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Research article

The counting formula for indecomposable modules over string algebra

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Abstract: Let $A = kQ/I$ be a string algebra. We show that, if for any vertex v of its bound quiver (Q, I), there exists at most one arrow (resp. at most two arrows) ending with v and there exist at most two arrows (resp. at most one arrow) starting with v, then the number of indecomposable modules over A is $\dim_k A + \Sigma$, where Σ is induced by $radP(v)$ (resp. $E(v)/\mathrm{soc}E(v)$) with decomposable socle (resp. top), where $P(v)$ (resp. $E(v)$) is the indecomposable projective (resp. injective) module corresponded by the vertex v.

Keywords: representation-finite; representations of quivers; indecomposable modules
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1. Introduction

String algebras play an important role in representation theory; they are closely related to many algebras, such as biserial algebras, gentle algebras, and Nakayama algebras. In [17], Wald and Waschbüsch described all indecomposable modules over tame biserial algebras by V-sequences and primitive V-sequences; and provided an important theorem that shows an arbitrary indecomposable module over tame biserial algebra can be corresponded by some V-sequences or some pairs of primitive V-sequences and Jordan blocks. But this corresponding is not bijective, except for the case all projective–injective modules over tame biserial algebras to be uniserial. In [3], Butler and Ringel introduced strings and bands on bound quiver, which are special V-sequences and primitive V-sequences, and show that if the tame biserial algebras are string algebras, then the above descriptions given by Wald and Waschbüsch provide a bijection $\mathfrak{M}$ to describe the indecomposable modules over string algebra. This result points out that a string algebra is representation-finite, that is, the number of isoclasses of an indecomposable module is finite; if and only if its bound quiver does not contain bands.

With the research of Wald, Waschbüsch, Butler, and Ringel, we can conduct further research on string algebras and gentle algebras. For example, the tensor algebras and Clebsch–Gordan problems of string algebras [11,14] derived representation-types of gentle algebras [18,19], the (co)homologies and homological dimensions of string and gentle algebras [8,12], Cohen–Macaulay–Auslander algebras of string and gentle algebras [4,5,13], the tilting and silting theories of gentle and skew-gentle algebras [1,6,10] and so on.

In [7], Gabriel showed that a finite-dimensional connected basic hereditary algebra is representation-finite if and only if the underlying graph of its quiver is one of the Dynkin diagrams $\mathbb{A}_m$ with $m \geq 1$, $\mathbb{D}_n$ with $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$, that also appear in Lie theory (see, for example, [9]). Later, Bernstein, Gelfand, and Ponomarev [2] gave a very elegant and conceptual proof underlining the links between the two theories by applying the nice concept of reflection functors, and, furthermore, they showed that the number of isoclasses of indecomposable modules over A, a path algebra of Dynkin quiver $\mathbb{A}_m$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$, equals to $\frac{m(m+1)}{2}$, $n^2 - n$, 36, 63, and 120, respectively. c.f. [16, Chap IIV, Theorem 5.10, (c)]. In this paper, we focus on how to compute the number of isoclasses of indecomposable modules over string algebra and show the following result:

Theorem 1.1. *Let A be a basic and connected string algebra.*

(1) (Theorem 3.7) *If for each vertex v of its bound quiver* $(\mathcal{Q}, \mathcal{I})$*, there exists at most one arrow ending at v and there exist at most two arrows starting from v, then the number of indecomposable modules over A is*
$$\dim_{\Bbbk} A + \sum_{\substack{\mathrm{soc}P(v) \text{ is a direct sum} \\ \text{of two simple modules}}} \dim_{\Bbbk} D_{v,1} \cdot \dim_{\Bbbk} D_{v,2},$$
where $P(v)$ *is the indecomposable projective module corresponded by* $v \in \mathcal{Q}_0$*, and* $D_{v,1}$ *and* $D_{v,2}$ *are direct summands of* $\mathrm{rad}P(v) = D_{v,1} \oplus D_{v,2}$.

(2) (Theorem 3.8) *If for each vertex v of its bound quiver* $(\mathcal{Q}, \mathcal{I})$*, there exist at most two arrows ending at v and there exists at most one arrow starting from v, then the number of indecomposable modules over A is*
$$\dim_{\Bbbk} A + \sum_{\substack{\mathrm{top}E(v) \text{ is a direct sum} \\ \text{of two simple modules}}} \dim_{\Bbbk} D_{v,1} \cdot \dim_{\Bbbk} D_{v,2},$$
where $E(v)$ *is the indecomposable injective module corresponding to* $v \in \mathcal{Q}_0$*, and* $D_{v,1}$ *and* $D_{v,2}$ *are direct summands of* $E(v)/\mathrm{soc}E(v) = D_{v,1} \oplus D_{v,2}$.

