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Research article

Dynamics of a damped quintic wave equation with time-dependent coefficients

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Abstract: We present a comprehensive investigation of the long-term dynamics generated by a semilinear wave equation with time-dependent coefficients and quintic nonlinearity on a bounded domain subject to Dirichlet boundary conditions. By employing rescaling techniques for time and utilizing the Strichartz estimates applicable to bounded domains, we initially study the global well-posedness of the Shatah–Struwe (S–S) solutions. Subsequently, we establish the existence of a uniform weak global attractor consisting of points on complete bounded trajectories through an approach based on evolutionary systems. Finally, we prove that this uniformly weak attractor is indeed strong by means of a backward asymptotic a priori estimate and the so-called energy method. Moreover, the smoothness of the obtained attractor is also shown with the help of a decomposition technique.

Keywords: wave equation; time-dependent coefficients; quintic nonlinearity; attractor; evolutionary systems
Mathematics Subject Classification: 37L30, 35L70, 47H20

1. Introduction

In this paper, we are concerned with the following semilinear damped wave model

$$\begin{cases} \partial_t(\alpha(t)\partial_t u) + \beta(t)Au + \gamma(t)\partial_t u + \beta(t)g(u) = \beta(t)f(x), \\ u|_{\partial\Omega} = 0, \\ u(x,\tau) = u_{0\tau}, \ \partial_t u(x,\tau) = u_{1\tau}. \end{cases} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary, $A = -\Delta$, $g(u)$ is a given source term, and the coefficients $\alpha(t), \beta(t)$, and $\gamma(t)$ all depend on time. *From now on, we assume the external force* $f \in L^2(\Omega)$.

The semilinear wave equations with time-dependent coefficients have been investigated quite extensively by several authors in recent years, with particular regard to its long-term behavior. For example, wave equations with time-dependent speed of propagation were investigated by Conti et al. [12, 13], and very recently this model was generalized to the hyperbolic equations with time-dependent memory kernel in [15]; Uesaka et al. [19] made some significant progress in the oscillation property of semilinear wave equations with time-dependent coefficients, and subsequently the dynamics of this equation were studied by Aragão et al. in [1], including the continuity of pullback attractors.

It is worth observing that the nonlinear term g satisfies $g(u) \sim u|u|^{q-1}$ with $1 \leq q \leq 3$ in the aforementioned papers. Thus, it seemed natural to extend these results to the sup-cubic case. The case of sup-cubic growth rate is a bit more complicated since the uniqueness of energy-weak solutions is unknown as $q > 3$, see, e.g., [3, 20]. In order to overcome these difficulties, the authors in [4] studied the semilinear oscillon equation with the growth index $3 < q < 5$ by using parabolic approximations governed by the fractional powers of the wave operator. Another effective way to deal with sup-cubic nonlinearity is using S–S solutions, which have more delicate space-time integrability, such as $u \in L^4_{loc}(\mathbb{R}, L^{12}(\Omega))$. Very recently, based on the recent extension of Strichartz estimates for the bounded domains, the Eq (1.1) in the case $\alpha = \beta = 1$ with a sign changing damping and sub-quintic nonlinearity $(0 < q < 5)$ was discussed in detail in [5].

In this paper, motivated by the studies in [1, 5, 14], we consider the problem of the existence of a uniform global attractor for Eq (1.1) with quintic nonlinearity in the natural energy phase space $H^1_0(\Omega) \times L^2(\Omega)$. The difficulties with this problem mainly stem from the following aspects:

- How do you generalize the usual Strichartz estimates to wave equations with time-dependent coefficients? In the case of the whole space $\Omega = \mathbb{R}^3$, the Strichartz type estimates for variable coefficient wave equations have been studied by many authors, one can refer to [16] and the references cited therein. However, when $\Omega \subset \mathbb{R}^3$ is a general bounded domain, as far as we known, the corresponding results are still lacking.
- How to establish the asymptotic compactness of the system generated by S–S solutions of Eq (1.1) with quintic nonlinearity? In the sub-quintic case, one can establish the so-called energy-to-Strichartz (ETS) estimate (2.2), and based on the ETS estimate, one can obtain the well-posedness, dissipativity, asymptotic compactness, and existence of attractors in way that is similar to the classical cubic case. In contrast to this, in the quintic case, the ETS estimate is only proved in the case when $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{T}^3$ with periodic boundary conditions. Since the ETS estimate for a general domain is still an open problem, it is impossible to deduce the asymptotic compactness by giving any control of the Strichartz norm in terms of the initial data, and the control of this norm may be a priori lost when passing to the limit $t \to \infty$, and the attractor may contain solutions that are less regular than the S–S ones, for which we may not have the energy equality.

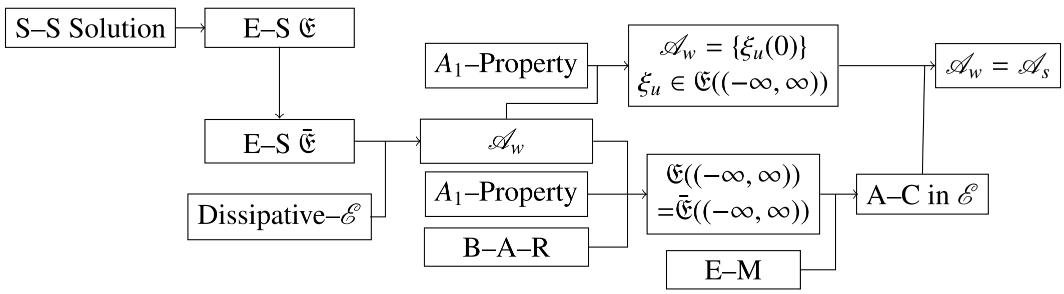
In this paper, in order to circumvent the difficulties mentioned above, we present a new scheme to study the dynamics of the wave equations (1.1), and summarize the main method in Figure 1 for clarity.

(1) By rescaling techniques for time, we can reduce the Eq (1.1) to an equation of simple form (2.3),

and then the well-posedness as well as the energy dissipativity of S–S solutions can be proved by the usual method.

(2) We apply a newly developed framework named evolutionary systems (see [7]) for studying the asymptotic dynamics of S–S solutions, and obtain the existence and structure of the uniform weak global attractor $\mathscr{A}_w$. Since the evolutionary systems $\mathfrak{E}$ (3.3) generated by S–S solutions may not be closed with respect to weak topology on the phase space, we follow an interesting technique initiated by Cheskidov and Lu in [10], which is based on taking a closure of the evolution systems $\bar{\mathfrak{E}}$ (3.4). In what follows, our main task is to show that $\mathfrak{E}((-\infty,\infty)) = \bar{\mathfrak{E}}((-\infty,\infty))$ via a newly developed approach presented in [20].

