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Research article

Silting objects and recollements of extriangulated categories

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Abstract: In this paper, let $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ be a recollement of extriangulated categories. We construct a silting object and a tilting object from the two end terms of a recollement. We also show that the reverse direction holds under natural assumptions. Moreover, we show that our gluing preserves cotorsion pairs.

Keywords: extriangulated category; recollement; silting object; tilting object; cotorsion pair
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1. Introduction

Belinson et al. [1] introduced the notions of recollements of abelian and triangulated categories, which were in connection with derived categories of sheaves on topological spaces, with the idea that one triangulated category may be glued together from two others. The recollements of abelian and triangulated categories are closely related, and they play an important role in algebraic geometry and representation theory; see, for instance, [2–4].

Gluing techniques concerning the recollement of triangulated or abelian categories have been investigated by more and more experts and scholars. For example, the glued cotorsion pairs [5], the glued torsion pairs [6], and so on (e.g., [7, 8]). In particular, Liu et al. [9] presented explicit constructions of the gluing of silting objects for a recollement of triangulated categories.

Nakaoka and Palu [10] introduced an extriangulated category that extracts properties from triangulated categories and exact categories. In particular, exact categories and triangulated categories are extriangulated categories. There are a lot of examples of extriangulated categories, which are neither exact nor triangulated categories; see [10, 11]. In the work of [12], Wang et al. defined the recollement of extriangulated categories, which generalized recollements of abelian categories and triangulated categories.

Motivated by the applications of gluing techniques in recollements, we want to consider gluing a silting object and a tilting object from two other silting objects and tilting objects in a recollement of extriangulated categories.

In this paper, let $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ be a recollement of extriangulated categories. Our first main result describes how to glue together silting objects σ in $\mathscr{A}$ and ω in $\mathscr{B}$ to obtain a silting object ρ in $\mathscr{C}$; see Theorem 3.3. In the reverse direction, our second main result gives sufficient conditions for a silting object ρ of $\mathscr{C}$, relative to the functors involved in the recollement, to induce silting objects in $\mathscr{A}$ and $\mathscr{B}$; see Theorem 3.5, which partially extends the results of [9, Theorem 3.1], [13, Theorem 2.10] and [14, Theorem 2.5] in the framework of extriangulated categories. Our third main result constructs a glued tilting object from tilting objects in $\mathscr{A}$ and in $\mathscr{B}$; see Theorems 4.2 and 4.3, which extend the result in [15, Theorem 3.5] to extriangulated categories.

2. Preliminaries

This section briefly recalls some definitions and basic properties of extriangulated categories from [10, 11]. We omit some details here, but we refer the readers to [10].

Let $\mathscr{C}$ be an additive category. All subcategories considered are full additive subcategories closed under isomorphisms. Let

$$\mathbb{E} : \mathscr{C}^{op} \times \mathscr{C} \to Ab$$

be a biadditive functor, where Ab is an abelian category. For any pair of objects $A, C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-*extension*.

Let $\mathfrak{s}$ be a correspondence that associates an equivalence class

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$$

to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a realization of $\mathbb{E}$ if it makes the diagrams in [10, Definition 2.9] commutative.

We call a triplet $(\mathscr{C}, \mathbb{E}, \mathfrak{s})$ an extriangulated category if it satisfies:

(1) $\mathbb{E} : \mathscr{C}^{op} \times \mathscr{C} \to Ab$ is a biadditive functor.

(2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

(3) $\mathbb{E}$ and $\mathfrak{s}$ satisfy the compatibility conditions $(ET3)$, $(ET3)^{op}$, $(ET4)$, and $(ET4)^{op}$ in [10, Definition 2.12].

Remark 2.1. By [10, Example 2.13 and Proposition 3.22], triangulated categories and exact categories (with a condition concerning the smallness) are typical examples of extriangulated categories.

For an extriangulated category $\mathscr{C}$, we use the following notations in [10, 11]:

- A sequence $A \xrightarrow{a} B \xrightarrow{b} C \overset{\delta}{\dashrightarrow}$ is called an $\mathbb{E}$-*triangle*.
- Let $A \xrightarrow{a} B \xrightarrow{b} C \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle. A is called the *CoCone* of b and denoted by $CoCone(B, C)$; C is called the *Cone* of a and denoted by $Cone(A, B)$.
- An object P is called projective if, for any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and any morphism $c \in \mathscr{C}(P, C)$, there exists a morphism $b \in \mathscr{C}(P, B)$, satisfying $y \circ b = c$. The class of all the projective objects in $\mathscr{C}$ is denoted by $\mathcal{P}_{\mathscr{C}}$. An injective object can be dually defined, and the class of all the injective objects in $\mathscr{C}$ is denoted by $\mathcal{I}_{\mathscr{C}}$.

- We say that $\mathscr{C}$ has enough projective objects if, for any object $C \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle $K \overset{x}{\to} P \overset{y}{\to} C \overset{\delta}{\dashrightarrow}$ satisfying $P \in \mathcal{P}_{\mathscr{C}}$. Dually, we can define $\mathscr{C}$ as having enough injective objects.

Let $\mathcal{X}$, $\mathcal{Y}$ be subcategories of $\mathscr{C}$.