Furthermore, we obtain a corollary from the above theorem as follows:

Corollary 1.2 (Example 4.2)**.** *The number of isoclasses of indecomposable modules over* $A = \Bbbk\mathcal{Q}/\mathcal{I}$ *is* $\dim_{\Bbbk} A$*, where the underlying graph of* $\mathcal{Q}$ *is type* $\mathbb{A}$ *and* $\mathcal{I}$ *is an arbitrary admissible ideal of* $\Bbbk\mathcal{Q}$.

2. Preliminaries

In this section, we recall the definition and some properties of string algebras. We refer the readers to [3] for more details. Throughout this paper, we always assume that: $\Bbbk$ is an algebraically closed; $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, \mathfrak{s}, \mathfrak{t})$ is a finite connected quiver; $\mathfrak{s}$ and $\mathfrak{t}$ are the functions $\mathcal{Q}_1 \to \mathcal{Q}_0$ sending any arrow in the arrow set $\mathcal{Q}_1$ of $\mathcal{Q}$ to its starting point and its ending point lying in the vertex set $\mathcal{Q}_0$ of $\mathcal{Q}$; for any two paths p_1 and p_2 with $\mathfrak{t}(p_1) = \mathfrak{s}(p_2)$, the composition is denoted by p_1p_2; $\mathcal{I}$, the ideal of the path algebra $\Bbbk Q$ of Q, is admissible; and, for the algebra $A = \Bbbk Q/\mathcal{I}$ of bound quiver $(Q, \mathcal{I})$, all modules we considered are right A-module.

The bound quiver $(Q, \mathcal{I})$ is said to be a *string pair* if it satisfies the following conditions:

(1) Any vertex of Q is the source and target of at most two arrows;
(2) For each arrow β, there is at most one arrow γ such that $\beta\gamma \notin \mathcal{I}$;
(3) For each arrow β, there is at most one arrow α such that $\alpha\beta \notin \mathcal{I}$;
(4) $\mathcal{I}$ is generated by paths of length great than or equal to 2.

Furthermore, $(Q, \mathcal{I})$ is said to be a *gentle pair* if it is a string pair such that the following conditions hold:

(5) For each arrow β, there is at most one arrow γ such that $\beta\gamma \in \mathcal{I}$;
(6) For each arrow β, there is at most one arrow α such that $\alpha\beta \in \mathcal{I}$.
(7) $\mathcal{I}$ is generated by paths of length 2.

Definition 2.1. A finite-dimensional algebra $A = \Bbbk Q/\mathcal{I}$ is called a *string (resp. gentle) algebra* if its bound quiver is a string (resp. gentle) pair $(Q, \mathcal{I})$.

For any arrow $a \in Q_1$, we denote by a^{-1} the formal inverse of a. Then $\mathfrak{s}(a^{-1}) = \mathfrak{t}(a)$, and $\mathfrak{t}(a^{-1}) = \mathfrak{s}(a)$. We denote by $Q_1^{-1} := \{a^{-1} \mid a \in Q_1\}$ the set of all formal inverses of arrows. Any path $p = a_1a_2\cdots a_\ell$ in $(Q, \mathcal{I})$ naturally provides a formal inverse path $p^{-1} = a_\ell^{-1}a_{\ell-1}^{-1}\cdots a_1^{-1}$ of p. For any path e_v of length one corresponding to $v \in Q_0$, we define $e_v^{-1} = e_v$.

Definition 2.2. A *string* on a string pair $(Q, \mathcal{I})$ is a sequence $s = (p_1, p_2, \ldots, p_n)$ such that:

(1) For any $1 \le i \le n$, p_i or p_i^{-1} is a path in $(Q, \mathcal{I})$;
(2) If p_i is a path, then p_{i+1} is a formal inverse path;
(3) If p_i is a formal inverse path, then p_{i+1} is a path;
(4) $\mathfrak{t}(p_i) = \mathfrak{s}(p_{i+1})$ holds for all $1 \le i \le n-1$, which are called *turning points*.

A *band* $b = (p_1, p_2, \ldots, p_n)$ is a string such that:

(5) $\mathfrak{t}(p_n) = \mathfrak{s}(p_1)$, and if p_n and p_1 are paths, then $p_np_1 \notin \mathcal{I}$, if p_n and p_1 are formal inverse paths, then $(p_np_1)^{-1} \notin \mathcal{I}$;
(6) b is not a non-trivial power of some strings, i.e., there is no string s such that $b = s^m$ for some $m \ge 2$.

A vertex v on a string s is called a *source* if one of the following conditions holds:

- v is a turning point $\mathfrak{t}(p_i) = \mathfrak{s}(p_{i+1})$ such that p_i is a formal inverse path and p_{i+1} is a path;
- p_1 is a path, and $v = \mathfrak{s}(s) = \mathfrak{s}(p_1)$;
- p_n is a formal inverse path, and $v = \mathfrak{t}(s) = \mathfrak{t}(p_n)$.

