m})f^T(y(t-r(t))) \\ &\quad \times Q_4f(y(t-r(t))) - \alpha y^T(t)Q_1y(t) + \alpha V_2(t) \\ &\le \chi^T(t)\Theta_2\chi(t) + \alpha V_2(t). \end{aligned}$$

The derivative of $V_3(t)$ is calculated to obtain

$$\dot{V}_3(t) \le x^T(t)(h_m^2R_1)x(t) + y^T(t)(r_m^2R_2)y(t) - h_m\int_{t-h_m}^{t} x^T(s)Q_1x(s)ds$$

$$-r_m \int_{t-r_m}^{t} y^T(s)Q_3y(s)ds + \alpha V_3(t).$$

We utilize Lemma 2.5 to estimate the amount, leading to the subsequent inequality.

$$\begin{aligned}\dot{V}_3(t) \quad \leq \quad & \chi^T(t)\Big(e_1^T(h_m^2R_1)e_1^T + e_6^T(r_m^2R_2)e_6 - \aleph_{11}^T(h_m^2R_1)\aleph_{11} - 3\aleph_{12}^T(h_m^2R_1)\aleph_{12} \\ & -\aleph_{21}^TQ(r_m^2R_2)\aleph_{21} - 3\aleph_{22}^T(r_m^2R_2)\aleph_{22}\Big)\chi(t) + \alpha V_3(t) = \chi^T(t)\Theta_3\chi(t) + \alpha V_3(t). \qquad (3.5)\end{aligned}$$

The derivative of $V_4(t)$ is calculated to obtain

$$\begin{aligned}\dot{V}_4(t) \quad = \quad & \dot{x}^T(t)(h_m^2S_1 + h_{Mm}^2e^{\alpha h_m}S_2)\dot{x}(t) + \dot{y}^T(t)(r_m^2S_3 + r_{Mm}^2e^{\alpha r_m}S_4)\dot{y}(t) \\ & -h_m\int_{t-h_m}^{t}\dot{x}^T(s)S_1\dot{x}(s)ds - r_m\int_{t-r_m}^{t}\dot{y}^T(s)S_3\dot{y}(s)ds \\ & -h_{Mm}e^{\alpha h_m}\int_{t-h_M}^{t-h_m}\dot{x}^T(s)S_2\dot{x}(s)ds - r_{Mm}e^{\alpha r_m}\int_{t-r_M}^{t-r_m}\dot{y}^T(s)S_4\dot{y}(s)ds + \alpha V_3(t).\end{aligned}$$

To facilitate our analysis, we decompose and examine the inequality in the derivative of $V_4(t)$ step by step as follows. In the first part, we use Lemma 2.5 to approximate the value, resulting in the following inequality.

$$\begin{aligned}& -h_m\int_{t-h_m}^{t}\dot{x}^T(s)S_1\dot{x}(s)ds - r_m\int_{t-r_m}^{t}\dot{y}^T(s)S_3\dot{y}(s)ds \\ & \leq \chi^T(t)\Big(-\aleph_{31}^TS_1\aleph_{31} - 3\aleph_{41}^TS_1\aleph_{41} - 5\aleph_{51}^TS_1\aleph_{51} - \aleph_{32}^TS_3\aleph_{32} - 3\aleph_{42}^TS_3\aleph_{42} - 5\aleph_{52}^TS_3\aleph_{52}\Big)\chi(t). \quad (3.6)\end{aligned}$$

Then, we estimate the values of the second part of the inequality as follows.

$$\begin{aligned}& -h_{Mm}e^{\alpha h_m}\int_{t-h_M}^{t-h_m}\dot{x}^T(s)S_2\dot{x}(s)ds - r_{Mm}e^{\alpha r_m}\int_{t-r_M}^{t-r_m}\dot{y}^T(s)S_4\dot{y}(s)ds \\ = \quad & -\frac{h_{Mm}e^{\alpha h_m}}{(h(t)-h_m)}(h(t)-h_m)\int_{t-h(t)}^{t-h_m}\dot{x}^T(s)S_2\dot{x}(s)ds - \frac{h_{Mm}e^{\alpha h_m}}{(h_M-h(t))}(h_M-h(t))\int_{t-h_M}^{t-h(t)}\dot{x}^T(s)S_2\dot{x}(s)ds \\ & -\frac{r_{Mm}e^{\alpha r_m}}{(r(t)-r_m)}(r(t)-r_m)\int_{t-r(t)}^{t-r_m}\dot{y}^T(s)S_4\dot{y}(s)ds - \frac{r_{Mm}e^{\alpha r_m}}{(r_M-r(t))}(r_M-r(t))\int_{t-r_M}^{t-r(t)}\dot{y}^T(s)S_4\dot{y}(s)ds.\end{aligned}$$

By utilizing Lemma 2.5 together with Lemma 2.6, we can approximate the inequality as follows

$$\begin{aligned}& -h_{Mm}e^{\alpha h_m}\int_{t-h_M}^{t-h_m}\dot{x}^T(s)Q_2\dot{x}(s)ds - r_{Mm}e^{\alpha r_m}\int_{t-r_M}^{t-r_m}\dot{y}^T(s)Q_4\dot{y}(s)ds \\ & \leq \chi^T(t)\Big(e^{\alpha h_m}(-\aleph_{61}^TQ\aleph_{61} - 3\aleph_{71}^TQ\aleph_{71} - 5\aleph_{81}^TQ\aleph_{81} - \aleph_{62}^TQ\aleph_{62} - 3\aleph_{72}^TQ\aleph_{72} - 5\aleph_{82}^TQ\aleph_{82}) \\ & \quad +e^{\alpha r_m}(-\aleph_{91}^TQ\aleph_{91} - 3\aleph_{101}^TQ\aleph_{101} - 5\aleph_{111}^TQ\aleph_{111} - \aleph_{92}^TQ\aleph_{92} - 3\aleph_{102}^TQ\aleph_{102} - 5\aleph_{112}^TQ\aleph_{112}\Big) \\ & \quad +2e^{\alpha h_m}\Big(-\aleph_{61}^TX_{11}\aleph_{62} - 3\aleph_{71}^TX_{12}\aleph_{72} - 5\aleph_{81}^TX_{13}\aleph_{82}) \\ & \quad +2e^{\alpha r_m}(-\aleph_{91}^TX_{21}\aleph_{92} - 3\aleph_{101}^TX_{22}\aleph_{102} - 5\aleph_{111}^TX_{23}\aleph_{112})\Big)\chi(t). \qquad (3.7)\end{aligned}$$

