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Research article

A note on 2D Navier-Stokes system in a bounded domain

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Abstract: This paper poses a new question and proves a related result. Particularly, the nonexistence of a nontrivial time-periodic solution to the Navier–Stokes system is proved in a bounded domain in $\mathbb{R}^2$.

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Mathematics Subject Classification: 35Q30, 76D03, 76D05

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain with a smooth boundary $\partial\Omega$ whose outward unit normal vector is denoted by n. For any $L > 0$, we consider the following problem for the 2D Navier–Stokes system: Find $\{u, p, h(t)\}$ that satisfy the following conditions:

$$\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = 0, \ \ \operatorname{div} u = 0 \ \text{ in } \ \Omega \times (0, \infty), \tag{1.1}$$

$$u \cdot n = 0, \omega := \operatorname{curl} u = h(t) \ \text{ on } \ \partial\Omega \times (0, \infty), \tag{1.2}$$

$$\int_\Omega \omega \mathrm{d}x = L \ \text{ in } \ (0, \infty) \ \text{ and } \ u(\cdot, 0) = u_0 \ \text{ in } \ \Omega. \tag{1.3}$$

Here $u : \Omega \times (0, \infty) \to \mathbb{R}^2$ is the velocity, $p : \Omega \times (0, \infty) \to \mathbb{R}$ is the pressure, and $\omega : \Omega \times (0, \infty) \to \mathbb{R}$ is the vorticity given by $\omega := \operatorname{curl} u := \partial_1 u_2 - \partial_2 u_1$ for $u := \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, and $h(t)$ is the unknown applied vorticity on the boundary which is independent of the space variable $x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$.

In fact, the problem (1.1)–(1.3) is an inverse problem. We think it is also a control problem. Can we find a simple boundary control (1.2) such that (1.3) holds true?

We refer to [1] for a study of the direct problem of the 2D Navier–Stokes in a bounded domain.

The first aim of this short note is to formulate an open problem:
Open problem: *Show the well-posedness of solutions to the boundary value problem* (1.1)–(1.3).

Remark 1.1. *A similar problem has been solved for the 2D time-dependent Ginzburg–Landau model in superconductivity [2, 3], in which the magnetic potential A satisfying $A \cdot n = 0, \operatorname{curl} A = h(t)$ on $\partial\Omega$ and $\int_\Omega \operatorname{curl} A \mathrm{d}x = L$.*

Next, we consider the time-periodic solutions to the problem (1.1)–(1.3). We will prove it.

Theorem 1.1. *Let $\{u, p, h(t)\}$ be time-periodic smooth solutions with period $T > 0$ to the problem* (1.1)–(1.3)*. Then we have*

$$\partial_t u = 0, h'(t) = 0, \text{ and } \omega(x,t) = h(t) = \frac{L}{|\Omega|} \text{ for any } (x,t) \in \Omega \times (0,\infty). \tag{1.4}$$

Remark 1.2. *Here we assume that*

$$\begin{aligned} &u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2), \partial_t u \in L^2(0,T;L^2), \\ &\nabla p \in L^2(0,T;L^2), \text{ and } h \in L^\infty(0,T). \end{aligned}$$

Corollary 1.1. *Let $\Omega := \{(x_1,x_2); x_1^2 + x_2^2 < 1\}$. Then the unique time-periodic (stationary) solutions are given by*

$$u = \frac{h}{2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, p = \frac{h^2}{4}(x_1^2 + x_2^2), h = \frac{L}{\pi}. \tag{1.5}$$

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.
First, taking the curl of (1.1), we see that

$$\partial_t \omega + u \cdot \nabla\omega - \Delta\omega = 0. \tag{2.1}$$

Using the fact that $h(t)$ is independent of x, we have

$$\partial_t(\omega - h) + u \cdot \nabla(\omega - h) - \Delta(\omega - h) = -h'(t). \tag{2.2}$$

Testing (2.2) by $(\omega - h)$ over $\Omega \times [0,T]$, we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^T \int_\Omega \partial_t(\omega-h)^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T \int_\Omega \operatorname{div}[u(\omega-h)^2]\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega |\nabla(\omega-h)|^2 \mathrm{d}x\mathrm{d}t \\ = \quad & -\int_0^T \int_\Omega h'(\omega-h)\mathrm{d}x\mathrm{d}t \\ = \quad & -\int_0^T (h'L - |\Omega|hh')\mathrm{d}t \\ = \quad & -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(hL - \frac{|\Omega|}{2}h^2\right)\mathrm{d}t = 0, \end{aligned}$$

where we have used the time periodicity of h.

By the time periodicity of ω and h,

$$\int_0^T \int_\Omega \partial_t(\omega - h)^2 dxdt = 0.$$

By the Gauss integral formula and (1.2),

$$\int_\Omega \mathrm{div}[u(\omega - h)^2]dx = 0.$$

Therefore,

$$\int_0^T \int_\Omega |\nabla(\omega - h)|^2 \mathrm{d}x\mathrm{d}t = 0. \tag{2.3}$$

Using Poincaré inequality

$$\|\omega - h\|_{L^2} \lesssim \|\nabla(\omega - h)\|_{L^2} = 0, \tag{2.4}$$

we conclude that

$$\omega = h(t) \text{ in } \Omega \times (0, \infty). \tag{2.5}$$

It follows from (1.3) and (2.5) that

$$h(t) = \frac{L}{|\Omega|} \tag{2.6}$$

and thus

$$\partial_t \omega = h'(t) = 0. \tag{2.7}$$

Now using the vector Poincaré type inequality [4, Page 75]:

$$\|u\|_{L^2} \lesssim \|\mathrm{div}\, u\|_{L^2} + \|\mathrm{curl}\, u\|_{L^2} \tag{2.8}$$

with $u \cdot n = 0$ on $\partial\Omega$ applied to $\partial_t u$ shows

$$\partial_t u = 0. \tag{2.9}$$

This completes the proof.

□

Proof of Corollary 1.1: By Theorem 1.1, it is easy to show the uniqueness of stationary solutions, and then it is easy to show that (1.5) is the only solution. In fact, the stationary solutions satisfy

$$\mathrm{div}\, u = 0, \mathrm{curl}\, u = \frac{L}{|\Omega|} \text{ in } \Omega \text{ and } u \cdot n = 0 \text{ on } \partial\Omega,$$

which yields that u is unique due to (2.8).

□

3. Conclusions

To summarize, in this paper, we first formulated an open question: Show the existence of strong solutions to the initial boundary value problem (1.1)–(1.3). Next, we proved a Liouville-type theorem for the time-periodic solutions to the problem. Finally, we show that the stationary problem has a unique solution and give the exact form when Ω is a unit ball in $\mathbb{R}^2$.

Author contributions

Fan and Ozawa: Writing-original draft, Writing-review and editing. All authors have read and approved the final version of manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Prof. Tohru Ozawa is an editorial board member for AIMS Mathematics and was not involved in the editorial review and the decision to publish this article. The authors declare that they have no conflicts of interest.

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