- Denote $Cone(\mathcal{X}, \mathcal{Y})$ by the subcategory of $\mathscr{C}$ consisting of $M \in \mathscr{C}$, which admits an $\mathbb{E}$-triangle $X \to Y \to M \dashrightarrow$ in $\mathscr{C}$, with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We say that $\mathcal{X}$ is closed under *Cones* if $Cone(\mathcal{X}, \mathcal{X}) \subseteq \mathcal{X}$.
- Denote $CoCone(\mathcal{X}, \mathcal{Y})$ by the subcategory of $\mathscr{C}$ consisting of $M \in \mathscr{C}$, which admits an $\mathbb{E}$-triangle $M \to X \to Y \dashrightarrow$ in $\mathscr{C}$, with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We say that $\mathcal{X}$ is closed under *CoCones* if $CoCone(\mathcal{X}, \mathcal{X}) \subseteq \mathcal{X}$.
- We denote
$$\Omega\mathcal{X} = CoCone(\mathcal{P}_{\mathscr{C}}, \mathcal{X}).$$
And Ω is called the syzygy of $\mathcal{X}$. Dually, we define the cosyzygy of $\mathcal{X}$ by
$$\Sigma\mathcal{X} = Cone(\mathcal{X}, \mathcal{P}_{\mathscr{C}}),$$
see [16, Definition 4.2, and Proposition 4.3].

For any subcategory $\mathcal{X}$ of $\mathscr{C}$, put
$$\Omega^0\mathcal{X} = \mathcal{X},$$
for $k \geq 1$, define $\Omega^k\mathcal{X}$ inductively by
$$\Omega^k\mathcal{X} = \Omega(\Omega^{k-1}\mathcal{X}) = CoCone(\mathcal{P}_{\mathscr{C}}, \Omega^{k-1}\mathcal{X}).$$
$\Omega^k\mathcal{X}$ is called the k-th syzygy of $\mathcal{X}$. Dually the k-th cosyzygy $\Sigma^k\mathcal{X}$ by
$$\Sigma^k\mathcal{X} = Cone(\Sigma^{k-1}, \mathcal{I}_{\mathscr{C}})$$
for $k \geq 1$ can be defined.

For any $k > 0$ and any objects X, Y, Liu and Nakaoka [16, Proposition 5.2] defined the higher extension groups in an extriangulated category having enough projective and injective objects as
$$\mathbb{E}^{k+1}(X, Y) = \mathbb{E}(\Omega^k X, Y) = \mathbb{E}(X, \Sigma^k Y).$$
They showed the following result:

Lemma 2.2. *[16, Proposition 5.2] Let $A \overset{f}{\to} B \overset{g}{\to} C \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle. For any object $X \in \mathscr{C}$, there are long exact sequences*
$$\mathscr{C}(X, A) \overset{\mathscr{C}(X,f)}{\to} \mathscr{C}(X, B) \overset{\mathscr{C}(X,g)}{\to} \mathscr{C}(X, C) \overset{(\delta_\sharp)_X}{\to} \mathbb{E}(X, A) \overset{f_*}{\to} \mathbb{E}(X, B) \overset{g_*}{\to} \mathbb{E}(X, C) \to \mathbb{E}^2(X, A) \to \cdots$$
and
$$\mathscr{C}(C, X) \overset{\mathscr{C}(g,X)}{\to} \mathscr{C}(B, X) \overset{\mathscr{C}(f,X)}{\to} \mathscr{C}(A, X) \overset{(\delta^\sharp)_X}{\to} \mathbb{E}(C, X) \overset{g^*}{\to} \mathbb{E}(B, X) \overset{f^*}{\to} \mathbb{E}(A, X) \to \mathbb{E}^2(C, X) \to \cdots.$$

We define

$$\mathcal{X}^{\perp} = \{Y \in \mathscr{C} \mid \mathbb{E}^k(\mathcal{X}, Y) = 0, \forall k \geq 1\}.$$

Dually, we define

$$^{\perp}\mathcal{X} = \{Y \in \mathscr{C} \mid \mathbb{E}^k(Y, \mathcal{X}) = 0, \forall k \geq 1\}.$$

Definition 2.3. *[17, Definition 3.1]* Let $\mathscr{C}$ be an extriangulated category, $\mathcal{X}$ and $\mathcal{Y}$ subcategories of $\mathscr{C}$. We call a pair $(\mathcal{X}, \mathcal{Y})$ a *hereditary cotorsion pair* if:

(1) Both $\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands;
(2) $\mathbb{E}^k(X, Y) = 0$ for any $k \geq 1$ and $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;
(3) $\mathscr{C} = Cone(\mathcal{Y}, \mathcal{X})$;
(4) $\mathscr{C} = CoCone(\mathcal{Y}, \mathcal{X})$.

We recall the concepts and basic properties of recollements of extriangulated categories from [12].