Estimating the derivatives of $V_4(t)$ along with approximating the values, in inequalities (3.6) and (3.7) is a crucial part of this analysis. We obtained consistent equations in this process as follows:

$$\dot{V}_4(t) \leq \chi^T(t)\Theta_4\chi(t) + \alpha V_4(t). \tag{3.8}$$

Regarding $V_5(t)$, we derived its derivative and applied Lemma 2.4 for estimation. Additionally, when approximating the exponential function where $1 = e^0 \leq e^{\alpha\rho_a}$, $1 = e^0 \leq e^{\alpha\rho_c}$, $-1 = -e^0 \geq -e^{\alpha\rho_a}$, $-1 = -e^0 \geq -e^{\alpha\rho_c}$, we obtain the following results

$$\begin{aligned}\dot{V}_5(t) \;\; &\leq \;\; \rho_a^2\dot{x}^T(t)U_1\dot{x}(t) + \rho_c^2\dot{y}^T(s)U_2\dot{y}(t) - \int_{t-\rho_a}^{t}\dot{x}^T(s)dsU_1\int_{t-\rho_a}^{t}\dot{x}(s)ds \\ &\quad - \int_{t-\rho_c}^{t}\dot{y}^T(s)dsU_2\int_{t-\rho_c}^{t}\dot{y}(s)ds + \alpha V_5(t) = \chi^T(t)\Theta_5\chi(t) + \alpha V_5(t).\end{aligned} \tag{3.9}$$

Next, by adopting Eqs (2.6) and (2.7) and let $\Gamma_i > 0,\ i = 1,2$ be diagonal matrices, we obtain

$$\begin{aligned}&\begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix}^T \begin{bmatrix} -\Gamma_1\Xi_1 & \Gamma_1\Xi_2 \\ \Gamma_1\Xi_2 & -\Gamma_1 \end{bmatrix} \begin{bmatrix} y(t) \\ f(y(t)) \end{bmatrix} + \begin{bmatrix} y(t-r(t)) \\ f(y(t-r(t))) \end{bmatrix}^T \begin{bmatrix} -\Gamma_2\Xi_1 & \Gamma_2\Xi_2 \\ \Gamma_2\Xi_2 & -\Gamma_2 \end{bmatrix} \begin{bmatrix} y(t-r(t)) \\ f(y(t-r(t))) \end{bmatrix} \\ &= \chi^T(t)\Theta_6\chi(t) \;\; > \;\; 0.\end{aligned}$$

Additionally, we utilize the zero equation derived from the Newton-Leibniz equation: $x(t) -x(t-h(t)) - \int_{t-h(t)}^{t}\dot{x}(s)ds = 0$. Let $N_i, i = 1,2,3,4$ be appropriate dimensional matrices, resulting in the equation:

$$\begin{aligned}&\chi^T(t)\Big(2(e_1 - e_3 - e_{21})^T M_1(-C(e_6 - e_{14}) + De_3 - e_{10})^T + 2(e_1 - e_3 - e_{21})^T M_2 e_1 \\ &+2(e_1 - e_3 - e_{21})^T M_3 e_3 + 2(e_1 - e_3 - e_{21})^T M_4 e_{21}\Big)\chi(t) = \chi^T(t)\Theta_7\chi(t) = 0.\end{aligned} \tag{3.10}$$

It becomes evident that, upon estimating $\dot{V}(t)$ after recalling (3.1), we receive

$$\dot{V}(t) - \alpha V(t) \leq 0, \tag{3.11}$$

where $\dot{V}(t) \leq \chi^T(t)\Theta\chi(t)+\alpha V(t)$,
$\chi^T(t) = \Big[x^T(t),\ x^T(t-h_m),\ x^T(t-h(t)),\ x^T(t-h_M),\ \dot{x}^T(t),\ y^T(t),\ y^T(t-r_m),\ y^T(t-r(t)),\ y^T(t-r_M),$
$\dot{y}^T(t),\ f^T(y(t)),\ f^T(y(t-r(t))),\ \int_{t-\rho_a}^{t}\dot{x}^T(s)ds,\ \int_{t-\rho_c}^{t}\dot{y}^T(s)ds,\ \frac{1}{h_m}\int_{t-h_m}^{t}x^T(s)ds,\ \frac{1}{r_m}\int_{t-r_m}^{t}y^T(s)ds,$
$\frac{1}{h(t)-h_m}\int_{t-h(t)}^{t-h_m}x^T(s)ds,\ \frac{1}{h_M-h(t)}\int_{t-h_M}^{t-h(t)}x^T(s)ds,\ \frac{1}{r(t)-r_m}\int_{t-r(t)}^{t-r_m}y^T(s)ds,\ \frac{1}{r_M-r(t)}\int_{t-r_M}^{t-r(t)}y^T(s)ds,\ \int_{t-h(t)}^{t}\dot{x}^T(s)ds,$
$\frac{1}{h_m^2}\int_{t-h_m}^{t}\int_{\theta}^{t}x^T(s)dsd\theta,\ \frac{1}{r_m^2}\int_{t-r_m}^{t}\int_{\theta}^{t}y^T(s)dsd\theta,\ \frac{1}{(h(t)-h_m)^2}\int_{t-h(t)}^{t-h_m}\int_{\theta}^{t-h_m}x^T(s)dsd\theta,\ \frac{1}{(h_M-h(t))^2}\int_{t-h_M}^{t-h(t)}$
$\times\int_{\theta}^{t-h(t)}x^T(s)dsd\theta,\ \frac{1}{(r(t)-r_m)^2}\int_{t-r(t)}^{t-r_m}\int_{\theta}^{t-r_m}y^T(s)dsd\theta,\ \frac{1}{(r_M-r(t))^2}\int_{t-r_M}^{t-r(t)}\int_{\theta}^{t-r(t)}y^T(s)dsd\theta\Big].$