We can define *sink* in a dual way.

If $n = 1$, then we call s a *direct string*. In particular, s is called a *trivial string* if it is empty. Two strings s and s' are called *equivalent* if $s' = s$ or $s' = s^{-1}$; two bands $b = \alpha_1\cdots\alpha_n$ and $b' = \alpha'_1\cdots\alpha'_t$ are called *equivalent* if $b[t] = b'$ or $b[t]^{-1} = b'$, where $b[t] = \alpha_{1+t}\cdots\alpha_n\alpha_1\cdots\alpha_{1+t-1}$. We denote by $\mathsf{Str}(A)$

the set of all equivalent classes of strings and by $\mathsf{Band}(A)$ the set of all equivalent classes of bands on the bound quiver of A. In [3], Butler and Ringel showed that all indecomposable modules over a string algebra can be described by strings and bands. To be more precise, we have the following result:

Theorem 2.3 (Butler–Ringel)**.** *Let $A = \Bbbk Q/\mathcal{I}$ be a string algebra. Then there is a bijection*

$$\mathfrak{M} : \mathsf{Str}(A) \cup (\mathsf{Band}(A) \times \mathscr{J}) \to \mathsf{ind}(\mathsf{mod}(A)),$$

where $\mathsf{ind}(\mathsf{mod}A)$ is the set of all isoclasses of indecomposable A-modules and $\mathscr{J}$ is the set of all indecomposable $\Bbbk[x, x^{-1}]$-modules.

Usually, if $\mathfrak{M}^{-1}(N)$ is a (direct) string, then we say N is a *(direct) string module*; if $\mathfrak{M}^{-1}(N)$ is a band with some pairs (n, λ), we say it is a *band module*. The original definition of string and band modules over string algebra can be referred to [3].

3. The number of all isoclasses of indecomposable modules

Now, for simplification, we always assume that all bound quivers we considered are string pairs whose underlying graph $\overline{Q}$ is a tree (i.e., $\overline{Q}$ does not contain a cycle), and all algebras we considered are string algebras, which are of the form $\Bbbk Q/I$ (up to isomorphism) in this section.

3.1. The string with unique source/sink

Lemma 3.1.

(1) *If a string s on $(Q, \mathcal{I})$ has a unique source, then it is one of the following:*

(A) *s is a direct string, that is,*

$$s = \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet;$$

(B) *$s = p_1p_2$, where p_1 is a formal inverse path and p_2 is a path, that is,*

$$s = \bullet \longleftarrow \cdots \longleftarrow \bullet \longrightarrow \cdots \longrightarrow \bullet.$$

(2) *Dually, if a string s on $(Q, \mathcal{I})$ has a unique sink, then it is one of the following forms:*

(A′) *s is a direct string, that is,*

$$s = \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet;$$

(B′) *$s = p_1p_2$, where p_1 is a path and p_2 is a formal inverse path, that is,*

$$s = \bullet \longrightarrow \cdots \longrightarrow \bullet \longleftarrow \cdots \longleftarrow \bullet.$$

Proof. We only prove (1); the proof of (2) is similar. If s has at least two sources, then it has at least one sink. So, it has a substring, which is of the form

$$u \longrightarrow v \longleftarrow w.$$

It is easy to see that s has at least two sources in this case, a contradiction. □

Let $p = a_r^{-1} \cdots a_2^{-1} a_1^{-1} b_1 b_2 \cdots b_t$ be a string with $r, t \geq 0$, $a_1^{-1}, \ldots, a_r^{-1} \in Q_1^{-1}$ and $b_1, \ldots, b_t \in Q_1$. If it satisfies the following conditions:

(P1) $\mathfrak{t}(a_1^{-1}) = \mathfrak{s}(b_1)$;
(P2) For any $\alpha \in Q_1$ with $\mathfrak{t}(a_r) = \mathfrak{s}(\alpha)$, $a_{r'} a_{r'+1} \cdots a_r \alpha \in \mathcal{I}$ for some $1 \leq r' \leq r$;
(P3) For any $\beta \in Q_1$ with $\mathfrak{t}(b_t) = \mathfrak{s}(\beta)$, $b_{t'} b_{t'+1} \cdots b_t \beta \in \mathcal{I}$ for some $1 \leq t' \leq t$.

Then $\mathfrak{M}(p)$ is an indecomposable projective A-module; we call that p is a *projective string* in this case. Dually, we can define any indecomposable *injective string*.

