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Research article

On neutrosophic ideals and prime ideals in rings

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Abstract: This article aims to introduce and explore the concept of neutrosophic ideals in the context of rings. Besides, we investigate how the property of regularity in a ring can be understood through the lens of neutrosophic ideals. We further present the concept of neutrosophic prime ideals and systematically identify all neutrosophic prime ideals in $\mathbb{Z}$.

Keywords: neutrosophic set; regular ring; neutrosophic ideal; neutrosophic prime ideal
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1. Introduction

Uncertainty permeates every aspect of human life. Zadeh [24] first presented the concept of a fuzzy set in response to the limitations of classical set theory in accommodating such uncertainties. These studies characterized a fuzzy set by a membership function graded across a unit interval. However, subsequent scrutiny revealed this definition's inadequacy when confronted with membership and non-membership degrees. To address this ambiguity, Atanassov [2] formulated intuitionistic fuzzy theory as an extension of the fuzzy set framework. Despite its broad approach and real-world application [1, 23], this theory encounters several challenges in practical applications. Consequently, Smarandache [19] proposed a neutrosophic set to confront ambiguous and inconsistent data problems. Since then, research has dealt with it, whether in the study of algebraic structures [5, 7, 11, 21] or life applications, for example, see [9, 10].

The exploration of algebraic constructions in the realm of neutrosophics has garnered considerable interest from researchers. For instance, Kandasamy et al. [14] delved into the study of neutrosophic rings. In [18], Salama et al. defined the notion of neutrosophic ideal topology, while Saber et al. introduced the notion of single-valued neutrosophic ideals set in Sostak's sense, which is considered a generalization of fuzzy ideals in $\check{S}$ostak's sense and intuitionistic fuzzy ideals in [17]. In the context of the notion of neutrosophic subalgebras in BCK/BCI-algebras introduced in [12], Young et al.

introduced neutrosophic ideals in BCK/BCI-algebras, see [16].

At the same time, Vildan et al. [5] proposed an approach to a single-valued neutrosophic ideal within a classical ring, extending the discourse on neutrosophic algebraic structures outlined in [4, 6, 13]. This approach uses min/ min/ max and $\geqslant$ / $\geqslant$ / $\leqslant$, respectively. Due to the importance and diversity of neutrosophic applications in life and the processing of uncertain and unlimited data, it was necessary to develop this approach, which has become urgent in light of these applications. Recently, Elrawy et al. [7] introduced a novel perspective on defining and analyzing neutrosophic sub-rings and ideals. This approach utilizes min/ max/ max and $\geqslant$ / $\leqslant$ / $\leqslant$. This consideration is better since the component μ represents a positive quality. In contrast, γ and ζ denote negative qualities, thus justifying the employment of max/ max and $\geqslant$ / $\leqslant$, respectively, for γ and ζ.

Our study of the concepts of ideal and prime ideal in the context of neutrosophic sets has two primary motivations. First, we explore the possibilities and limits of extending classical ideal and prime ideal theory to the neutrosophic ideal and the neutrosophic prime ideal by allowing elements to satisfy the prime condition with varying degrees of truth, indeterminacy, and falsity. Second, the neutrosophic ideals and neutrosophic prime ideal provide a more flexible framework for dealing with uncertain, incomplete, or conflicting information, which is essential and significant in applications such as artificial intelligence, economics, social sciences, and decision-making where data or conditions are uncertain. Classical ideals and prime ideals impose strict membership criteria, whereas neutrosophic and prime ideals allow partial and uncertain membership. The latter leads to more affluent and adaptable algebraic structures that sufficiently reflect the complexity of the real world. Also, this article builds on and extends the approach taken in [7]. Moreover, we delve into applying neutrosophic ideals within regular rings. Additionally, we present and scrutinize neutrosophic prime ideals. Furthermore, we deduce various properties and characterizations of neutrosophic ideals over the set of integers $\mathbb{Z}$.

The subsequent section of this article is structured as follows: In Section 2, fundamental definitions and outcomes are delineated, which serve as valuable groundwork for the main findings of this article. Section 3 presents the concept of a neutrosophic ideal within a regular ring. Furthermore, we explore the notion of a neutrosophic prime ideal and analyze all instances of neutrosophic prime ideals in $\mathbb{Z}$. To culminate our discussion, Section 3 encapsulates the findings and conclusions drawn throughout this article.

2. Some basic concepts

In this part, we will explore certain principles and outcomes that serve as the foundation for the subsequent section.

Definition 2.1. [20] A neutrosophic set $\mathcal{N}$ on $\mathbb{R}$ is given by:

$$\mathcal{N} = \{< \Theta, \mu(\Theta), \gamma(\Theta), \zeta(\Theta) >: \Theta \in \mathbb{R}\},$$

where $\mathbb{R}$ is a universe set, and $\mu, \gamma, \zeta : \mathbb{R} \to [0, 1]$.

Definition 2.2. [19, 21] Presume $\mathcal{N}_1 = \{< \Theta, \mu_1(\Theta), \gamma_1(\Theta), \zeta_1(\Theta) >: \Theta \in \mathbb{R}\}$, and $\mathcal{N}_2 = \{< \Theta, \mu_2(\Theta), \gamma_2(\Theta), \zeta_2(\Theta) >: \Theta \in \mathbb{R}\}$ are two neutrosophic sets on $\mathbb{R}$. Then,

1) $\mathcal{N}_1 \subseteq \mathcal{N}_2 = \{< \Theta, \mu_1(\Theta) \leq \mu_2(\Theta), \gamma_1(\Theta) \geq \gamma_2(\Theta), \zeta_1(\Theta) \geq \zeta_2(\Theta) >: \Theta \in \mathbb{R}\}$;

2) $\mathcal{N}_1 \cup \mathcal{N}_2 = \{<\Theta, \mu_1(\Theta) \vee \mu_2(\Theta), \gamma_1(s) \wedge \gamma_2(\Theta), \zeta_1(\Theta) \wedge \zeta_2(\Theta) >: \Theta \in \mathbb{R}\}$;
3) $\mathcal{N}_1 \cap \mathcal{N}_2 = \{<\Theta, \mu_1(\Theta) \wedge \mu_2(\Theta), \gamma_1(\Theta) \vee \gamma_2(\Theta), \zeta_1(\Theta) \vee \zeta_2(\Theta) >: \Theta \in \mathbb{R}\}$.

