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***Research article***

# Global attractors for a class of viscoelastic plate equations with past history

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**Abstract:** This paper is concerned with a class of viscoelastic plate equations with past history. We first transform the original initial-boundary value problem into an equivalent one by means of the history space framework. Then we use the perturbed energy method to establish a stabilizability estimate. By employing the gradient property and quasi-stability of the dynamical system, we obtain the existence of a global attractor with finite fractal dimension.

**Keywords:** viscoelastic plate equations; past history; global attractors; gradient dynamical system; quasi-stability
**Mathematics Subject Classification:** 35B41, 35L75, 37N15

## 1. Introduction

In this paper, we study the following viscoelastic plate equation of Kirchhoff-like type governing the small-deflection vibrations of viscoelastic thin homogeneous and isotropic plates

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \int_{-\infty}^{t} g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \mathrm{div}H(\nabla u) - \Delta u_t + f(u) = \phi(x), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

with simply supported boundary condition

$$u = \Delta u = 0, \quad x \in \partial\Omega,\ t \in \mathbb{R} \tag{1.2}$$

and initial conditions

$$u(x,t) = u_0(x,t),\ u_t(x,0) = u_1(x), \quad x \in \Omega,\ t \le 0, \tag{1.3}$$

where $u := u(x,t)$ represents the deflection at time $t$ of a filament having position $x$ in a given reference configuration, $u_0 : \Omega \times (-\infty, 0] \to \mathbb{R}$ is a prescribed past history, $u_1(x) = \partial_t u_0(x,t)|_{t=0}$, and $\Omega$ is a bounded domain of $\mathbb{R}^N$ ($N \geq 1$) with a smooth boundary $\partial\Omega$. The memory kernel $g$, the vector field $H : \mathbb{R}^N \to \mathbb{R}^N$, the nonlinear source term $f$, and the external force $\phi$ will be specified later. By variable substitution, Eq (1.1) is sometimes written in the equivalent form

$$u_{tt} - \Delta u_{tt} + \Delta^2 u - \int_0^\infty g(\tau)\Delta^2 u(t-\tau)\,\mathrm{d}\tau - \mathrm{div}H(\nabla u) - \Delta u_t + f(u) = \phi(x).$$

Equation (1.1) arises from the following Mindlin-Timoshenko plate model [1,2] taking into account transverse shear deformation that is usually neglected by the viscoelastic plate equation of Kirchhoff type

$$\begin{cases} \rho\hbar u_{tt} - K\mathrm{div}(\nabla u + \Psi) = f, \\ \dfrac{\rho\hbar^3}{12}\Psi_{tt} - DS + K(\nabla u + \Psi) = H, \end{cases}$$

where $\rho$ is the density, $\hbar$ is the thickness, $K = kY\hbar/(2(1+r))$ is the shear modulus, $D = Y\hbar^3/(12(1-r^2))$ is the flexural rigidity, $0 < r < 1/2$ is Poisson's ratio, $Y$ is Young's modulus, and $k$ is the shear correction coefficient. In the two-dimensional case, $\Psi = (\psi, \varphi)$, where $\psi = \psi(x_1, x_2, t)$ and $\varphi = \varphi(x_1, x_2, t)$ correspond to rotation angles of the filament. According to the theory of elasticity, the stress tensor $S = A\Psi$, where

$$A = \begin{bmatrix} \partial_{x_1x_1} + \dfrac{1-r}{2}\partial_{x_2x_2} & \dfrac{1+r}{2}\partial_{x_1x_2} \\ \dfrac{1+r}{2}\partial_{x_1x_2} & \dfrac{1-r}{2}\partial_{x_1x_1} + \partial_{x_2x_2} \end{bmatrix}.$$

For viscoelastic thin plates, $S$ can be expressed in the form [3]

$$S = A\Psi - \int_0^\infty g(s)A\Psi(t-s)\,\mathrm{d}s.$$

Thus, in nonconservative systems, we can arrive at the following viscoelastic Mindlin-Timoshenko plate model

$$\begin{cases} \rho\hbar u_{tt} - K\mathrm{div}(\nabla u + \Psi) + f(u) = \phi(x), \\ \dfrac{\rho\hbar^3}{12}\Psi_{tt} - D\left(A\Psi - \displaystyle\int_0^\infty g(s)A\Psi(t-s)\,\mathrm{d}s\right) + K(\nabla u + \Psi) = H(-\Psi) - \nu\Psi_t, \end{cases} \tag{1.4}$$

where $\nu \geq 0$ is the damping coefficient. Substitution of $(1.4)_2$ into $(1.4)_1$ gives

$$\rho\hbar u_{tt} + \frac{\rho\hbar^3}{12}\mathrm{div}\Psi_{tt} - D\mathrm{div}\left(A\Psi - \int_0^\infty g(s)A\Psi(t-s)\,\mathrm{d}s\right) - \mathrm{div}H(-\Psi) + \nu\mathrm{div}\Psi_t + f(u) = \phi(x). \tag{1.5}$$

Taking the Kirchhoff limit $k \to \infty$, we have $\Psi = -\nabla u$. Under a normalization of coefficients, (1.5) becomes

$$u_{tt} - \sigma\Delta u_{tt} + \Delta^2 u - \int_{-\infty}^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \mathrm{div}H(\nabla u) - \nu\Delta u_t + f(u) = \phi(x),$$

where $\sigma = \hbar^2/12$. Since the results of this paper are independent of the coefficients in the equation, we take $\sigma = \nu = 1$ for simplicity. Thus, Eq (1.1) is derived. The above derivation is still valid for the higher-dimensional case. As for the one-dimensional case, Eq (1.1) can be obtained from the Timoshenko beam model [4]. As shown above, Eq (1.1) is a simple form of the viscoelastic Mindlin-Timoshenko plate model and the viscoelastic Timoshenko beam model. From the point of view of applications, when the viscoelastic Mindlin-Timoshenko plate model or the viscoelastic Timoshenko beam model is too complex to be used to deal with practical problems, we could attempt to apply Eq (1.1). And, the global well-posedness and long-time dynamics of solution are important steps in studying Eq (1.1).

Viscoelastic plate equations belong to so-called equations with memory [5]. Under certain assumptions on the memory kernel (and the nonlinear source term), the global well-posedness and decay of solutions to viscoelastic plate equations of Kirchhoff type have been widely studied. Alabau-Boussouira and Cannarsa [6] provided a general condition on which the total energy function of solutions is shown to decay at least as fast as the memory kernel at infinity. Muñoz Rivera et al. [7] considered a viscoelastic plate equation of Kirchhoff type with rotational inertia term, and proved that the energy decays to zero with the same rate as the memory kernel. Muñoz Rivera and Fatori [8] obtained the global existence, uniqueness, and exponential decay of solution to a viscoelastic plate equation of Kirchhoff type with strong damping term. Cavalcanti et al. [9] got the global existence and uniqueness of solution and the exponential decay of the energy for a viscoelastic plate equation of Kirchhoff type with weak damping term. Alabau-Boussouira et al. [10] derived the exponential and polynomial energy decay with the same rate as the memory kernel for a viscoelastic plate equation of Kirchhoff type with nonlinear source term. Cannarsa and Sforza [11] obtained the existence and uniqueness of mild and strong solutions and the exponential decay of the energy for a viscoelastic plate equation of Kirchhoff type with nonlinear source term. Hajjej [12] established the global existence and uniqueness of solution and the general energy decay depending on the memory kernel for a viscoelastic plate equation of Kirchhoff type with nonlinear damping and source terms. Regarding viscoelastic plate equations of Kirchhoff-like type similar to (1.1), Jorge Silva et al. [13] studied

$$u_{tt} - \sigma\Delta u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \mathrm{div}H(\nabla u) = 0, \tag{1.6}$$

where $\sigma \geq 0$ is a constant. They obtained the global well-posedness and regularity of solution, and employed the perturbed energy method to derive the general rates of energy decay. Gomes Tavares et al. [14] also dealt with Eq (1.6), and obtained sharp energy decay rate. In the case $H(s) = |s|^{p-2}s$, Jorge Silva and Ma [15] investigated a viscoelastic plate equation of Kirchhoff-like type

$$u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t + f(u) = \phi(x),$$

where $\alpha > 0$ is a constant. They established the global well-posedness and regularity of solution, and proved the exponential decay of the energy by using the perturbed energy method when $\phi = 0$. Pereira et al. [16] studied a viscoelastic equation of Kirchhoff-like type with logarithmic source term

$$u_{tt} + \Delta^2 u + \int_0^t g(t-\tau)\Delta u(\tau)\,\mathrm{d}\tau - \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t = |u|^{q-2}u\ln|u|,$$

and obtained the global existence, uniqueness, exponential decay, and blow-up of solution by using the theory of potential wells. In the case $H(s) = (H_1, H_2, \cdots, H_N)$ with $H_i(s) = |s_i|^{p_i(x,t)-2}s_i$ for all $s =$