(3) Using the energy method developed in [3] and remembering the backward regularity of complete trajectories contained in $\mathfrak{E}((-\infty,\infty))$, we can prove that the uniform weak global attractor $\mathscr{A}_w$ constructed in Step 2 is in fact a strongly compact strong global attractor $\mathscr{A}_s$. Moreover, we obtain a bounded uniform attractor $\mathscr{A}_s^{\Sigma} \subset \mathscr{E}^1$ by a decomposition technique.



1. S–S: Shatah–Struwe. 2. E–S: evolutionary system. 3. $\mathscr{A}_w$: weak attractor. 4. $\mathscr{A}_s$: strong attractor. 5. B–A–R: backward asymptotic regularity. 6. A–C: asymptotic compact. 7. E–M: energy method. 8. A_1–Property: $\mathfrak{E}([0,\infty))$ is pre-compact in $C([0,\infty);\mathscr{E}_w)$.

Figure 1. Overview of the technique.

We impose the following standing assumptions on the nonlinear damping and coefficients:

Assumption 1.1. *(**G**) $g \in C^2(\mathbb{R})$ with $g(0) = 0$ and satisfies*

$$|g''(s)| \le C_g(1 + |s|^{q-2}),\ g'(s) \ge -\kappa_1 + \delta|s|^{q-1}, \tag{1.2}$$

$$g(s)s - 4G(s) \ge -\kappa_2,\ G(s) \ge \kappa_3|s|^{q+1} - \kappa_4,\ \forall s \in \mathbb{R}. \tag{1.3}$$

Here $3 \le q \le 5$, $G(s) = \int_0^s g(\tau)d\tau$, κ_i ($i = 1,2,3,4$), δ, and C_g are given positive constants.
*(**COEF**) $\alpha,\beta,\gamma \in C_b^2(\mathbb{R})$ satisfying*

$$\alpha_0 \le \alpha(t) \le \alpha_1, \beta_0 \le \beta(t) \le \beta_1, \gamma_0 \le \gamma(t) \le \gamma_1, \tag{1.4}$$

$$|\alpha'(t)| \le \alpha_2, |\beta'(t)| \le \beta_2, \gamma'(t) \le \gamma_2, \quad \forall t \in \mathbb{R}, \tag{1.5}$$

where α_i, β_i and γ_i ($i = 0,1,2$) are all positive constants, and $2\gamma_0\beta_0 > \alpha_2\beta_1 + \alpha_1\beta_2$.

The outline of our paper is given below. In Section 2, the property of the S–S solutions of Eq (1.1) is discussed in Theorem 2.2. In Section 3, the existence and structure of the uniform weak global attractor are studied in Theorem 3.7, and the backward asymptotic regularity of complete trajectories contained in $\bar{\mathfrak{E}}((-\infty,\infty))$ are proved in Theorem 3.11. Finally, the existence and regularity of the uniformly strong global attractor is established in Theorems 4.3 and 4.4.

2. Well-posedness of S–S solutions

Let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ be the usual norm and inner product in $L^2(\Omega)$. For convenience, we denote $\mathcal{H}^s = D(A^{\frac{s}{2}})$, $\mathscr{E}^s = \mathcal{H}^{s+1}\times\mathcal{H}^s$, $s\in\mathbb{R}$. Then, $\mathcal{H}^0 = L^2(\Omega)$, $\mathcal{H}^1 = H_0^1(\Omega)$, $\mathcal{H}^2 = H^2(\Omega)\cap H_0^1(\Omega)$, and $\mathcal{H}^{-1}$ is the dual space to $H_0^1(\Omega)$. In particular, we denote $\mathscr{E} := \mathscr{E}^0 = H_0^1(\Omega)\times L^2(\Omega)$.

Definition 2.1. *A function $u(t)$ is a weak solution of Eq (1.1) iff $\xi_u := (u,\partial_t u)\in L^\infty(\tau,T;\mathscr{E})$ and Eq (1.1) is satisfied in the sense of distribution, i.e.,*

$$
\begin{aligned}
-\int_\tau^T \langle\alpha(t)\partial_t u,\partial_t\phi\rangle dt + \int_\tau^T \beta(t)\langle\nabla u\cdot\nabla\phi,1\rangle dt \\
+\int_\tau^T \gamma(t)\langle\partial_t u,\phi\rangle dt + \int_\tau^T \beta(t)\langle g(u),\phi\rangle dt = \int_\tau^T \beta(t)\langle f,\phi\rangle dt
\end{aligned}
$$

for any $\phi\in C_0^\infty((\tau,T)\times\Omega)$. A weak solution is a Shatah–Struwe (S–S) solution of Eq (1.1) on the interval $[\tau,T]$ iff $u\in L^4(\tau,T;L^{12}(\Omega))$.

Theorem 2.2. *Under Assumption 1.1, then for every $\xi_u(\tau) = (u_{0\tau},u_{1\tau})\in\mathscr{E}$, the Eq (1.1) admits a unique global S–S solution $u(t)$ with the estimate*

$$
\|\xi_u(t)\|_{\mathscr{E}} \le e^{-\varpi(t-\tau)}Q(\|\xi_u(\tau)\|_{\mathscr{E}}) + Q(\|f\|^2), \quad \forall t\ge\tau, \tag{2.1}
$$

where the positive constant ϖ and the monotone increasing function Q are independent of u, t and τ. In the sub-quintic case, we have, in addition, the estimate

$$
\|\xi_u(t)\|_{\mathscr{E}} + \|u\|_{L^4(t,t+1;L^{12}(\Omega))} \le e^{-\varpi(t-\tau)}Q(\|\xi_u(\tau)\|_{\mathscr{E}}) + Q(\|f\|^2), \quad \forall t\ge\tau. \tag{2.2}
$$

Moreover, if $\xi_u(\tau)\in\mathscr{E}^1$, then Eq. (1.1) admits a unique global strong solution and estimate (2.1) also holds.

