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*Research article*

# The diameter of the nil-clean graph of $\mathbb{Z}_n$

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**Abstract:** Let $R$ be a ring with identity. An element $r$ was called to be nil-clean if $r$ was a sum of an idempotent and a nilpotent element in $R$. The nil-clean graph of $R$ was a simple graph, denoted by $G_{NC}(R)$, whose vertex set was $R$, where two distinct vertices $x$ and $y$ were adjacent if, and only if, $x+y$ was a nil-clean element of $R$. In the absence of the condition that vertex $x$ is not the same as $y$, the graph defined in the same way was called the closed nil-clean graph of $R$, which may contain loops, and was denoted by $\overline{G_{NC}}(R)$. In this short note, we completely determine the diameter of $G_{NC}(\mathbb{Z}_n)$.

**Keywords:** nil-clean graph; integers modulo $n$; diameter; Chinese remainder theorem
**Mathematics Subject Classification:** 05C25, 13A99

## 1. Introduction

Using special elements to characterize the properties and structure of rings is a very common method, and it is also a very popular research field, which has attracted widespread attention by many researchers. Idempotents, nilpotent elements, and units are three very important types of elements that play a crucial role in characterizing the properties and structure of rings, and have also sparked many new concepts and the classes of rings, for example, clean rings, nil-clean rings, 2-good rings, and fine rings (see [8, 9, 13, 17]).

On the other hand, using graphic properties and invariants to characterize the structure and properties of rings has been a hot research field in recent decades. After Beck [6] introduced the zero-divisor graph of a commutative ring in 1998, especially after Anderson and Livingston [3] modified the definition in 1999, various graph structures on rings were defined and studied, for example, unit graphs of rings, total graphs of rings, comaximal ideal graph of rings, zero-divisor graphs with respect to ideals of rings, and cozero divisor graphs (see [2, 5, 12, 14, 18]). This greatly enriches the methods for studying the properties and structures of rings. In 2017, Basnet and Bhattacharyya [7] introduced the nil-clean graph of a ring and studied its basic properties. Due to the complexity of the nil-clean elements of a ring, the properties of the nil-clean graph of the ring are less known. Even the diameter of the nil-clean graph for a basic finite commutative ring, such as the residue class ring of integers modulo $n$, is not completely clear. As usual, we use $\mathbb{Z}_n$ to denote the residue class ring of integers modulo $n$. In [7], the authors have only showed that $\text{diam}(G_{NC}(\mathbb{Z}_{2^k})) = 1$; $\text{diam}(G_{NC}(\mathbb{Z}_{2^k 3^l})) = 2$ for $k \geq 0$ and $l \geq 1$; $\text{diam}(G_{NC}(\mathbb{Z}_p)) = p - 1$ for a prime $p$; and $\text{diam}(G_{NC}(\mathbb{Z}_{2p})) = \text{diam}(G_{NC}(\mathbb{Z}_{3p})) = p - 1$ for an odd prime $p$. Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be a prime factorization and $p_1 < \cdots < p_s$. We show that $\text{diam}(G_{NC}(\mathbb{Z}_n)) = p_s - 1$.

Diameter is one of the important invariants of a graph. Many papers are devoted to the diameter of the resulting graph in this research area (see [1, 4, 11, 15]). For the unit graph of a ring, Heydari and Nikmehr [10] proved that the diameter of the unit graph of an Artinian ring only has four possibilities: $1, 2, 3, \infty$, and they classified all Artinian rings via its diameter of unit graphs. In 2019, Su and Wei generalized the result to self-injective rings in [16]. They also proved that there exists a ring such that the diameter of its unit graph is more than three.

Let $R$ be a ring with identity. We use $Id(R)$ and $Nil(R)$ to denote the set of all idempotents and the set of all nilpotent elements of $R$, respectively. An element $r$ is said to be a nil-clean element if $r$ is a sum of an idempotent and a nilpotent element in $R$. The set of all nil-clean elements of $R$ is denoted as the notation $NC(R)$. The nil-clean graph of $R$, denoted by $G_{NC}(R)$, is a simple graph with $R$ as its vertex set, and two distinct vertices $x$ and $y$ are adjacent if, and only if, $x + y \in NC(R)$. In the absence of the condition that vertex $x$ is not same as $y$, the graph defined in the same way is called the closed nil-clean graph of $R$, which may contain loops, and is denoted by $\overline{G_{NC}}(R)$. Some basic properties are studied in [7], for example, they have showed that $G_{NC}(R)$ is a complete graph if, and only if, $R$ is a nil-clean ring; the degree of a vertex $x$ in $G_{NC}(R)$ is either $| NC(R) | -1$ or $| NC(R) |$ depending on if $2x$ belongs to $NC(R)$ or not; and $G_{NC}(R)$ is a bipartite graph if, and only if, $R$ is a field.