$(s_1, s_2, \cdots, s_N) \in \mathbb{R}^N$ and $i = 1, 2, \cdots, N$, Merah and Mesloub [17] studied the following viscoelastic plate equation of Kirchhoff-like type with nonstandard growth conditions and time delay

$$u_{tt} + \Delta^2 u - \int_0^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x,t)-2}\frac{\partial u}{\partial x_i}\right) - \nu\Delta u_t - \nu'\Delta u_t(t-s) = |u|^{q-2}u,$$

and proved the blow-up of solution with non-positive and positive initial energy, respectively.

As is well known, attractors are an effective way to describe the long-time dynamics of solutions to nonlinear evolution equations. In an autonomous infinite-dimensional dynamical system, the existence of a global attractor can be derived by verifying the existence of an absorbing set and the compactness of the semigroup. The commonly used compactness mainly includes: uniform compactness [18], asymptotic compactness [19,20], asymptotic smoothness [21], and Condition (C) [22], which could be chosen according to the characteristics of the problem under consideration. Peng et al. [23] studied the following viscoelastic plate equation of Kirchhoff-like type

$$u_{tt} - \Delta u_{tt} + \alpha\Delta^2 u - \int_0^\infty g(\tau)\Delta^2 u(t-\tau)\,\mathrm{d}\tau - \mathrm{div}H(\nabla u) - \Delta u_t + f(u) = \phi(x), \tag{1.7}$$

and established the existence of a global attractor with finite fractal dimension by verifying the dissipativity and asymptotic smoothness of the corresponding dynamical system. In addition, the gradient property and quasi-stability of the dynamical system discussed by Chueshov and Lasiecka [24] have been used to investigate the existence of a global attractor. Feng et al. [25] considered

$$u_{tt} - \Delta u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \mathrm{div}H(\nabla u) = \phi(x),$$

and obtained the existence of a global attractor and its properties by verifying the gradient property and quasi-stability. In the case $H(s) = |s|^{p-2}s$, Liu and Jorge Silva [26] investigated the following viscoelastic plate equation of Kirchhoff-like type with nonlocal damping term

$$u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau - \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) - M\left(\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x\right)\Delta u_t + f(u) = \phi(x).$$

They established the global well-posedness of solution and obtained the existence of a global attractor as well as its properties by verifying the gradient property and quasi-stability of the corresponding dynamical system.

Equation (1.1) considered in this paper is essentially the same as Eq (1.7) in [23], except for slight differences in the coefficient normalization. The result of [23] mentioned above requires that one of the assumptions on the nonlinear source term $f$ is $-\mu \le F(u) \le uf(u)$ for some constant $\mu \ge 0$, where $F(u) = \int_0^u f(s)\,\mathrm{d}s$ and $u \in \mathbb{R}$. Thus, a natural question is whether the corresponding dynamical system possess a global attractor under the assumption $F(u) > uf(u)$. Motivated by this question, we aim to handle the existence of a global attractor for problem (1.1)–(1.3) under assumption $(\mathrm{A}_f)$ that will be seen in Section 2. Compared with previous work [23], assumption $(\mathrm{A}_f)$ on the nonlinear source term $f$ in the present paper is different, which provides another sufficient condition for the existence of a global attractor for problem (1.1)–(1.3). It is worth mentioning that it can be seen from (2.5) and (2.6) in $(\mathrm{A}_f)$ that there is no requirement for the size relationship between $F(u)$ and $uf(u)$. In other words, (2.5) and (2.6) in ($A_f$) not only include $F(u) \le uf(u)$, but also allow $F(u) > uf(u)$. On the other hand, (2.5) and (2.6) in ($A_f$) provide a more detailed characterization for lower bounds of $F(u)$ and $uf(u)$. Our main technical tools are the gradient property and quasi-stability of the dynamical system. In the gradient dynamical system, the quasi-stability can conveniently induce the asymptotic smoothness, which further allows us to readily obtain the existence of the global attractor. At the same time, geometric structure and finite fractal dimension of the global attractor can be obtained along with the existence. Our main result shows that even if $f$ satisfies ($A_f$), the corresponding dynamical system still possesses a global attractor with finite fractal dimension. Thus, we supplement the results of previous work [23], and add new content to the qualitative study on viscoelastic plate equations of Kirchhoff-like type.

The rest of this paper is organized as follows. In Section 2, we display some notations and assumptions on $g$, $H$, and $f$. By means of the history space framework [27, 28], we transform (1.1)–(1.3) into an equivalent problem. In Section 3, we introduce several definitions and properties on the gradient dynamical system and quasi-stability. In Section 4, we prove the existence of a global attractor with finite fractal dimension. In Section 5, we summarize our work.

## 2. Preliminaries

### 2.1. Notations and assumptions

Throughout the paper, for the sake of simplicity, we denote

$$\|\cdot\|_p := \|\cdot\|_{L^p(\Omega)}, \ \ \|\cdot\| := \|\cdot\|_2.$$

Moreover, $(\cdot,\cdot)$ denotes either the $L^2$-inner product or a duality pairing between a space and its dual space, $|\Omega|$ represents the Lebesgue measure of $\Omega$, $C$ represents a generic positive constant that may be different even in the same formula, $C(\cdot,\cdots,\cdot)$ stands for a positive constant depending on the quantities appearing in the parenthesis, and $\mathfrak{C}_1, \mathfrak{C}_2$ represent the embedding constants for inequalities

$$\|u\| \le \mathfrak{C}_1\|\nabla u\|, \ \ \|u\| \le \mathfrak{C}_2\|\Delta u\|, \ \ \|\nabla u\| \le \mathfrak{C}_3\|\Delta u\|.$$

As in [15], we make the following assumptions on $g$.
($A_g$): $g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $g(t) \ge 0$ and $g'(t) \le 0$ for all $t \in [0,\infty)$, and

$$\kappa := 1 - \int_0^\infty g(t)\,\mathrm{d}t > 0. \tag{2.1}$$

Concerning $H$, we adopt the following assumptions suggested in [13].
($A_H$): there exist $a_i > 0$ and $p_i$ satisfying

$$2 \le p_i < \infty \text{ if } N \le 2, \ 2 \le p_i \le \frac{2N-2}{N-2} \text{ if } N > 2,$$

such that the $C^1$-vector field $H = (H_1, H_2, \cdots, H_N)$ satisfies

$$|\nabla H_i(s)| \le a_i\left(|s|^{p_i-2} + 1\right), \ \ i = 1,2,\cdots,N, \ s \in \mathbb{R}^N. \tag{2.2}$$

Moreover, $H$ is a conservative vector field, that is, there exists a real valued function $h : \mathbb{R}^N \to \mathbb{R}$ such that $H = \nabla h$. And,

$$-\frac{\lambda_1}{2}|s|^2 - \lambda_2 \le h(s) \le H(s)\cdot s + \frac{\lambda_1}{2}|s|^2, \tag{2.3}$$

where $0 \le \lambda_1 < \kappa/(2\mathfrak{C}_3^2)$ and $\lambda_2 \ge 0$.

Now we give the following assumptions on $f$.

