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Research article

Self-adjoint and hyponormal weighted composition operators on the Fock space

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Abstract: One major aim of this paper is to characterize the self-adjointness of some special weighted composition operators on Fock space, and another major aim is to characterize the hyponormality of some composition operators on such space.

Keywords: Fock space; weighted composition operator; self-adjointness; hyponormality
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1. Introduction

Let $\mathbb{C}$ be the complex plane. Denote by $\mathbb{C}^N$ the N-dimensional complex Euclidean space with the inner product $\langle z, w\rangle = \sum_{j=1}^{N} z_j\overline{w}_j$; by $|z|^2 = \langle z, z\rangle$; by $H(\mathbb{C}^N)$ the set of all holomorphic functions on $\mathbb{C}^N$; and by I the identity operator on $\mathbb{C}^N$.

The Fock space $\mathcal{F}^2(\mathbb{C}^N)$ is a Hilbert space of all holomorphic functions $f \in H(\mathbb{C}^N)$ with the inner product

$$\langle f, g\rangle = \frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} f(z)\overline{g(z)}e^{-\frac{1}{2}|z|^2}d\nu(z),$$

where $\nu(z)$ denotes Lebesgue measure on $\mathbb{C}^N$. To simplify notation, we will often use $\mathcal{F}^2$ instead of $\mathcal{F}^2(\mathbb{C}^N)$, and we will denote by $\|f\|$ the corresponding norm of f. The reproducing kernel functions of the Fock space are given by

$$K_w(z) = e^{\frac{\langle z,w\rangle}{2}}, \quad z \in \mathbb{C}^N,$$

which means that if $f \in \mathcal{F}^2$, then $f(z) = \langle f, K_z \rangle$ for all $z \in \mathbb{C}^N$. It is easy to see that $\|K_w\| = e^{|w|^2/4}$. Therefore, the following evaluation holds:

$$|f(z)| \leq e^{\frac{|z|^2}{4}} \|f\|$$

for $f \in \mathcal{F}^2$ and $z \in \mathbb{C}^N$. If k_w is the normalization of K_w, then

$$k_w(z) = e^{\frac{\langle z,w \rangle}{2} - \frac{|w|^2}{4}}, \quad z \in \mathbb{C}^N.$$

Indeed, $\mathcal{F}^2$ is used to describe systems with varying numbers of particles in the states of quantum harmonic oscillators. On the other hand, the reproducing kernels in $\mathcal{F}^2$ are used to describe the coherent states in quantum physics. See [17] for more about the Fock space, and see [1, 7, 11] for the studies of some operators on the Fock space.

For a given holomorphic mapping $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ and $u \in H(\mathbb{C}^N)$, the *weighted composition operator*, usually denoted by $W_{u,\varphi}$, on or between some subspaces of $H(\mathbb{C}^N)$ is defined by

$$W_{u,\varphi} f(z) = u(z) f(\varphi(z)).$$

When $u = 1$, it is the *composition operator*, usually denoted by C_φ. While $\varphi(z) = z$, it is the *multiplication operator*, usually denoted by M_u.

Forelli in [8] proved that the isometries on Hardy space H^p defined on the open unit disk (for $p \neq 2$) are certain weighted composition operators, which can be regarded as the earliest presence of the weighted composition operators. Weighted composition operators have also been used in descriptions of adjoints of composition operators (see [4]). An elementary problem is to provide function-theoretic characterizations for which the symbols u and φ induce a bounded or compact weighted composition operator on various holomorphic function spaces. There have been many studies of the weighted composition operators and composition operators on holomorphic function spaces. For instance, several authors have recently worked on the composition operators and weighted composition operators on Fock space. For the one-variable case, Ueki [13] characterized the boundedness and compactness of weighted composition operators on Fock space. As a further work of [13], Le [10] found the easier criteria for the boundedness and compactness of weighted composition operators. Recently, Bhuia in [2] characterized a class of C-normal weighted composition operators on Fock space.