By multiplying the inequality (3.11) by $e^{-\alpha t}$ and integrating from 0 to t where t belongs to the interval $[0,T]$, we derive the following result:

$$V(t) \;\; \leq \;\; e^{\alpha t}V(0),$$

where

$$V(\phi(0),\xi(0)) \;\; = \;\; \phi^T(0)P_1\phi(0) + \int_{-h_m}^{0}e^{-\alpha s}\phi^T(s)P_2\phi(s)ds + \int_{-h_M}^{-h_m}e^{-\alpha s}\phi^T(s)P_3\phi(s)ds$$

$$
\begin{aligned}
&+\xi^T(0)Q_1\xi(0)+\int_{-r_m}^{0}e^{-\alpha s}\xi^T(s)Q_2\xi(s)ds+\int_{-r_M}^{-r_m}e^{-\alpha s}\xi^T(s)Q_3\xi(s)ds\\
&+\int_{-r(0)}^{0}e^{-\alpha s}f^T(\xi(s))Q_4f(\xi(s))ds+h_m\int_{-h_m}^{0}\int_{\theta}^{0}e^{-\alpha s}\phi^T(s)R_1\phi(s)dsd\theta\\
&+r_m\int_{-r_m}^{0}\int_{\theta}^{0}e^{-\alpha s}\xi^T(s)R_2\xi(s)dsd\theta+h_m\int_{-h_m}^{0}\int_{\theta}^{0}e^{-\alpha s}\dot{\phi}^T(s)S_1\dot{\phi}(s)dsd\theta\\
&+h_{Mm}\int_{-h_M}^{-h_m}\int_{\theta}^{0}e^{-\alpha-s}\dot{\phi}^T(s)S_2\dot{\phi}(s)dsd\theta+r_m\int_{-r_m}^{0}\int_{\theta}^{0}e^{-\alpha s}\dot{\xi}^T(s)S_3\dot{\xi}(s)dsd\theta\\
&+r_{Mm}\int_{-r_M}^{-r_m}\int_{\theta}^{0}e^{-\alpha s}\dot{\xi}^T(s)S_4\dot{\xi}(s)dsd\theta+\rho_a\int_{-\rho_a}^{0}\int_{\theta}^{0}e^{-\alpha s}\dot{\phi}^T(s)U_1\dot{\phi}(s)dsd\theta\\
&+\rho_c\int_{-\rho_c}^{0}\int_{\theta}^{0}e^{-\alpha s}\dot{\xi}^T(s)U_2\dot{\xi}(s)dsd\theta
\end{aligned}
$$

$$
\begin{aligned}
\le\ &\lambda_{\max}(N^{-1}P_1)\phi^T(0)N\phi^T(0)+\int_{-h_m}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}P_2)\phi^T(s)N\phi(s)ds+\int_{-h_M}^{-h_m}e^{-\alpha s}\lambda_{\max}(N^{-1}P_3)\\
&\times\phi^T(s)N\phi(s)ds+\lambda_{\max}(N^{-1}Q_1)\xi^T(0)N\xi(0)+\int_{-r_m}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}Q_2)\xi^T(s)N\xi(s)ds\\
&+\int_{-r_M}^{-r_m}e^{-\alpha s}\lambda_{\max}(N^{-1}Q_3)\xi^T(s)N\xi(s)ds+\int_{-r(0)}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}Q_4)f^T(\xi(s))Nf(\xi(s))ds\\
&+h_m\int_{-h_m}^{0}\int_{\theta}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}R_1)\phi^T(s)N\phi(s)dsd\theta+r_m\int_{-r_m}^{0}\int_{\theta}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}R_2)\xi^T(s)N\xi(s)dsd\theta\\
&+h_m\int_{-h_m}^{0}\int_{\theta}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}S_1)\dot{\phi}^T(s)N\dot{\phi}(s)dsd\theta+h_{Mm}\int_{-h_M}^{-h_m}\int_{\theta}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}S_2)\dot{\phi}^T(s)N\dot{\phi}(s)dsd\theta\\
&+r_m\int_{-r_m}^{0}\int_{\theta}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}S_3)\dot{\xi}^T(s)N\dot{\xi}(s)dsd\theta+r_{Mm}\int_{-r_M}^{-r_m}\int_{\theta}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}S_4)\dot{\xi}^T(s)N\dot{\xi}(s)dsd\theta\\
&+\rho_a\int_{-\rho_a}^{0}\int_{\theta}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}U_1)\dot{\phi}^T(s)N\dot{\phi}(s)dsd\theta+\rho_c\int_{-\rho_c}^{0}\int_{\theta}^{0}e^{-\alpha s}\lambda_{\max}(N^{-1}U_2)\dot{\xi}^T(s)N\dot{\xi}(s)dsd\theta\\
\le\ &\lambda_{\max}(N^{-1}P_1)||\phi(t)||_N^2+\beta_{1h}\lambda_{\max}(N^{-1}P_2)||\phi(t)||_N^2+\beta_{2h}\lambda_{\max}(N^{-1}P_3)||\phi(t)||_N^2+\lambda_{\max}(N^{-1}Q_1)||\xi(t)||_N^2\\
&+\beta_{1r}\lambda_{\max}(N^{-1}Q_2)||\xi(s)||_N^2+\beta_{2r}\lambda_{\max}(N^{-1}Q_3)||\xi(t)||_N^2+\beta_{3r}\lambda_{\max}(N^{-1}Q_4)\lambda_{\max}(ð^Tð)||\xi(t)||_N^2\\
&+\beta_{3h}\lambda_{\max}(N^{-1}R_1)||\phi(t)||_N^2+\beta_{4r}\lambda_{\max}(N^{-1}R_2)||\phi(t)||_N^2+\beta_{3h}\lambda_{\max}(N^{-1}S_1)||\xi(t)||_N^2+\beta_{4r}\lambda_{\max}(N^{-1}S_2)||\xi(t)||_N^2\\
&+\beta_{4h}\lambda_{\max}(N^{-1}S_3)||\phi||_N^2+\beta_{5r}\lambda_{\max}(N^{-1}S_4)||\xi||_N^2+\beta_{1\rho}\lambda_{\max}(N^{-1}U_1)||\phi(t)||_N^2+\beta_{2\rho}\lambda_{\max}(N^{-1}U_2)||\xi(s)||_N^2\\
\le\ &\lambda_1(||\Phi(t)||_N^2+||\Psi(t)||_N^2), \qquad (3.12)
\end{aligned}
$$