($\mathrm{A}_f$): $f \in C^1(\mathbb{R})$, and there exists a constant $b > 0$ such that

$$|f'(u)| \le b\left(|u|^{q-2} + 1\right), \quad u \in \mathbb{R}, \tag{2.4}$$

where

$$2 < q < \infty \text{ if } N \le 4;\ 2 < q \le \frac{2N-4}{N-4} \text{ if } N > 4.$$

Moreover, there exist constants $0 \le \eta < \kappa/(2\mathfrak{C}_2^2)$ and $\mu > 0$ such that

$$uf(u) \ge -\eta u^2 - \mu \tag{2.5}$$

and

$$F(u) \ge -\frac{\eta}{2}u^2 - \mu, \tag{2.6}$$

where

$$F(u) = \int_0^u f(s)\,\mathrm{d}s.$$

*2.2. Reformulation of the problem*

We define an auxiliary function

$$v := v^t(x,\tau) = u(x,t) - u(x,t-\tau), \quad x \in \Omega,\ \tau > 0,\ t \ge 0. \tag{2.7}$$

Then, from (2.1) in ($\mathrm{A}_g$), the memory term in (1.1) can be rewritten as

$$\begin{aligned} -\int_{-\infty}^t g(t-\tau)\Delta^2 u(\tau)\,\mathrm{d}\tau &= -\int_0^\infty g(\tau)\Delta^2 u(t-\tau)\,\mathrm{d}\tau \\ &= -(1-\kappa)\Delta^2 u + \int_0^\infty g(\tau)\Delta^2 v(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Consequently, (1.1) is transformed into the system

$$\begin{cases} u_{tt} - \Delta u_{tt} + \kappa\Delta^2 u + \displaystyle\int_0^\infty g(\tau)\Delta^2 v(\tau)\,\mathrm{d}\tau - \mathrm{div}H(\nabla u) - \Delta u_t + f(u) = \phi(x), & x \in \Omega,\ t > 0, \\ v_t = u_t - v_\tau, \quad x \in \Omega,\ \tau > 0,\ t > 0, \end{cases} \tag{2.8}$$

with boundary conditions

$$\begin{cases} u = \Delta u = 0, & x \in \partial\Omega,\ t > 0, \\ v = \Delta v = 0, & x \in \partial\Omega,\ \tau > 0,\ t > 0, \end{cases} \tag{2.9}$$

and initial conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v^0(x,\tau) = v_0(x,\tau), \quad x \in \Omega,\ \tau > 0, \end{cases} \tag{2.10}$$

where

$$\begin{aligned} &u_0(x) := u_0(x,0), \quad x \in \Omega, \\ &v_0(x,\tau) := u_0(x,0) - u_0(x,-\tau), \quad x \in \Omega,\ \tau > 0. \end{aligned}$$

We introduce

$$H_3(\Omega) := \left\{ u \in H^3(\Omega) \cap H_0^1(\Omega) | \Delta u \in H_0^1(\Omega) \right\},$$

which is a Hilbert space equipped with inner product and norm (see [13, 15])

$$(u,w)_{H_3(\Omega)} := (\nabla\Delta u, \nabla\Delta w), \quad \|u\|_{H_3(\Omega)} := \|\nabla\Delta u\|.$$

We introduce two weighted $L^2$-spaces

$$\begin{aligned} L_g :=& L_g^2\left(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega)\right) \\ =& \left\{ v : \mathbb{R}^+ \to H^2(\Omega) \cap H_0^1(\Omega) \,\middle|\, \int_0^\infty g(\tau)\|\Delta v(\tau)\|^2 \,\mathrm{d}\tau < \infty \right\} \end{aligned}$$

and

$$L_{g,3} := L_g^2(\mathbb{R}^+; H_3(\Omega)) = \left\{ v : \mathbb{R}^+ \to H_3(\Omega) \,\middle|\, \int_0^\infty g(\tau)\|\nabla\Delta v(\tau)\|^2 \,\mathrm{d}\tau < \infty \right\}.$$

They are Hilbert spaces equipped with inner products and norms (see [13, 15])

$$(v,w)_g := \int_0^\infty g(\tau)(\Delta v(\tau), \Delta w(\tau)) \,\mathrm{d}\tau, \quad \|v\|_g^2 := \int_0^\infty g(\tau)\|\Delta v(\tau)\|^2 \,\mathrm{d}\tau,$$

$$(v,w)_{g,3} := \int_0^\infty g(\tau)(\nabla\Delta v(\tau), \nabla\Delta w(\tau)) \,\mathrm{d}\tau, \quad \|v\|_{g,3}^2 := \int_0^\infty g(\tau)\|\nabla\Delta v(\tau)\|^2 \,\mathrm{d}\tau.$$

**Definition 2.1** (Weak solutions)**.** *We say that* $(u, u_t, v)$ *is a weak solution to problem* (2.8)–(2.10) *if* $u \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$*,* $u_t \in C([0,T]; H_0^1(\Omega))$*,* $v \in C([0,T]; L_g)$*,* $u(0) = u_0$ *in* $H^2(\Omega) \cap H_0^1(\Omega)$*,* $u_t(0) = u_1$ *in* $H_0^1(\Omega)$*,* $v^0 = v_0$ *in* $L_g$*, and*

$$\begin{cases} (u_t(t), w_1) + (\nabla u_t(t), \nabla w_1) + \kappa \int_0^t (\Delta u(s), \Delta w_1) \,\mathrm{d}s + \int_0^t (v^s, w_1)_g \,\mathrm{d}s \\ \qquad + \int_0^t (H(\nabla u(s)), \nabla w_1) \,\mathrm{d}s + (\nabla u(t), \nabla w_1) + \int_0^t (f(u(s)), w_1) \,\mathrm{d}s \\ \qquad = \int_0^t (\phi, w_1) \,\mathrm{d}s + (u_1, w_1) + (\nabla u_1, \nabla w_1) + (\nabla u_0, \nabla w_1), \\ (v^t, w_2)_g = (u(t), w_2)_g - (u_0, w_2)_g - \int_0^t (v_\tau^s, w_2)_g \,\mathrm{d}s + (v_0, w_2)_g, \end{cases} \tag{2.11}$$

*for any* $w_1 \in H^2(\Omega) \cap H_0^1(\Omega)$*,* $w_2 \in L_g$*, and* $t \in (0,T]$*.*

**Remark 2.2.** *System* (2.11) *implies that*

$$\begin{cases}(u_{tt}(t), w_1) + (\nabla u_{tt}(t), \nabla w_1) + \kappa(\Delta u(t), \Delta w_1) + (v^t, w_1)_g \\ \qquad\qquad +(H(\nabla u(t)), \nabla w_1) + (\nabla u_t(t), \nabla w_1) + (f(u(t)), w_1) = (\phi, w_1), \\ (v^t_t, w_2)_g = (u_t(t), w_2)_g - (v^t_\tau, w_2)_g,\end{cases} \tag{2.12}$$

*for a.e.* $t \in (0, T)$.

By using analogous arguments to those used in the proof of [13, Theorem 2.3], we have the following global well-posedness of solution to problem (2.8)–(2.10).

**Theorem 2.3** (Global well-posedness)**.** *Let* $(A_g)$*,* $(A_H)$*, and* $(A_f)$ *be fulfilled. Assume that* $\phi \in L^2(\Omega)$*. Then,*

*(i) if* $(u_0, u_1, v_0) \in Y := \left(H^2(\Omega) \cap H^1_0(\Omega)\right) \times H^1_0(\Omega) \times L_g$*, then for any* $T > 0$ *problem* (2.8)–(2.10) *admits a unique solution depending continuously on the initial data, where*

$$u \in C\left([0, T]; H^2(\Omega) \cap H^1_0(\Omega)\right), \quad u_t \in C\left([0, T]; H^1_0(\Omega)\right), \quad v^t \in C\left([0, T]; L_g\right); \tag{2.13}$$

*(ii) if* $(u_0, u_1, v_0) \in H_3(\Omega) \times \left(H^2(\Omega) \cap H^1_0(\Omega)\right) \times L_{g,3}$*, then for any* $T > 0$ *problem* (2.8)–(2.10) *has a unique solution depending continuously on the initial data, which possesses higher regularity*

$$u \in C\left([0, T]; H_3(\Omega)\right), \quad u_t \in C\left([0, T]; H^2(\Omega) \cap H^1_0(\Omega)\right), \quad v^t \in C\left([0, T]; L_{g,3}\right). \tag{2.14}$$

Define an operator $S(t) : Y \to Y$ by

$$S(t)(u_0, u_1, v_0) := (u(t), u_t(t), v^t).$$

Then, it is easy to see from Theorem 2.3 that $\{S(t)\}_{t\geq 0}$ is a $C^0$-semigroup generated by problem (2.8)–(2.10).