For the several-variable case, Carswell et al. [3] studied the boundedness and compactness of composition operators. From [3], we see that the one-variable case composition operator C_φ is bounded on Fock space if and only if $\varphi(z) = az + b$, where $|a| \leq 1$, and if $|a| = 1$, then $b = 0$. Let $A : \mathbb{C}^N \to \mathbb{C}^N$ be a linear operator. Zhao [14–16] characterized the unitary, invertible, and normal weighted composition operator $W_{u,\varphi}$ on Fock space, when $\varphi(z) = Az + b$ and $u = k_c$. Interestingly enough, Zhao [15] proved that for $\varphi(z) = Az + b$ and $u(z) = K_c(z)$, weighted composition operator $W_{u,\varphi}$ is bounded on Fock space if and only if $\|A\| \leq 1$ and $\langle A\zeta, b + Ac \rangle = 0$ whenever $|A\zeta| = |\zeta|$ for $\zeta \in \mathbb{C}^N$.

Motivated by the above-mentioned interesting works, for the special symbols $\varphi(z) = Az + b$ and $u = K_c$, here we study the adjoint, self-adjointness, and hyponormality of weighted composition operators on Fock space. Such properties of the abstract or concrete operators (for example, Toeplitz operators, Hankel operators, and composition operators) have been extensively studied on some other holomorphic function spaces. This paper can be regarded as a continuation of the weighted composition operators on Fock space.

2. Adjoint of the weighted composition operators

In this section, we characterize the adjoints of weighted composition operators $W_{u,\varphi}$ on Fock space, where $\varphi(z) = Az + b$ and $u = K_c$.

We first have the following result:

Lemma 2.1. *Let $A, B : \mathbb{C}^N \to \mathbb{C}^N$ be linear operators with $\|A\| \leq 1$ and $\|B\| \leq 1$, $\varphi(z) = Az + a$, $\psi(z) = Bz + b$ for $a, b \in \mathbb{C}^N$, and the operators C_φ and C_ψ be bounded on $\mathcal{F}^2$. Then*

$$C_\varphi^* C_\psi = W_{K_a, BA^*z+b},$$

where A^* is the adjoint operator of A.

Proof. From Lemma 2 in [3], it follows that

$$C_\varphi^* C_\psi = M_{K_a} C_{A^*z} C_{Bz+b} = M_{K_a} C_{(Bz+b)\circ A^*z} = M_{K_a} C_{BA^*z+b} = W_{K_a, BA^*z+b},$$

from which the result follows. The proof is complete. □

In Lemma 2.1, we prove that the product of the adjoint of a composition operator and another composition operator is expressed as a weighted composition operator. Next, we will see that in some sense, the converse of Lemma 2.1 is also true. Namely, we will prove that if $\varphi(z) = Az + b$, where $A : \mathbb{C}^N \to \mathbb{C}^N$ is a linear operator with $\|A\| < 1$, and $u = K_c$, then the operator $W_{u,\varphi}$ on $\mathcal{F}^2$ can be written as the product of the adjoint of a composition operator and another composition operator.

Lemma 2.2. *Let $A : \mathbb{C}^N \to \mathbb{C}^N$ be a linear operator with $\|A\| < 1$. If A and c satisfy the condition $\langle A^*\zeta, c\rangle = 0$ whenever $|A^*\zeta| = |\zeta|$, then there exists a positive integer n such that the operator $W_{u,\varphi}$ on $\mathcal{F}^2$ defined by $\varphi(z) = Az + b$ and $u(z) = K_c(z)$ is expressed as*

$$W_{u,\varphi} = C^*_{\frac{n+1}{n}A^*z+c} C_{\frac{n}{n+1}z+b}.$$

Proof. From Theorem 2 in [3], we see that the operator C_{A^*z+c} is bounded on $\mathcal{F}^2$. Since $\|A^*\| < 1$, there exists a large enough positive integer n such that

$$\left\|(1 + \frac{1}{n})A^*\right\| \leq 1.$$

Also, by Theorem 2 in [3], the operator $C_{\frac{n+1}{n}A^*z+c}$ is bounded on $\mathcal{F}^2$, which implies that the operator $C^*_{\frac{n+1}{n}A^*z+c}$ is also bounded on $\mathcal{F}^2$. Since $|\frac{n}{n+1}I\zeta| = |\zeta|$ if and only if $\zeta = 0$, $\langle \frac{n}{n+1}I\zeta, b\rangle = 0$ whenever $|\frac{n}{n+1}I\zeta| = |\zeta|$. By Theorem 2 in [3], the operator $C_{\frac{n}{n+1}Iz+b}$ is bounded on $\mathcal{F}^2$. Then, it follows from Lemma 2.1 that

$$C^*_{\frac{n+1}{n}A^*z+c} C_{\frac{n}{n+1}Iz+b} = W_{K_c, Az+b}.$$

The proof is complete. □

Now, we can obtain the adjoint for some weighted composition operators.