Definition 2.3. [8] Presume $\mathcal{W} \subseteq \mathcal{N}$. For $0 \leqslant \alpha \leqslant 1$, then the set

$$\mathcal{W}_\alpha = \{<\Theta, \mu(\Theta), \gamma(\Theta), \zeta(\Theta) >: \Theta \in \mathcal{R},\ \mu(\Theta) \geq \alpha, \gamma(\Theta) \leq \alpha, \zeta(\Theta) \leq \alpha\},$$

is a level subset of $\mathcal{W}$.

Theorem 2.4. *[8] Suppose that $\mathcal{G}$ is a group with identity e and $\mathcal{W}$ is a neutrosophic sub-group of $\mathcal{G}$, then the level subset $\mathcal{W}_\alpha$, for $\alpha \in [0,1]$, $\alpha \leqslant \mu(e)$, $\alpha \geqslant \gamma(e)$, and $\alpha \geqslant \zeta(e)$ is a sub-group of $\mathcal{G}$.*

Certainly, $\mathcal{W}_{\alpha_1} \subseteq \mathcal{W}_{\alpha_2}$, when $\alpha_1 > \alpha_2$, and $\mathcal{W}_{\alpha_1}, \mathcal{W}_{\alpha_2}$ sub-groups of $\mathcal{G}$ with the same conditions in Theorem 2.4.

Definition 2.5. [7] A neutrosophic subset $\mathcal{S}$ of $\mathbb{R}$ is said to be a neutrosophic sub-ring of $\mathbb{R}$ when the following conditions are met:
1) $\mu(\Theta_1 - \Theta_2) \geqslant \min(\mu(\Theta_1), \mu(\Theta_2))$,
2) $\mu(\Theta_1\Theta_2) \geqslant \min(\mu(\Theta_1), \mu(\Theta_2))$,
3) $\gamma(\Theta_1 - \Theta_2) \leqslant \max(\gamma(\Theta_1), \gamma(\Theta_2))$,
4) $\gamma(\Theta_1\Theta_2) \leqslant \max(\gamma(\Theta_1), \gamma(\Theta_2))$,
5) $\zeta(\Theta_1 - \Theta_2) \leqslant \max(\zeta(\Theta_1), \zeta(\Theta_2))$,
6) $\zeta(\Theta_1\Theta_2) \leqslant \max(\zeta(\Theta_1), \zeta(\Theta_2))$,
where $\Theta_1, \Theta_2 \in \mathbb{R}$.

Example 2.6. Let $(\mathbb{R}, +, .)$ be a classical ring of real numbers. Then, a neutrosophic subset $\mathcal{R} = \{<\Theta, \mu(\Theta), \gamma(\Theta), \zeta(\Theta) >: \Theta \in \mathbb{R}\}$ is defined as follows:

$$\mu(\Theta) = \begin{cases} 0.6 & \text{if } \Theta = 0, \\ 0.8 & \text{if } \Theta \neq 0, \end{cases}$$

$$\gamma(\Theta) = \begin{cases} 0.5 & \text{if } \Theta \neq 0, \\ 0.2 & \text{if } \Theta = 0, \end{cases}$$

$$\zeta(\Theta) = \begin{cases} 0.6 & \text{if } \Theta \neq 0, \\ 0.4 & \text{if } \Theta = 0, \end{cases}$$

is a neutrosophic sub-ring for all conditions of Definition 2.5 are held.

Definition 2.7. [7] A neutrosophic sub-ring $\mathcal{I}$ is called a neutrosophic left ideal if the next conditions are met:
1) $\mu(\Theta_1\Theta_2) \geqslant \mu(\Theta_2)$,
2) $\gamma(\Theta_1\Theta_2) \leqslant \gamma(\Theta_2)$,
3) $\zeta(\Theta_1\Theta_2) \leqslant \zeta(\Theta_2)$.
Again, $\mathcal{I}$ is called a neutrosophic right ideal when
1) $\mu(\Theta_1\Theta_2) \geqslant \mu(\Theta_1)$,
2) $\gamma(\Theta_1\Theta_2) \leqslant \gamma(\Theta_1)$,
3) $\zeta(\Theta_1\Theta_2) \leqslant \zeta(\Theta_1)$.
Also, $\mathcal{I}$ is called a neutrosophic ideal when it is a neutrosophic right and left ideal.

Proposition 2.8. *[7] A neutrosophic subset $\mathcal{I}$ of $\mathbb{R}$ is called a neutrosophic ideal when the following axioms are met:*

1) $\mu(\Theta_1 - \Theta_2) \geqslant \min(\mu(\Theta_1), \mu(\Theta_2))$,
2) $\mu(\Theta_1\Theta_2) \geqslant \max(\mu(\Theta_1), \mu(\Theta_2))$,
3) $\gamma(\Theta_1 - \Theta_2) \leqslant \max(\gamma(\Theta_1), \gamma(\Theta_2))$,
4) $\gamma(\Theta_1\Theta_2) \leqslant \min(\gamma(\Theta_1), \gamma(\Theta_2))$,
5) $\zeta(\Theta_1 - \Theta_2) \leqslant \max(\zeta(\Theta_1), \zeta(\Theta_2))$,
6) $\zeta(\Theta_1\Theta_2) \leqslant \min(\zeta(\Theta_1), \zeta(\Theta_2))$,
where $\Theta_1, \Theta_2 \in \mathbb{R}$.

Example 2.9. Presume $(Z_8, \oplus_8, \otimes_8)$ is a ring. Then, a neutrosophic subset $\mathcal{I} = \{< \Theta, \mu(\Theta), \gamma(\Theta), \zeta(\Theta) >: \Theta \in Z_8\}$ is defined as follows:

$$\mu(\Theta) = \begin{cases} 0.8 & \text{if } \Theta = 0, \\ 0.4 & \text{if } \Theta \in \{2,4,6\}, \\ 0.5 & \text{otherwise,} \end{cases}$$

$$\gamma(\Theta) = \begin{cases} 0.4 & \text{if } \Theta = 0, \\ 0.6 & \text{if } \Theta \in \{2,4,6\}, \\ 0.7 & \text{otherwise,} \end{cases}$$

$$\zeta(\Theta) = \begin{cases} 0.3 & \text{if } \Theta = 0, \\ 0.7 & \text{if } \Theta \in \{2,4,6\}, \\ 0.6 & \text{otherwise,} \end{cases}$$

is a neutrosophic ideal for all conditions of Proposition 2.8 are held.

Definition 2.10. [7] Let $\mathcal{S}$ be a neutrosophic ideal (sub-ring) of a ring $\mathbb{R}$ with $0 \leq \alpha \leq \mu(0)$ and $0 \leq \gamma(0), \zeta(0) \leq \alpha$. The ideal (sub-ring) $\mathcal{S}_\alpha$ is said to be a level ideal (level sub-ring) of $\mathcal{S}$.