At the end of this subsection, we recall and list the above definitions and notations that will continue to be used in Section 4.

$$v := v^t(x, \tau) = u(x, t) - u(x, t - \tau),$$

$$L_g := \left\{ v : \mathbb{R}^+ \to H^2(\Omega) \cap H^1_0(\Omega) \,\middle|\, \int_0^\infty g(\tau) \|\Delta v(\tau)\|^2 \,\mathrm{d}\tau < \infty \right\},$$

$$(v, w)_g := \int_0^\infty g(\tau)(\Delta v(\tau), \Delta w(\tau)) \,\mathrm{d}\tau,$$

$$\|v\|^2_g := \int_0^\infty g(\tau) \|\Delta v(\tau)\|^2 \,\mathrm{d}\tau,$$

$$Y := \left(H^2(\Omega) \cap H^1_0(\Omega)\right) \times H^1_0(\Omega) \times L_g.$$

## 3. Materials and methods

We will employ the gradient property and quasi-stability of the dynamical system to prove the existence of a global attractor with finite fractal dimension. For the convenience of the reader, we first introduce several definitions and properties of the gradient dynamical system and quasi-stability in [24], which will play a crucial role in the proof of our main result.

Generally, in terms of [24, Definition 7.5.3], the gradient dynamical system is defined as follows.

**Definition 3.1** ( [24]). *A dynamical system $(Y, S(t))$ is said to be gradient if there exists a strict Lyapunov functional $L$ for $(Y, S(t))$ on the whole phase space $Y$, that is:*

*(i) the function $t \mapsto L(S(t)y)$ is non-increasing for any $y \in Y$;*
*(ii) the equation $L(S(t)y) = L(y)$ for all $t > 0$ and some $y \in Y$ implies that $S(t)y = y$ for all $t > 0$.*

Under appropriate conditions, the existence and structure of global attractors for a gradient and asymptotically smooth dynamical system can be provided by [24, Corollary 7.5.7], namely, the following theorem.

**Theorem 3.2** ( [24]). *Assume that $(Y, S(t))$ is a gradient and asymptotically smooth dynamical system, and its Lyapunov function $L(x)$ is bounded from above on any bounded subset of $Y$. In addition, assume that the set $L_R := \{x \in Y | L(x) \leq R\}$ is bounded for every $R$. If the set $\mathcal{N}$ of stationary points of $(Y, S(t))$, that is,*

$$\mathcal{N} := \{\varphi \in Y | S(t)\varphi = \varphi \text{ for all } t \geq 0\},$$

*is bounded, then $(Y, S(t))$ possesses a compact global attractor $A = \mathcal{M}^z(\mathcal{N})$.*

In order to better introduce the quasi-stability of a dynamical system, we have the following assumptions.

(A): Let $U$, $V$, and $W$ be three reflexive Banach spaces with $U$ compactly embedded in $V$. We endow the space $Y := U \times V \times W$ with the norm

$$\|(u(t), u_t(t), v(t))\|_Y^2 := \|(u(t)\|_U^2 + \|u_t(t)\|_V^2 + \|v(t)\|_W^2, \quad (u(t), u_t(t), v(t)) \in Y.$$

We assume that $(Y, S(t))$ is a dynamical system on $Y$ with an evolution operator of the form

$$S(t)y := (u(t), u_t(t), v(t)), \quad y := (u_0, u_1, v_0) \in Y,$$

where

$$u \in C(\mathbb{R}^+; U) \cap C^1(\mathbb{R}^+; V), \quad v \in C(\mathbb{R}^+; W).$$

The definition of the quasi-stability of a dynamical system is given by [24, Definition 7.9.2], namely, the following definition.

**Definition 3.3** ( [24]). *The dynamical system $(Y, S(t))$ satisfying* (A) *is said to be quasi-stable on a set $B \subset Y$ if there exist a compact seminorm $n_U(\cdot)$ on the space $U$ and non-negative functions $\varsigma_i(t)$ $(i = 1, 2, 3)$ such that*

*(i) $\varsigma_1(t)$ and $\varsigma_3(t)$ are locally bounded on $[0, \infty)$;*
*(ii) $\varsigma_2 \in L^1(\mathbb{R}^+)$ and $\lim_{t\to\infty} \varsigma_2(t) = 0$;*

*(iii) the relations*

$$\|S(t)\bar{y} - S(t)y\|_Y^2 \le \varsigma_1(t)\|\bar{y} - y\|_Y^2 \tag{3.1}$$

*and*

$$\|S(t)\bar{y} - S(t)y\|_Y^2 \le \varsigma_2(t)\|\bar{y} - y\|_Y^2 + \varsigma_3(t) \sup_{0<s<t} (n_U(\bar{u}(s) - u(s)))^2 \tag{3.2}$$

*hold for every* $y, \bar{y} \in B$ *and* $t > 0$*, where* $S(t)\bar{y} := (\bar{u}(t), \bar{u}_t(t), \bar{v}(t))$ *and* $\bar{y} := (\bar{u}_0, \bar{u}_1, \bar{v}_0)$*.*

A quasi-stable dynamical system possesses the following properties from [24, Proposition 7.9.4 and Theorem 7.9.6].

**Proposition 3.4** ( [24])**.** *Let* (A) *be fulfilled. Assume that the dynamical system* $(Y, S(t))$ *is quasi-stable on every bounded positively invariant set* $B$ *of* $Y$*. Then,* $(Y, S(t))$ *is asymptotically smooth.*

**Theorem 3.5** ( [24])**.** *Let* (A) *be fulfilled. Assume that the dynamical system* $(Y, S(t))$ *possesses a compact global attractor* $A$ *and is quasi-stable on* $A$*. Then,* $A$ *has finite fractal dimension.*

## 4. Results

In this section, concerning problem (2.8)–(2.10), we write

$$\|S(t)y\|_Y^2 := \kappa\|\Delta u(t)\|^2 + \|u_t(t)\|^2 + \|\nabla u_t(t)\|^2 + \|v^t\|_g^2, \quad y := (u_0, u_1, v_0).$$

We start with the following lemma, which will help us carry out the energy estimates on $f$.

**Lemma 4.1.** *Let* $f$ *satisfy* $(\mathrm{A}_f)$*. Then, for any two functions* $u(x,t)$ *and* $\bar{u}(x,t)$ *with* $(x,t) \in \Omega \times [0,T)$*, it holds that*

$$f(\bar{u}) - f(u) \le C\left(|u|^{q-2} + |\bar{u}|^{q-2} + 1\right)|\bar{u} - u|.$$

*Proof.* Set $\tilde{u} := \bar{u} - u$. Then, by the property of the Gâteaux derivative, we have

$$f(\bar{u}) - f(u) = f(u + \tilde{u}) - f(u) = \int_0^1 \mathrm{d}f(u + s\tilde{u}; \tilde{u})\,\mathrm{d}s$$

for all $s \in (0,1)$. From

$$\mathrm{d}f(u + s\tilde{u}; \tilde{u}) = \lim_{\tau \to 0} \frac{f(u + s\tilde{u} + \tau\tilde{u}) - f(u + s\tilde{u})}{\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau} f(u + s\tilde{u} + \tau\tilde{u})\Big|_{\tau=0},$$

we further get

$$f(\bar{u}) - f(u) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} f(u + s\tilde{u} + \tau\tilde{u})\Big|_{\tau=0} \mathrm{d}s,$$

which together with (2.4) gives

$$\begin{aligned} f(\bar{u}) - f(u) \le& b\int_0^1 \left(|u + s\tilde{u}|^{q-2} + 1\right)\tilde{u}\,\mathrm{d}s \\ =& b\int_0^1 \left(|(1-s)u + s\bar{u}|^{q-2} + 1\right)(\bar{u} - u)\,\mathrm{d}s \\ \le& C\left(|u|^{q-2} + |\bar{u}|^{q-2} + 1\right)|\bar{u} - u|. \end{aligned}$$

□

In order to verify that the dynamical system $(Y, S(t))$ corresponding to problem (2.8)–(2.10) is gradient, we need to seek a strict Lyapunov functional $L$ in terms of Definition 3.1. To this end, we define the total energy function corresponding to problem (2.8)–(2.10) as

$$\begin{aligned} E(t) :=& \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{\kappa}{2}\|\Delta u(t)\|^2 + \frac{1}{2}\|v^t\|_g^2 \\ &+ \int_\Omega h(\nabla u(t))\,\mathrm{d}x + \int_\Omega F(u(t))\,\mathrm{d}x - (\phi, u(t)), \quad t \geq 0. \end{aligned} \tag{4.1}$$

The following lemma provides the monotonicity of $E(t)$.