Definition 2.4. *[12, Definition 3.1]* Let $\mathscr{A}$, $\mathscr{C}$, and $\mathscr{B}$ be three extriangulated categories. A recollement of $\mathscr{C}$ relative to $\mathscr{A}$ and $\mathscr{B}$, denoted by $(\mathscr{A}, \mathscr{C}, \mathscr{B})$, is a diagram

$$\mathscr{A} \underset{\overset{\longleftarrow i^! \longleftarrow}{}}{\overset{\overset{\longleftarrow i^* \longleftarrow}{}}{\xrightarrow{\quad i_* \quad}}} \mathscr{C} \underset{\overset{\longleftarrow j_* \longleftarrow}{}}{\overset{\overset{\longleftarrow j_! \longleftarrow}{}}{\xrightarrow{\quad j^* \quad}}} \mathscr{B} \tag{2.1}$$

given by two exact functors i_*, j^*, two right exact functors i^*, $j_!$, and two left exact functors $i^!$, j_*, which satisfies the following conditions:

(R1) $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples.
(R2) Im i_* =Ker j^*.
(R3) i_*, $j_!$, j_* are fully faithful.
(R4) For each $C \in \mathscr{C}$, there exists a left exact $\mathbb{E}_{\mathscr{C}}$-triangle sequence

$$i_*i^!C \to C \to j_*j^*C \to i_*A$$

with $A \in \mathscr{A}$.
(R5) For each $C \in \mathscr{C}$, there exists a right exact $\mathbb{E}_{\mathscr{C}}$-triangle sequence

$$i_*A_1 \to j_!j^*C \to C \to i_*i^*C$$

with $A_1 \in \mathscr{A}$.

Remark 2.5. (1) If the categories $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are triangulated, then Definition 2.4 coincides with the definition of a recollement of triangulated categories [1].

(2) If the categories $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are abelian, then Definition 2.4 coincides with the definition of a recollement of abelian categories [3, 4, 6].

Lemma 2.6. *[12, Lemma 3.3] Let $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ be a recollement of extriangulated categories as* (2.1) *in Definition 2.4.*

(1) All the natural transformations $i^*i_* \Rightarrow Id_{\mathscr{A}}$, $Id_{\mathscr{A}} \Rightarrow i^!i_*$, $Id_{\mathscr{B}} \Rightarrow j^*j_!$, $j^*j_* \Rightarrow Id_{\mathscr{B}}$ are natural isomorphisms.

(2) $i^*j_! = 0 = i^!j_*$.

(3) i^* preserves projective objects, and $i^!$ preserves injective objects.

(4) $j_!$ preserves projective objects, and j_* preserves injective objects.

(5) If $i^!$ (resp. j_*) is exact, then i_* (resp. j^*) preserves projective objects.

(6) If i^* (resp. $j_!$) is exact, then i_* (resp. j^*) preserves injective objects.

(7) If $\mathscr{C}$ has enough projectives, then $\mathscr{A}$ has enough projectives; if $\mathscr{C}$ has enough injectives, then $\mathscr{A}$ has enough injectives.

(8) If $\mathscr{C}$ has enough projectives and j_* is exact, then $\mathscr{B}$ has enough projectives ; if $\mathscr{C}$ has enough injectives and $j_!$ is exact, then $\mathscr{B}$ has enough injectives.

(9) If i^* is exact, then $j_!$ is exact.

(10) If $i^!$ is exact, then j_* is exact.

The following results will be used frequently in this paper; their proof is very similar to [18, Proposition 2.8], we omit it.

Lemma 2.7. *Let $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ be a recollement of extriangulated categories as* (2.1) *in Definition 2.4.*

(1) If $\mathscr{A}$ has enough projectives and $i^!$ is exact, then

$$\mathbb{E}^k_{\mathscr{C}}(i_*X, Y) \cong \mathbb{E}^k_{\mathscr{A}}(X, i^!Y)$$

for any $k \geq 1$, $X \in \mathscr{A}$ and $Y \in \mathscr{C}$.

(2) If $\mathscr{A}$ has enough injectives and i^* is exact, then

$$\mathbb{E}^k_{\mathscr{A}}(i^*X, Y) \cong \mathbb{E}^k_{\mathscr{C}}(X, i_*Y)$$

for any $k \geq 1$, $X \in \mathscr{C}$ and $Y \in \mathscr{A}$.

(3) If $\mathscr{B}$ has enough projectives and $j_!$ is exact, then

$$\mathbb{E}^k_{\mathscr{C}}(j_!Z, Y) \cong \mathbb{E}^k_{\mathscr{B}}(Z, j^*Y)$$

for any $k \geq 1$, $Z \in \mathscr{B}$ and $Y \in \mathscr{C}$.

3. Gluing silting objects

In this section, we always assume $\mathscr{A}, \mathscr{C}$ and $\mathscr{B}$ are three extriangulated categories with enough projective and injective objects.

A subcategory $\mathcal{X}$ of $\mathscr{C}$ is called *thick* if it is closed under extensions, *Cones*, *CoCones*, and direct summands. Let *thick*$\mathcal{X}$ denote the smallest thick subcategory containing $\mathcal{X}$. When $\mathcal{X}$ is a single object M, then *thick*(M) = *thick* (add M).

Definition 3.1. An object σ in an extriangulated category $\mathscr{C}$ is silting if

(1) $\mathbb{E}^k(\sigma, \sigma) = 0$ for all $k \geq 1$;

(2) $thick(\sigma) = \mathscr{C}$.

For an object σ, add σ denotes the closure of σ under finite direct sums and summands. Clearly, when σ is a silting object, then add σ is a silting subcategory in the sense of [17, Definition 5.1]. When $\mathscr{C}$ is a well generated triangulated category, Definition 3.1 coincides with [13, Definition 1.11].