Definition 2.11. [7] Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be two neutrosophic ideals of $\mathbb{R}$. Then, the product of $\mathcal{I}_1$ and $\mathcal{I}_2$ is defined as follows:

$$(\mu_1 \bullet \mu_2)(\ltimes) = \sup_{\ltimes = \sum_i s_i t_i} (min_i(min(\mu_1(s_i), \mu_2(t_i)))),$$
$$(\gamma_1 \bullet \gamma_2)(\ltimes) = \inf_{\ltimes = \sum_i s_i t_i} (max_i(max(\gamma_1(s_i), \gamma_2(t_i)))),$$
$$(\zeta_1 \bullet \zeta_2)(\ltimes) = \inf_{\ltimes = \sum_i s_i t_i} (max_i(max(\zeta_1(s_i), \zeta_2(t_i)))),$$

where $\ltimes, s_i, t_i \in \mathbb{R}$.

Definition 2.12. [22] A ring $\mathbb{R}$ is termed regular if for every element $a \in \mathbb{R}$ $\exists$ $s \in \mathbb{R}$ such that $a = asa$.

Theorem 2.13. *[3] A ring $\mathbb{R}$ is regular iff $IJ = I \cap J$, where I is the right ideal and J is the left ideal over $\mathbb{R}$.*

3. Main results

In this part, we introduce the idea of a neutrosophic ideal within a conventional ring. Additionally, we investigate the notion of a neutrosophic prime ideal and examine all occurrences of neutrosophic prime ideals in $\mathbb{Z}$. We start with the following subsection:

3.1. Neutrosophic ideal over regular ring

Next, we explain that the property characteristic of regularity in a classical ring can be defined with the help of neutrosophic ideals of the ring.

Lemma 3.1. *Let $\mathcal{I}_1$ be a neutrosophic left ideal and $\mathcal{I}_2$ be the neutrosophic right ideal; then $\mathcal{I}_1 \bullet \mathcal{I}_2 \subseteq \mathcal{I}_1 \cap \mathcal{I}_2$.*

Proof. Since

$$(\mu_1 \bullet \mu_2)(\ltimes) = \sup_{\ltimes=\sum_i s_i t_i} (min_i(min(\mu_1(s_i), \mu_2(t_i)))),$$
$$(\gamma_1 \bullet \gamma_2)(\ltimes) = \inf_{\ltimes=\sum_i s_i t_i} (max_i(max(\gamma_1(s_i), \gamma_2(t_i)))),$$
$$(\zeta_1 \bullet \zeta_2)(\ltimes) = \inf_{\ltimes=\sum_i s_i t_i} (max_i(max(\zeta_1(s_i), \zeta_2(t_i)))),$$

and $\mathcal{I}_1$ is a neutrosophic left ideal and $\mathcal{I}_2$ is a neutrosophic right ideal, then we obtain

$$\mu_1(t_i) \leqslant \mu_1(s_i t_i) = \mu_1(\ltimes),$$
$$\gamma_1(t_i) \geqslant \gamma_1(s_i t_i) = \gamma_1(\ltimes),$$
$$\zeta_1(t_i) \geqslant \zeta_1(s_i t_i) = \zeta_1(\ltimes),$$

and

$$\mu_2(s_i) \leqslant \mu_2(s_i t_i) = \mu_2(\ltimes),$$
$$\gamma_2(s_i) \geqslant \gamma_2(s_i t_i) = \gamma_2(\ltimes),$$
$$\zeta_2(s_i) \geqslant \zeta_2(s_i t_i) = \zeta_2(\ltimes),$$

so,

$$(\mu_1 \bullet \mu_2)(\ltimes) = \sup_{\ltimes=\sum_i s_i t_i} (min_i(min(\mu_1(s_i), \mu_2(t_i)))) \leqslant \min(\mu_1(\ltimes), \mu_2(\ltimes)),$$
$$(\gamma_1 \bullet \gamma_2)(\ltimes) = \inf_{\ltimes=\sum_i s_i t_i} (max_i(max(\gamma_1(s_i), \gamma_2(t_i)))) \geqslant \max(\gamma_1(\ltimes), \gamma_2(\ltimes)),$$
$$(\zeta_1 \bullet \zeta_2)(\ltimes) = \inf_{\ltimes=\sum_i s_i t_i} (max_i(max(\zeta_1(s_i), \zeta_2(t_i)))) \geqslant \max(\zeta_1(\ltimes), \zeta_2(\ltimes)),$$

Therefore, $\mathcal{I}_1 \bullet \mathcal{I}_2 \subseteq \mathcal{I}_1 \cap \mathcal{I}_2$. □

Theorem 3.2. *A ring $\mathbb{R}$ is considered regular iff $\mathcal{I}_1 \bullet \mathcal{I}_2 = \mathcal{I}_1 \cap \mathcal{I}_2$, for any neutrosophic left ideal $\mathcal{I}_1$ and any neutrosophic right ideal $\mathcal{I}_2$ over $\mathbb{R}$.*

Proof. Assume that $\mathbb{R}$ is regular, and let $\ltimes, \iota, \varsigma \in \mathbb{R}$ with $\ltimes = \iota\varsigma$. Then,

$$(\mu_1 \bullet \mu_2)(\ltimes) = \sup_{\ltimes=\iota\varsigma} (min(\mu_1(\iota), \mu_2(\varsigma))),$$
$$(\gamma_1 \bullet \gamma_2)(\ltimes) = \inf_{\ltimes=\iota\varsigma} (max(\gamma_1(\iota), \gamma_2(\varsigma))),$$
$$(\zeta_1 \bullet \zeta_2)(\ltimes) = \inf_{\ltimes=\iota\varsigma} (max(\zeta_1(\iota), \zeta_2(\varsigma))).$$

Since $\ltimes = \ltimes \rtimes \ltimes$ for some $\ltimes \in \mathbb{R}$, then,

$$\mu_2(\ltimes) = \mu_2(\ltimes \rtimes \ltimes) \geqslant \mu_2(\ltimes\rtimes) \geqslant \mu_2(\ltimes),$$
$$\gamma_2(\ltimes) = \gamma_2(\ltimes \rtimes \ltimes) \leqslant \gamma_2(\ltimes\rtimes) \leqslant \gamma_2(\ltimes),$$
$$\zeta_2(\ltimes) = \zeta_2(\ltimes \rtimes \ltimes) \leqslant \zeta_2(\ltimes\rtimes) \leqslant \zeta_2(\ltimes).$$