We recall some necessary notions in graph theory. Let $G$ be a simple graph, meaning it has no loops and multi-edges. We use the symbol $x \sim y$ to denote when two vertices $x$ and $y$ in a graph $G$ are adjacent. A walk of lengths $k$ with endpoints $v_0$ and $v_k$ in $G$ is a sequence of vertices $(v_0, v_1, \ldots, v_{k-1}, v_k)$, in which $v_{i-1} \sim v_i$ for every $i = 1, \ldots, k$. A path in a graph is a walk that has all distinct vertices (except the endpoints). A graph $G$ is connected if there is a path between each pair of the vertices of $G$; otherwise, $G$ is disconnected. The distance between two vertices $x$ and $y$, denoted by $d(x, y)$, is the length a shortest path between $x$ and $y$. If there is no path connecting two vertices, the distance between them is defined as infinite. The longest distance between all pairs of vertices of $G$ is called the diameter of $G$, and is denoted by $\text{diam}(G)$. Let $G_1, G_2$ be two graphs. Their tensor product, denoted by $G_1 \otimes G_2$, is a graph with vertex set $G_1 \times G_2$, where $(x_1, x_2) \sim (y_1, y_2)$ if, and only if, $x_1 \sim y_1$ in $G_1$ and $x_2 \sim y_2$ in $G_2$.

## 2. Main results and proofs

To begin, we decompose the graph $\overline{G_{NC}}(\mathbb{Z}_n)$ into $\overline{G_{NC}}(\mathbb{Z}_{p_1^{\alpha_1}}) \otimes \cdots \otimes \overline{G_{NC}}(\mathbb{Z}_{p_s^{\alpha_s}})$ for the prime factorization $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$. On this basis, we mainly provide a complete characterization of $\text{diam}(\mathbb{Z}_n)$. In addition, a mechanical way to find a path between two vertices $x$ and $y$ in $G_{NC}(\mathbb{Z}_n)$ is given.

For two isomorphic rings, their nil-clean graphs are clearly isomorphic as well. Due to the Chinese remainder theorem, there is an isomorphism of the rings, $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$, where $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$

is the prime factorization. Therefore, we are now investigating the nil-clean graph of the ring $\mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$, and denoting it by $G_{NC}(\mathbb{Z}_n)$ by abuse of notation.

Let $R_1$ and $R_2$ be two rings. We have that $Nil(R_1\times R_2) = Nil(R_1)\times Nil(R_2)$ and $Id(R_1\times R_2) = Id(R_1)\times Id(R_2)$. For a prime $p$ and a positive integer $\alpha$, $\mathbb{Z}_{p^\alpha}$ is a commutative local ring, having the unique maximal ideal $Nil(\mathbb{Z}_{p^\alpha}) = (p)$, and $Id(\mathbb{Z}_{p^\alpha})$ is trivial. Then, $NC(\mathbb{Z}_{p^\alpha})$ is the disjoint union $(p) \cup 1 + (p)$, where $1 + (p) = \left\{1 + x \in \mathbb{Z}_{p^\alpha} \mid x \in (p)\right\}$. We have that $NC(\mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}}) = NC(\mathbb{Z}_{p_1^{\alpha_1}}) \times NC(\mathbb{Z}_{p_2^{\alpha_2}})$. The following proposition characterizes the form of nil-clean elements in $\mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$.

**Proposition 2.1.** *Give $x = (x_1, \ldots, x_s) \in \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$. Then, $x$ is a nil-clean element of $\mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$ if, and only if, $x_i$ is a nil-clean element of $\mathbb{Z}_{p_i^{\alpha_i}}$, for every $i = 1, \ldots, s$.*

*Proof.* Suppose $x = (x_1, \ldots, x_s)$ is a nil-clean element in $\mathbb{Z}_{p_1^{\alpha_1}}\times\cdots\times\mathbb{Z}_{p_s^{\alpha_s}}$, then there exists an idempotent $y = (y_1, \ldots, y_s)$ and a nilpotent element $z = (z_1, \ldots, z_s)$, such that $x = y + z = (y_1 + z_1, \ldots, y_s + z_s)$. Note that $y_i$ is an idempotent element of $\mathbb{Z}_{p_i^{\alpha_i}}$ and $z_i$ is a nilpotent element of $\mathbb{Z}_{p_i^{\alpha_i}}$, for every $i = 1, \ldots, s$. Thus, $x_i$ is a nil-clean element of $\mathbb{Z}_{p_i^{\alpha_i}}$, for every $i = 1, \ldots, s$.