Example 3.2. *Let R be any ring, and let $\mathcal{P}^{\wedge}(R)$ denote the category of right R-modules of finite projective dimension. Since $\mathcal{P}^{\wedge}(R)$ is closed under extensions, it is an extriangulated category. We can easily check that R is a silting object of $\mathcal{P}^{\wedge}(R)$.*

For any subcategory $\mathcal{X}$ of $\mathscr{C}$, let

$$\mathcal{X}_0^{\vee} = \mathcal{X}_0^{\wedge} = \mathcal{X}.$$

We denote the full subcategories $\mathcal{X}_i^{\vee}$ and $\mathcal{X}_i^{\wedge}$ for $i > 0$ inductively by

$$\mathcal{X}_i^{\vee} = CoCone(\mathcal{X}, \mathcal{X}_{i-1}^{\vee}), \quad \mathcal{X}_i^{\wedge} = Cone(\mathcal{X}_{i-1}^{\wedge}, \mathcal{X}).$$

Put

$$\mathcal{X}^{\vee} := \bigcup_{i\geq 0} \mathcal{X}_i^{\vee} \text{ and } \mathcal{X}^{\wedge} := \bigcup_{i\geq 0} \mathcal{X}_i^{\wedge}.$$

Proposition 3.3. *Assume that $\mathscr{C}$ admits a recollement relative to $\mathscr{A}$ and $\mathscr{B}$ as* (2.1) *in Definition 2.4 and σ, ω are silting objects in $\mathscr{A}$ and $\mathscr{B}$, respectively. If i^* is exact, then*

$$thick(i_*\sigma \oplus j_!\omega) = \mathscr{C}.$$

Proof. Let $C \in \mathscr{C}$. As i^* is exact, there exists an $\mathbb{E}$-triangle by [12, Proposition 3.4],

$$(1) \;\; j_!j^*C \to C \to i_*i^*C \dashrightarrow .$$

Clearly, $i^*C \in \mathscr{A}$ and $j^*C \in \mathscr{B}$. By [17, Theorem 5.7], there exist two hereditary cotorsion pairs ((add $\sigma)^{\vee}$, (add $\sigma)^{\wedge}$) in $\mathscr{A}$ and ((add $\omega)^{\vee}$, (add $\omega)^{\wedge}$) in $\mathscr{B}$, respectively. So we get two $\mathbb{E}$-triangles associated with the above two hereditary cotorsion pairs:

$$(2) \;\; N \to M \to i^*C \dashrightarrow \quad \text{and} \quad (3) \;\; V \to U \to j^*C \dashrightarrow,$$

where $N \in$(add $\sigma)^{\wedge}$, $M \in$(add $\sigma)^{\vee}$ and $V \in$(add $\omega)^{\wedge}$, $U \in$(add $\omega)^{\vee}$.

Since i_* is exact, i_*(add $\sigma)^{\wedge} \subseteq$ (add $i_*\sigma)^{\wedge}$ and i_*(add $\sigma)^{\vee} \subseteq$ (add $i_*\sigma)^{\vee}$. So $i_*i^*C \in thick(i_*\sigma)$ by the $\mathbb{E}$-triangle (2). Similarly, since i^* is exact, $j_!$ is exact. So $j_!j^*C \in thick(j_!\omega)$ by the $\mathbb{E}$-triangle (3). Hence $C \in thick(j_!\omega \oplus i_*\sigma)$ by the $\mathbb{E}$-triangle (1). So

$$\mathscr{C} = thick(i_*\sigma \oplus j_!\omega).$$

This completes the proof. □

Theorem 3.4. *Let $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ be a recollement as* (2.1) *in Definition 2.4. Assume σ and ω are silting objects in $\mathscr{A}$ and $\mathscr{B}$, respectively. If*

$$\mathbb{E}_C^k(i_*\sigma, j_!w) = 0$$

for any $k \geq 1$ and i^* is exact, then $j_!w \oplus i_*\sigma$ is a silting object in $\mathscr{C}$.

Proof. Since i^* is exact, $j_!$ is exact by Lemma 2.6(9). So

$$\mathbb{E}^k_{\mathscr{C}}(j_!\omega, j_!\omega) \cong \mathbb{E}^k_{\mathscr{B}}(\omega, j^* j_!\omega) \cong \mathbb{E}^k_{\mathscr{B}}(\omega, \omega) = 0$$

and

$$\mathbb{E}^k_{\mathscr{C}}(j_!\omega, i_*\sigma) \cong \mathbb{E}^k_{\mathscr{B}}(\omega, j^* i_*\sigma) = 0$$

for any $k \geq 1$ by Lemmas 2.6(1) and 2.7(3). Since i^* is exact, i_* preserves injective objects. So

$$\mathbb{E}^k_{\mathscr{C}}(i_*\sigma, i_*\sigma) \cong \mathbb{E}^k_{\mathscr{A}}(i^* i_*\sigma, \sigma) \cong \mathbb{E}^k_{\mathscr{A}}(\sigma, \sigma) = 0$$

by Lemmas 2.6(1) and 2.7(2). So for any $k \geq 1$, we get

$$\mathbb{E}^k_{\mathscr{C}}(i_*\sigma \oplus j_!\omega, i_*\sigma \oplus j_!\omega) = \mathbb{E}^k_{\mathscr{C}}(i_*\sigma, i_*\sigma) \oplus \mathbb{E}^k_{\mathscr{C}}(i_*\sigma, j_!\omega) \oplus \mathbb{E}^k_{\mathscr{C}}(j_!\omega, i_*\sigma) \oplus \mathbb{E}^k_{\mathscr{C}}(j_!\omega, j_!\omega) = 0.$$