Proof. (***Sketch***) Using the change of variable $s=\phi(t)=\int_0^t\sqrt{\frac{\beta(\omega)}{\alpha(\omega)}}d\omega$ and chain rule (see [1, Section 3] for more details), we can rewrite the Eq (1.1) as follows:

$$
\begin{cases}
\partial_s^2 v - \Delta v + \eta(s)\partial_s v + g(v) = f(x),\\
v|_{\partial\Omega} = 0,\\
v(x,\mu) = u_{0\tau},\ \partial_t v(x,\mu) = \sqrt{\frac{\alpha(\tau)}{\beta(\tau)}}u_{1\tau},
\end{cases} \tag{2.3}
$$

where $u(x,t) = u(x,\phi^{-1}(s)) = v(x,s)$, $\eta(s) = \frac{(\sqrt{\alpha\beta})_s+\gamma}{\sqrt{\alpha\beta}}$ and $\mu=\phi(\tau)$. The local well-posedness of Eq (2.3) can be verified by using the Galerkin method and Strichartz estimate, and the global existence and regularity of the S–S solution can be proved by Morawetz–Pohozhaev identity and a prior estimate; see [17] for further details. Analyzing the term $\eta(s)$ and recalling the assumption (**COEF**), we have

$$
\eta_0 \le \eta(s) = \frac{(\sqrt{\alpha\beta})_s+\gamma}{\sqrt{\alpha\beta}} = \frac{\alpha_t\beta+\alpha\beta_t+2\beta\gamma}{2\beta\sqrt{\alpha\beta}} \le \eta_1, \quad \forall s\ge\mu. \tag{2.4}
$$

Here, we can choose $\eta_0 = \frac{2\beta_0\gamma_0-\alpha_2\beta_1-\alpha_1\beta_2}{2\beta_1\sqrt{\alpha_1\beta_1}}$ and $\eta_1 = \frac{2\beta_1\gamma_1+\alpha_2\beta_1+\alpha_1\beta_2}{2\beta_0\sqrt{\alpha_0\beta_0}}$. Taking the multiplier $\partial_s v + \varepsilon v$ ($\varepsilon > 0$ is small enough) in Eq (2.3), and applying dissipative assumptions (1.3) and (1.4) and Gronwall's inequality, we can obtain the estimate (2.1) for v, e.g., see [5]. In the sub-quintic case, we can also obtain an estimate (2.2) for v by using the standard bootstrapping method in [17]. Finally, in view of

$$\|\xi_u(t)\|_{\mathscr{E}} \cong \|\xi_v(\phi(t))\|_{\mathscr{E}} \quad \text{and} \quad \|u\|_{L^4(\tau,T;L^{12}(\Omega))} \cong \|v\|_{L^4(\phi(\tau),\phi(T);L^{12}(\Omega))}, \tag{2.5}$$

then the theorem is completed. □

3. Uniform weak global attractors

3.1. Evolutionary systems

Here we recall some basic ideas and results from the abstract theory of evolutionary systems; see [7–10] for details. Let $(X, d_s(\cdot,\cdot))$ be a metric space endowed with a metric d_s, which will be referred to as a strong metric. Let $d_w(\cdot,\cdot)$ be another metric on X satisfying the following conditions:

(1) X is d_w-compact.

(2) If $d_s(u_n, v_n) \to 0$ as $n \to \infty$ for some $u_n, v_n \in X$, then $d_w(u_n, v_n) \to 0$.

Due to the property 2, $d_w(\cdot,\cdot)$ and $d_s(\cdot,\cdot)$ will be referred to as weak metric, and strong metric respectively. Let $C([a,b]; X_\bullet)$, where $\bullet = s$ or w, be the space of $d_\bullet$-continuous X-valued functions on $[s,t]$ endowed with the metric

$$d_{C([a,b];X_\bullet)}(u,v) := \sup_{t\in[a,b]} d_\bullet(u(t), v(t)).$$

Let also $C([a,\infty); X_\bullet)$ be the space of $d_\bullet$-continuous X-valued functions on $[a,\infty)$ endowed with the metric

$$d_{C([a,\infty);X_\bullet)}(u,v) := \sum_{K\in\mathbb{N}} \frac{1}{2^K} \frac{d_{C([a,a+K];X_\bullet)}(u,v)}{1 + d_{C([a,a+K];X_\bullet)}(u,v)}. \tag{3.1}$$

To define an evolutionary system, first let

$$\mathcal{T} := \{I : I = [T,\infty) \subset \mathbb{R}, \text{ or } I = (-\infty,\infty)\},$$

and for each $I \in \mathcal{T}$, let $\mathfrak{F}(I)$ denote the set of all X-valued functions on I.

Definition 3.1. *A map $\mathfrak{E}$ that associates with each $I \in \mathcal{T}$ a subset $\mathfrak{E}(I) \subset \mathfrak{F}(I)$ will be called an evolutionary system if the following conditions are satisfied:*

(1) $\mathfrak{E}([0,\infty)) \neq \emptyset$.

(2) $\mathfrak{E}(I+s) = \{u(\cdot) : u(\cdot - s) \in \mathfrak{E}(I)\}$ *for all* $s \in \mathbb{R}$.

(3) $\{u(\cdot)|_{I_2} : u(\cdot) \in \mathfrak{E}(I_1)\} \subset \mathfrak{E}(I_2)$ *for all pairs* I_1, $I_2 \subset \mathcal{T}$, *such that* $I_2 \subset I_1$.

(4) $\mathfrak{E}((-\infty,\infty)) = \{u(\cdot) : u(\cdot)|_{[T,\infty)} \in \mathfrak{E}([T,\infty)), \forall T \in \mathbb{R}\}$.