where $||\Phi(t)||_N=\sup_{-\max\{\rho_1,h_M\}\le t\le 0}\{||\phi(t)||_N,||\dot{\phi}(t)||_N\}$ and $||\Psi(t)||_N=\sup_{-\max\{\rho_2,r_M\}\le t\le 0}\{||\xi(t)||_N,||\dot{\xi}(t)||_N\}$, $\lambda_1=\lambda_{\max}\Big((N^{-1}P_1)+\beta_{1h}(N^{-1}P_2)+\beta_{2h}(N^{-1}P_3)+(N^{-1}Q_1)+\beta_{1r}(N^{-1}Q_2)+\beta_{2r}(N^{-1}Q_3)+\beta_{3r}(N^{-1}Q_4)(ð^Tð)$ $+\beta_{3h}(N^{-1}R_1)+\beta_{4r}(N^{-1}R_2)+\beta_{3h}(N^{-1}S_1)+\beta_{4r}(N^{-1}S_2)+\beta_{4h}(N^{-1}S_3)+\beta_{5r}(N^{-1}S_4)+\beta_{1\rho}(N^{-1}U_1)$ $+\beta_{2\rho}(N^{-1}U_2)\Big)$, and $N>0$.

In the interim,

$$
V(t)\ge\min\{\lambda_{\min}(P_1N^{-1})||x(t)||_N^2+\lambda_{\min}(Q_1N^{-1})||y(t)||_N^2\}\ge\lambda_2(||x(t)||_N^2+||y(t)||_N^2), \qquad (3.13)
$$

where $\lambda_2 = \lambda_{\min}(N^{-1}P_1 + N^{-1}Q_1)$. Let c_1 be a positive real number where

$$\|\Phi(t)\|_N^2 + \|\Psi(t)\|_N^2 \leq c_1.$$

Combining inequalities (3.11)–(3.13) we acquire:

$$\begin{aligned}(\|x(t)\|_N^2 + \|y(t)\|_N^2) &\leq \frac{1}{\lambda_2} e^{\alpha T} V(\phi(0), \xi(0)) \\ &\leq \frac{1}{\lambda_2} e^{\alpha T} \lambda_1 c_1 \\ &\leq \frac{\lambda_1}{\lambda_2} e^{\alpha T} c_1 \leq c_2,\end{aligned}$$

where c_2 be a positive real number. In accordance with Definition 2.1 of stability, finite-time stability with respect to c_1, c_2, T and N can be found for the GRNs (2.4). The proof is complete. □

When the interval time-varying delays $h(t)$ and $r(t)$ adhere to the conditions (2.2) and (2.3), and $\rho_a = \rho_c = 0$, and $V_5(t) = 0$, the subsequent Corollary can be applied to assess the FTS of GRNs in the following system form:

$$\begin{aligned}\dot{x}(t) &= -Ax(t) + Wf(y(t - r(t))) \\ \dot{y}(t) &= -Cy(t) + Dx(t - h(t)), \\ x(t) &= \phi(t), \qquad y(t) = \xi(t), \quad t \in (-\widetilde{\tau}, 0), \quad \widetilde{\tau} = \max\{h_M, r_M\},\end{aligned} \tag{3.14}$$

as describe in Corollary 3.2.

Corollary 3.2. *Given that* **Assumption 1** *valid. For positive scalars* c_1, c_2, T, h_m, h_M, h_{dm}, h_{dM}, r_m, r_M, r_{dm} *and* r_{dM} *according conditions* (2.2) *and* (2.3), *if there existmatrices* $P_i > 0, i = 1, 2, 3$, $Q_i > 0, i = 1, 2, 3, 4$, $R_i > 0$, $i = 1, 2$, $S_i > 0$, $i = 1, 2, 3, 4$, *any diagonal matrices* $\Gamma_i > 0, i = 1, 2$, *any appropriate dimensional matrices* $E_i, N_i, i = 1, 2, 3, 4$, *satisfying the following conditions:*

$$\begin{aligned}\begin{bmatrix} S_2 & X_{2i} \\ * & S_2 \end{bmatrix} &\geq 0, \quad n = 1, 2, i = 1, 2, 3, \\ \begin{bmatrix} S_4 & X_{2i} \\ * & S_4 \end{bmatrix} &\geq 0, \quad n = 1, 2, i = 1, 2, 3, \\ \widetilde{\Theta} &< 0, \\ \frac{\widetilde{\lambda}_1}{\lambda_2} e^{\alpha T} c_1 &\leq c_2.\end{aligned}$$

Then, the system (3.14) *exhibits FTS concerning* $N > 0$ *and positive real numbers* (c_1, c_2, T), *where* $\widetilde{\Theta} = \sum_{i=1}^{6} \Theta_i$ *is defined:*

$$\widetilde{\Theta}_1 = \Theta_1 - (e_1^T E_1^T + e_5^T E_2^T)(Ae_{13}), \quad \widetilde{\Theta}_2 = -(e_6^T E_3^T + e_{10}^T E_4^T)(Ce_{14}),$$
$$\widetilde{\Theta}_3 = \Theta_3, \quad \widetilde{\Theta}_4 = \Theta_4, \quad \widetilde{\Theta}_5 = \Theta_6, \quad \widetilde{\Theta}_6 = \Theta_7, x \quad \widetilde{\lambda}_1 = \lambda_1 - (\beta_{1\rho}\lambda_{\max}(N^{-1}U_1) + \beta_{2\rho}\lambda_{\max}(N^{-1}U_2)).$$

Proof. The Corollary 3.2 can be derived using a similar reasoning as that employed in the proof of Theorem 3.1.