Hence $j_!w \oplus i_*\sigma$ is a silting object in $\mathscr{C}$ by Proposition 3.2 and Definition 3.1. □

Let σ and ω be silting objects; then there exist hereditary cotorsion pairs $((\mathrm{add}\sigma)^\vee, (\mathrm{add}\sigma)^\wedge)$ and $((\mathrm{add}\ \omega)^\vee, (\mathrm{add}\ \omega)^\wedge)$ in $\mathscr{A}$ and $\mathscr{B}$ by [17, Theorem 5.7], respectively. Define

$$\mathcal{U} := \{C \in \mathscr{C} \mid i^*C \in (add\sigma)^\vee, j^*C \in (add\omega)^\vee\}$$

and

$$\mathcal{V} := \{C \in \mathscr{C} \mid i^!C \in (add\sigma)^\wedge, j^*C \in (add\omega)^\wedge\}.$$

By [12, Theorem 4.4], $(\mathcal{U}, \mathcal{V})$ is a glued hereditary cotorsion pair in $\mathscr{C}$.

On the other hand, by Theorem 3.3, we obtain a silting object

$$\rho = j_!\omega \oplus i_*\sigma$$

in $\mathscr{C}$. So there exists a hereditary cotorsion pair $((\mathrm{add}\,\rho)^\vee, (\mathrm{add}\,\rho)^\wedge)$ in $\mathscr{C}$.

The following result indicates that our gluing preserves the cotorsion pair:

Theorem 3.5. *In the situation above, let*

$$\rho = j_!\omega \oplus i_*\sigma.$$

Then

$$(\mathcal{U}, \mathcal{V}) = ((add\rho)^\vee, (add\rho)^\wedge).$$

Proof. Since σ and ω are silting objects,

$$(add\sigma)^\wedge = \sigma^\perp$$

and

$$(add\omega)^\wedge = \omega^\perp$$

by [17, Lemma 4.11]. By Theorem 3.3, ρ is a silting object in $\mathscr{C}$. So

$$thick(\rho) = \mathscr{C}.$$

Hence

$$\rho^{\perp} = (add\rho)^{\perp} = (add\rho)^{\wedge}$$

by [17, Lemma 4.11]. So $C \in \mathcal{V}$ if and only if $i^{!}C \in \sigma^{\perp}$ and $j^{*}C \in \omega^{\perp}$. By Lemma 2.7, the latter holds if and only if $C \in (i_{*}\sigma)^{\perp}$ and $C \in (j_{!}\omega)^{\perp}$ if and only if

$$C \in \rho^{\perp} = (add\rho)^{\wedge}.$$

So

$$(\mathcal{U}, \mathcal{V}) = ((add\rho)^{\vee}, (add\rho)^{\wedge}).$$

□

For the converse of Theorem 3.3, we have

Theorem 3.6. *Let* $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ *be a recollement as* (2.1) *in Definition 1.4. Assume that* ρ *is a silting object in* $\mathscr{C}$ *and* i^{*} *is exact,*

(1) If $i_{*}i^{*}\rho \in add\,\rho$*, then* $i^{*}\rho$ *is a silting object in* $\mathscr{A}$*.*

(2) If $j_{!}j^{*}\rho \in add\,\rho$*, then* $j^{*}\rho$ *is a silting object in* $\mathscr{B}$*.*

In this case, ρ *is a glued silting object with respect to two silting objects,* σ *in* $\mathscr{A}$ *and* ω *in* $\mathscr{B}$*.*

Proof. (1) Since i^{*} is exact,

$$\mathbb{E}^{k}_{\mathscr{A}}(i^{*}\rho, i^{*}\rho) \cong \mathbb{E}^{k}_{\mathscr{C}}(\rho, i_{*}i^{*}\rho)$$

for any $k \geq 1$ by Lemma 2.11. Since $i_{*}i^{*}\rho \in \text{add}\,\rho$,

$$\mathbb{E}^{k}_{\mathscr{C}}(\rho, i_{*}i^{*}\rho) = 0$$

by Definition 3.1. So

$$\mathbb{E}^{k}_{\mathscr{A}}(i^{*}\rho, i^{*}\rho) = 0.$$

For any $A \in \mathscr{A}$, then $i_{*}A \in \mathscr{C}$. Since ρ is a silting object in $\mathscr{C}$, there exists a hereditary cotorsion pair $((\text{add}\,\rho)^{\vee}, (\text{add}\,\rho)^{\wedge})$ induced by ρ. So there exists an $\mathbb{E}$-triangle $V_1 \to U_1 \to i_{*}A \dashrightarrow$ with $U_1 \in(\text{add}\,\rho)^{\vee}$ and $V_1 \in(\text{add}\,\rho)^{\wedge}$. Since i^{*} is exact, we get another $\mathbb{E}$-triangle: $i^{*}V_1 \to i^{*}U_1 \to i^{*}i_{*}A \dashrightarrow$, Clearly, $i^{*}V_1 \in (i^{*}(\text{add}\,\rho))^{\wedge} \subseteq thick(i^{*}\rho)$ and $i^{*}U_1 \in (i^{*}(\text{add}\,\rho))^{\vee} \subseteq thick(i^{*}\rho)$. Also

$$i^{*}i_{*}A \cong A,$$

so $A \in thick(i^{*}\rho)$ and

$$thick(i^{*}\rho) = \mathscr{A}.$$

Hence, $i^{*}\rho$ is a silting object in $\mathscr{A}$.