Conversely, suppose that $x_i$ is a nil-clean element of $\mathbb{Z}_{p_i^{\alpha_i}}$, for each $i = 1, \ldots, s$. Then, there exist an idempotent element $y_i \in \mathbb{Z}_{p_i^{\alpha_i}}$ and a nilpotent element $z_i \in \mathbb{Z}_{p_i^{\alpha_i}}$, such that $x_i = y_i + z_i$, for each $i = 1, \ldots, s$. Note that $y = (y_1, \ldots, y_s)$ is an idempotent and $z = (z_1, \ldots, z_s)$ is a nilpotent element in $\mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$. The equality $x = y + z$ implies that $x$ is a nil-clean element in $\mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$. □

We can determine the adjacency relation of $G_{NC}(\mathbb{Z}_n)$ completely by Proposition 2.1.

**Corollary 2.1.** *Suppose that $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$ are distinct vertices of $G_{NC}(\mathbb{Z}_n)$. Then, $x$ and $y$ are adjacent if, and only if, $x_i + y_i$ is a nil-clean element of $\mathbb{Z}_{p_i^{\alpha_i}}$, for every $i = 1, \ldots, s$. More precisely, $x$ and $y$ are adjacent if, and only if, $x_i + y_i \in (p_i) \cup 1 + (p_i)$, for every $i = 1, \ldots, s$. In short, $\overline{G_{NC}}(\mathbb{Z}_n) \cong \overline{G_{NC}}(\mathbb{Z}_{p_1^{\alpha_1}}) \otimes \cdots \otimes \overline{G_{NC}}(\mathbb{Z}_{p_s^{\alpha_s}})$.*

It is a fact that $\mathbb{Z}_{p^\alpha}/(p) \cong \mathbb{Z}_p$, and it is easy to study the nil-clean graph of $\mathbb{Z}_p$. In particular, we can determine the diameter of $G_{NC}(\mathbb{Z}_p)$, as stated in Proposition 2.2 below, which has been shown in [7].

**Proposition 2.2.** *Let $p$ be a prime. Then, $G_{NC}(\mathbb{Z}_p)$ is a path with $p$ vertices and its diameter is $p - 1$.*

*Proof.* $\mathbb{Z}_p$ is a field with characteristic $p$. There are only two nil-clean elements, 0 and 1, in $\mathbb{Z}_p$. For any vertex $x \in G_{NC}(\mathbb{Z}_p)$, the only adjacent vertices are $p - x$ and $p - x + 1$. This implies that $G_{NC}(\mathbb{Z}_p)$ is a path. For two vertices $x$ and $y$ in $G_{NC}(\mathbb{Z}_p)$, the path between them is unique. Note that $d(0, \frac{p+1}{2}) = p-1$. Therefore, $\mathrm{diam}(G_{NC}(\mathbb{Z}_p)) = p - 1$. □

**Proposition 2.3.** *Let $p$ be a prime and $\alpha$ be a positive integer. Then,* $\mathrm{diam}(G_{NC}(\mathbb{Z}_{p^\alpha})) = p - 1$.

*Proof.* The facts that $\mathbb{Z}_{p^\alpha}/(p) \cong \mathbb{Z}_p$ and $Nil(\mathbb{Z}_{p^\alpha}) = (p)$ are evident. Let $x, y \in \mathbb{Z}_{p^\alpha}$ and $\overline{x}, \overline{y} \in \mathbb{Z}_{p^\alpha}/(p)$. Thus, by Proposition 2.2, there is a unique path from $\overline{x}$ to $\overline{y}$ in the graph $G_{NC}(\mathbb{Z}_p)$. We may assume that the path is $(x_1, x_2, \ldots, x_{l-1}, x_l\}$, where $x_1 = \overline{x}$ and $x_l = \overline{y}$. It is easy to see that $(x, x_2, \ldots, x_{l-1}, y\}$ is a path from $x$ to $y$ in $G_{NC}(\mathbb{Z}_{p^\alpha})$. Thus, $\mathrm{diam}(G_{NC}(\mathbb{Z}_{p^\alpha})) \leq \mathrm{diam}(G_{NC}(\mathbb{Z}_p)) = p - 1$.