4. Numerical examples

In this section, we employ a numerical instance to showcase the efficiency of the criteria outlined as follows:

Example 4.1. Consider the GRNs (2.4) and (3.14) with the following parameters:

$$A = diag(2,2,2), \quad C = diag(3,3,3), \quad D = diag(1,1,1), \quad \text{and } W = 1.5 \times \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

The gene regulation function is represented by $f_i(y_i) = \frac{y_i^2}{1+y_i^2}$, $\eth^- = diag(0,0,0)$, and $\eth^+ = diag(0.65, 0.65, 0.65)$. The time delays $h(t)$ and $r(t)$ are assumed as follows: $r(t) = 0.5 + 0.7 sin^2 t$, $h(t) = 1 + cos^2 t$. We can derive the parameters as follows: $r_m = 0.7$, $r_M = 1.2$, $r_{dm} = -0.7$, $r_{dM} = 0.7$, $h_m = 1$, $h_M = 2$ $h_{dm} = -1$ $h_{dM} = 1$. Furthermore, we demonstrate that the system exhibits FTS by defining the parameters as follows: $\alpha = 0.01$, $T = 5$, $c_1 = 0.4$, and $c_2 = 6.0$. It is crucial to emphasize that the Theorem 3.1 becomes feasible by utilizing MATLAB to solve LMIs (3.1) and (3.2), which enables us to attain a viable solution. As a result, the GRNs (2.4) exhibit FTS, with $\rho_a = \rho_c = 0.1$ representing the allowable value. From this, we can derive the feasible solutions as follows:

$$P_1 = \begin{bmatrix} 0.8300 & 0.0002 & 0.0002 \\ 0.0002 & 0.8300 & 0.0002 \\ 0.0002 & 0.0002 & 0.8300 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.2037 & 0.0026 & 0.0026 \\ 0.0026 & 1.2037 & 0.0026 \\ 0.0026 & 0.0026 & 1.2037 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.0067 & -0.0000 & -0.0000 \\ -0.0000 & 0.0067 & -0.0000 \\ -0.0000 & -0.0000 & 0.0067 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.5386 & 0.0003 & 0.0003 \\ 0.0003 & 0.5386 & 0.0003 \\ 0.0003 & 0.0003 & 0.5386 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.1992 & 0.0008 & 0.0008 \\ 0.0008 & 0.1992 & 0.0008 \\ 0.0008 & 0.0008 & 0.1992 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.0246 & -0.0000 & -0.0000 \\ -0.0000 & 0.0246 & -0.0000 \\ -0.0000 & -0.0000 & 0.0246 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 3.0795 & -0.0004 & -0.0004 \\ -0.0004 & 3.0795 & -0.0004 \\ -0.0004 & -0.0004 & 3.0795 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 0.3466 & 0.0004 & 0.0004 \\ 0.0004 & 0.3466 & 0.0004 \\ 0.0004 & 0.0004 & 0.3466 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.2473 & 0.0005 & 0.0005 \\ 0.0005 & 0.2473 & 0.0005 \\ 0.0005 & 0.0005 & 0.2473 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.0297 & -0.0000 & -0.0000 \\ -0.0000 & 0.0297 & -0.0000 \\ -0.0000 & -0.0000 & 0.0297 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0.2547 & 0.0001 & 0.0001 \\ 0.0001 & 0.2547 & 0.0001 \\ 0.0001 & 0.0001 & 0.2547 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0.0446 & -0.0000 & -0.0000 \\ -0.0000 & 0.0446 & -0.0000 \\ -0.0000 & -0.0000 & 0.0446 \end{bmatrix},$$

$$S_4 = \begin{bmatrix} 0.1249 & 0.0001 & 0.0001 \\ 0.0001 & 0.1249 & 0.0001 \\ 0.0001 & 0.0001 & 0.1249 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 10.8856 & -0.0428 & -0.0428 \\ -0.0428 & 10.8856 & -0.0428 \\ -0.0428 & -0.0428 & 10.8856 \end{bmatrix},$$

$$U_2 = \begin{bmatrix} 10.8619 & 0.0117 & 0.0117 \\ 0.0117 & 10.8619 & 0.0117 \\ 0.0117 & 0.0117 & 10.8619 \end{bmatrix}, \quad N = \begin{bmatrix} 16.0427 & 0 & 0 \\ 0 & 16.0427 & 0 \\ 0 & 0 & 16.0427 \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} 6.4410 & 0 & 0 \\ 0 & 6.4410 & 0 \\ 0 & 0 & 6.4410 \end{bmatrix}, \qquad \Gamma_2 = \begin{bmatrix} 0.3886 & 0 & 0 \\ 0 & 0.3886 & 0 \\ 0 & 0 & 0.3886 \end{bmatrix},$$

$$X_1 = \begin{bmatrix} -0.0090 & 0.0001 & 0.0001 \\ 0.0001 & -0.0090 & 0.0001 \\ 0.0001 & 0.0001 & -0.0090 \end{bmatrix}, \qquad X_2 = \begin{bmatrix} -0.0127 & -0.0002 & -0.0002 \\ -0.0002 & -0.0127 & -0.0002 \\ -0.0002 & -0.0002 & -0.0127 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 0.0098 & 0.0000 & 0.0000 \\ 0.0000 & 0.0098 & 0.0000 \\ 0.0000 & 0.0000 & 0.0098 \end{bmatrix}, \qquad X_4 = \begin{bmatrix} -0.0283 & 0.0003 & 0.0003 \\ 0.0003 & -0.0283 & 0.0003 \\ 0.0003 & 0.0003 & -0.0283 \end{bmatrix},$$