(2) Since i^{*} is exact, j^{*} preserves injective objects by Lemma 2.10(6). So

$$\mathbb{E}^{k}_{\mathscr{B}}(j^{*}\rho, j^{*}\rho) \cong \mathbb{E}^{k}_{\mathscr{C}}(j_{!}j^{*}\rho, \rho)$$

for any $k \geq 1$, by Lemma 2.7. Since $j_{!}j^{*}\rho \in \text{add}\,\rho$,

$$\mathbb{E}^{k}_{\mathscr{B}}(j^{*}\rho, j^{*}\rho) = 0.$$

For any $B \in \mathscr{B}$, using the same proof method, we can also obtain

$$thick(j^*\rho) = \mathscr{B}.$$

Hence, $j^*\rho$ is a silting object in $\mathscr{B}$.

Denote $i^*\rho$ by σ and $j^*\rho$ by ω. Then σ is a silting object in $\mathscr{A}$, and ω is a silting object in $\mathscr{B}$. Moreover, $i_*\sigma \in \operatorname{add}\rho$ and $j_!\omega \in \operatorname{add}\rho$. So

$$\mathbb{E}_{\mathscr{C}}^{k\geq 1}(i_*\sigma, j_!\omega) = 0.$$

By Theorem 3.3, $j_!\omega \oplus i_*\sigma$ is a silting object. Since i^* is exact, there exists an $\mathbb{E}_{\mathscr{C}}$-triangle $j_!\omega \to \rho \to i_*\sigma \dashrightarrow$ by [12, Proposition 3.4]. So

$$\rho \cong j_!\omega \oplus i_*\sigma$$

and ρ glued from σ and ω. □

By applying Theorem 3.5 to triangulated categories and using the fact that any triangulated category can be viewed as an extriangulated category, we get the following result:

Corollary 3.7. *When $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ is a recollement of triangulated categories, assume that i^* is exact. The silting object ρ glued from two silting objects σ in $\mathscr{A}$ and ω in $\mathscr{B}$ if and only if $i_*i^*\rho \in \operatorname{add}\rho$ and $j_!j^*\rho \in \operatorname{add}\rho$, which is a special case of [13, Theorem 2.26].*

4. Gluing tilting objects

In this section, we restrict the gluing procedures to tilting objects. We always assume that $\mathscr{A}, \mathscr{C}, \mathscr{B}$ are three extriangulated categories with enough projective and injective objects.

The notion of tilting objects in an extriangulated category was introduced in [19, Definition 5.2], and we recall it.

Definition 4.1. Let T be an object in an extriangulated category $\mathscr{C}$. It is called tilting if

(1) $T \in (\mathcal{P}_{\mathscr{C}})^{\wedge}$;

(2) $\mathbb{E}^k(T, T) = 0$, for all $k \geq 1$;

(3) $\mathcal{P}_{\mathscr{C}} \subseteq (\operatorname{add} T)^{\vee}$.

Clearly, if T is a tilting object, then

$$thick(T) = thick(\mathcal{P}_{\mathscr{C}}).$$

By [17, Proposition 4.10],

$$thick(\mathcal{P}_{\mathscr{C}}) = (\mathcal{P}_{\mathscr{C}})^{\wedge}$$

and by [19, Lemma 5.4], any tilting object of $\mathscr{C}$ is silting of $(\mathcal{P}_{\mathscr{C}})^{\wedge}$ and they coincide when every object in $\mathscr{C}$ has finite projective dimension by [19, Theorem 5.6].

Theorem 4.2. *Let $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ be a recollement of extriangulated categories as* (2.1) *in Definition 2.4 and i^*, $i^!$ exact. Assume that T_1 and T_2 are tilting objects in $\mathscr{A}$ and $\mathscr{B}$, respectively, such that*

$$\mathbb{E}_{\mathscr{C}}^{k}(i_*T_1, j_!T_2) = 0$$

for all $k \geq 1$. Then there exists a glued tilting object ρ with respect to T_1 and T_2 in $\mathscr{C}$.