**Lemma 4.2.** *Under the assumptions of Theorem 2.3, $E(t)$ is non-increasing, and*

$$E'(t) = -\|\nabla u_t(t)\|^2 + \frac{1}{2}\int_0^\infty g'(\tau)\|\Delta v^t(\tau)\|^2\,\mathrm{d}\tau. \tag{4.2}$$

*Proof.* For the solution with higher regularity (2.14), by taking $w_1 = u_t(t)$ in $(2.12)_1$ and $w_2 = v^t$ in $(2.12)_2$ and adding the two results, we obtain

$$E'(t) = -\|\nabla u_t(t)\|^2 - (v^t_\tau, v^t)_g. \tag{4.3}$$

For the second term on the right-hand side of (4.3), we have

$$(v^t_\tau, v^t)_g = \frac{1}{2}\int_0^\infty \frac{\partial}{\partial \tau}\left(g(\tau)\|\Delta v^t(\tau)\|^2\right)\mathrm{d}\tau - \frac{1}{2}\int_0^\infty g'(\tau)\|\Delta v^t(\tau)\|^2\,\mathrm{d}\tau.$$

In view of (2.7), we have $\lim_{\tau\to 0} v^t(x,\tau) = 0$. From $(\mathrm{A}_g)$ we are in a position to get $\lim_{\tau\to\infty} g(\tau) = 0$. Hence,

$$(v^t_\tau, v^t)_g = -\frac{1}{2}\int_0^\infty g'(\tau)\|\Delta v^t(\tau)\|^2\,\mathrm{d}\tau,$$

which combined with (4.3) gives (4.2). By the assumption $g'(t) \leq 0$ in $(\mathrm{A}_g)$, we have

$$\int_0^\infty g'(\tau)\|\Delta v^t(\tau)\|^2\,\mathrm{d}\tau \leq 0. \tag{4.4}$$

Hence, $E'(t) \leq 0$, which means that $E(t)$ is non-increasing. We infer from the density arguments [13] that this lemma remains valid for the solution satisfying (2.3). □

**Lemma 4.3** (Gradient property)**.** *In addition to the assumptions of Theorem 2.3, suppose that there exists a constant $\rho > 0$ such that $g'(t) + \rho g(t) \leq 0$ for all $t \in [0,\infty)$. Then, the dynamical system $(Y, S(t))$ corresponding to problem* (2.8)–(2.10) *is gradient.*

*Proof.* For any $y \in Y$, we take $L(S(t)y)$ as $E(t)$. Then, we see from Lemma 4.2 that $L(S(t)y)$ is non-increasing.

Let $L(S(t)y) = L(y)$ for all $t > 0$ and some $y \in Y$. Then, again by Lemma 4.2, we obtain

$$E(t) + \int_0^t \|\nabla u_s(s)\|^2\,\mathrm{d}s - \frac{1}{2}\int_0^t\int_0^\infty g'(\tau)\|\Delta v^s(\tau)\|^2\,\mathrm{d}\tau\mathrm{d}s = E(0). \tag{4.5}$$

Consequently,

$$\int_0^t \|\nabla u_s(s)\|^2 \,\mathrm{d}s - \frac{1}{2}\int_0^t \int_0^\infty g'(\tau)\|\Delta v^s(\tau)\|^2 \,\mathrm{d}\tau\mathrm{d}s = 0.$$

Thus,

$$\int_0^t \|\nabla u_s(s)\|^2 \,\mathrm{d}s = 0$$

and

$$-\int_0^t \int_0^\infty g'(\tau)\|\Delta v^s(\tau)\|^2 \,\mathrm{d}\tau\mathrm{d}s = 0. \tag{4.6}$$

By the assumption $g'(t) + \rho g(t) \le 0$, we discover

$$-\int_0^t \int_0^\infty g'(\tau)\|\Delta v^s(\tau)\|^2 \,\mathrm{d}\tau\mathrm{d}s \ge \rho \int_0^t \|v^s\|_g^2 \,\mathrm{d}s,$$

which together with (4.6) gives $v^t = 0$ for all $t \ge 0$. From (2.7), we further get $u(t) = u_0$ for all $t \ge 0$. Hence, $(u(t), u_t(t), v^t) = (u_0, 0, 0)$, i.e., $S(t)y = y$ for all $t \ge 0$. By Definition 3.1, we easily see that $(Y, S(t))$ is gradient. □

To show that the dynamical system $(Y, S(t))$ corresponding to problem (2.8)–(2.10) is quasi-stable, we first use the perturbed energy method [29, 30], with some modifications, to establish the following stabilizability estimate.

**Lemma 4.4** (Stabilizability estimate)**.** *Under the assumptions of Lemma 4.3, for a given bounded set $B \subset Y$, there exist constants $\alpha, \beta > 0$ and $\sigma > 0$ depending on $B$ such that*

$$\begin{aligned} \|S(t)\bar{y} - S(t)y\|_Y^2 \le& \alpha e^{-\beta t}\|\bar{y} - y\|_Y^2 + \sigma \int_0^t e^{-\beta(t-s)} \\ &\cdot \left( \sum_{i=1}^N \|\nabla\bar{u}(s) - \nabla u(s)\|_{2p_i-2}^2 + \|\bar{u}(s) - u(s)\|_{2q-2}^2 \right) \mathrm{d}s \end{aligned} \tag{4.7}$$

*for every $y, \bar{y} \in B$ and $t > 0$, where $S(t)\bar{y} = (\bar{u}(t), \bar{u}_t(t), \bar{v}(t))$ and $\bar{y} = (\bar{u}_0, \bar{u}_1, \bar{v}_0)$.*

*Proof.* Set $\tilde{u} := \bar{u} - u$, $\tilde{v} := \bar{v} - v$ and

$$\Psi(t) := \widetilde{E}(t) + \varepsilon\psi(t), \tag{4.8}$$

where

$$\begin{aligned} \widetilde{E}(t) &:= \kappa\|\Delta\tilde{u}(t)\|^2 + \|\tilde{u}_t(t)\|^2 + \|\nabla\tilde{u}_t(t)\|^2 + \|\tilde{v}^t\|_g^2, \\ \psi(t) &:= (\tilde{u}(t), \tilde{u}_t(t)) + (\nabla\tilde{u}(t), \nabla\tilde{u}_t(t)), \end{aligned} \tag{4.9}$$

and $\varepsilon > 0$ is a constant to be determined later.