$$X_5 = \begin{bmatrix} -0.0017 & -0.0000 & -0.0000 \\ -0.0000 & -0.0017 & -0.0000 \\ -0.0000 & -0.0000 & -0.0017 \end{bmatrix}, \qquad X_6 = \begin{bmatrix} 0.0024 & 0.0000 & 0.0000 \\ 0.0000 & 0.0024 & 0.0000 \\ 0.0000 & 0.0000 & 0.0024 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} -0.0159 & -0.0001 & -0.0001 \\ -0.0001 & -0.0159 & -0.0001 \\ -0.0001 & -0.0001 & -0.0159 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0.4132 & -0.0006 & -0.0006 \\ -0.0006 & 0.4132 & -0.0006 \\ -0.0006 & -0.0006 & 0.4132 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} -0.0070 & 0.0001 & 0.0001 \\ 0.0001 & -0.0070 & 0.0001 \\ 0.0001 & 0.0001 & -0.0070 \end{bmatrix}, \qquad E_4 = \begin{bmatrix} 0.1779 & 0.0002 & 0.0002 \\ 0.0002 & 0.1779 & 0.0002 \\ 0.0002 & 0.0002 & 0.1779 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} -0.0590 & -0.0009 & -0.0009 \\ -0.0009 & -0.0590 & -0.0009 \\ -0.0009 & -0.0009 & -0.0590 \end{bmatrix}, \qquad N_2 = \begin{bmatrix} -2.6266 & -0.0009 & -0.0009 \\ -0.0009 & -2.6266 & -0.0009 \\ -0.0009 & -0.0009 & -2.6266 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 2.6944 & 0.0007 & 0.0007 \\ 0.0007 & 2.6944 & 0.0007 \\ 0.0007 & 0.0007 & 2.6944 \end{bmatrix}, \qquad N_4 = \begin{bmatrix} 2.6951 & -0.0002 & -0.0002 \\ -0.0002 & 2.6951 & -0.0002 \\ -0.0002 & -0.0002 & 2.6951 \end{bmatrix}.$$

Moreover, we found that system (2.4), under the specified conditions and Theorem 3.1, remains FTS up to $T = 9.9225$ ($\alpha = 0.1$).

The simulation results are illustrated in Figures 1–6, depicting the trajectories of variables $x(t)$ and $y(t)$ for the GRNs (3.14) and (2.4). These figures showcase the impact of leakage delays on system stability. The initial conditions are set as $x(0) = (0.2, 0.4, 0.6)$ and $y(0) = (0.1, 0.3, 0.5)$.

Figure 1 presents the solutions for GRN (3.14) in the absence of leakage delays. We observe that the solution for $x(t)$ approaches 0 as t approaches 4, and the solution for y(t) approaches 0 as t approaches 5. In this case, the system converges to 0.

Figures 2–4 present the solutions for GRN (2.4), incorporating leakage delays. These figures clearly illustrate the interference caused by the system's leakage delays, impacting the convergence to the equilibrium point. We found that an increase in the leakage parameter has led to a corresponding rise in the frequency of the oscillations.

Figure 5 demonstrates the solutions of the GRNs (3.14) over the time interval $t \in [0, 10]$. The solution for $x(t)$ approaches 0 as t approaches 4, and the solution for $y(t)$ approaches 0 as t approaches 5. In this case, the system converges to 0. While similar to Figure 1, the solution's behavior is observed over a shorter time interval in this instance.

Figure 6 shows the solutions of GRN (2.4) over the time interval $t \in [0, 15]$. Notably, a disturbance in the solution is observed around $t = 3$, where both $x(t)$ and $y(t)$ begin to oscillate. Despite this disturbance, the system remains stable. This scenario shows a similar solution trajectory to Figure 3 but over a shorter time span, which reduces the frequency of oscillations.

This comprehensive analysis of the simulation results provides valuable insights into the influence of leakage delays on the dynamics of GRNs (2.4) and (3.14). The analysis demonstrates the importance of considering such delays in the design and analysis of these systems. The leakage delay has a significant effect on the dynamic behaviors of the model, often leading to instability. It is therefore crucial to study stability while taking into account the impact of leakage delays.

Additionally, we examine the leakage delays effects by substituting the following values: In Case 1, we investigate the value of h_M at $r_m = 0.7$, $r_M = 1.2$, $r_{dm} = -0.7$, $r_{dM} = 0.7$, $h_m = 1$, $h_{dm} = -1$, $h_{dM} = 1$, $\alpha = 0.01$, $T = 5$, and varying leakage delays (ρ_a and ρ_c) as shown in Table 1. In Case 2, we will explore the value of r_M at $r_m = 0.7, r_{dm} = -0.7, r_{dM} = 0.7, h_m = 1, h_M = 2, h_{dm} = -1$, $h_{dM} = 1$, $\alpha = 0.01$, $T = 5$, and varying leakage delays as indicated in Table 2.

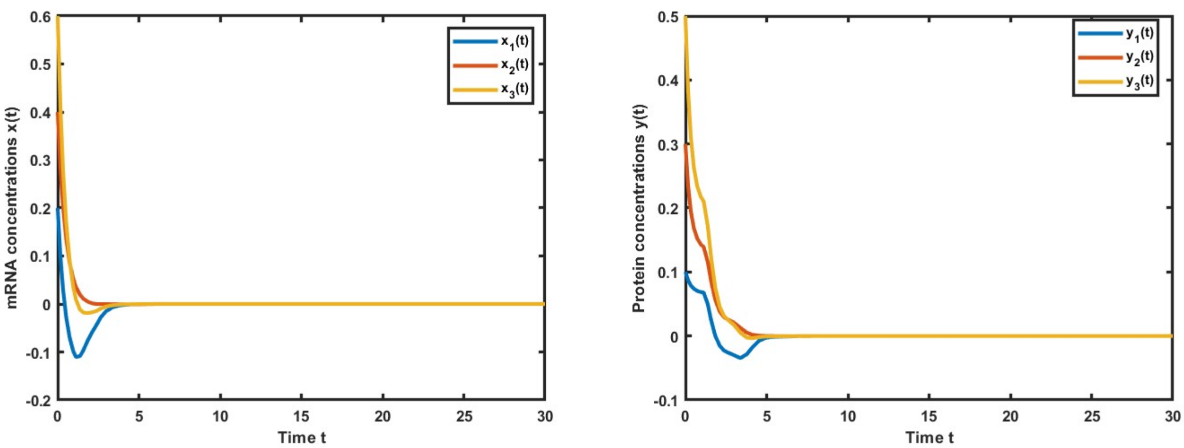
We observed that as the values of leakage delays ρ_a and ρ_c increases, the times delays h_M and r_M both decrease. Hence, it can be stated that the effects of leakage delays lead to a decrease in the FTS boundary.