Proof. By [19, Lemma 5.4], T_1 and T_2 are silting objects of $(\mathcal{P}_{\mathscr{A}})^{\wedge}$ and $(\mathcal{P}_{\mathscr{B}})^{\wedge}$, respectively, and we have

$$thick(T_1) = thick(\mathcal{P}_{\mathscr{A}}) = (\mathcal{P}_{\mathscr{A}})^{\wedge}$$

and

$$thick(T_2) = thick(\mathcal{P}_{\mathscr{B}}) = (\mathcal{P}_{\mathscr{B}})^{\wedge}.$$

Moreover, there exist two cotorsion pairs $((\text{add } T_1)^{\vee}, (\text{add } T_1)^{\wedge})$ in $(\mathcal{P}_{\mathscr{A}})^{\wedge}$ and $((\text{add } T_2)^{\vee}, (\text{add } T_2)^{\wedge})$ in $(\mathcal{P}_{\mathscr{B}})^{\wedge}$, respectively. Denote $j_!T_2 \oplus i_*T_1$ by ρ. Since i^* is exact, for any $k \geq 1$,

$$\mathbb{E}^k_{\mathscr{C}}(i_*T_1, i_*T_1) \cong \mathbb{E}^k_{\mathscr{A}}(T_1, T_1) = 0, \quad \mathbb{E}^k_{\mathscr{C}}(j_!T_2, j_!T_2) \cong \mathbb{E}^k_{\mathscr{B}}(T_2, T_2) = 0$$

and

$$\mathbb{E}^k_{\mathscr{C}}(j_!T_2, i_*T_1) \cong \mathbb{E}^k_{\mathscr{B}}(T_2, j^*i_*T_1) = 0$$

by Lemmas 2.6 and 2.7.

So

$$\mathbb{E}^k_{\mathscr{C}}(\rho, \rho) = 0,$$

and ρ is self-orthogonal. Since i^* is exact, $j_!$ is exact. Since $i^!$ is exact, i_* preserves projective objects. So

$$\rho = j_!T_2 \oplus i_*T_1 \in (\mathcal{P}_{\mathscr{C}})^{\wedge}$$

by [19, Lemma 5.1].

Since i^* is exact, for any $C \in (\mathcal{P}_{\mathscr{C}})^{\wedge}$, we have the following $\mathbb{E}$-triangle by [12, Proposition 3.4],

$$(1) \;\; j_!j^*C \to C \to i_*i^*C \dashrightarrow .$$

Since both i^* and j^* preserves projective objects, $i^*C \in (\mathcal{P}_{\mathscr{A}})^{\wedge}$ and $j^*C \in (\mathcal{P}_{\mathscr{B}})^{\wedge}$. So we get two $\mathbb{E}$-triangles associated with the above two cotorsion pairs:

$$(2) \;\; N \to M \to i^*C \dashrightarrow \quad \text{and} \quad (3) \;\; V \to U \to j^*C \dashrightarrow,$$

where $N \in (\text{add } T_1)^{\wedge}$, $M \in (\text{add } T_1)^{\vee}$ and $V \in (\text{add } T_2)^{\wedge}$, $U \in (\text{add } T_2)^{\vee}$.

Since i_* is exact, $i_*i^*C \in thick(i_*T_1)$ by the $\mathbb{E}$-triangle (2). Similarly, $j_!j^*C \in thick(j_!T_2)$ by the $\mathbb{E}$-triangle (3). Hence $C \in thick(j_!T_2 \oplus i_*T_1)$ by the $\mathbb{E}$-triangle (1). So

$$thick(\rho) = (\mathcal{P}_{\mathscr{C}})^{\wedge}$$

and thus ρ is a silting object in $(\mathcal{P}_{\mathscr{C}})^{\wedge}$. Hence $((\text{add } \rho)^{\vee}, (\text{add } \rho)^{\wedge})$ is a cotorsion pair in $(\mathcal{P}_{\mathscr{C}})^{\wedge}$. Since $\mathcal{P}_{\mathscr{C}} \subseteq (\mathcal{P}_{\mathscr{C}})^{\wedge}$, for any $M \in \mathcal{P}_{\mathscr{C}}$, there exists an $\mathbb{E}$-triangle $V \to U \to M \dashrightarrow$, where $U \in (\text{add } \rho)^{\vee}$ and $V \in (\text{add } \rho)^{\wedge}$. Since M is a project object, the $\mathbb{E}$-triangle splits. So M is a direct summand of U, and thus $\mathcal{P}_{\mathscr{C}} \subseteq (\text{add } \rho)^{\vee}$. So ρ is a tilting object in $\mathscr{C}$ by Definition 4.1. □

For the converse direction,

Theorem 4.3. *Let $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ be a recollement of extriangulated categories as* (2.1) *in Definition 2.4 and i^*, $i^!$ are exact. Assume that ρ is a tilting object in $\mathscr{C}$.*

(1) If $i_*i^*\rho \in add\, \rho$, then $i^*\rho$ is a tilting object in $\mathscr{A}$.

(2) If $j_!j^*\rho \in \mathrm{add}\,\rho$, then $j^*\rho$ is a tilting object in $\mathscr{B}$.

In this case, ρ glued from two tilting objects in $\mathscr{A}$ and $\mathscr{B}$.