Theorem 2.1. *Let $\varphi(z) = Az + b$, $u(z) = K_c(z)$, and A and c satisfy $\langle A^*\zeta, c\rangle = 0$ whenever $|A^*\zeta| = |\zeta|$. Then it holds that*

$$W_{u,\varphi}^* = W_{K_b, A^*z+c}.$$

Proof. In Lemma 2.2, we have

$$W_{u,\varphi} = C^*_{\frac{n+1}{n}A^*z+c} C_{\frac{n}{n+1}Iz+b}. \tag{2.1}$$

It follows from (2.1) that

$$W_{u,\varphi}^* = C^*_{\frac{n}{n+1}Iz+b} C_{\frac{n+1}{n}A^*z+c}. \tag{2.2}$$

Therefore, from (2.2) and Lemma 2.1, the desired result follows. The proof is complete. □

By using the kernel functions, we can obtain the following result:

Lemma 2.3. *Let the operator $W_{u,\varphi}$ be a bounded operator on $\mathcal{F}^2$. Then it holds that*

$$W_{u,\varphi}^* K_w = \overline{u(w)} K_{\varphi(w)}.$$

Proof. Let f be an arbitrary function in $\mathcal{F}^2$. We see that

$$\langle W_{u,\varphi}^* K_w, f\rangle = \langle K_w, W_{u,\varphi} f\rangle = \overline{\langle W_{u,\varphi} f, K_w\rangle} = \overline{u(w) f(\varphi(w))} = \overline{u(w)} \langle K_{\varphi(w)}, f\rangle.$$

From this, we deduce that $W_{u,\varphi}^* K_w = \overline{u(w)} K_{\varphi(w)}$. The proof is complete. □

Here, we characterize the self-adjoint weighted composition operators.

Theorem 2.2. *Let $A : \mathbb{C}^N \to \mathbb{C}^N$ be a linear operator, $b, c \in \mathbb{C}^N$, $\varphi(z) = Az + b$, $u(z) = K_c(z)$, and the operator $W_{u,\varphi}$ be bounded on $\mathcal{F}^2$. Then the operator $W_{u,\varphi}$ is self-adjoint on $\mathcal{F}^2$ if and only if $A : \mathbb{C}^N \to \mathbb{C}^N$ is self-adjoint and $b = c$.*

Proof. In Lemma 2.3, we have

$$W_{u,\varphi}^* K_w(z) = \overline{u(w)} K_{\varphi(w)} = \overline{K_c(w)} e^{\frac{\langle z, \varphi(w)\rangle}{2}} = e^{\frac{\langle c, w\rangle}{2}} e^{\frac{\langle z, Aw+b\rangle}{2}}. \tag{2.3}$$

On the other hand,

$$W_{u,\varphi} K_w(z) = u(z) K_w(\varphi(z)) = e^{\frac{\langle z, c\rangle}{2}} e^{\frac{\langle Az+b, w\rangle}{2}}. \tag{2.4}$$

It is clear that operator $W_{u,\varphi}$ is self-adjoint on $\mathcal{F}^2$ if and only if

$$W_{u,\varphi}^* K_w = W_{u,\varphi} K_w.$$

From (2.3) and (2.4), it follows that

$$e^{\frac{\langle c,w\rangle}{2}} e^{\frac{\langle z,Aw+b\rangle}{2}} = e^{\frac{\langle z,c\rangle}{2}} e^{\frac{\langle Az+b,w\rangle}{2}}. \tag{2.5}$$

Letting $z = 0$ in (2.5), we obtain that $e^{\frac{\langle c,w\rangle}{2}} = e^{\frac{\langle b,w\rangle}{2}}$ which implies that

$$\langle c, w\rangle - \langle b, w\rangle = 4k\pi i, \tag{2.6}$$

where $k \in \mathbb{N}$. Also, letting $w = 0$ in (2.6), we see that $k = 0$. This shows that $\langle c, w\rangle - \langle b, w\rangle = 0$, that is, $\langle c, w\rangle = \langle b, w\rangle$. From this, we deduce that $b = c$. Therefore, (2.5) becomes $e^{\frac{\langle z,Aw\rangle}{2}} = e^{\frac{\langle Az,w\rangle}{2}}$. From this, we obtain that $\langle z, Aw\rangle = \langle Az, w\rangle$, which implies that $\langle A^*z, w\rangle = \langle Az, w\rangle$. This shows that $A = A^*$, that is, $A : \mathbb{C}^N \to \mathbb{C}^N$ is self-adjoint.