Lemma 3.2. *If a string s on $(Q, \mathcal{I})$ has a unique source, then it is a substring of some projective string p such that s and p have the same source.*

Proof. If s is projective, we have finished. If s is not projective, assume that s lies in Lemma 3.1 (B), i.e., $s = a_{r'}^{-1} \cdots a_1^{-1} b_1 \cdots b_{t'}$ $(a_1, \ldots, a_{r'}, b_1, \ldots, b_{t'} \in Q_1)$, then there are arrows $a_{r'+1}, \ldots, a_r$, $r \geq r'$, and $b_{t'+1}, \ldots, b_t$, $t \geq t'$ such that

$$p = a_r^{-1} \cdots a_{r'+1}^{-1} s b_{t'+1} \cdots b_t = a_{r'}^{-1} \cdots a_1^{-1} b_1 \cdots b_{t'} \tag{3.1}$$

is projective (consider the string corresponded by the indecomposable projective module $P(\mathfrak{s}(b_1)) = P(\mathfrak{s}(a_1))$), where at least one of $t - t'$ and $r - r'$ is positive. In this case, s is a subsrting of p, and the sources of s and p equal to $\mathfrak{t}(a_1^{-1})(= \mathfrak{s}(a_1)) = \mathfrak{s}(b_1)$ as required.

The case of s belongs to Lemma 3.1 (A) is similar. □

Denoted by $\wp(s)$ the projective string p is given by (3.1). If s is a string with a unique source on $(Q, \mathcal{I})$, then the projective cover of $\mathfrak{M}(s)$ is an A-homomorphism $\mathfrak{p}_M : \mathfrak{M}(\wp(s)) \cong P(v) \to \mathfrak{M}(s)$, and v is the unique source of $\wp(s)$. Indeed, the string module $\mathfrak{M}(\wp(s))$ corresponded by $\wp(s) = a_r^{-1} \cdots a_{r'+1}^{-1} s b_{t'+1} \cdots b_t$ is an indecomposable projective module whose top is isomorphic to the simple module $S(v)$, which corresponds to the unique source v of s. Notice that the top of $\mathfrak{M}(s)$ is isomorphic to $S(v)$, thus $\mathfrak{p}_M$ is of the form $\mathfrak{M}(\wp(s)) \to \mathfrak{M}(s)$. Furthermore, the kernel $\mathrm{Ker}(\mathfrak{p}_M)$ of $\mathfrak{p}_M$ is a direct sum of at most two direct string modules; see [15, Lemma 2.9] and [20, Lemma 3.4].

3.2. The indecomposable modules over string tree algebras

We call a vertex v of a quiver $(t_{v,1}^{\mathrm{in}}, t_{v,2}^{\mathrm{out}})$-*vertex* if the number of arrows ending at v is $t_{v,1}$ and that of arrows starting from v is $t_{v,2}$. We call a bound $(Q, \mathcal{I})$ is a *bound* $((\leq 1)^{\mathrm{in}}, (\leq 2)^{\mathrm{out}})$-*quiver* (resp. *bound* $((\leq 2)^{\mathrm{in}}, (\leq 1)^{\mathrm{out}})$-*quiver*) if any vertex v of Q_0 is a $(t_{v,1}^{\mathrm{in}}, t_{v,2}^{\mathrm{out}})$-vertex, where $t_{v,1}^{\mathrm{in}} \leq 1$ (resp. ≤ 2) and $t_{v,2}^{\mathrm{in}} \leq 2$ (resp. ≤ 1). Obviously, if $(Q, \mathcal{I})$ is a string pair, then $t_{v,1}^{\mathrm{in}} \leq 2$ and $t_{v,2}^{\mathrm{out}} \leq 2$ hold for all $v \in Q_0$. Furthermore, a string algebra A is said to be a *string* $((\leq 1)^{\mathrm{in}}, (\leq 2)^{\mathrm{out}})$-*algebra* (resp. *string* $((\leq 2)^{\mathrm{in}}, (\leq 1)^{\mathrm{out}})$-*algebra*) if its bound quiver is a bound $((\leq 1)^{\mathrm{in}}, (\leq 2)^{\mathrm{out}})$-quiver (resp. bound $((\leq 2)^{\mathrm{in}}, (\leq 1)^{\mathrm{out}})$-quiver).

Lemma 3.3. *If A is a string* $((\leq 1)^{\mathrm{in}}, (\leq 2)^{\mathrm{out}})$-*algebra, then any indecomposable module is isomorphic to $\mathfrak{M}(s)$, where s is either a string lying in Lemma 3.1* (A) *or a string lying in Lemma 3.1* (B).