Next, we show that $d(0, \frac{p+1}{2}) = p - 1$ in the graph $G_{NC}(\mathbb{Z}_{p^\alpha})$. Assume, to the contrary, there exists a path from 0 and $\frac{p+1}{2}$ to a length less than $p-1$. This forces that $d(0, \frac{p+1}{2}) < p-1$ in the graph $G_{NC}(\mathbb{Z}_p)$, which contradicts Proposition 2.2. This completes the proof. □

**Lemma 2.1.** *Let $p$ be a prime, $q$ be an odd positive integer, and $q \geq p$, $x$ and $y$ be two vertices in $G_{NC}(\mathbb{Z}_{p^\alpha})$. Then, there exists a sequence $\{\beta_k\}_{k=1}^{k=q}$ consisting $q$ elements (allowing repeat) of $\mathbb{Z}_{p^\alpha}$, where $\beta_1 = x, \beta_q = y$, such that $\beta_k + \beta_{k+1}$ is a nil-clean element of $\mathbb{Z}_{p^\alpha}$, for every $k = 1, \ldots, q-1$.*

*Proof.* In the case $p = 2$, $\mathbb{Z}_{2^\alpha}$ is a nil-clean ring, and $G_{NC}(\mathbb{Z}_{2^\alpha})$ is a complete graph with $2^\alpha$ vertices. Let $\beta_1 = \cdots = \beta_{q-1} = x, \beta_q = y$.

For $p$ is an odd prime, we discuss the case where $q = p$ since we just need to repeat the last two elements of the sequence when $q > p$. Without loss of generality, suppose that $d(0, \overline{x}) = \min\{d(0, \overline{x}), d(\overline{x}, \frac{p+1}{2}), d(0, \overline{y}), d(\overline{y}, \frac{p+1}{2})\}$ in $G_{NC}(\mathbb{Z}_p)$. There exists a unique shortest path from $\overline{x}$ to 0 in $G_{NC}(\mathbb{Z}_p)$ by Proposition 2.2, denoted by $(v_1, \ldots, v_{d(0,\overline{x})+1})$, where $\overline{x} = v_1$ and $v_{d(0,\overline{x})+1} = 0$. Similarly, there exists a unique shortest path from 0 to $\overline{y}$, denoted by $(u_1, \ldots, u_{d(0,\overline{y})+1})$, where $u_1 = 0$ and $u_{d(0,\overline{y})+1} = \overline{y}$.

Let $\beta_1 = x, \beta_2 = v_2, \cdots, \beta_{d(0,\overline{x})} = v_{d(0,\overline{x})} = 1, \beta_{d(0,\overline{x})+1} = \cdots = \beta_{q-d(0,\overline{y})} = 0, \beta_{q-d(0,\overline{y})+1} = u_2 = 1, \beta_{q-d(0,\overline{y})+2} = u_3 = p-1, \cdots, \beta_{q-1} = u_{d(0,\overline{y})}, \beta_q = y$. The sequence $\{\beta_k\}_{k=1}^{k=q}$ is desired. □

Now, we prove the main result in this note.

**Theorem 2.1.** *Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be the prime factorization, where $p_1 < \cdots < p_s$. Then,* $\operatorname{diam}(G_{NC}(\mathbb{Z}_n)) = p_s - 1$.

*Proof.* It holds for the case $n = 2^\alpha$, and $G_{NC}(\mathbb{Z}_{2^\alpha})$ is a complete graph. In other cases, give two vertices $(x_1, \ldots, x_s), (y_1, \ldots, y_s) \in \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$.

For $i = 1$, $p = p_1$, and $q = p_s$, there exists a sequence $\left\{\alpha_k^1\right\}_{k=1}^{k=p_s}$ by Lemma 2.1, where $\alpha_1^1 = x_1$, $\alpha_{p^s}^1 = y_1$, such that $\alpha_k^1 + \alpha_{k+1}^1 \in NC(\mathbb{Z}_{p_1^{\alpha_1}})$, for every $k = 1, \ldots, p_s - 1$.