Now, assume that A is a self-adjoint operator on $\mathbb{C}^N$ and $b = c$. A direct calculation shows that (2.5) holds. Then $W_{u,\varphi}$ is a self-adjoint operator on $\mathcal{F}^2$. The proof is complete. □

In [14], Zhao proved that the operator $W_{u,\varphi}$ on $\mathcal{F}^2$ is unitary if and only if there exist an unitary operator $A : \mathbb{C}^N \to \mathbb{C}^N$, a vector $b \in \mathbb{C}^N$, and a constant α with $|\alpha| = 1$ such that $\varphi(z) = Az - b$ and $u(z) = \alpha K_{A^{-1}b}(z)$. Without loss of generality, here we characterize the self-adjoint unitary operator $W_{u,\varphi}$ on $\mathcal{F}^2$ for the case $\alpha = 1$ and obtain the following result from Theorem 2.2.

Corollary 2.1. *Let $A : \mathbb{C}^N \to \mathbb{C}^N$ be a unitary operator and $b \in \mathbb{C}^N$ such that $\varphi(z) = Az - b$ and $u(z) = K_{A^{-1}b}(z)$. Then the operator $W_{u,\varphi}$ is self-adjoint on $\mathcal{F}^2$ if and only if $A : \mathbb{C}^N \to \mathbb{C}^N$ is self-adjoint and $Ab + b = 0$.*

3. Hyponormality of the weighted composition operators

First, we recall the definition of hyponormal operators. An operator T on a Hilbert space H is said to be hyponormal if $\|Ax\| \geq \|A^*x\|$ for all vectors $x \in H$. T is called co-hyponormal if T^* is hyponormal. In 1950, Halmos, in his attempt to solve the invariant subspace problem, extended the notion of normal operators to two new classes, one of which is now known as the hyponormal operator (see [9]). Clearly, every normal operator is hyponormal. From the proof in [6], it follows that T is hyponormal if and only if there exists a linear operator C with $\|C\| \leq 1$ such that $T^* = CT$. In some sense, this result can help people realize the characterizations of the hyponormality of some operators. For example, Sadraoui in [12] used this result to characterize the hyponormality of composition operators defined by the linear fractional symbols on Hardy space. On the other hand, some scholars studied the hyponormality of composition operators on Hardy space by using the fact that the operator C_φ on Hardy space is hyponormal if and only if

$$\|C_\varphi f\|^2 \geq \|C_\varphi^* f\|^2$$

for all f in Hardy space. For example, Dennis in [5] used the fact to study the hyponormality of composition operators on Hardy space. In particular, this inequality for norms is used when f is a reproducing kernel function K_w for any $w \in \mathbb{C}^N$. Actually, to the best of our knowledge, there are few studies on the hyponormality of weighted composition operators. Here, we consider this property of weighted composition operators on Fock space.

First, we have the following result, which can be proved by using the reproducing kernel functions.

Lemma 3.1. *Let $w \in \mathbb{C}^N$ and the operator $W_{u,\varphi}$ be bounded on $\mathcal{F}^2$. Then*

$$\|W_{u,\varphi}K_w\|^2 = W_{u,\varphi}^* W_{u,\varphi}K_w(w).$$

Proof. From the inner product, we have

$$\|W_{u,\varphi}K_w\|^2 = \langle W_{u,\varphi}K_w, W_{u,\varphi}K_w\rangle = \langle W_{u,\varphi}^* W_{u,\varphi}K_w, K_w\rangle = W_{u,\varphi}^* W_{u,\varphi}K_w(w).$$

The proof is complete. □

Theorem 3.1. *Let $A : \mathbb{C}^N \to \mathbb{C}^N$ be a linear operator, $\varphi(z) = Az + b$, $u = k_c$, and the operator $W_{u,\varphi}$ be bounded on $\mathcal{F}^2$. If the operator $W_{u,\varphi}$ is hyponormal on $\mathcal{F}^2$, then $A^*b - b = Ac - c$ and $|b| \leq |c|$.*