Proof. By Theorem 2.3, we show that any string s on $(Q, \mathcal{I})$ lies in either Lemma 3.1 (A) or 3.1 (B) in this proof. Since each vertex $v \in Q_0$ satisfies that $t_{v,1} \leq 1$ and $t_{v,2} \leq 2$, a string crossing v is one of

$$\text{(a)} \cdots \longrightarrow v \longrightarrow \cdots, \text{ (b)} \cdots \longleftarrow v \longleftarrow \cdots, \text{ and (c)} \cdots \longleftarrow v \longrightarrow \cdots.$$

If s lies in (a), then all sources of s are on the left of v. Indeed, if there is a source on the right of v, then we can find a vertex, say z, between v and w such that $t_{z,1} = 2$, a contradiction. Furthermore, if s has two sources, say w_1 and w_2, which are left to v, then s is of the form

$$\dashleftarrow w_2 \longrightarrow \cdots \longleftarrow w_1 \longrightarrow \cdots \longrightarrow v \longrightarrow \cdots .$$

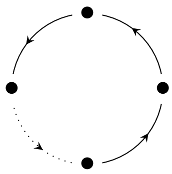
It is easy to see that there is a vertex z between w_1 and w_2 such that $t_{z,1} = 2$, a contradiction. Thus, s has a unique source that is left to v, and, if $v = \mathfrak{s}(s)$, then s lies in Lemma 3.1 (A); otherwise, s lies in Lemma 3.1 (B). The case of s lies in (b) can be obtained in a dual way.

For the case of s lies (c), we show that v is the unique source of s. Otherwise, there is another source w on the left (resp. right) of v, and then we can find a vertex z between v and w such that $t_{z,1} = 2$, a contradiction. In this case s is a string lying in Lemma 3.1 (B). □

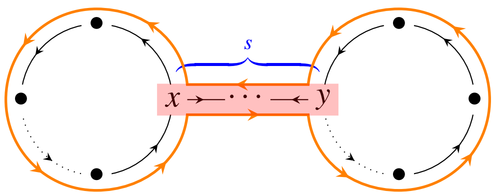
Lemma 3.4. *String* $((\leq 1)^{\mathrm{in}}, (\leq 2)^{\mathrm{out}})$*-algebras are representation-finite.*

Proof. Indeed, a string algebra is representation-finite if and only if its bound quiver does not contain bands. This well-known result can be proved by Theorem 2.3 and Brauer–Thrall Theorem.

Next, we show that the bound quiver $(Q, \mathcal{I})$ of any string $((\leq 1)^{\mathrm{in}}, (\leq 2)^{\mathrm{out}})$-algebra does not contain bands. First, if Q contains at least one cycle, then all cycles must be oriented cycles, i.e., the subquiver of the following form.



Otherwise, there are two vertex v and w on the cycle such that $t^{\mathrm{in}}_{v,1} = 2$ and $t^{\mathrm{out}}_{w,2} = 2$. This is a contradiction. In this case, if $(Q, \mathcal{I})$ contains a band b, then it can be seen in the following form:



(note that x and y are $((\leq 1)^{\mathrm{in}}, (\leq 2)^{\mathrm{out}})$-points). If b has a sink v, then v, as a vertex of Q, must be a vertex on the string s lying in the shadow part "▬" (x and y are two end points of s). In this case, v is a $(2^{\mathrm{in}}, t^{\mathrm{out}}_{v,2})$-vertex, this is a contradiction. □

For any projective string p, we define its *vertices pair* as the following:

- If p lies in Lemma 3.1 (A), then it is of the form

$$v_0 \xrightarrow{a_1} v_1 \xrightarrow{a_2} \cdots \xrightarrow{a_l} v_l. \tag{3.2}$$

For any v_i, $0 \leq i \leq l$, a vertices pair, written as $(v_i, v_i)_p$, is the triple (p, v_i, v_i).

- If p lying in Lemma 3.1 (B), then it is of the form

$$x_r \xleftarrow{a_r} x_{r-1} \xleftarrow{a_{r-1}} \cdots \xleftarrow{a_2} x_1 \xleftarrow{a_1} v \xrightarrow{b_1} y_1 \xrightarrow{b_2} \cdots \xrightarrow{b_{t-1}} y_{t-1} \xrightarrow{b_t} y_t. \tag{3.3}$$

For any x_i, y_j, where $0 \le i \le r$, $0 \le j \le t$, a vertices pair, written as $(x_i, y_j)_p$, is the triple (p, x_i, y_i). In this case, we define $(x_i, y_j)_p = (y_i, x_j)_p$.

Proposition 3.5. *If A is a string $((\le 1)^{\text{in}}, (\le 2)^{\text{out}})$-algebra, then there is a bijection*

$$\Psi : \mathsf{VP}(A) \to \mathsf{ind}(\mathsf{mod}A)$$

from the set $\mathsf{VP}(A)$ of all vertices pairs of projective strings to the set of all isoclasses of indecomposable A-modules.

Proof. By Lemma 3.4, A is representation-finite. Thus, the bijection $\mathfrak{M}$ given in Theorem 2.3 is a bijection between $\mathsf{ind}(\mathsf{mod}A)$ and $\mathsf{Str}(A)$. Then we need to construct a bijection from $\mathsf{VP}(A)$ to $\mathsf{Str}(A)$ in this proof.