For $i = 2$, $p = p_2$, and $q = p_s$, there exists a sequence $\left\{\alpha_k^2\right\}_{k=1}^{k=p_s}$ by Lemma 2.1, where $\alpha_1^2 = x_2$, $\alpha_{p^s}^2 = y_2$, such that $\alpha_k^2 + \alpha_{k+1}^2 \in NC(\mathbb{Z}_{p_2^{\alpha_2}})$, for every $k = 1, \ldots, p_s - 1$.

Continuing this process, for $i = s$, $p = p_s$, and $q = p_s$, there exists a sequence $\left\{\alpha_k^s\right\}_{k=1}^{k=p_s}$ by Lemma 2.1, where $\alpha_1^s = x_s$, $\alpha_{p^s}^s = y_s$, such that $\alpha_k^s + \alpha_{k+1}^s \in NC(\mathbb{Z}_{p_s^{\alpha_s}})$, for every $k = 1, \ldots, p_s - 1$.

We obtain the sequence $\left\{(\alpha_k^1, \ldots, \alpha_k^s)\right\}_{k=1}^{k=p_s}$, where $(\alpha_1^1, \ldots, \alpha_1^s) = (x_1, \ldots, x_s) = x$ and $(\alpha_{p_s}^1, \ldots, \alpha_{p_s}^s) = (y_1, \ldots, y_s) = y$. In addition, $(\alpha_k^1 + \alpha_{k+1}^1, \ldots, \alpha_k^s + \alpha_{k+1}^s)$ is a nil-clean element in $\mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}$, for every $k = 1, \ldots, p_s - 1$. Then, there exists a path from $x$ to $y$ according the above sequence by removing those consecutive duplicate vertices (if applicable). Thus, $\operatorname{diam}(G_{NC}(\mathbb{Z}_n)) \leq p_s - 1$. On the other hand, it is clear that $d((0, \ldots, 0, 0), (0, \ldots, 0, \frac{p_s+1}{2}))$ is $p_s - 1$ in $G_{NC}(\mathbb{Z}_n)$ by Proposition 2.3, and $\operatorname{diam}(G_{NC}(\mathbb{Z}_n)) \geq p_s - 1$. This completes the proof. □

We finish this note with an example to helping in understanding Theorem 2.1.

**Example 2.1.** *Let $n = 2^4 7^3 11^2 = 664048$, $\mathbb{Z}_{664048} \cong \mathbb{Z}_{2^4} \times \mathbb{Z}_{7^3} \times \mathbb{Z}_{11^2}$. Given $(1, 339, 20), (12, 55, 114) \in \mathbb{Z}_{2^4} \times \mathbb{Z}_{7^3} \times \mathbb{Z}_{11^2}$, then $\overline{1} = 1$, $\overline{12} = 0 \in \mathbb{Z}_2$, $\overline{339} = 3$, $\overline{55} = 6 \in \mathbb{Z}_7$, $\overline{20} = 9$, $\overline{114} = 4 \in \mathbb{Z}_{11}$.*

*For $p = 2$ and $q = 11$, we obtain the sequence $\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 12\}$ by Lemma 2.1.*

*For $p = 7$ and $q = 11$, then $d(3, 4) = \min\{d(0, 6), d(0, 3), d(3, 4), d(6, 4)\}$ in the graph $G_{NC}(\mathbb{Z}_7)$. We obtain the sequence $\{339, 4, 4, 3, 5, 2, 55, 2, 55, 2, 55\}$ by Lemma 2.1.*

*For $p = 11$ and $q = 11$, then $d(4, 5) = \min\{d(0, 9), d(0, 4), d(4, 5), d(9, 4)\}$ in the graph $G_{NC}(\mathbb{Z}_{11})$. We obtain the sequence $\{20, 3, 8, 4, 7, 6, 5, 5, 6, 7, 114\}$ by Lemma 2.1.*

*According to the proof of Theorem 2.1, we obtain the sequence:* $\beta_1 = (1, 339, 20)$, $\beta_2 = (1, 4, 3)$, $\beta_3 = (1, 4, 8)$, $\beta_4 = (1, 4, 4)$, $\beta_5 = (1, 4, 7)$, $\beta_6 = (1, 4, 5)$, $\beta_7 = (1, 4, 6)$, $\beta_8 = (1, 3, 6)$, $\beta_9 = (1, 5, 5)$, $\beta_{10} = (1, 2, 7)$, $\beta_{11} = (12, 55, 114)$. *We do not need to delete any vertices of this sequence, and obtain the path from* $(1, 339, 20)$ *to* $(12, 55, 114)$ *in the graph* $G_{NC}(\mathbb{Z}_{2^4} \times \mathbb{Z}_{7^3} \times \mathbb{Z}_{11^2})$.