So, $\mu_2(\ltimes\rtimes) = \mu_2(\ltimes)$, $\gamma_2(\ltimes\rtimes) = \gamma_2(\ltimes)$, and $\zeta_2(\ltimes\rtimes) = \zeta_2(\ltimes)$. Taking $\iota = \ltimes$ and $\varsigma = \ltimes\rtimes$ so that

$$\begin{aligned}
(\mu_1 \bullet \mu_2)(\ltimes) &= \sup_{\ltimes=\iota\varsigma}(min(\mu_1(\iota), \mu_2(\varsigma))) \\
&\geqslant \min(\mu_1(\ltimes), \mu_2(\ltimes\rtimes)) \\
&= \min(\mu_1(\ltimes), \mu_2(\ltimes)). \\
(\gamma_1 \bullet \gamma_2)(\ltimes) &= \inf_{\ltimes=\iota\varsigma}(max(\gamma_1(\iota), \gamma_2(\varsigma))) \\
&\leqslant \max(\gamma_1(\ltimes), \gamma_2(\ltimes\rtimes)) \\
&= \max(\gamma_1(\ltimes), \gamma_2(\ltimes)). \\
(\zeta_1 \bullet \zeta_2)(\ltimes) &= \inf_{\ltimes=\iota\varsigma}(max(\zeta_1(\iota), \zeta_2(\varsigma))) \\
&\leqslant \max(\zeta_1(\ltimes), \zeta_2(\ltimes\rtimes)) \\
&= \max(\zeta_1(\ltimes), \zeta_2(\ltimes)).
\end{aligned}$$

Hence, $\mathcal{I}_1 \cap \mathcal{I}_2 \subseteq \mathcal{I}_1 \bullet \mathcal{I}_2$. Again, $\mathcal{I}_1 \bullet \mathcal{I}_2 \subseteq \mathcal{I}_1 \cap \mathcal{I}_2$ from Lemma 3.1. Therefore, $\mathcal{I}_1 \bullet \mathcal{I}_2 = \mathcal{I}_1 \cap \mathcal{I}_2$. In the other direction, suppose that $\mathcal{I}_1 \bullet \mathcal{I}_2 = \mathcal{I}_1 \cap \mathcal{I}_2$ for any neutrosophic left ideal $\mathcal{I}_1$, and any neutrosophic right ideal $\mathcal{I}_2$, then by Theorem 2.13, it is found that $\mathbb{R}$ is regular. □

3.2. Neutrosophic prime ideal

Here, we define and study the neutrosophic prime ideal.

Definition 3.3. A neutrosophic ideal $\mathcal{P} = \{< \ltimes, \mu(\ltimes), \gamma(\ltimes), \zeta(\ltimes) >: \ltimes \in \mathbb{R}\}$ of $\mathbb{R}$ is called prime for any neutrosophic ideals $\mathcal{A} = \{< \ltimes, \mu_1(\ltimes), \gamma_1(\ltimes), \zeta_1(\ltimes) >: \ltimes \in \mathbb{R}\}$, and $\mathcal{B} = \{< \ltimes, \mu_2(\ltimes), \gamma_2(\ltimes), \zeta_2(\ltimes) >: \ltimes \in \mathbb{R}\}$,

$$\mu_1(\ltimes) \bullet \mu_2(\ltimes) \leqslant \mu(\ltimes) \Rightarrow \mu_1(\ltimes) \leqslant \mu(\ltimes) \text{ or } \mu_2(\ltimes) \leqslant \mu(\ltimes),$$

$$\gamma_1(\ltimes) \bullet \gamma_2(\ltimes) \geqslant \gamma(\ltimes) \Rightarrow \gamma_1(\ltimes) \geqslant \gamma(\ltimes) \text{ or } \gamma_2(\ltimes) \geqslant \gamma(\ltimes),$$
$$\zeta_1(\ltimes) \bullet \zeta_2(\ltimes) \geqslant \zeta(\ltimes) \Rightarrow \zeta_1(\ltimes) \geqslant \zeta(\ltimes) \text{ or } \zeta_2(\ltimes) \geqslant \zeta(\ltimes).$$

Equivalent formulations: for any two elements $\ltimes, \rtimes \in \mathbb{R}$, then $\mathcal{P}$ is a neutrosophic prime ideal if

$$< \ltimes, \mu(\ltimes), \gamma(\ltimes), \zeta(\ltimes) > \bullet < \rtimes, \mu(\rtimes), \gamma(\rtimes), \zeta(\rtimes) > \in \mathcal{P},$$

then either $< \ltimes, \mu(\ltimes), \gamma(\ltimes), \zeta(\ltimes) > \in \mathcal{P}$ or $< \rtimes, \mu(\rtimes), \gamma(\rtimes), \zeta(\rtimes) > \in \mathcal{P}$.

Example 3.4. Suppose that $(Z_6, \oplus_6, \otimes_6)$ is a ring. Then, define a neutrosophic subset $\mathcal{S} = \{< u, \mu(u), \gamma(u), \zeta(u) >: u \in Z_6\}$ as follows:

$$\mu(u) = \begin{cases} 0.8 & \text{if } u = 0, \\ 0.5 & \text{if } u \in \{2, 4\}, \\ 0.3 & \text{otherwise,} \end{cases}$$

$$\gamma(u)=\begin{cases}0.3 & \text{if } u=0,\\ 0.7 & \text{if } u\in\{2,4\},\\ 0.9 & \text{otherwise,}\end{cases}$$

$$\zeta(u)=\begin{cases}0.1 & \text{if } u=0,\\ 0.7 & \text{if } u\in\{2,4\},\\ 0.6 & \text{otherwise.}\end{cases}$$

Thus, $\mathcal{S}$ is a neutrosophic ideal. Also, we get $\mathcal{P}=\{<0,0.8,0.3,0.1>,<2,0.5,0.7,0.7>,<4,0.5,0.7,0.7>\}$ is a neutrosophic prime ideal.