For any projective string p lying in Lemma 3.1 (A), assume that it is of the form given by (3.2). By Lemma 3.2, we define a corresponding $\overline{\Psi}_{\mathrm{A}}$ that sends any vertices pair $(v_i, v_i)_p$, $(0 \le i \le l)$, to the direct string $a_1 \cdots a_i$, here, $\overline{\Psi}_{\mathrm{A}}((v_i, v_i)_p)$ is a substring of p and the sources of $\overline{\Psi}_{\mathrm{A}}((v_i, v_i)_p)$ and p coincide (if $i = 0$, take $\overline{\Psi}_{\mathrm{A}}((v_0, v_0)_p)$ the string e_{v_0} of length zero corresponded by the vertex v_0).

For any projective string p lying in Lemma 3.1 (B), assume that it is of the form given by (3.3). By Lemma 3.2, we define a corresponding $\overline{\Psi}_{\mathrm{B}}$ which sends any vertices pair $(x_i, y_j)_p$, $(0 \le i \le r, 0 \le j \le t)$, to the string $a_i^{-1} \cdots a_1^{-1} b_1 \cdots b_j$, here, $\overline{\Psi}_{\mathrm{B}}((x_i, y_i)_p)$ is a substring of p, and the sources of $\overline{\Psi}_{\mathrm{B}}((x_i, y_i)_p)$ and p coincide (if $i = 0$, then $\overline{\Psi}_{\mathrm{B}}((x_i, y_j)_p) = b_1 \cdots b_j$ is a direct string; the case for $j = 0$ is dual; if i and j are zero, then $\overline{\Psi}_{\mathrm{B}}((x_i, y_j)_p)$ is the string e_v of length zero corresponded by the vertex v).

Then, by Lemma 3.3, all strings on the string pair $(\mathcal{Q}, \mathcal{I})$, a bound $((\le 1)^{\text{in}}, (\le 2)^{\text{out}})$-quiver, of A are corresponded by vertices pair by

$$\overline{\Psi} : \mathsf{VP}(A) \to \mathsf{Str}(A), (x, y)_p \mapsto \begin{cases} \overline{\Psi}_{\mathrm{A}}(p), & \text{if } p \text{ lies in Lemma 3.1 (A);} \\ \overline{\Psi}_{\mathrm{B}}(p), & \text{if } p \text{ lies in Lemma 3.1 (B).} \end{cases} \tag{3.4}$$

One can check that $\overline{\Psi}$ is a bijection. Then, by Theorem 2.3, $\overline{\Psi} \circ \mathfrak{M}$ is a bijection from $\mathsf{VP}(A)$ to $\mathsf{ind}(\mathsf{mod}A)$, as required. □

3.3. The countering formula

Now we provide the main result of our paper. In this part, we use $\sharp S$ to represent the number of elements of the set S.

First, we show the following corollary:

For any projective string p, we define

$$\Psi_p = \{\Psi((x, y)_p) \mid (x, y)_p \text{ is a vertices pair of } p\}.$$

The following corollary describes all indecomposable A-modules that are quotients of $\mathfrak{M}(p)$.

Corollary 3.6. *Let p be a projective string on a given bound $((\leq 1)^{\text{in}}, (\leq 2)^{\text{out}})$-quiver $(\mathcal{Q}, \mathcal{I})$, and assume that v is the unique source of p. Then the following statements hold:*

(1) *Ψ_p is the set of all isoclasses of indecomposable A-modules whose top is isomorphic to the simple A-module $S(v)$ corresponding to v.*
(2) *If p is a projective string lying in Lemma 3.1* (A), *then* $\sharp\Psi_p = \dim_{\Bbbk} P(v)$*;*
(3) *If p is a projective string lying in Lemma 3.1* (B), *assume that p equals to of the form given by (3.3), then $\sharp\Psi_p = (r+1)(t+1)$.*

Proof. (1) is a direct corollary of the formula (3.4) in the proof of Proposition 3.5.

(2) Assume that p is of the form given by (3.2), then $\Psi_p = \{a_1 \cdots a_i \mid 0 \leq i \leq l\}$. Thus, $\sharp\Psi_p = l+1 = \dim_{\Bbbk} \mathfrak{M}(p) = \dim_{\Bbbk} P(v)$ ($v = \mathfrak{s}(a_1)$ in this case).

(3) By the definition of a vertices pair of p, we have $\sharp\Psi_p = \binom{r+1}{1}\binom{t+1}{1} = (r+1)(t+1)$ in the case of p lying in Lemma 3.1 (B). □

Let $\mathsf{pStr}(A)$ be the set of all equivalent classes of projective strings. Now we provide the main result of our paper.