Proof. From a direct calculation, we have

$$\begin{aligned} W_{u,\varphi}K_w(z) &= u(z)K_w(\varphi(z)) = k_c(z)K_w(Az + b) \\ &= e^{\frac{\langle z,c\rangle}{2} - \frac{|c|^2}{4}} e^{\frac{\langle Az+b,w\rangle}{2}} = e^{\frac{\langle z,A^*w+c\rangle + \langle b,w\rangle}{2} - \frac{|c|^2}{4}} \\ &= e^{\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4}} K_{A^*w+c}(z). \end{aligned} \tag{3.1}$$

From (3.1), it follows that

$$\begin{aligned} W_{u,\varphi}^* W_{u,\varphi}K_w(z) &= e^{\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4}} W_{u,\varphi}^* K_{A^*w+c}(z) \\ &= e^{\frac{\langle b,w\rangle}{2} - \frac{|c|^2}{4}} \overline{u(A^*w + c)} K_{\varphi(A^*w+c)}(z) \\ &= e^{\frac{\langle b,w\rangle}{2} + \frac{\langle c,A^*w+c\rangle}{2} + \frac{\langle z,AA^*w+Ac+b\rangle}{2} - \frac{|c|^2}{2}} \\ &= e^{\frac{\langle b+Ac,w\rangle}{2} + \frac{\langle z,AA^*w\rangle}{2} + \frac{\langle z,Ac+b\rangle}{2}}. \end{aligned} \tag{3.2}$$

On the other hand, we also have

$$\begin{aligned} W_{u,\varphi} W_{u,\varphi}^* K_w(z) &= \overline{u(w)} W_{u,\varphi}K_{\varphi(w)}(z) = \overline{u(w)} u(z) K_{\varphi(w)}(\varphi(z)) \\ &= e^{\frac{\langle c,w\rangle}{2} + \frac{\langle z,c\rangle}{2} + \frac{\langle Az+b,Aw+b\rangle}{2} - \frac{|c|^2}{2}} \\ &= e^{\frac{\langle c+A^*b,w\rangle}{2} + \frac{|b|^2}{2} + \frac{\langle z,AA^*w\rangle}{2} + \frac{\langle z,c+A^*b\rangle}{2} - \frac{|c|^2}{2}}. \end{aligned} \tag{3.3}$$

From Lemma 3.1, (3.2), and (3.3), it follows that

$$\|W_{u,\varphi}^* K_w\|^2 = W_{u,\varphi} W_{u,\varphi}^* K_w(w) = e^{\frac{\langle c+A^*b,w\rangle}{2} + \frac{|b|^2}{2} + \frac{|A^*w|^2}{2} + \frac{\langle w,c+A^*b\rangle}{2} - \frac{|c|^2}{2}}$$

and

$$\|W_{u,\varphi}K_w\|^2 = W_{u,\varphi}^* W_{u,\varphi}K_w(w) = e^{\frac{\langle b+Ac,w\rangle}{2} + \frac{|A^*w|^2}{2} + \frac{\langle w,Ac+b\rangle}{2}}.$$

Then, we have

$$\|W_{u,\varphi}^* K_w\|^2 - \|W_{u,\varphi}K_w\|^2 = e^{\frac{|A^*w|^2}{2}} \Big(e^{\frac{\langle c+A^*b,w\rangle}{2} + \frac{|b|^2}{2} + \frac{\langle w,c+A^*b\rangle}{2} - \frac{|c|^2}{2}} - e^{\frac{\langle b+Ac,w\rangle}{2} + \frac{\langle w,Ac+b\rangle}{2}} \Big),$$

which shows that

$$\|W_{u,\varphi}^* K_w\|^2 - \|W_{u,\varphi} K_w\|^2 \leq 0$$

for all $w \in \mathbb{C}^N$ if and only if

$$e^{\frac{\langle c+A^*b,w\rangle}{2}+\frac{|b|^2}{2}+\frac{\langle w,c+A^*b\rangle}{2}-\frac{|c|^2}{2}} \leq e^{\frac{\langle b+Ac,w\rangle}{2}+\frac{\langle w,Ac+b\rangle}{2}}. \tag{3.4}$$