We will refer to $\mathfrak{E}(I)$ as the set of all trajectories on the time interval I. Let $P(\mathcal{X})$ be the set of all subsets of $\mathcal{X}$. For every $t \geq 0$, define a map

$$R(t) : P(\mathcal{X}) \to P(\mathcal{X}),$$
$$R(t)A := \{u(t) : u(0) \in A, u \in \mathfrak{E}([0, \infty))\}, \quad A \subset \mathcal{X}.$$

Definition 3.2. *A set $\mathscr{A}_w \subset \mathcal{X}$ is a d_w-global attractor of $\mathfrak{E}$ if $\mathscr{A}_w$ is a minimal set that is*

(1) d_w-closed;

(2) d_w-attracting: for any $B \subset \mathcal{X}$ and $\epsilon > 0$, there exists t_0, such that

$$R(t)B \subset B_w(\mathscr{A}_w, \epsilon) := \left\{ u : \inf_{x \in \mathscr{A}_w} d_w(u, x) < \epsilon \right\}, \quad \forall t \geq t_0.$$

Definition 3.3. *The $\omega_\bullet$-limit set ($\bullet = s, w$) of a set $A \subset \mathcal{X}$ is*

$$\omega_\bullet(A) := \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} R(t)A}^{\bullet}.$$

In order to extend the notion of invariance from a semiflow to an evolutionary system, we will need the following mapping:

$$\widetilde{R}(t)A := \{u(t) : u(0) \in A, u \in \mathfrak{E}((-\infty, \infty))\}, \quad A \subset \mathcal{X}, \ t \in \mathbb{R}.$$

Definition 3.4. *A set $A \subset \mathcal{X}$ is positively invariant if*

$$\widetilde{R}(t)A \subset A, \quad \forall t \geq 0.$$

A is invariant if

$$\widetilde{R}(t)A = A, \quad \forall t \geq 0.$$

A is quasi-invariant if, for every $a \in A$, there exists a complete trajectory $u \in \mathfrak{E}((-\infty, \infty))$ with $u(0) = a$ and $u(t) \in A$ for all $t \in \mathbb{R}$.

As shown in [7, 10], a semiflow $\{S(t)\}$ or a family of a processes $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$, defines an evolutionary system. In order to investigate the existence and structure of $\mathscr{A}_w$, we use a new method initiated by Cheskidov and Lu in [10] by taking a closure of the evolutionary system $\mathfrak{E}$. Let

$$\bar{\mathfrak{E}}([\tau, \infty)) := \overline{\mathfrak{E}([\tau, \infty))}^{C([\tau,\infty);\mathcal{X}_w)}, \quad \forall \tau \in \mathbb{R}.$$

Obviously, $\bar{\mathfrak{E}}$ is also an evolutionary system. We call $\bar{\mathfrak{E}}$ the closure of the evolutionary system $\mathfrak{E}$, and add the top-script $^-$ to the corresponding notations. Below is an important assumption that we will impose on $\mathfrak{E}$ in some cases.

- ◇ **A1** $\mathfrak{E}([0, \infty))$ is pre-compact in $C([0, \infty); \mathcal{X}_w)$.

Theorem 3.5. *[10] Let $\mathfrak{E}$ be an evolutionary system. Then the weak global attractor $\mathscr{A}_w$ exists. Furthermore, assume that $\mathfrak{E}$ satisfies **A1**. Let $\bar{\mathfrak{E}}$ be the closure of $\mathfrak{E}$. Then*

(1) $\mathscr{A}_w = \omega_w(X) = \bar{\omega}_w(X) = \bar{\mathscr{A}_w} = \{u_0 \in X : u_0 = u(0)$ *for some* $u \in \bar{\mathfrak{E}}((-\infty,\infty))\}$.

(2) $\mathscr{A}_w$ *is the maximal invariant and maximal quasi-invariant set w.r.t.* $\bar{\mathfrak{E}}$.

(3) (Weak uniform tracking property) For any $\epsilon > 0$*, there exists* t_0*, such that for any* $t^* > t_0$*, every trajectory* $u \in \mathfrak{E}([0,\infty))$ *satisfies*

$$d_{C([t^*,\infty);X_w)}(u,v) \le \epsilon,$$

for some complete trajectory $v \in \bar{\mathfrak{E}}((-\infty,\infty))$.

3.2. The existence and properties of $\mathscr{A}_w$

Let $U_\sigma(t,\tau) : \mathscr{E} \to \mathscr{E}$, $t \ge \tau$ be the S–S solution operator of Eq (1.1), where $\sigma = (\alpha,\beta,\gamma) \in \Sigma := [T_h\sigma, h \in \mathbb{R}]_{C_b^2(\mathbb{R})}$, $(T_h\sigma)(\cdot) := \sigma(\cdot + h)$, then $T_h\Sigma = \Sigma$. We construct the skew product flow by

$$\mathbb{S}(t)(\xi,\sigma) := (U_\sigma(t,0)\xi, T_t\sigma),\ t \ge 0, \tag{3.2}$$

where $(\xi,\sigma) \in \mathbb{E} = \mathscr{E} \times \Sigma$ with norm $\|(\xi,\sigma)\|_{\mathbb{E}} = (\|\xi\|_{\mathscr{E}}^2 + \|\sigma\|_{C_b^2(\mathbb{R})}^2)^{\frac{1}{2}}$. Then $\{\mathbb{S}(t)\}_{t\ge0}$ forms a semigroup. Now define an evolutionary systems (ES) on $\mathbb{E}$ by

$$\mathfrak{E}([0,\infty)) := \{(\xi_u(\cdot),\sigma(\cdot)) : (\xi_u(t),\sigma(t)) = \mathbb{S}(t)(\xi,\sigma),\ \xi_u(t) \in X,\ \sigma \in \Sigma,\ \forall t \ge 0\}, \tag{3.3}$$

where $X := \{\xi_u \in \mathscr{E} : \|\xi_u\|_{\mathscr{E}}^2 \le 2Q(\|f\|^2)\}$. Let

$$\bar{\mathfrak{E}}([0,\infty)) := \overline{\mathfrak{E}([0,\infty))}^{C([0,\infty);\mathbb{X}_w)}, \tag{3.4}$$

where $\mathbb{X}_w = X_w \times C_b^2(\mathbb{R})$ and the metric on $C([0,\infty);\mathbb{X}_\bullet)$ defined in the same manner as (3.1).

Lemma 3.6. *Suppose* σ *is translation compact in* $C_b^2(\mathbb{R})$*, and let* $\xi_{u_n} = (u_n, \partial_t u_n)$ *be a sequence of S–S solutions of Eq* (1.1) *with symbols* σ_n *such that* $(\xi_{u_n}(t), \sigma_n(t)) \in \mathbb{X}$ *for all* $t \ge t_0$*. Then*

$$\xi_{u_n} \text{ is bounded in } L^\infty([t_0,T];\mathscr{E}), \quad \partial_t\xi_{u_n} \text{ is bounded in } L^\infty([t_0,T];\mathscr{E}^{-1}),\ \ \forall T > t_0. \tag{3.5}$$

Moreover, there exists a subsequence n_j *such that* σ_{n_j} *converges in* $C_b^2(\mathbb{R})$ *to some* $\sigma \in \Sigma$ *and* $\xi_{u_{n_j}}$ *converges to some* ξ_u *in* $C([t_0,T];\mathscr{E}_w)$*, i.e.,* $(\xi_{u_{n_j}},\phi) \to (\xi_u,\phi)$ *uniformly on* $[t_0,T]$ *as* $n_j \to \infty$ *for all* $\phi \in \mathscr{E}$.