Theorem 3.7. *If A is a string $((\leq 1)^{\text{in}}, (\leq 2)^{\text{out}})$-algebra, then*

$$\sharp\mathsf{ind}(\mathsf{mod}A) = \dim_{\Bbbk} A + \sum_{\substack{p\in\mathsf{pStr}(A) \\ \textit{lies in Lemma 3.1 } (\mathrm{B})}} \dim_{\Bbbk} D_{p,1} \cdot \dim_{\Bbbk} D_{p,2},$$

where $D_{p,1}$ and $D_{p,2}$ are the direct summands of $\mathrm{rad}(\mathfrak{M}(p))$ ($p \in \mathsf{pStr}(A)$ *lies in Lemma 3.1* (B)).

Proof. All projective strings can be divided in two parts:

(I) The set $\mathsf{pStr}(A)_{\mathrm{I}}$ of all equivalent classes of projective strings lying in Lemma 3.1 (A);
(II) The set $\mathsf{pStr}(A)_{\mathrm{II}}$ of all equivalent classes of projective strings lying in Lemma 3.1 (B).

By Proposition 3.5, we have $\sharp\mathsf{ind}(\mathsf{mod}A) = \sharp\mathsf{VP}(A)$. Let $\mathsf{VP}_p(A)$ be the set of all vertices pairs of p, then

$$\mathsf{VP}(A) = \bigcup_{p \text{ is a projective string}} \mathsf{VP}_p(A)$$

is a disjoint union, see Corollary 3.6 (1), and so,

$$\sharp\mathsf{ind}(\mathsf{mod}A) = \sharp\mathsf{VP}(A) = \sum_{p\in\mathsf{pStr}(A)_{\mathrm{I}}} \sharp\mathsf{VP}_p(A) + \sum_{p\in\mathsf{pStr}(A)_{\mathrm{II}}} \sharp\mathsf{VP}_p(A).$$

By Corollary 3.6 (2), we have:

- $\sharp\mathsf{VP}_p(A) = \dim_{\Bbbk} \mathfrak{M}(p)$ if p is a projective string lying in $\mathsf{pStr}(A)_{\mathrm{I}}$;
- Let $r_p = \dim_{\Bbbk} D_{p,1}$ and $t_p = \dim_{\Bbbk} D_{p,2}$. By Corollary 3.6 (3), we have $\sharp\mathsf{VP}_p(A) = (r_p+1)(t_p+1) = \dim_{\Bbbk} \mathfrak{M}(p) + r_p t_p$ if p is a projective string

$$x_{r_p} \xleftarrow{a_{r_p}} x_{r_p-1} \xleftarrow{a_{r_p-1}} \cdots \xleftarrow{a_2} x_1 \xleftarrow{a_1} v \xrightarrow{b_1} y_1 \xrightarrow{b_2} \cdots \xrightarrow{b_{t_p-1}} y_{t_p-1} \xrightarrow{b_{t_p}} y_{t_p}.$$

lying in $\mathsf{pStr}(A)_{\mathrm{II}}$.

Therefore,

$$
\begin{aligned}
&\sharp\mathsf{ind}(\mathsf{mod}A) = \sharp\mathsf{VP}(A)\\
=& \sum_{p\in\mathsf{pStr}(A)_{\mathrm{I}}} \dim_{\Bbbk}\mathfrak{M}(p) + \sum_{p\in\mathsf{pStr}(A)_{\mathrm{II}}} (\dim_{\Bbbk}\mathfrak{M}(p) + r_p t_p)\\
=& \dim_{\Bbbk} A + \sum_{p\in\mathsf{pStr}(A)_{\mathrm{II}}} r_p t_p.
\end{aligned}
$$

□

Let $\mathsf{iStr}(A)$ be the set of all equivalent classes of injection strings. We can prove the following result in a dual way.

Theorem 3.8. *If A is a string $((\leq 2)^{\mathrm{in}}, (\leq 1)^{\mathrm{out}})$-algebra, then*

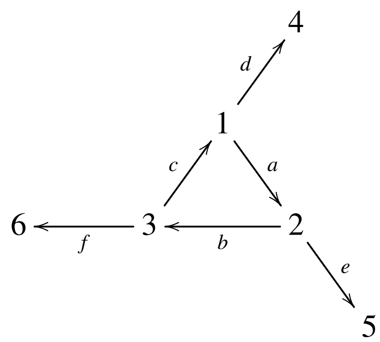
$$
\sharp\mathsf{ind}(\mathsf{mod}A) = \dim_{\Bbbk} A + \sum_{\substack{\iota\in\mathsf{iStr}(A)\\ \textit{lies in Lemma 3.1 }(\mathrm{B'})}} \dim_{\Bbbk} D_{\iota,1}\cdot \dim_{\Bbbk} D_{\iota,2},
$$

where $D_{p,1}$ and $D_{p,2}$ are the direct summands of $\mathfrak{M}(\iota)/\mathrm{sco}(\mathfrak{M}(\iota))$ ($\iota\in\mathsf{iStr}(A)$ lies in Lemma 3.1 (B')).