It is clear that (3.4) holds if and only if

$$\langle c + A^*b, w\rangle + |b|^2 + \langle w, c + A^*b\rangle - |c|^2 \leq \langle b + Ac, w\rangle + \langle w, Ac + b\rangle. \tag{3.5}$$

From (3.5), we see that (3.4) holds if and only if

$$\langle A^*b - Ac + c - b, w\rangle + \langle w, A^*b - Ac + c - b\rangle + |b|^2 - |c|^2 \leq 0. \tag{3.6}$$

Therefore, we deduce that (3.4) holds for all $w \in \mathbb{C}^N$ if and only if $|b| \leq |c|$ and $A^*b - b = Ac - c$. The proof is complete. □

If $b = c = 0$ in Theorem 3.1, then $W_{u,\varphi}$ is reduced into the composition operator C_{Az}. For this case, Theorem 3.1 does not provide any useful information on the operator $A : \mathbb{C}^N \to \mathbb{C}^N$ when C_{Az} is hyponormal on $\mathcal{F}^2$. However, we have the following result, which completely characterizes the hyponormal composition operators:

Theorem 3.2. *Let $A : \mathbb{C}^N \to \mathbb{C}^N$ be a linear operator such that C_{Az} is bounded on $\mathcal{F}^2$. Then the operator C_{Az} is hyponormal on $\mathcal{F}^2$ if and only if $A : \mathbb{C}^N \to \mathbb{C}^N$ is co-hyponormal.*

Proof. Assume that $A : \mathbb{C}^N \to \mathbb{C}^N$ is co-hyponormal. Then there exists an operator $B : \mathbb{C}^N \to \mathbb{C}^N$ with $\|B\| \leq 1$ such that $A = BA^*$. We therefore have

$$C_{Az}^* = C_{A^*z} = C_{AB^*z} = C_{B^*z}C_{Az}.$$

Next, we want to show that $\|C_{B^*z}\| = 1$. By Theorem 4 in [3], we have

$$\|C_{B^*z}\| = e^{\frac{1}{4}(|w_0|^2-|B^*w_0|^2)}, \tag{3.7}$$

where w_0 is any solution to $(I - BB^*)w = 0$. From this, we obtain that $w_0 = BB^*w_0$, and then

$$|B^*w_0|^2 = \langle B^*w_0, B^*w_0\rangle = \langle w_0, BB^*w_0\rangle = \langle w_0, w_0\rangle = |w_0|^2. \tag{3.8}$$

Thus, by considering (3.7) and (3.8), we see that $\|C_{B^*z}\| = 1$. It follows that the operator C_{Az} is hyponormal on $\mathcal{F}^2$.

Now, assume that the operator C_{Az} is hyponormal on $\mathcal{F}^2$. Then there exists a linear operator C on $\mathcal{F}^2$ with $\|C\| \leq 1$ such that $C_{Az}^* = CC_{Az}$. By Lemma 2 in [3], we have $C_{Az}^* = C_{A^*z}$. This shows that CC_{Az} is a composition operator. This result shows that there exists a holomorphic mapping $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ such that $C = C_\varphi$. So $A^*z = A(\varphi(z))$ for all $z \in \mathbb{C}^N$, which implies that there exists a linear operator $B : \mathbb{C}^N \to \mathbb{C}^N$ such that $\varphi(z) = B^*z$, and then $C = C_{B^*z}$. Therefore, $A^* = AB^*$, that is, $A = BA^*$. Since $\|C\| \leq 1$, this shows that the operator $C = C_{B^*z}$ is bounded on $\mathcal{F}^2$. From Lemma 2.3 in [15], we obtain that $\|B^*\| \leq 1$, which also shows that $\|B\| \leq 1$ since $\|B^*\| = \|B\|$. We prove that $A : \mathbb{C}^N \to \mathbb{C}^N$ is co-hyponormal. The proof is complete. □

Remark 3.1. In the paper, we only obtain a necessary condition for the hyponormality of weighted composition operators on Fock space. We hope that the readers can continuously consider the problem in Fock space.

4. Conclusions

In this paper, I give a proper description of the adjoint $W_{u,\varphi}^*$ on Fock space for the special symbol functions $u(z) = K_c(z)$ and $\varphi(z) = Az + b$. However, it is difficult to give a proper description of the general symbols. On the other hand, I consider the hyponormal weighted composition operators on Fock space and completely characterize hyponormal composition operators on this space. I hope that people are interested in the research in this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no competing interests.

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