Theorem 3.5. *Let P be a proper prime ideal in $\mathbb{R}$; then a neutrosophic sub-ring $\mathcal{P}$ over P is a neutrosophic prime ideal.*

Proof. Suppose that $\mathcal{P}_1=\{<\ltimes,\mu_1(\ltimes),\gamma_1(\ltimes),\zeta_1(\ltimes)>:\ltimes\in\mathbb{R}\}$, and $\mathcal{P}_2=\{<\ltimes,\mu_2(\ltimes),\gamma_2(\ltimes),\zeta_2(\ltimes)>:\ltimes\in\mathbb{R}\}$ are neutrosophic ideals over $\mathbb{R}$ with

$$\mu_1(\ltimes)\bullet\mu_2(\ltimes)\leqslant\mu_(\ltimes),$$
$$\gamma_1(\ltimes)\bullet\gamma_2(\ltimes)\geqslant\gamma_(\ltimes),$$
$$\zeta_1(\ltimes)\bullet\zeta_2(\ltimes)\geqslant\zeta_(\ltimes),$$

where $\mathcal{P}=\{<\ltimes,\mu(\ltimes),\gamma(\ltimes),\zeta(\ltimes)>:\ltimes\in P\}$, $\mathcal{P}_1\nsubseteq_1\mathcal{P}$, and $\mathcal{P}_2\nsubseteq_1\mathcal{P}$. So, $\exists\ \iota,\varsigma\in\mathbb{R}$ such that one of the following held $\mu_1(\iota)\nleq\mu(\iota)$, $\gamma_1(\iota)\ngeq\gamma(\iota)$, $\zeta_1(\iota)\ngeq\zeta(\iota)$, and $\mu_2(\varsigma)\nleq\mu(\varsigma)$, $\gamma_2(\varsigma)\ngeq\gamma(\varsigma)$, $\zeta_2(\varsigma)\ngeq\zeta(\varsigma)$. Now, $\mu_1(\iota)\neq0$, $\gamma_1(\iota)\neq0$, $\zeta_1(\iota)\neq0$, and $\mu_2(\varsigma)\neq0$, $\gamma_2(\varsigma)\neq0$, $\zeta_2(\varsigma)\neq0$, but $\mu(\iota)=0,\gamma(\iota)=0,\zeta(\iota)=0$, and $\mu(\varsigma)=0,\gamma(\varsigma)=0,\zeta(\varsigma)=0$. Thus, $\iota\notin P$ and $\varsigma\notin P$. Suppose that $\varrho=\iota\varepsilon\varsigma$, with $\varepsilon\in\mathbb{R}$, now, $\mu(\varrho)=0,\gamma(\varrho)=0,\zeta(\varrho)=0$, thus,

$$(\mu_1\bullet\mu_2)(\varrho)=0,$$
$$(\gamma_1\bullet\gamma_2)(\varrho)=0,$$
$$(\zeta_1\bullet\zeta_2)(\varrho)=0,$$

but

$$\begin{aligned}(\mu_1\bullet\mu_2)(\varrho)&=\sup_{\varrho=lk}(\min(\mu_1(l),\mu_2(k)))\\&\geqslant\min(\mu_1(\iota),\mu_2(\varepsilon\varsigma))\\&\geqslant\min(\mu_1(\iota),\mu_2(\varsigma)),\\(\gamma_1\bullet\gamma_2)(\varrho)&=\inf_{\varrho=lk}(\max(\gamma_1(l),\gamma_2(k)))\\&\leqslant\max(\gamma_1(\iota),\gamma_2(\varepsilon\varsigma))\\&\leqslant\max(\gamma_1(\iota),\gamma_2(\varsigma)),\\(\zeta_1\bullet\zeta_2)(\varrho)&=\inf_{\varrho=lk}(\max(\zeta_1(l),\zeta_2(k)))\\&\leqslant\max(\zeta_1(\iota),\zeta_2(\varepsilon\varsigma))\\&\leqslant\max(\zeta_1(\iota),\zeta_2(\varsigma)),\end{aligned}$$

which gives contradictions, this means either $\mathcal{P}_1\subseteq_1\mathcal{P}$ or $\mathcal{P}_2\subseteq_1\mathcal{P}$. □

Example 3.6. Consider Z_6 as a classical ring, since $P=\{0,3\}$ prime ideal, the neutrosophic ideal $\mathcal{P}=\{<0,0.8,0.2.0.1>,<3,0.3,0.5,0.4>\}$ is also a neutrosophic prime ideal.

Theorem 3.7. *Let $P \subseteq \mathbb{R}$ be an ideal where $\mathcal{P}$ is a neutrosophic prime ideal over P, then P is a prime ideal.*

Proof. Suppose that $\mathcal{P}$ is a neutrosophic prime ideal over $P \neq \mathbb{R}$ and $P_1 \subseteq \mathbb{R}$, $P_2 \subseteq \mathbb{R}$ are two ideals with $P_1P_2 \subseteq \mathbb{R}$. Then, we can define two neutrosophic ideals $\mathcal{P}_1$ and $\mathcal{P}_2$ over P_1 and P_2, respectively. Let $(\mu_1 \bullet \mu_2)(\iota) \neq 0$, $(\gamma_1 \bullet \gamma_2)(\iota) \neq 0$, and $(\zeta_1 \bullet \zeta_2)(\iota) \neq 0$. This leads to

$$(\mu_1 \bullet \mu_2)(\iota) = \sup_{\iota=lk}(\min(\mu_1(l), \mu_2(k))) \neq 0,$$
$$(\gamma_1 \bullet \gamma_2)(\iota) = \inf_{\iota=lk}(\max(\gamma_1(l), \gamma_2(k))) \neq 0,$$
$$(\zeta_1 \bullet \zeta_2)(\iota) = \inf_{\iota=lk}(\max(\zeta_1(l), \zeta_2(k))) \neq 0.$$

So, $\mu_1(l), \gamma_1(l), \zeta_1(l) \neq 0$, and $\mu_2(k), \gamma_2(k), \zeta_2(k) \neq 0$, then $\mu_1(l), \gamma_1(l), \zeta_1(l) = 1$, and $\mu_2(k), \gamma_2(k), \zeta_2(k) = 1$, this we drive that $l \in P_1$ and $k \in P_2$ and $\iota \in P_1P_2 \subseteq P$, hence $\mu(\iota), \gamma(\iota), \zeta(\iota) = 1$. Therefore, $\mathcal{P}$ is a neutrosophic prime ideal, and either $\mathcal{P}_1 \subseteq \mathcal{P}$ or $\mathcal{P}_2 \subseteq \mathcal{P}$. This also leads to either $P_1 \subseteq P$ or $P_2 \subseteq P$. Again, presume $\iota \in \mathbb{R}$, and when $(\mu_1 \bullet \mu_2)(\iota) = 0$, $(\gamma_1 \bullet \gamma_2)(\iota) = 0$ and $(\zeta_1 \bullet \zeta_2)(\iota) = 0$ then we obtain

$$(\mu_1 \bullet \mu_2)(\iota) \leqslant \mu(\iota),$$
$$(\gamma_1 \bullet \gamma_2)(\iota) \geqslant \gamma(\iota),$$
$$(\zeta_1 \bullet \zeta_2)(\iota) \geqslant \zeta(\iota).$$