We first claim that there exist two constants $\gamma_1, \gamma_2 > 0$, depending on $\varepsilon$, such that

$$\gamma_1\widetilde{E}(t) \le \Psi(t) \le \gamma_2\widetilde{E}(t). \tag{4.10}$$

To confirm this, we deduce from Schwarz's and Cauchy's inequalities that

$$|\psi(t)| \le \frac{1}{2}\|\tilde{u}(t)\|^2 + \frac{1}{2}\|\tilde{u}_t(t)\|^2 + \frac{1}{2}\|\nabla\tilde{u}(t)\|^2 + \frac{1}{2}\|\nabla\tilde{u}_t(t)\|^2$$

$$\leq \frac{\mathfrak{C}_2^2 + \mathfrak{C}_3^2}{2} \|\Delta \tilde{u}(t)\|^2 + \frac{1}{2} \left( \|\tilde{u}_t(t)\|^2 + \|\nabla \tilde{u}_t(t)\|^2 \right).$$

Combining this inequality with (4.9), we infer that there exists a constant $Q > 0$ such that $|\psi(t)| \leq Q\widetilde{E}(t)$, which together with (4.8) gives

$$(1 - \varepsilon Q)\widetilde{E}(t) \leq \Psi(t) \leq (1 + \varepsilon Q)\widetilde{E}(t).$$

Thus, assertion (4.10) is demonstrated, and $\gamma_1 > 0$ will be guaranteed by the selection of $\varepsilon$ later.

Next we claim that there exist constants $\gamma_3 > 0$ and $\gamma_4 > 0$ depending on $B$ such that

$$\Psi'(t) \leq -\gamma_3 \widetilde{E}(t) + \gamma_4 \|\tilde{u}(t)\|^2. \tag{4.11}$$

Indeed, by the density arguments, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left( \|\tilde{u}_t(t)\|^2 + \|\nabla \tilde{u}_t(t)\|^2 + \kappa \|\Delta \tilde{u}(t)\|^2 + \|\tilde{v}^t\|_g^2 \right) + \|\nabla \tilde{u}_t(t)\|^2 \\ =&\frac{1}{2}\int_0^\infty g'(\tau)\|\Delta \tilde{v}^t(\tau)\|^2 \,\mathrm{d}\tau - (H(\nabla \bar{u}(t)) - H(\nabla u(t)), \nabla \tilde{u}_t(t)) - (f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)), \end{aligned}$$

i.e.,

$$\begin{aligned} \widetilde{E}'(t) =& -2\|\nabla \tilde{u}_t(t)\|^2 + \int_0^\infty g'(\tau)\|\Delta \tilde{v}^t(\tau)\|^2 \,\mathrm{d}\tau \\ & - 2(H(\nabla \bar{u}(t)) - H(\nabla u(t)), \nabla \tilde{u}_t(t)) - 2(f(\bar{u}(t)) - f(u(t)), \tilde{u}_t(t)). \end{aligned} \tag{4.12}$$

Concerning the second term on the right-hand side of (4.12), we deduce from the assumption $g'(t) + \rho g(t) \leq 0$ that

$$\int_0^\infty g'(\tau)\|\Delta \tilde{v}^t(\tau)\|^2 \,\mathrm{d}\tau \leq -\rho \|\tilde{v}^t\|_g^2. \tag{4.13}$$

We now estimate the third term on the right-hand side of (4.12). In terms of [13, p.190], (2.2) in $(\mathrm{A}_H)$ implies that there exists a constant $\tilde{a} := \tilde{a}(a_i, p_i, N) > 0$ $(i = 1, 2, \cdots, N)$ such that

$$|H(s_2) - H(s_1)| \leq \tilde{a} \sum_{i=1}^N \left( |s_1|^{p_i - 2} + |s_2|^{p_i - 2} + 1 \right) |s_2 - s_1|, \quad s_1, s_2 \in \mathbb{R}^N.$$

Hence, from Hölder's inequality with $(p_i - 2)/(2p_i - 2) + 1/(2p_i - 2) + 1/2 = 1$, Minkowski's inequality, the Sobolev inequality for the embedding $H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow W_0^{1, 2p_i - 2}(\Omega)$, and Cauchy's inequality, we deduce that

$$\begin{aligned} & -2(H(\nabla \bar{u}(t)) - H(\nabla u(t)), \nabla \tilde{u}_t(t)) \\ \leq& 2\tilde{a} \sum_{i=1}^N \int_\Omega \left( |\nabla u(t)|^{p_i - 2} + |\nabla \bar{u}(t)|^{p_i - 2} + 1 \right) |\nabla \bar{u}(t) - \nabla u(t)| |\nabla \tilde{u}_t(t)| \,\mathrm{d}x \\ \leq& 2\tilde{a} \sum_{i=1}^N \left( \|\nabla u(t)\|_{2p_i - 2}^{p_i - 2} + \|\nabla \bar{u}(t)\|_{2p_i - 2}^{p_i - 2} + |\Omega|^{\frac{p_i - 2}{2p_i - 2}} \right) \|\nabla \tilde{u}(t)\|_{2p_i - 2} \|\nabla \tilde{u}_t(t)\| \end{aligned}$$

$$\leq C(B)\sum_{i=1}^{N}\|\nabla\tilde{u}(t)\|_{2p_i-2}^2+\frac{1}{2}\|\nabla\tilde{u}_t(t)\|^2, \tag{4.14}$$

Likewise, for the last term on the right-hand side of (4.12), we deduce from Lemma 4.1 that

$$\begin{aligned}-2(f(\bar{u}(t))-f(u(t)),\tilde{u}_t(t)) \leq & C\int_\Omega\left(|u(t)|^{q-2}+|\bar{u}(t)|^{q-2}+1\right)|\tilde{u}(t)||\tilde{u}_t(t)|\,\mathrm{d}x\\ \leq & C\left(\|u(t)\|_{2q-2}^{q-2}+\|\bar{u}(t)\|_{2q-2}^{q-2}+|\Omega|^{\frac{q-2}{2q-2}}\right)\|\tilde{u}(t)\|_{2q-2}\|\tilde{u}_t(t)\|\\ \leq & C(B)\|\tilde{u}(t)\|_{2q-2}\|\nabla\tilde{u}_t(t)\|\\ \leq & C(B)\|\tilde{u}(t)\|_{2q-2}^2+\frac{1}{2}\|\nabla\tilde{u}_t(t)\|^2.\end{aligned} \tag{4.15}$$

As a consequence, by plugging (4.13)–(4.15) into (4.12), we derive

$$\widetilde{E}'(t)\leq C(B)\left(\sum_{i=1}^{N}\|\nabla\tilde{u}(t)\|_{2p_i-2}^2+\|\tilde{u}(t)\|_{2q-2}^2\right)-\|\nabla\tilde{u}_t(t)\|^2-\rho\|\tilde{v}^t\|_g^2. \tag{4.16}$$

Since

$$\psi'(t)=\|\tilde{u}_t(t)\|^2+\|\nabla\tilde{u}_t(t)\|^2+(\tilde{u}_{tt}(t),\tilde{u}(t))+(\nabla\tilde{u}_{tt}(t),\nabla\tilde{u}(t)),$$

we deduce from $(2.12)_1$ that

$$\begin{aligned}\psi'(t)=&\|\tilde{u}_t(t)\|^2+\|\nabla\tilde{u}_t(t)\|^2-\kappa\|\Delta\tilde{u}(t)\|^2-(\tilde{v}^t,\tilde{u}(t))_g\\ &-(H(\nabla\bar{u}(t))-H(\nabla u(t)),\nabla\tilde{u}(t))-(\nabla\tilde{u}_t(t),\nabla\tilde{u}(t))\\ &-(f(\bar{u}(t))-f(u(t)),\tilde{u}(t)).\end{aligned}$$

Thus, there exists a constant $0<\theta<1$ such that

$$\begin{aligned}\psi'(t)=&-\theta\widetilde{E}(t)+(1+\theta)\|\tilde{u}_t(t)\|^2+(1+\theta)\|\nabla\tilde{u}_t(t)\|^2-\kappa(1-\theta)\|\Delta\tilde{u}(t)\|^2\\ &+\theta\|\tilde{v}^t\|_g^2-(\tilde{v}^t,\tilde{u}(t))_g-(H(\nabla\bar{u}(t))-H(\nabla u(t)),\nabla\tilde{u}(t))\\ &-(\nabla\tilde{u}_t(t),\nabla\tilde{u}(t))-(f(\bar{u}(t))-f(u(t)),\tilde{u}(t)).\end{aligned} \tag{4.17}$$