Table 1. The upper bound for h_M with different values of ρ_a and ρ_c.

$\rho_a = \rho_c$	0.00	0.01	0.05	0.10	0.12
Corollary 3.2	2.8741	-	-	-	-
Theorem 3.1	2.8734	2.8067	2.5490	2.2491	2.1371

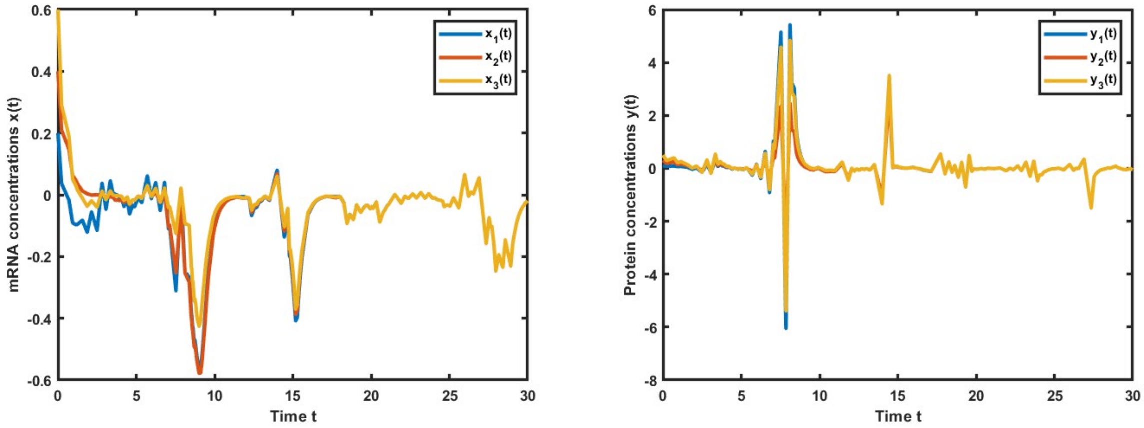
Table 2. The upper bound for r_M with different values of ρ_a and ρ_c.

$\rho_a = \rho_c$	0.00	0.01	0.05	0.10	0.12
Corollary 3.2	6.6890	-	-	-	-
Theorem 3.1	6.6890	6.6243	6.3692	6.0493	5.9209



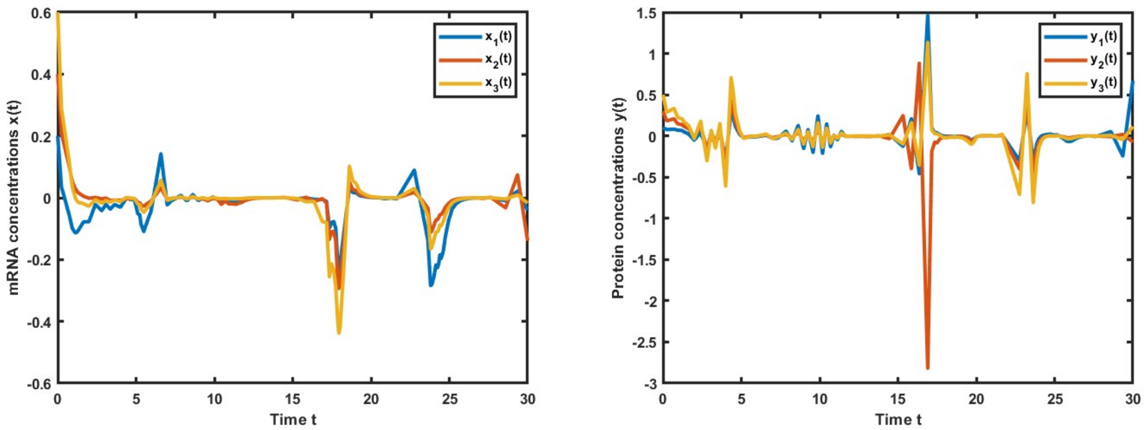
(i) (ii)

Figure 1. (i) mRNA concentrations $x(t)$. (ii) Protein concentrations $y(t)$.



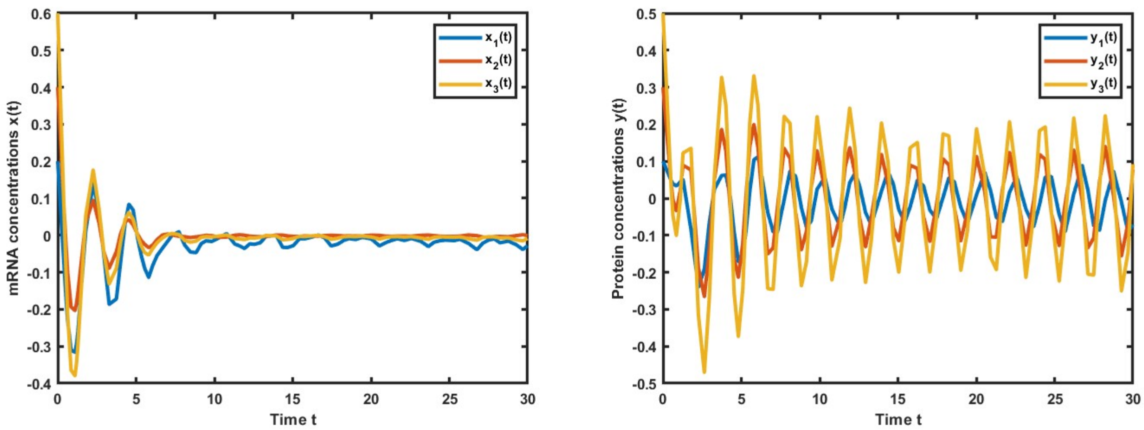
(iii) (iv)

Figure 2. (iii) mRNA concentrations $x(t)$ with $\rho_a = 0.05$. (iv) Protein concentrations $y(t)$ with $\rho_c = 0.05$.