**Remark 2.1.** For a finite commutative ring $R$, we have the Artin's decomposition in local rings of $R \cong R_1 \times \cdots \times R_s$. Next, we may study the nil-clean graph of the finite product of finite fields, ignoring the impact of nilpotent elements on calculating the diameter. For a finite field $GF(p^k)$, $G_{NC}(GF(p^k))$ is the union of a path with $p$ vertices and $\frac{p^{k-1}-1}{2}$ $2p$-cycles [7]. In particular, $\text{diam}(G_{NC}(GF(p^k))) = \infty$ when $k > 1$. Note that graph $G \otimes H$ is disconnected when one of $G$ and $H$. In other words, $\text{diam}(G_{NC}(R)) = p - 1$ for some prime $p$ or $\infty$.

## Author contributions

All authors contributed to the study conception and design. The first version of the manuscript was written by H. Su and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare that they have no conflict of interest.

## References

1. D. F. Anderson, On the diameter and girth of a zero-divisor graph, II, *Houston J. Math.*, **34** (2008), 361–371.
2. D. F. Anderson, A. Badawi, The total graph of a commutative ring, *J. Algebra*, **320** (2008), 2706–2719. https://doi.org/10.1016/j.jalgebra.2008.06.028
3. D. F. Anderson, P. S. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, **217** (1999), 434–447. https://doi.org/10.1006/jabr.1998.7840
4. D. F. Anderson, S. B. Mulay, On the diameter and girth of a zero-divisor graph, *J. Pure Appl. Algebra*, **210** (2007), 543–550. https://doi.org/10.1016/j.jpaa.2006.10.007

5. N. Ashrafi, H. R. Maimani, M. R. Pournaki, S. Yassemi, Unit graphs associated with rings, *Commun. Algebra*, **38** (2010), 2851–2871. https://doi.org/10.1080/00927870903095574
6. I. Beck, Coloring of commutative rings, *J. Algebra*, **116** (1988), 208–226. https://doi.org/10.1016/0021-8693(88)90202-5
7. D. K. Basnet, J. Bhattacharyya, Nil clean graphs of rings, *Algebra Colloq.*, **24** (2017), 481–492. https://doi.org/10.1142/S1005386717000311
8. G. Călugăreanu, T. Y. Lam, Fine rings: A new class of simple rings, *J. Algebra Appl.*, **15** (2016), 1650173. https://doi.org/10.1142/S0219498816501735
9. A. J. Diesl, Nil clean rings, *J. Algebra*, **383** (2013), 197–211. https://doi.org/10.1016/j.jalgebra.2013.02.020
10. F. Heydari, M. J. Nikmehr, The unit graph of a left Artinian ring, *Acta Math. Hung.*, **139** (2013), 134–146. https://doi.org/10.1007/s10474-012-0250-3
11. T. G. Lucas, The diameter of a zero divisor graph, *J. Algebra*, **301** (2006), 174–193. https://doi.org/10.1016/j.jalgebra.2006.01.019
12. H. R. Maimani, M. R. Pournaki, S. Yassemi, Zero-divisor graph with respect to an ideal, *Commun. Algebra*, **34** (2006), 923–929. https://doi.org/10.1080/00927870500441858
13. W. K. Nicholson, Lifting idempotents and exchange rings, *T. Am. Math. Soc.*, **229** (1977), 269–278. https://doi.org/10.2307/1998510
14. B. A. Rather, Independent domination polynomial for the cozero divisor graph of the ring of integers modulo *n*, *Discrete Math. Lett.*, **13** (2024), 36–43. https://doi.org/10.47443/dml.2023.215
15. H. Su, On the diameter of unitary Cayley graphs of rings, *Can. Math. Bull.*, **59** (2016), 652–660. https://doi.org/10.4153/CMB-2016-014-7
16. H. Su, Y. Wei, The dimaeter of unit graphs of rings, *Taiwanese J. Math.*, **23** (2019), 1–10. https://doi.org/10.11650/tjm/180602
17. P. Vámos, 2-good rings, *Q. J. Math.*, **56** (2005), 417–430. https://doi.org/10.1093/qmath/hah046
18. H. J. Wang, Graphs associated to co-maximal ideals of commutative rings, *J. Algebra*, **320** (2008), 2917–2933. https://doi.org/10.1016/j.jalgebra.2008.06.020