Regarding the sixth and eighth terms on the right-hand side of (4.17), from Schwarz's inequality and Cauchy's inequalities with $\epsilon_1,\epsilon_2>0$, it follows that

$$\begin{aligned}-(\tilde{v}^t,\tilde{u}(t))_g\leq&\|\Delta\tilde{u}(t)\|\int_0^\infty g(\tau)\|\Delta\tilde{v}^t(\tau)\|\,\mathrm{d}\tau\\ \leq&\epsilon_1\|\Delta\tilde{u}(t)\|^2+\frac{1}{4\epsilon_1}\|\tilde{v}^t\|_g^2\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}-(\nabla\tilde{u}_t(t),\nabla\tilde{u}(t))\leq&\|\nabla\tilde{u}(t)\|\|\nabla\tilde{u}_t(t)\|\\ \leq&\epsilon_2\mathfrak{C}_3^2\|\Delta\tilde{u}(t)\|^2+\frac{1}{4\epsilon_2}\|\nabla\tilde{u}_t(t)\|^2.\end{aligned} \tag{4.19}$$

For the seventh term on the right-hand side of (4.17), we deduce from arguments similar to those used in the proof of (4.14) that

$$-\left(H(\nabla\bar{u}(t))-H(\nabla u(t)),\nabla\tilde{u}(t)\right)\leq C(B,\epsilon_3)\sum_{i=1}^{N}\|\nabla\tilde{u}(t)\|_{2p_i-2}^2+\epsilon_3\|\Delta\tilde{u}(t)\|^2. \tag{4.20}$$

Moreover, for the last term on the right-hand side of (4.17), we infer from arguments similar to those used in the proof of (4.15) that

$$-\left(f(\bar{u}(t))-f(u(t)),\tilde{u}(t)\right)\leq C(B,\epsilon_4)\|\tilde{u}(t)\|_{2q-2}^2+\epsilon_4\|\Delta\tilde{u}(t)\|^2. \tag{4.21}$$

Hence, by inserting (4.18)–(4.21) into (4.17), we obtain

$$\begin{aligned}\psi'(t)\leq&-\theta\widetilde{E}(t)+(1+\theta)\|\tilde{u}_t(t)\|^2+\left(1+\theta+\frac{1}{4\epsilon_2}\right)\|\nabla\tilde{u}_t(t)\|^2\\&-\left(\kappa(1-\theta)-\epsilon_1-\epsilon_2\mathfrak{C}_3^2-\epsilon_3-\epsilon_4\right)\|\Delta\tilde{u}(t)\|^2+\left(\theta+\frac{1}{4\epsilon_1}\right)\|\tilde{v}^t\|_g^2\\&+C(B,\epsilon_3)\sum_{i=1}^{N}\|\nabla\tilde{u}(t)\|_{2p_i-2}^2+C(B,\epsilon_4)\|\tilde{u}(t)\|_{2q-2}^2.\end{aligned}$$

Choosing sufficiently small $\epsilon_i$ ($i=1,2,3,4$) such that

$$\kappa(1-\theta)-\epsilon_1-\epsilon_2\mathfrak{C}_3^2-\epsilon_3-\epsilon_4>0,$$

we arrive at

$$\begin{aligned}\psi'(t)\leq&-\theta\widetilde{E}(t)+\left((1+\theta)\left(1+\mathfrak{C}_1^2\right)+\frac{1}{4\epsilon_2}\right)\|\nabla\tilde{u}_t(t)\|^2\\&+\left(\theta+\frac{1}{4\epsilon_1}\right)\|\tilde{v}^t\|_g^2+C(B)\left(\sum_{i=1}^{N}\|\nabla\tilde{u}(t)\|_{2p_i-2}^2+\|\tilde{u}(t)\|_{2q-2}^2\right).\end{aligned} \tag{4.22}$$

Therefore, from (4.8), (4.16), and (4.22), we deduce that

$$\begin{aligned}\Psi'(t)\leq&-\varepsilon\theta\widetilde{E}(t)+(C(B)+\varepsilon C(B))\left(\sum_{i=1}^{N}\|\nabla\tilde{u}(t)\|_{2p_i-2}^2+\|\tilde{u}(t)\|_{2q-2}^2\right)\\&-\left(1-\varepsilon\left((1+\theta)\left(1+\mathfrak{C}_1^2\right)+\frac{1}{4\epsilon_2}\right)\right)\|\nabla\tilde{u}_t(t)\|^2\\&-\left(\rho-\varepsilon\left(\theta+\frac{1}{4\epsilon_1}\right)\right)\|\tilde{v}^t\|_g^2.\end{aligned} \tag{4.23}$$

For fixed $\epsilon_1$ and $\epsilon_2$, we choose

$$\varepsilon<\min\left\{\frac{1}{Q},\ \frac{4\epsilon_2}{4\epsilon_2(1+\theta)\left(1+\mathfrak{C}_1^2\right)+1},\ \frac{4\rho\epsilon_1}{4\theta\epsilon_1+1}\right\}$$

such that the last two terms on the right-hand side of (4.23) are non-positive and could be neglected. Thus, assertion (4.11) is proved. Here, $\varepsilon < 1/Q$ ensures $\gamma_1 > 0$ in assertion (4.10).

By assertion (4.11) and the second inequality in assertion (4.10), we get

$$\Psi'(t) \le -\frac{\gamma_3}{\gamma_2}\Psi(t) + \gamma_4\left(\sum_{i=1}^{N} \|\nabla\tilde{u}(t)\|_{2p_i-2}^2 + \|\tilde{u}(t)\|_{2q-2}^2\right).$$

Hence,

$$\Psi(t) \le \Psi(0)e^{-\beta t} + \gamma_4\int_0^t e^{-\beta(t-s)}\left(\sum_{i=1}^{N} \|\nabla\tilde{u}(s)\|_{2p_i-2}^2 + \|\tilde{u}(s)\|_{2q-2}^2\right) \mathrm{d}s, \tag{4.24}$$

where $\beta = \gamma_3/\gamma_2$. Again by the second inequality in assertion (4.10), we have $\Psi(0) \le \gamma_2\widetilde{E}(0)$, which, combined with (4.24) and the first inequality in assertion (4.10), yields

$$\widetilde{E}(t) \le \alpha\widetilde{E}(0)e^{-\beta t} + \sigma\int_0^t e^{-\beta(t-s)}\left(\sum_{i=1}^{N} \|\nabla\tilde{u}(s)\|_{2p_i-2}^2 + \|\tilde{u}(s)\|_{2q-2}^2\right) \mathrm{d}s,$$

where $\alpha = \gamma_2/\gamma_1$ and $\sigma = \gamma_4/\gamma_1$. Thus, (4.7) follows from (4.9) immediately. □

**Lemma 4.5** (Quasi-stability)**.** *Under the assumptions of Lemma 4.3, the dynamical system* $(Y, S(t))$ *corresponding to problem* (2.8)–(2.10) *is quasi-stable on any bounded positively invariant set* $B \subset Y$*.*

*Proof.* Let $U = H^2(\Omega) \cap H_0^1(\Omega)$, $V = H_0^1(\Omega)$, and $W = L_g$. Then, Theorem 2.3 implies that the dynamical system $(Y, S(t))$ satisfies (A). Since the solution depends continuously on the initial data, we get (3.1). Moreover, by taking

$$n_U(\tilde{u})^2 = \sum_{i=1}^{N} \|\nabla\tilde{u}(s)\|_{2p_i-2}^2 + \|\tilde{u}(s)\|_{2q-2}^2, \quad \varsigma_2(t) = \alpha e^{-\beta t}, \quad \varsigma_3(t) = \sigma\int_0^t e^{-\beta(t-s)}\,\mathrm{d}s,$$

we conclude from Lemma 4.4 that (3.2) holds. Thus, by Definition 3.3, the proof of Lemma 4.5 is finished. □

**Theorem 4.6** (Existence of global attractors)**.** *Under the assumptions of Lemma 4.3, the dynamical system* $(Y, S(t))$ *corresponding to problem* (2.8)–(2.10) *possesses a compact global attractor* $A = \mathcal{M}^z(\mathcal{N})$ *with finite fractal dimension, where* $\mathcal{N}$ *is the set of stationary points of the dynamical system* $(Y, S(t))$*, that is,*

$$\mathcal{N} := \{(u, 0, 0) | \kappa\Delta^2 u - \mathrm{div}H(\nabla u) + f(u) = \phi\},$$

*and* $\mathcal{M}^z(\mathcal{N})$ *is an unstable manifold emanating from the set* $\mathcal{N}$ *as a set of all* $y := (u_0, u_1, v_0) \in Y$ *such that there exists a full trajectory* $\{z(t) | t \in \mathbb{R}\}$ *with the properties* $z(0) = y$ *and* $\lim_{t\to-\infty} \mathrm{dist}_Y(z(t), \mathcal{N}) = 0$.