Proof. Applying Theorem 2.2 and remembering that ξ_{u_n} are the S–S solutions of Eq (2.3), thus we obtain (3.5). Now applying the Alaoglu compactness theorem to extract a subsequence $\xi_{u_{n_j}}$ which w*-converges to some function $\xi_u \in L^\infty([t_0,T];\mathscr{E})$, i.e.,

$$\xi_{u_{n_j}} \rightharpoonup \xi_u \text{ weakly-}* \text{ in } L^\infty([t_0,T];\mathscr{E}). \tag{3.6}$$

Recalling the following compact embedding

$$\begin{aligned}&\{(u,\partial_t u) \in L^\infty([t_0,T];\mathscr{E})\} \cap \{\partial_t^2 u \in L^\infty([t_0,T];\mathcal{H}^{-1}(\Omega))\}\\ &\subset\subset \{(u,\partial_t u) \in C([t_0,T];\mathcal{H}^{1-\iota}(\Omega) \times \mathcal{H}^{-\iota}(\Omega))\}\end{aligned}$$

for some $0 < \iota \le 1$, we deduce that the weak-$*$ convergence (3.6) implies the strong convergence $\xi_{u_{n_j}} \to \xi_u$ in $C([t_0,T];\mathscr{E}_w)\}$. The proof is completed. □

Theorem 3.7. *Under Assumption 1.1 assume that σ is translation compact in $C_b^2(\mathbb{R})$. Then the uniform weak global attractor $\mathscr{A}_w$ for ES $\mathfrak{E}$ defined by* (3.3) *exists. In addition, $\mathfrak{E}$ satisfies **A1** and $\mathscr{A}_w := \{(\xi_{u_0}, \sigma_0) : (\xi_{u_0}, \sigma_0) = (\xi_u(0), \sigma(0))$ for some $\Xi = (\xi_u, \sigma) \in \bar{\mathfrak{E}}((-\infty, \infty))\}$. Moreover, $\forall \epsilon > 0$ there is some $t_0 := t_0(\epsilon)$ such that for any $t^* > t_0$ and every trajectory $\Xi \in \mathfrak{E}([0, +\infty))$ satisfies $d_{C([0,\infty):\mathbb{X}_w)}(\Xi, \Xi^*) < \epsilon$ for some complete trajectory $\Xi^* \in \bar{\mathfrak{E}}((-\infty, \infty))$.*

Proof. The existence of the attractor $\mathscr{A}_w$ can be established by using Theorem 3.5 directly. Let Ξ_n be a sequence in $\mathfrak{E}([0, \infty))$. Using Lemma 3.6, we extract a subsequence (still denoting by Ξ_n) that converges to some $\Xi^1 \in C([0, 1]; \mathbb{X}_w)$ as $n \to \infty$. Passing to a subsequence and still denote ξ_{u_n} once more, we obtain that $\Xi_n \to \Xi^2 \in C([0, 2]; \mathbb{X}_w)$ as $n \to \infty$ for some $\Xi^2 \in C([0, 2]; \mathbb{X}_w)$ with $\Xi^1 = \Xi^2$ on $[0, 1]$. Continuing this diagonalization process, we get a subsequence Ξ_{n_j} converges to $\Xi \in C([0, \infty); \mathbb{X}_w)$, and **A1** is proven. The other statement contained in the above theorem can be proved by applying Theorem 3.5 again. □

Theorem 3.8. *Under Assumption 1.1 assume that σ is translation compact in $C_b^2(\mathbb{R})$. Then the complete trajectory $\Xi = (\xi_u, \sigma) \in \bar{\mathfrak{E}}((-\infty, \infty))$ iff there exists a sequence of times $t_n \to -\infty$ and a sequence of S–S solutions $\xi_{u_n}(t)$ of Eq* (1.1)*:*

$$\begin{cases} \partial_t(\alpha_n(t)\partial_t u_n) - \beta_n(t)\Delta u_n + \gamma_n(t)\partial_t u_n + \beta_n(t)g(u_n) = \beta_n(t)f(x), \\ \xi_{u_n}(t_n) = \xi_n^0 \in \mathcal{X}, \ t \geq t_n, \end{cases} \tag{3.7}$$

such that $(\xi_{u_n}, \sigma_n) \rightharpoonup (\xi_u, \sigma)$ in $C([-T, \infty); \mathbb{X}_w)$ for any $T > 0$, where $\sigma_n = (\alpha_n, \beta_n, \gamma_n) \in \Sigma$.

Proof. Let $\Xi = (\xi_u, \sigma) \in \bar{\mathfrak{E}}((-\infty, \infty))$ and denote $\Xi_n = \Xi|_{[t_n,\infty)} \in \bar{\mathfrak{E}}([t_n, \infty))$, where $t_n \to -\infty$ as $n \to \infty$. Obviously $\Xi_n \rightharpoonup \Xi$ in $C([-T, \infty); \mathbb{X}_w)$, $\forall T > 0$. Since $\Xi_n \in \bar{\mathfrak{E}}([t_n, \infty))$, then there exists a sequence $\{\Xi_n^{(k)}\}_{k=1}^\infty \in \mathfrak{E}([t_n, \infty))$ such that $\Xi_n^{(k)} \rightharpoonup \Xi_n$ in $C([t_n, \infty); \mathbb{X}_w)$ as $k \to \infty$. By a standard diagonalization process, we obtain that there exists a sequence $\Xi_n^{(n)}$ (denoted by $\Xi_n = (\xi_{u_n}, \sigma_n)$) such that $\Xi_n \rightharpoonup \Xi$ in $C([-T, \infty); \mathbb{X}_w)$ for any $T > 0$. Recalling the definition of $\mathfrak{E}$ and Ξ, we know that ξ_{u_n} is the S–S solution of Eq (1.1).