Hence, $\mathcal{P}_1 \bullet \mathcal{P}_2 \subseteq \mathcal{P}$. Since $\mathcal{P}$ is a neutrosophic ideal, either $\mathcal{P}_1 \subseteq \mathcal{P}$ or $\mathcal{P}_2 \subseteq \mathcal{P}$. This leads to either $P_1 \subseteq P$ or $P_2 \subseteq P$. □

Theorem 3.8. *Let $\mathcal{P} = \{< \ell, \mu(\ell), \gamma(\ell), \zeta(\ell) >: \ell \in \mathbb{R}\}$ be a neutrosophic ideal over $\mathbb{R}$, and define $P_0 = \{\varsigma : \varsigma \in \mathbb{R}, \mu(\varsigma) = \mu(0), \gamma(\varsigma) = \gamma(1), \zeta(\varsigma) = \zeta(1)\}$. Then, $P_0 \subseteq \mathbb{R}$ is a prime ideal.*

Proof. First, we show that P_0 is an ideal. Suppose that $\ell, \iota \in P_0$, then we explain $\ell - \iota \in P_0$ as follows:

$$\mu(\ell - \iota) \geqslant \min(\mu(\ell), \mu(\iota)) = \mu(0),$$
$$\gamma(\ell - \iota) \leqslant \max(\gamma(\ell), \gamma(\iota)) = \gamma(1),$$
$$\zeta(\ell - \iota) \leqslant \max(\zeta(\ell), \zeta(\iota)) = \zeta(1),$$

conversely,

$$\begin{aligned}
\mu(0) &= \mu(0.(\ell - \iota)) \\
&\geqslant \min(\mu(0), \mu(\ell - \iota)) \\
&= \mu(\ell - \iota), \\
\gamma(1) &= \gamma((\ell - \iota)(\ell - \iota)^{-1}) \\
&\leqslant \max(\gamma(\ell - \iota), \gamma(\ell - \iota)^{-1}) \\
&= \gamma(\ell - \iota), \\
\zeta(1) &= \zeta((\ell - \iota)(\ell - \iota)^{-1}) \\
&\leqslant \max(\zeta(\ell - \iota), \zeta(\ell - \iota)^{-1}) \\
&= \zeta(\ell - \iota).
\end{aligned}$$

Now, we show $\varsigma\ell, \ell\varsigma \in P_0$ for all $\varsigma \in \mathcal{R}$ and $\ell \in P_0$ as follows:

$$\mu(\varsigma\ell) \geqslant \mu(\ell) = \mu(0),$$
$$\gamma(\varsigma\ell) \leqslant \gamma(\ell) = \gamma(1),$$
$$\zeta(\varsigma\ell) \leqslant \zeta(\ell) = \zeta(1),$$

similarly, $\ell\varsigma \in P_0$. Assume that I_1, I_2 are two ideals of $\mathbb{R}$ with $I_1 I_2 \subseteq P_0$. Define the neutrosophic ideals $\mathcal{I}_1 = \{<\iota, \mu(0)\mu_1(\iota), \gamma(1)\gamma_1(\iota), , \zeta(1)\zeta_1(\iota)>: \iota \in I_1\}$, and $\mathcal{I}_2 = \{<\iota, \mu(0)\mu_2(\iota), \gamma(1)\gamma_2(\iota), , \zeta(1)\zeta_2(\iota)>: \iota \in I_2\}$. Next, we show that $\mathcal{I}_1 \bullet \mathcal{I}_2(\varsigma) \subseteq \mathcal{P}(\varsigma)$ $\forall \varsigma \in \mathcal{R}$.

$$(\mu(0)\mu_1 \bullet \mu(0)\mu_2)(\varsigma) = \sup_{\varsigma=uv}(\min(\mu(0)\mu_1(u), \mu(0)\mu_2(v))),$$
$$(\gamma(1)\gamma_1 \bullet \gamma(1)\gamma_2)(\varsigma) = \inf_{\varsigma=uv}(\max(\gamma(1)\gamma_1(u), \gamma(1)\gamma_2(v))),$$
$$(\zeta(1)\zeta_1 \bullet \zeta(1)\zeta_2)(\varsigma) = \inf_{\varsigma=uv}(\max(\zeta(1)\zeta_1(u), \zeta(1)\zeta_2(v))),$$

then, we obtain $\mu(0)\mu_1(u) = \mu(0)\mu_2(v) = \mu(0)$, $\gamma(1)\gamma_1(u) = \gamma(1)\gamma_2(v) = \gamma(1)$, and $\zeta(1)\zeta_1(u) = \zeta(1)\zeta_2(v) = \zeta(1)$, also this implies $\mu_1(u) = 1, \gamma_1(u) = \zeta_1(u) = 0$, and $\mu_2(v) = 1, \gamma_2(v) = \zeta_2(v) = 0$. Hence, $u \in I_1$ and $v \in I_2$, so that $\varsigma \in I_1 I_2 \subseteq P_0$, so $\mu(\varsigma) = \mu(0), \gamma(\varsigma) = \gamma(1), \zeta(\varsigma) = \zeta(1)$, and $\mathcal{I}_1 \bullet \mathcal{I}_2(\varsigma) \subseteq \mathcal{P}(\varsigma)$. Again, since $\mathcal{P}$ is a neutrosophic ideal and $\mathcal{I}_1, \mathcal{I}_2$ are two neutrosophic ideals, then either $\mathcal{I}_1 \subseteq \mathcal{P}$ or $\mathcal{I}_2 \subseteq \mathcal{P}$. Presume $\mathcal{I}_1 \subseteq \mathcal{P}$, then,

$$\mu(0)\mu_1(\iota) \leqslant \mu(\iota), \gamma(1)\gamma_1(\iota) \geqslant \gamma(\iota), \text{ and } \zeta(1)\zeta_1(\iota) \geqslant \zeta(\iota).$$

Next, we explain that $I_1 \subseteq P_0$. by contradiction. Assume that $I_1 \nsubseteq P_0$ then we obtain $u \in I_1$ and $u \notin P_0$, which means one of the following:

$$\mu(u) \neq \mu(0), \gamma(u) \neq \gamma(1), \text{ and } \zeta(u) \neq \zeta(1).$$

Now,

$$\mu(0.u) \geqslant \mu(u), \gamma(u.u^{-1}) \leqslant \gamma(u), \text{ and } \zeta(u.u^{-1}) \leqslant \zeta(u).$$

So, $\mu(u) < \mu(0)$, $\gamma(u) > \gamma(1)$, and $\zeta(u) > \zeta(1)$, then $\mu(0)\mu_1(u) = \mu(0) > \mu(u)$, $\gamma(1)\gamma_1(u) < \gamma(u)$, and $\zeta(1)\zeta_1 < \zeta(u)$ this gives a contradiction; therefore, $I_1 \subseteq P_0$. The same way, $I_2 \subseteq P_0$. Hence, the theorem claimed. □

3.3. Neutrosophic prim ideal over $\mathbb{Z}$

In this part, our aim is to define the neutrosophic prime ideals over $\mathbb{Z}$. Suppose $\mathcal{P}$ represents a neutrosophic ideal over $\mathbb{Z}$. As per Theorem 3.8, it can be inferred that P_0 is indeed an ideal in $\mathbb{Z}$. Consequently, there exists a positive integer n with P_0 equals $n\mathbb{Z}$.