*Proof.* First of all, since $L$ is defined as $E$ given by (4.1), we easily see that $L(y)$ is bounded from above on any bounded subset of $Y$.

We now claim that there exist two constants $M_1, M_2 > 0$ such that

$$E(t) \ge \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u_t(t)\|^2 + M_1\|\Delta u(t)\|^2 + \frac{1}{2}\|v^t\|_g^2 - M_2. \tag{4.25}$$

Indeed, for the fifth term on the right-hand side of (4.1), we deduce from (2.3) in ($A_H$) that

$$\begin{aligned}\int_\Omega h(\nabla u(t))\,\mathrm{d}x \geq & -\frac{\lambda_1}{2}\|\nabla u(t)\|^2 - \lambda_2|\Omega| \\ \geq & -\frac{\lambda_1\mathfrak{C}_3^2}{2}\|\Delta u(t)\|^2 - \lambda_2|\Omega|. \end{aligned} \tag{4.26}$$

Concerning the sixth term on the right-hand side of (4.1), it follows from (2.6) in ($A_f$) that

$$\begin{aligned}\int_\Omega F(u(t))\,\mathrm{d}x \geq & -\frac{\eta}{2}\|u(t)\|^2 - \mu|\Omega| \\ \geq & -\frac{\eta\mathfrak{C}_2^2}{2}\|\Delta u(t)\|^2 - \mu|\Omega|. \end{aligned} \tag{4.27}$$

For the last term on the right-hand side of (4.1), from Schwarz's inequality and Cauchy's inequality with $\epsilon > 0$, we have

$$\begin{aligned}-(\phi, u(t)) \geq & -\|\phi\|\|u(t)\| \\ \geq & -\epsilon\mathfrak{C}_2^2\|\Delta u(t)\|^2 - \frac{1}{4\epsilon}\|\phi\|^2. \end{aligned} \tag{4.28}$$

Consequently, by choosing sufficiently small $\epsilon$ such that

$$M_1 := \frac{\kappa}{2} - \frac{\lambda_1\mathfrak{C}_3^2}{2} - \frac{\eta\mathfrak{C}_2^2}{2} - \epsilon\mathfrak{C}_2^2 > 0$$

and

$$M_2 := (\lambda_2 + \mu)|\Omega| + \frac{1}{4\epsilon}\|\phi\|^2,$$

assertion (4.25) follows from (4.1) and (4.26)–(4.28).

Regarding the solution $(u, u_t, v)$ to problem (2.8)–(2.10) such that $L(y) \leq R$, according to Lemma 4.2, we again obtain (4.5) and (4.4) here. Hence,

$$E(t) + \int_0^t \|\nabla u_s(s)\|^2\,\mathrm{d}s \leq E(0),$$

which together with assertion (4.25) and Lemma 4.2 gives

$$C\|(u(t), u_t(t), v^t)\|_Y^2 - M_2 \leq R,$$

i.e., $\|(u(t), u_t(t), v^t)\|_Y^2 \leq C$. Thus, $L_R$ is bounded.

For the stationary solution $(u, 0, 0)$ to problem (2.8)–(2.10), we have

$$\kappa\|\Delta u\|^2 = -(H(\nabla u), \nabla u) - (f(u), u) + (\phi, u). \tag{4.29}$$

Owing to (2.3) in ($A_H$), we have

$$\begin{aligned}-(H(\nabla u), \nabla u) \leq & \lambda_1\|\nabla u\|^2 + \lambda_2|\Omega| \\ \leq & \lambda_1\mathfrak{C}_3^2\|\Delta u\|^2 + \lambda_2|\Omega|. \end{aligned} \tag{4.30}$$

It follows from (2.5) in ($\mathrm{A}_f$) that

$$\begin{aligned}-(f(u),u) \leq& \eta\|u\|^2+\mu|\Omega|\\ \leq& \eta\mathfrak{C}_2^2\|\Delta u\|^2+\mu|\Omega|.\end{aligned} \tag{4.31}$$

Moreover, from Schwarz's inequality and Cauchy's inequality with $\epsilon>0$, we have

$$\begin{aligned}(\phi,u) \leq& \epsilon\|u\|^2+\frac{1}{4\epsilon}\|\phi\|^2\\ \leq& \epsilon\mathfrak{C}_2^2\|\Delta u\|^2+\frac{1}{4\epsilon}\|\phi\|^2.\end{aligned} \tag{4.32}$$

Consequently, by inserting (4.30)–(4.32) into (4.29), we derive

$$\kappa\|\Delta u\|^2 \leq \left(\lambda_1\mathfrak{C}_3^2+\eta\mathfrak{C}_2^2+\epsilon\mathfrak{C}_2^2\right)\|\Delta u\|^2+(\lambda_2+\mu)|\Omega|+\frac{1}{4\epsilon}\|\phi\|^2.$$

Choosing sufficiently small $\epsilon$ such that

$$\kappa-\lambda_1\mathfrak{C}_3^2-\eta\mathfrak{C}_2^2-\epsilon\mathfrak{C}_2^2>0,$$

we further obtain $\|\Delta u\|^2 \leq C$. Hence, $\mathcal{N}$ is bounded. In light of Lemma 4.5 and Proposition 3.4, it is easy to see that $(Y,S(t))$ is asymptotically smooth. According to Theorem 3.2 and Lemma 4.3, it is obvious that $(Y,S(t))$ possesses a compact global attractor $A=\mathcal{M}^z(\mathcal{N})$. Finally, we conclude from Lemma 4.5 and Theorem 3.5 that $A$ has finite fractal dimension. □

## 5. Conclusions

In this paper, we considered the initial-boundary value problem for a class of viscoelastic plate equations with past history, namely, (1.1)–(1.3). In order to handle the long-time dynamics for problem (1.1)–(1.3), we transformed problem (1.1)–(1.3) into the equivalent problem (2.8)–(2.10) by means of the history space framework. By exploiting the properties of the total energy function, we obtained the gradient property of the dynamical system $(Y,S(t))$ corresponding to problem (2.8)–(2.10). Moreover, by using the perturbed energy method, we established a stabilizability estimate, which enabled us to get the quasi-stability of the dynamical system $(Y,S(t))$ corresponding to problem (2.8)–(2.10). Then, we further obtained the existence of a global attractor with finite fractal dimension, namely, Theorem 4.6. This theorem showed that even if $f$ satisfies ($\mathrm{A}_f$), the dynamical system $(Y,S(t))$ corresponding to problem (2.8)–(2.10) still possesses a global attractor with finite fractal dimension. Thus, we supplemented the results of [23].

Finally, we point out the applicability of the main result of this paper. On the whole, Theorem 4.6, the main result of this paper, depends on the prescribed simply supported boundary condition (1.2) and initial conditions (1.3). More precisely, for the prescribed (1.2) and (1.3), the corresponding dynamical system $(Y,S(t))$ possesses a global attractor for any $(u_0,u_1,v_0)\in Y$. In this case, no matter how the initial value $(u_0,u_1,v_0)$ changes in $Y$, the existence of the global attractor is universal. Nevertheless, when (1.2) is replaced by the other boundary conditions from certain a physical meaning, further research is needed to determine whether a result similar to Theorem 4.6 can be obtained.

## Author contributions

Quan Zhou: Investigation, Writing-original draft; Yang Liu: Methodology, Writing-review & editing, Funding acquisition; Dong Yang: Investigation. All authors have read and approved the final version of the manuscript for publication.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

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