Conversely, let $\Xi_n = (\xi_{u_n}, \sigma_n) \in \mathfrak{E}([t_n, \infty))$ and $\Xi_n \rightharpoonup \Xi$ in $C([-T, \infty); \mathbb{X}_w)$, $\forall T > 0$. So $\{\Xi_n|_{[-T,\infty)} : \Xi_n \in \mathfrak{E}([t_n, \infty))\} \subset \mathfrak{E}([-T, \infty))$ converges to $\Xi|_{[-T,\infty)} \in C([-T, \infty); \mathcal{X}_w)$. Thus $\Xi \in \bar{\mathfrak{E}}([-T, \infty))$ for any $T > 0$. By definition, this implies $\Xi \in \bar{\mathfrak{E}}((-\infty, \infty))$. □

Remark 3.9. *Since every S–S solution ξ_{u_n} can be obtained as a limit of Galerkin approximations (see [14, 17] for more detail), then for any $\Xi = (\xi_u, \sigma) \in \bar{\mathfrak{E}}((-\infty, \infty))$, we can extract a sequence $\xi_{u_k}^{(k)}$ by using a standard diagonalization process again such that $\xi_{u_k}^{(k)} \rightharpoonup \xi_u$ in $C([-T, \infty); \mathcal{X}_w)$ for any $T > 0$, and $u_k^{(k)} = \sum_{l=1}^k d_l^k(t)e_l$ satisfies*

$$\begin{cases} \partial_t(\alpha_k(t)\partial_t u_k^{(k)}) - \beta_k(t)\Delta u_k^{(k)} + \gamma_k(t)\partial_t u_k^{(k)} + \beta_k(t)\mathrm{P}_k g(u_k^{(k)}) = \beta_k(t)\mathrm{P}_k f(x), \\ \xi_{u_k}^{(k)}(t_k) = \mathrm{P}_k \xi_{u_k}(t_k), \ t \geq t_k, \end{cases} \tag{3.8}$$

where $t_k \to -\infty$ as $k \to \infty$, $\{e_k\}_{i=1}^\infty$ be the orthonormal system of eigenvectors of the Laplacian $-\Delta$ with Dirichlet boundary conditions and P_k is the projector from $L^2(\Omega)$ to $E_k := span\{e_1, e_2, \cdots, e_k\}$.

Corollary 3.10. *Let the assumptions of Theorem 3.8 be satisfied. Then, for any $\Xi = (\xi_u, \sigma) \in \bar{\mathfrak{E}}((-\infty, \infty))$, we have*

$$\int_{-\infty}^{\infty} \|\partial_t u(r)\|^2 dr \leq \mathcal{Q}(\|f\|^2), \quad \partial_t u \in C_b(\mathbb{R}, \mathcal{H}^{-\iota}) \text{ and } \lim_{t\to\pm\infty} \|\partial_t u(t)\|_{\mathcal{H}^{-\iota}} = 0 \tag{3.9}$$

for any $0 < \iota \leq 1$*, where* $Q(\cdot)$ *is a monotone increasing function.*

Proof. Let $\Xi = (\xi_u, \sigma) \in \bar{\mathfrak{E}}((-\infty, \infty))$, taking the multiplier $\partial_t v$ in (2.3) and combining Theorem 2.2 and Remark 3.9 to deduce that $\int_{-\infty}^{\infty} \|\partial_t u(r)\|^2 dr \leq Q(\|f\|^2)$. In order to prove convergence in (3.9), we note that $\bar{\mathfrak{E}}((-\infty, \infty))$ is bounded in $(C_b(\mathbb{R}; \mathscr{E}) \cap C_b^1(\mathbb{R}; \mathscr{E}^{-1})) \times C_b^2(\mathbb{R})$ and σ is translation compact in $C_b^2(\mathbb{R})$, then the convergence is a standard corollary of dissipative integral in (3.9) and the compact embedding $[C_b(\mathbb{R}; \mathscr{E}) \cap C_b^1(\mathbb{R}; \mathscr{E}^{-1})] \subset\subset C_{loc}(\mathbb{R}; \mathcal{H}^{1-\iota} \times \mathcal{H}^{-\iota})$ for every $0 < \iota \leq 1$. □

3.3. Backward smoothness of complete trajectories

The following Theorem 3.11 discusses the backward smoothing property of the complete trajectory included in $\bar{\mathfrak{E}}((-\infty, \infty))$, and the proof is similar to the ones given in [20, Theorem 2.1], and for this reason we give a sketch of the main steps of the proof for the reader's convenience.

Theorem 3.11. *Under Assumption 1.1 and assume that* σ *is translation compact in* $C_b^2(\mathbb{R})$*, then for every complete trajectory* $\Xi = (\xi_u, \sigma) \in \bar{\mathfrak{E}}((-\infty, \infty))$*, there exists a time* $T = T(u, \sigma)$ *such that* $\xi_u \in C_b((-\infty, T]; \mathscr{E}^1)$ *and* $\|\xi_u\|_{C_b((-\infty,T];\mathscr{E}^1)} \leq Q(\|f\|^2, \|\sigma\|_{C_b^2(\mathbb{R})})$.

Proof. We divide the proof into several steps.

Step 1. Rewrite Eq (2.3) as follows:

$$\partial_s^2 v - \Delta v + \eta(s)\partial_s v + L(-\Delta)^{-1} v + g(v) = h(s) := L(-\Delta)^{-1} v + f(x).$$

From the definition of h and applying Theorem 2.2, we have $\|h(s)\|^2 \leq Q(\|f\|^2)$ and $\int_S^{S+1} \|\partial_s h(s)\|_{\mathcal{H}^2}^2 ds = \int_S^{S+1} \|\partial_s v(s)\|^2 ds \leq Q(\|f\|^2)$. Using Corollary 3.10, we infer that

$$\partial_t h \in C_b(\mathbb{R}; \mathcal{H}^{2-\iota}), \quad \lim_{s \to -\infty} \|\partial_s h(s)\|_{\mathcal{H}^{2-\iota}} = 0, \quad \forall 0 < \iota \leq 1. \tag{3.10}$$

Step 2. Applying Lemma 2.2 in [20], we know that for sufficiently large L (depending on the coefficients in Assumption 1.1), the parabolic equation

$$\partial_s w - \Delta w + g(w) + L(-\Delta)^{-1} w = h(s), s \in \mathbb{R} \tag{3.11}$$

possesses a unique solution $w(s)$ in the class $C_b(\mathbb{R}; \mathcal{H}^2)$ with the following estimates:

$$\|w(s)\|_{H^2} \leq Q(\|f\|^2), \quad \partial_s w \in C_b(\mathbb{R}; \mathcal{H}^2), \quad \partial_s^2 w \in L^2([S, S+1]; \mathcal{H}^1), \quad \forall S \in \mathbb{R}, \tag{3.12}$$

and the following convergence

$$\lim_{S \to -\infty} \{\|\partial_s w(S)\|_{\mathcal{H}^2} + \|\partial_s^2 w\|_{L^2([S,S+1];\mathcal{H}^1)}\} = 0. \tag{3.13}$$