Theorem 3.9. *Let $\mathcal{P} = \{<\ell, \mu(\ell), \gamma(\ell), \zeta(\ell)>: \ell \in \mathbb{Z}\}$ be a neutrosophic ideal over $\mathbb{Z}$, and define $P_0 = \{\varsigma : \varsigma \in \mathbb{Z}, \mu(\varsigma) = \mu(0), \gamma(\varsigma) = \gamma(1), \zeta(\varsigma) = \zeta(1)\} = n\mathbb{Z} \neq (0)$, then for ϱ a positive divisor of n, $\mathcal{P}$ can accept most values of ϱ.*

Proof. Suppose that $\kappa \in \mathbb{Z}$ and $\ltimes = \kappa\varrho + no$ for some $\varrho, o \in \mathbb{Z}$. We obtain

$$\begin{aligned}\mu(\ltimes) = \mu(\kappa\varrho - (-no)) &\geqslant \min(\mu(\kappa\varrho), \mu(-no))\\ &\geqslant \min(\mu(\kappa), \mu(n)),\\ \gamma(\ltimes) = \gamma(\kappa\varrho - (-no)) &\leqslant \max(\gamma(\kappa\varrho), \gamma(-no))\\ &\leqslant \max(\gamma(\kappa), \gamma(n)),\\ \zeta(\ltimes) = \zeta(\kappa\varrho - (-no)) &\leqslant \max(\zeta(\kappa\varrho), \zeta(-no))\\ &\leqslant \max(\zeta(\kappa), \zeta(n)).\end{aligned}$$

Since $n \in P_0 = n\mathbb{Z}$, then we obtain:

$$\mu(n) = \mu(0) = \mu(0\ltimes) \geqslant \mu(\ltimes),$$

$$\gamma(n) = \mu(1) = \mu(\ltimes\ltimes^{-1}) \leqslant \gamma(\ltimes),$$

$$\zeta(n) = \zeta(1) = \zeta(\ltimes\ltimes^{-1}) \leqslant \zeta(\ltimes).$$

Thus, $\mu(\ltimes) \geqslant \mu(\kappa)$, $\gamma(\ltimes) \leqslant \gamma(\kappa)$, and $\zeta(\ltimes) \leqslant \zeta(\kappa)$. Also, $\ltimes | \kappa$ this leads to $\kappa = l\ltimes$, $l \in \mathbb{Z}$, now,

$$\begin{aligned}\mu(\kappa) &= \mu(l\ltimes) \geqslant \mu(\ltimes),\\ \gamma(\kappa) &= \gamma(l\ltimes) \leqslant \gamma(\ltimes),\\ \zeta(\kappa) &= \zeta(l\ltimes) \leqslant \zeta(\ltimes).\end{aligned}$$

Therefore, $\mu(\kappa) = \mu(\ltimes)$, $\gamma(\kappa) = \gamma(\ltimes)$ and $\zeta(\kappa) = \zeta(\ltimes)$. So, for every integer κ there is a positive factor $\ltimes$ of n where $\mu(\kappa) = \mu(\ltimes)$, $\gamma(\kappa) = \gamma(\ltimes)$, and $\zeta(\kappa) = \zeta(\ltimes)$. □

Theorem 3.10. *Let $\mathcal{P} = \{< \ell, \mu(\ell), \gamma(\ell), \zeta(\ell) >: \ell \in \mathbb{Z}\}$ be a neutrosophic ideal over $\mathbb{Z}$ and define $P_0 = \{\varsigma : \varsigma \in \mathbb{Z}, \mu(\varsigma) = \mu(0), \gamma(\varsigma) = \gamma(1), \zeta(\varsigma) = \zeta(1)\} \neq (0)$. Then, $\mathcal{P}$ has two distinct values.*

Proof. Assume $\mathcal{P}$ is a neutrosophic ideal over $\mathbb{Z}$, with $P_0 = p\mathbb{Z} \neq (0)$. By Theorem 3.8, we obtain P_0 is a prime ideal; this leads to p being a prime integer. Now, p has exacted two distinct positive divisors. Again, by Theorem 3.9, we find that $\mathcal{P}$ has at most two distinct values. So, $\mathcal{P}$ has two distinct values. □

Theorem 3.11. *Let $\mathcal{P} = \{< \ell, \mu(\ell), \gamma(\ell), \zeta(\ell) >: \ell \in \mathbb{Z}\}$ be a neutrosophic subset over $\mathbb{Z}$ such that $\mu(n) = \gamma(n) = \zeta(n) = \ell_1$ when $p|n$ and $\mu(n) = \gamma(n) = \zeta(n) = \ell_2$ when $p \nmid n$, where p is a prime integral and $\ell_2 < \ell_1$. Then, $\mathcal{P}$ is a neutrosophic prime ideal on $\mathbb{Z}$ with $P_0 = \{\varsigma : \varsigma \in \mathbb{Z}, \mu(\varsigma) = \mu(0), \gamma(\varsigma) = \gamma(1), \zeta(\varsigma) = \zeta(1)\} \neq (0)$.*

Proof. Assume $\mathcal{P}$ is a neutrosophic subset over $\mathbb{Z}$ with the above conditions. Suppose that $x, y \in \mathbb{Z}$, then we have:
Case 1: When $x - y$ is a multiple of p, then

$$\begin{aligned}\ell_1 &= \mu(x - y) \geqslant \min(\mu(x), \mu(y)),\\ \ell_1 &= \gamma(x - y) \leqslant \max(\gamma(x), \gamma(y)),\\ \ell_1 &= \zeta(x - y) \leqslant \max(\zeta(x), \zeta(y)),\end{aligned}$$

for $\ell_2 < \ell_1$.
Case 2: When $x-y$ is not a multiple of p, then either $p \nmid x$ or $p \nmid y$, this drives $\mu(x) = \gamma(x) = \zeta(x) = \ell_2$ or $\mu(y) = \gamma(y) = \zeta(y) = \ell_2$. So,

$$\mu(x-y) \geqslant \min(\mu(x), \mu(y)),$$
$$\gamma(x-y) \leqslant \max(\gamma(x), \gamma(y)),$$
$$\zeta(x-y) \leqslant \max(\zeta(x), \zeta(y)).$$