Proof. By Theorem 3.5, both $i^*\rho$ and $j^*\rho$ are self-orthogonal. Since i^* is exact and i^* preserves projective objects,

$$i^*\rho \in (i^*\mathcal{P}_{\mathscr{C}})^\wedge \subseteq (\mathcal{P}_{\mathscr{A}})^\wedge.$$

Since $i^!$ is exact, j^* preserves projective objects. So

$$j^*\rho \in (j^*\mathcal{P}_{\mathscr{C}})^\wedge \subseteq (\mathcal{P}_{\mathscr{B}})^\wedge.$$

On the other hand, since ρ is a tilting object in $\mathscr{C}$, $\mathcal{P}_{\mathscr{C}} \subseteq (\mathrm{add}\rho)^\vee$. For any $M \in \mathcal{P}_{\mathscr{A}}$, then $i_*M \in \mathcal{P}_{\mathscr{C}}$. So

$$M \cong i^*i_*M \in (\mathrm{add}i^*\rho)^\vee.$$

For any $N \in \mathcal{P}_{\mathscr{B}}$, then $j_!N \in \mathcal{P}_{\mathscr{C}}$. So

$$N \cong j^*j_!N \in (\mathrm{add}\,j^*\rho)^\vee.$$

Hence, $i^*\rho$ is a tilting object in $\mathscr{A}$, and $j^*\rho$ is a tilting object in $\mathscr{B}$, by Definition 4.1.

Since i^* is exact, we have an $\mathbb{E}$-triangle by [12, Proposition 3.4],

$$(1)\ \ j_!j^*\rho \to \rho \to i_*i^*\rho \dashrightarrow .$$

Since $i_*i^*\rho \in \mathrm{add}\,\rho$ and $j_!j^*\rho \in \mathrm{add}\,\rho$,

$$\mathbb{E}^k_{\mathscr{C}}(i_*i^*\rho, j_!j^*\rho) = 0.$$

So

$$\rho \cong i_*i^*\rho \oplus j_!j^*\rho$$

and thus ρ glued from two tilting objects $i^*\rho$ in $\mathscr{A}$ and $j^*\rho$ in $\mathscr{B}$. □

Corollary 4.4. (1) If we let $\mathscr{C}$ be a triangulated category that is K-linear over a field K, $\mathscr{B}$ a localizing subcategory of $\mathscr{C}$, and

$$\mathscr{A} = ker(\mathscr{C}(\mathscr{A}, -)).$$

Then Theorem 4.2 is exactly [14, Theorem 2.5] when viewing triangulated categories as extriangulated categories.

(2) Let Λ_1, Λ, and Λ_2 be artin algebras,

$$\mathscr{A} = mod\Lambda_1, \quad \mathscr{C} = mod\Lambda \quad \text{and} \mathscr{B} = mod\Lambda_2.$$

Then Theorem 4.2 is [15, Theorem 3.5].

5. Conclusions

In this section, we present the conclusions of the paper.

Let $(\mathscr{A}, \mathscr{C}, \mathscr{B})$ be a recollement as (2.1) in Definition 2.4. Our first main result describes how to glue together silting objects σ in $\mathscr{A}$ and ω in $\mathscr{B}$ to obtain a silting object ρ in $\mathscr{C}$; see Theorem 3.3.

Theorem 5.1. *Assume σ and ω are silting objects in $\mathscr{A}$ and $\mathscr{B}$, respectively. If*

$$\mathbb{E}_C^k(i_*\sigma, j_!w) = 0$$

for any $k \geq 1$ and i^* is exact, then $j_!w \oplus i_*\sigma$ is a silting object in $\mathscr{C}$.

In the reverse direction, our second main result gives sufficient conditions for a silting object ρ of $\mathscr{C}$, relative to the functors involved in the recollement, to induce silting objects in $\mathscr{A}$ and $\mathscr{B}$; see Theorem 3.5, which partially extends the results of [9, Theorem 3.1], [13, Theorem 2.10] and [14, Theorem 2.5] in the framework of extriangulated categories.

Theorem 5.2. *Assume that ρ is a silting object in $\mathscr{C}$ and i^* is exact,*

(1) If $i_*i^*\rho \in add\,\rho$, then $i^*\rho$ is a silting object in $\mathscr{A}$.

(2) If $j_!j^*\rho \in add\,\rho$, then $j^*\rho$ is a silting object in $\mathscr{B}$.

In this case, ρ glued from two silting objects, σ in $\mathscr{A}$ and ω in $\mathscr{B}$.

Our third main result constructs a glued tilting object from tilting objects in $\mathscr{A}$ and in $\mathscr{B}$; see Theorems 4.2 and 4.3, which extend the result in [15, Theorem 3.5] to extriangulated categories.

Theorem 5.3. *Let i^* and $i^!$ be exact. Assume that T_1 and T_2 are tilting objects in $\mathscr{A}$ and $\mathscr{B}$, respectively, such that*

$$\mathbb{E}_{\mathscr{C}}^k(i_*T_1, j_!T_2) = 0$$

for all $k \geq 1$. Then there exists a glued tilting object ρ with respect to T_1 and T_2 in $\mathscr{C}$.

Theorem 5.4. *Let i^*, $i^!$ be exact. Assume that ρ is a tilting object in $\mathscr{C}$.*

(1) If $i_*i^*\rho \in add\,\rho$, then $i^*\rho$ is a tilting object in $\mathscr{A}$.

(2) If $j_!j^*\rho \in add\,\rho$, then $j^*\rho$ is a tilting object in $\mathscr{B}$.

In this case, ρ glued from two tilting objects in $\mathscr{A}$ and $\mathscr{B}$.

Author contributions

Zhen Zhang: contributed the creative ideals and proof techniques for this paper; Shance Wang: consulted the relevant background of the paper and composed the article, encompassing the structure of the article and the modification of grammar. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declared that they have no conflicts of interest.

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