Step 3. For a sufficiently large L, there exists a time $S = S(v, L, \sigma)$ such that the problem

$$\partial_s^2 z - \Delta z + \eta(s)\partial_s z + L(-\Delta)^{-1} z + g(z) = h(s), \quad s \leq S \tag{3.14}$$

possesses a unique regular bounded backward solution $\xi_z \in \mathscr{E}^1$, which satisfies

$$\|\partial_s z(s)\|_{H^2} + \|z(s)\|_{H^2} \leq Q(\|f\|^2, \|\sigma\|_{C_b^2}), s \leq S \text{ and } \lim_{s \to -\infty} \|\partial_s z(s)\|_{L^\infty(\Omega)} = 0. \tag{3.15}$$

To see this, let $z = w + Z$, where w satisfying Eq (3.11), then Z satisfies

$$\partial_s^2 Z - \Delta Z + \eta(s)\partial_s Z + L(-\Delta)^{-1}Z + g(w+Z) - g(w) = F_w(s) := -\partial_s^2 w - (\eta(s)-1)\partial_s w. \tag{3.16}$$

We can apply the implicit function theorem in order to solve Eq (3.16) in the space

$$\Phi_S := C_b((-\infty, S], \mathscr{E}^1)), \tag{3.17}$$

where S is small enough. Applying Step 2, we have

$$F_w \in L^2([s, s+1], \mathcal{H}^1)\ \forall s \in \mathbb{R} \text{ and } \lim_{S\to-\infty} \|F_w\|_{L^2([S,S+1],\mathcal{H}^1)} = 0.$$

Now, we intend to verify that the variation equation at $Z = 0$

$$\partial_s^2 Z - \Delta Z + \eta(s)\partial_s Z + L(-\Delta)^{-1}Z + g'(w)Z = H(s), \qquad s \le S \tag{3.18}$$

is uniquely solvable for every $H \in L^2_{loc}((-\infty, S], \mathcal{H}^1)$ such that

$$\|H\|_{L^2_b((-\infty,S],\mathcal{H}^1)} := \sup_{s\in(-\infty,S-1)} \|H\|_{L^2((s,s+1],\mathcal{H}^1)} < \infty$$

if S is small enough. Firstly, taking the multiplier $\partial_s Z + \varepsilon Z$ in (3.18) yields

$$\frac{d}{dt}\mathcal{E}_Z + Q_Z = 2\langle H, \partial_s Z + \varepsilon Z\rangle + \langle g''(w)\partial_s w, Z^2\rangle, \tag{3.19}$$

where

$$\begin{aligned}
\mathcal{E}_Z &= \|\partial_s Z\|^2 + \|Z\|^2_{\mathcal{H}^1} + \langle g'(w)Z, Z\rangle + 2\varepsilon\langle \partial_s Z, Z\rangle + L\|Z\|^2_{\mathcal{H}^{-1}},\\
Q_Z &= 2(\eta-\varepsilon)\|\partial_s Z\|^2 + 2\varepsilon\|Z\|^2_{\mathcal{H}^1} + 2\varepsilon\eta\langle \partial_s Z, Z\rangle + 2\varepsilon L\|Z\|^2_{\mathcal{H}^{-1}} + 2\varepsilon\langle g'(w)Z, Z\rangle.
\end{aligned}$$

Choosing $L \ge 4C^2\kappa_1^2$, we obtain

$$\begin{aligned}
-2\langle g'(w)Z, Z\rangle &\le 2\kappa_1\|Z\|^2 \le 2C\kappa_1\|Z\|_{\mathcal{H}^1}\|Z\|_{\mathcal{H}^{-1}}\\
&\le \frac{1}{2}(\|Z\|^2_{\mathcal{H}^1} + 4C^2\kappa_1^2\|Z\|^2_{\mathcal{H}^{-1}}) \le \frac{1}{2}(\|Z\|^2_{\mathcal{H}^1} + L\|Z\|^2_{\mathcal{H}^{-1}}).
\end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20) and recalling (3.12), there exists a sufficiently small parameter $\varepsilon > 0$ such that

$$C\|\xi_Z(s)\|^2_{\mathscr{E}} \le \mathcal{E}_Z(s) \le C\|\xi_Z(s)\|^2_{\mathscr{E}}$$

and

$$\frac{d}{ds}\mathcal{E}_Z(s) + \varepsilon\mathcal{E}_Z(s) \le C\|H(s)\|^2 + \langle g''(w)\partial_s w, Z^2\rangle - \frac{\alpha}{4}\|Z\|^2_{\mathcal{H}^1}.$$

Using (3.12), (3.13), and embedding $\mathcal{H}^2(\Omega) \subset L^\infty(\Omega)$, we have

$$\frac{d}{ds}\mathcal{E}_Z(s) + \varepsilon\mathcal{E}_Z(s) \le C\|H(s)\|^2, \qquad \text{if } s \le S \text{ and } S \text{ is small enough.}$$

Applying Gronwall's inequality, we deduce

$$\|\partial_s Z(s)\|^2 + \|Z(s)\|^2_{\mathcal{H}^1} \le C\int_{-\infty}^{s} e^{-\varepsilon(s-r)}\|H(r)\|^2 dr, \qquad s \le S. \tag{3.21}$$

Thus, the solution to (3.18) is unique. Secondly, taking the multiplier $-\Delta(\partial_s Z + \varepsilon Z)$ in (3.18), interpreting $g'(w)Z$ as an external force, and using (3.21) yields

$$\|\partial_s Z(s)\|^2_{\mathcal{H}^1} + \|Z(s)\|^2_{\mathcal{H}^2} \le C\int_{-\infty}^{s} e^{-\varepsilon(s-r)}\|H(r)\|^2_{\mathcal{H}^1} dr \le C\|H\|^2_{L^2_b((-\infty,S],\mathcal{H}^1)}, \qquad s \le S. \tag{3.22}$$