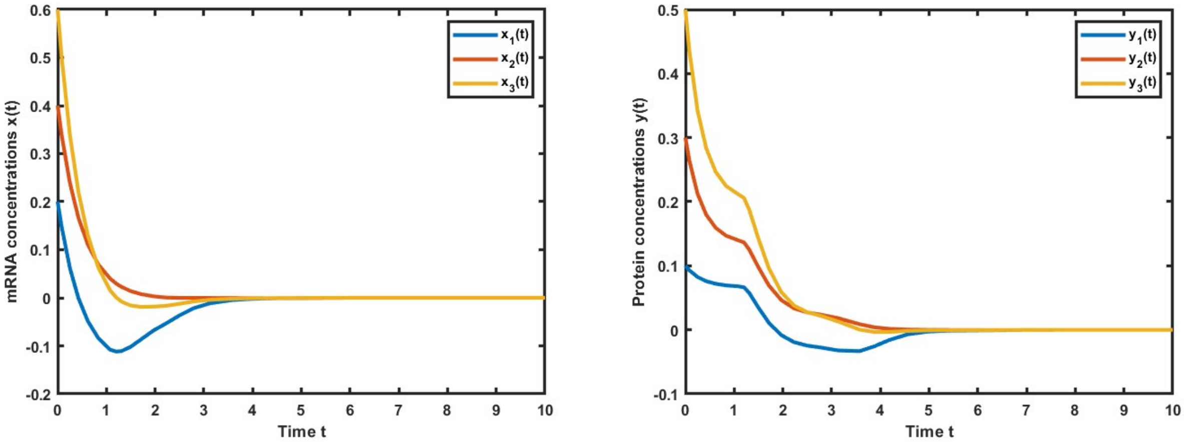
(v) (vi)

Figure 3. (v) mRNA concentrations $x(t)$ with $\rho_a = 0.1$. (vi) Protein concentrations $y(t)$ with $\rho_c = 0.1$.



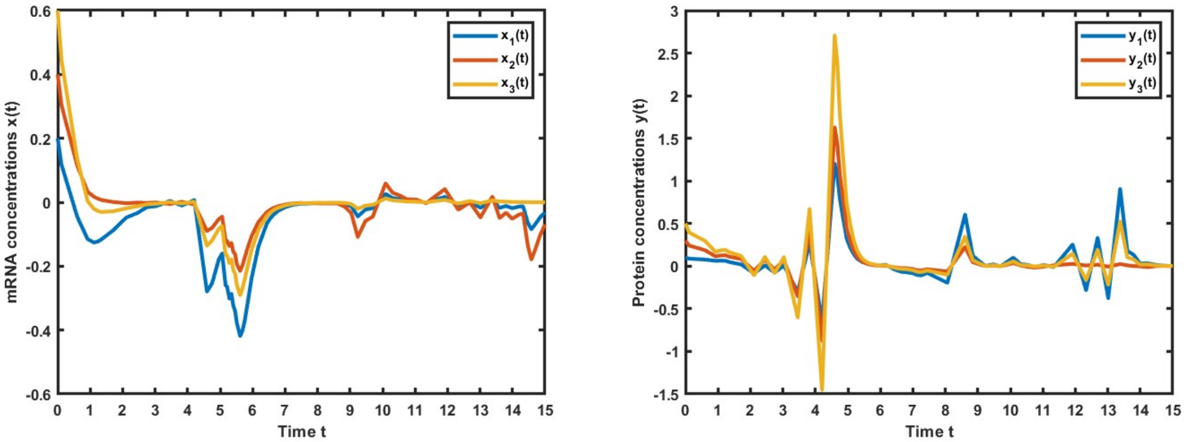
(vii) (viii)

Figure 4. (vii) mRNA concentrations $x(t)$ with $\rho_a = 0.5$. (viii) Protein concentrations $y(t)$ with $\rho_c = 0.5$.



(ix) (x)

Figure 5. (ix) mRNA concentrations $x(t)$. (x) Protein concentrations $y(t)$.



(xi) (xii)

Figure 6. (xi) mRNA concentrations $x(t)$ with $\rho_a = 0.1$. (xii) Protein concentrations $y(t)$ with $\rho_c = 0.1$.

5. Conclusions

In this paper, our focus is on investigating how leakage delays affect FTS in GRNs characterized by interval time-varying delays. To begin, we introduce GRNs that incorporate interval time-varying delays as well as leakage delays. These models consider lower bounds on delays, which may be either positive or zero, and allow for the derivatives of delays to be either positive or negative. Subsequently, we delve into the consequences of leakage delays through the construction of a LK function. We then enhance the criteria for FTS by employing estimates of integral inequalities and a reciprocally convex technique. This refinement enables us to express the new finite-time stability criteria for genetic regulatory networks in the form of LMIs. Finally, we present a numerical example to demonstrate the effect of leakage delays and validate the significance of our theoretical findings.

Author contributions

N.S. and I.K.: Conceptualization; N.S. and I.K.: Methodology; N.S. and I.K.: Software; N.S., K.M., and I.K.: Validation; I.K., N.S., and K.M.: Formal analysis; N.S., K.M., and I.K.: Investigation; N.S. and I.K.: Writing-original draft preparation; N.S., K.M., and I.K.: Writing-review and editing; N.S.and I.K.: Visualization; N.S., K.M., and I.K.: Supervision; N.S.: Project administration; N.S.: Funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by research and innovation funding from the Faculty of Agriculture and Technology, Nakhon Phanom University (Grant AG004/2566), and the Faculty of Engineering, Rajamangala University of Technology Isan Khon Kaen Campus.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript

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