Case 3: When $p|xy$, then we obtain:

$$\mu(xy) \geqslant \mu(y), \quad \gamma(xy) \leqslant \gamma(y), \quad \zeta(xy) \leqslant \zeta(y).$$

Case 4: When $p \nmid xy$, then neither $p|x$ nor $p|y$. So,

$$\mu(xy) \geqslant \mu(y), \quad \gamma(xy) \leqslant \gamma(y), \quad \zeta(xy) \leqslant \zeta(y).$$

Therefore, $\mathcal{P}$ is a neutrosophic ideal with $P_0 = p\mathbb{Z} \neq (0)$. The proof is complete in Theorem 3.5. □

Some results of the classical theories of ideals and prime ideals are not fully applicable to the theories of neutrosophic ideals and neutrosophic prime ideals. The main difference is that neutrosophic ideals integrate the independent aspects of truth, indeterminacy, and falsity, introducing a level of complexity that is missing in the binary structure of classical ideals. The following remark explains one of the differences.

Remark 3.12. In the context of classical prime ideals, the prime avoidance lemma [15] asserts that in a commutative ring, if an ideal is contained in a finite union of prime ideals, then it must be contained in one of them. However, this result does not hold in neutrosophic settings. The following example illustrates this discrepancy.

Example 3.13. Define three neutrosophic prime ideals over Z_6 as follows:
$\mathcal{S}_i = \{< u, \mu_i(u), \gamma_i(u), \zeta_i(u) >: u \in Z_6\}$, where $i = 1, 2, 3$,

$$\mu_1(u) = \begin{cases} 0.8 & \text{if } u = \{0,2,4\} \\ 0.3 & \text{otherwise} \end{cases}, \quad \mu_2(u) = \begin{cases} 0.8 & \text{if } u = \{0,3\} \\ 0.3 & \text{otherwise} \end{cases}, \quad \mu_3(u) = \begin{cases} 0.8 & \text{if } u = \{0,5\} \\ 0.3 & \text{otherwise} \end{cases},$$

$$\gamma_1(u) = \begin{cases} 0.1 & \text{if } u = \{0,2,4\} \\ 0.4 & \text{otherwise} \end{cases}, \quad \gamma_2(u) = \begin{cases} 0.1 & \text{if } u = \{0,3\} \\ 0.4 & \text{otherwise} \end{cases}, \quad \gamma_3(u) = \begin{cases} 0.1 & \text{if } u = \{0,5\} \\ 0.4 & \text{otherwise} \end{cases},$$

$$\zeta_1(u) = \begin{cases} 0.2 & \text{if } u = \{0,2,4\} \\ 0.4 & \text{otherwise} \end{cases}, \quad \zeta_2(u) = \begin{cases} 0.2 & \text{if } u = \{0,3\} \\ 0.4 & \text{otherwise} \end{cases}, \quad \zeta_3(u) = \begin{cases} 0.2 & \text{if } u = \{0,5\} \\ 0.4 & \text{otherwise} \end{cases}.$$

Now, consider the neutrosophic ideal $\mathcal{S}_0 = \{< u, \mu_0(u), \gamma_0(u), \zeta_i(u) >: u \in Z_6\}$, defined as follows:

$$\mu_0(u) = \max(\mu_1, \mu_2, \mu_3), \quad \gamma_0(u) = \min(\gamma_1, \gamma_2, \gamma_3), \quad \zeta_0(u) = \min(\zeta_1, \zeta_2, \zeta_3).$$

It is obvious that μ_0 is not fully contained in μ_1, for $\mu_0(5) = 0.8$ and $\mu_1(5) = 0.3$. Again, μ_0 is not fully contained in μ_2 and μ_3, and so on in case $\gamma_0(\zeta_0)$ are not fully contained in $\gamma_1(\zeta_1)$, $\gamma_2(\zeta_2)$, and $\gamma_3(\zeta_3)$, respectively. Therefore, $\mathcal{S}_0$ is not contained in any single $\mathcal{S}_i$, $i = \{1, 2, 3\}$.

Remark 3.14. Finally, we compare our results to classical, fuzzy, and intuitive fuzzy set theories, emphasizing the distinctive contributions of neutrosophic sets.

First: Neutrosophic sets provide a more comprehensive representation because they account for truth, indeterminacy, and falsity independently. This allows for a more nuanced and flexible modeling of uncertainties compared to classical and intuitionistic fuzzy sets.

Second: The ability to handle a wider range of uncertainties makes neutrosophic ideals particularly useful in complex real-world scenarios where the data is not always precise or complete. This includes applications in areas such as artificial intelligence, decision-making, and information systems.

Third: By extending classical algebraic structures to the neutrosophic context, we open up new avenues for theoretical research. Neutrosophic ideals and prime ideals offer new perspectives and solutions to long-standing problems in algebra under conditions of uncertainty.

4. Conclusions

The study of neutrosophic ideals and prime ideals represents a significant advance in the field of algebra and provides powerful tools for modeling and analyzing uncertain environments. This work introduces and thoroughly explores the concept of neutrosophic ideals within the framework of ring theory. We have successfully extended the classical, fuzzy, and intuitionistic fuzzy ideal concepts to neutrosophic sets, thus accounting for a wider range of uncertainties through the independent treatment of truth, indeterminacy, and falsity. We also investigated the correlation between the regularity property of a ring and neutrosophic ideals and showed how the regularity condition interacts with the neutrosophic components. This investigation provided new insights into how traditional algebraic properties can be reinterpreted and extended in a neutrosophic context. In addition, we provided a comprehensive characterization of all neutrosophic prime ideals in the ring of integers $\mathbb{Z}$. This concrete enumeration illustrates how neutrosophic prime ideals can be identified and analyzed within a familiar algebraic structure.

The ability to manage degrees of truth, indeterminacy, and falsity independently not only increases the flexibility and applicability of algebraic structures, such as the study of neutrosophic prime ideals in Noetherian rings and Artinian rings, but also paves the way for future research and practical applications in various fields such as cryptography, coding theory, and computational algebra, which could provide valuable innovations and insights.

Author contributions

Ali Yahya Hummdi: Writing–review & editing, visualisation, funding; Amr Elrawy: Conceptualization, formal analysis, investigation, methodology, visualisation, writing–original draft acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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