4. Examples

We provide some examples in this section.

Example 4.1. Let $A = \Bbbk Q/\mathcal{I}$ given by $Q =$



and $\mathcal{I} = \langle abc, bca, cab, ae, bf, cd\rangle$. Then $P(1) = \left(\begin{smallmatrix} & 1 & \\ 2 & & 4 \\ 3 & & \end{smallmatrix}\right)$, $P(2) = \left(\begin{smallmatrix} & 2 & \\ 3 & & 5 \\ 1 & & \end{smallmatrix}\right)$, $P(3) = \left(\begin{smallmatrix} & 3 & \\ 1 & & 6 \\ 2 & & \end{smallmatrix}\right)$, $P(4) = S(4) = (4)$, $P(5) = S(5) = (5)$, and $P(6) = S(6) = (6)$. Furthermore, the Auslander–Reiten quiver of A is shown in Figure 1, and then $\sharp\mathsf{ind}(\mathsf{mod}A) = 21$. On the other hand, $\dim_{\Bbbk} A = \dim_{\Bbbk} \bigoplus_{i=1}^{6} P(i) = 15$, and $\mathrm{rad}P(1) = \binom{2}{3}\oplus(4)$, $\mathrm{rad}P(2) = \binom{3}{1}\oplus(5)$ and $\mathrm{rad}P(3) = \binom{1}{2}\oplus(6)$. We obtain

$$
\begin{aligned}
&\dim_{\Bbbk} A + \dim_{\Bbbk}\tbinom{2}{3}\cdot\dim_{\Bbbk}(4) + \dim_{\Bbbk}\tbinom{3}{1}\cdot\dim_{\Bbbk}(5) + \dim_{\Bbbk}\tbinom{1}{2}\cdot\dim_{\Bbbk}(6)\\
=\ & 15 + 2 + 2 + 2 = 21 = \sharp\mathsf{ind}(\mathsf{mod}A).
\end{aligned}
$$

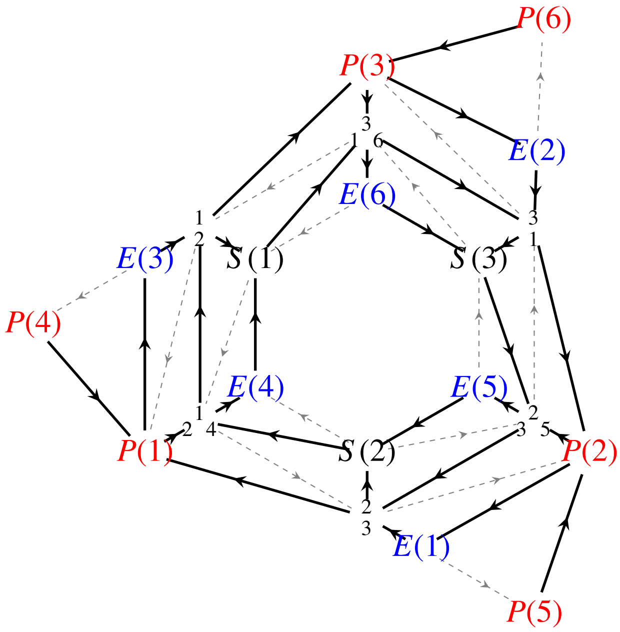


Figure 1. The Auslander–Reiten quiver of A given in Example 4.1.

Example 4.2. Take $A = \Bbbk Q/\mathcal{I}$ with $Q = 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} n$ and $\mathcal{I}$ be an arbitrary admissible ideal. Then the number of indecomposable A-modules equals $\dim_{\Bbbk} A$ (up to isomorphism). In particular, if $\mathcal{I} = \mathrm{rad}^t(\Bbbk Q)$ $(2 \le t \le n)$, then we have

$$\sharp\mathsf{ind}(\mathsf{mod}A) = (n-t+1)t + \sum_{k=1}^{t-1}(t-k) = \frac{2nt+t-t^2}{2}$$

by Theorem 3.7. In particular, if $t = n$, then $\Bbbk Q$ is hereditary; we have $\sharp\mathsf{ind}(\mathsf{mod}A) = \frac{n(n+1)}{2}$ in this case.

5. Conclusions

We obtained the counting formula for indecomposable modules over some string algebras in this paper.

Author contributions

Haicun Wen: Conceptualization, Methodology, Validation, Writing-review and editing, Visualization; Mian-Tao Liu: Methodology, Validation, Writing-original draft preparation; Yu-Zhe Liu: Conceptualization, Methodology, Validation, Writing-review and editing; All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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