Thus, the Eq (3.18) is uniquely solvable in space (3.17) if S is small enough. Now, applying the implicit function theorem for Eq (3.16), for a sufficiently small $S \in \mathbb{R}$, there exists a unique solution $\xi_Z \in \Phi_S$ of problem (3.16) satisfying

$$\|\partial_s Z(s)\|^2_{\mathcal{H}^1} + \|Z(s)\|^2_{\mathcal{H}^2} \le Q(\|f\|^2, \|\sigma\|_{C^2_b}), \quad s \le S \quad \text{and} \lim_{s\to-\infty}\|\partial_s Z(s)\|_{\mathcal{H}^1} = 0. \tag{3.23}$$

Combining the estimates of w in Step 2 and (3.23) we have

$$\|\partial_s z(s)\|^2_{\mathcal{H}^1} + \|z(s)\|^2_{\mathcal{H}^2} \le Q(\|f\|^2, \|\sigma\|_{C^2_b}), \quad s \le S(L, \nu, \sigma) \quad \text{and} \lim_{s\to-\infty}\|\partial_s z(s)\|_{\mathcal{H}^1} = 0. \tag{3.24}$$

Finally, differentiate equation (3.14) and set $\partial_s z = \zeta$, we have

$$\partial_s^2\zeta - \Delta\zeta + \eta(s)\partial_s\zeta + L(-\Delta)^{-1}\zeta = h_z(s) := h'(s) - g'(z)\partial_s z - \eta'(s)\partial_s z. \tag{3.25}$$

Recalling (3.10) and (3.24), we obtain

$$\lim_{s\to-\infty}\|h_z(s)\|_{\mathcal{H}^1} = 0. \tag{3.26}$$

Similar to (3.22), we have

$$\|\partial_s\zeta(s)\|^2_{\mathcal{H}^1} + \|\zeta(s)\|^2_{\mathcal{H}^2} \le C\int_{-\infty}^{s} e^{-\varepsilon(s-r)}\|h_z(r)\|^2 dr, \quad s \le S. \tag{3.27}$$

Since $\mathcal{H}^2 \subset C(\overline{\Omega})$, then (3.26) and (3.27) imply (3.15).

Step 4. We need to prove $z = v$, for $s \le S$. Applying Remark 3.9, there exists a sequence of Galerkin approximations such that $\xi^{(k)}_{v_k} \rightharpoonup \xi_v$ in $C([-T,\infty); \mathcal{X}_w)$ for any $T > 0$, and

$$\begin{aligned} &\partial_s^2 v_k^{(k)} - \Delta v_k^{(k)} + \eta_k(s)\partial_s v_k^{(k)} + L(-\Delta)^{-1}v_k^{(k)} + \mathrm{P}_k g(v_k^{(k)}) \\ =&h_k(s) := L(-\Delta)^{-1}v_k^{(k)} + \mathrm{P}_k f, \ s \ge s_k. \end{aligned} \tag{3.28}$$

Here $s_k = \int_0^{t_k}\sqrt{\frac{\beta(\omega)}{\alpha(\omega)}}d\omega$, $\eta_k \in C^1_b(\mathbb{R})$, so there exits a subsequence (still denote) $\eta_k \to \eta$ in $C^1_b(\mathbb{R})$. Now let $z_k(s) = \mathrm{P}_k z(s)$, $s \le S$. According to Step 3, the solution $\xi_z(s)$ is bounded in $\mathscr{E}^1$ when $s \le S$, and consequently

$$\lim_{k\to\infty}\|\xi_{z_k} - \xi_z\|_{C_b((-\infty,s],\mathscr{E})} = 0, \ \lim_{k\to\infty}\|\xi_{z_k} - \xi_z\|_{C_b((-\infty,S]\times\Omega)} = 0. \tag{3.29}$$

Here we used the fact that $\mathcal{H}^2 \subset\subset C(\overline{\Omega})$ again, and the convergence of the Fourier series is uniform on compact sets. Now denote $V(s) := v(s) - z(s)$ and $V_k(s) := v_k^{(k)}(s) - z_k(s)$, recalling (3.28) yields the following equation:

$$\partial_s^2 V_k - \Delta V_k + \eta_k(s)\partial_s V_k + L(-\Delta)^{-1}V_k + \mathrm{P}_k[g(V_k + z_k) - g(z_k)] = A_k, \tag{3.30}$$

where $A_k(s) = [\eta(s) - \eta_k(s)]\partial_s z_k(s) + \mathrm{P}_k[g(z(s)) - g(z_k(s))]$. Taking the inner product between (3.30) and $\partial_s V_k + \varepsilon V_k$, we have

$$\frac{d}{ds}\mathcal{E}_{V_k}(s) + \varepsilon\mathcal{E}_{V_k}(s) = \langle A_k(s), 2\partial_s V_k(s) + \varepsilon V_k(s)\rangle - Q_{V_k}(s) + \mathcal{G}_{V_k}(s), \tag{3.31}$$

where

$$\begin{aligned}
\mathcal{E}_{V_k} =& \|\partial_s V_k\|^2 + \|V_k\|^2_{\mathcal{H}^1} + 2\varepsilon\langle\partial_s V_k, V_k\rangle + L\|V_k\|^2_{\mathcal{H}^{-1}} + 2\langle G(V_k + z_k) - G(z_k) - g(z_k)V_k, 1\rangle,\\
Q_{V_k} =& 2\varepsilon\eta_k(s)\langle\partial_s V_k, V_k\rangle + \varepsilon L\|V_k\|^2_{\mathcal{H}^{-1}} + \varepsilon\|V_k\|^2_{\mathcal{H}^1} - 2\varepsilon^2\langle\partial_s V_k, V_k\rangle + (2\eta_k(s) - 3\varepsilon)\|\partial_s V_k\|^2,\\
\mathcal{G}_{V_k} =& 2\varepsilon[\langle G(V_k + z_k) - G(z_k) - g(z_k)V_k, 1\rangle - \langle g(V_k + z_k) - g(z_k), V_k\rangle]+\\
&+ 2\langle g(V_k + z_k) - g(z_k) - g'(z_k)V_k, \partial_t z_k\rangle.
\end{aligned}$$

Using the fact that

$$\begin{aligned}
G(v + w) - G(v) - g(v)w - (g(v + w) - g(v))w &\le \frac{\kappa_1}{2}|w|^2,\\
|g(v + w) - g(v) - g'(v)w| &\le C|w|^2(1 + |v|^{q-2} + |w|^{q-2}),
\end{aligned}$$

we have

$$\mathcal{G}_{V_k} \le \frac{\kappa_1\varepsilon}{2}\|V_k\|^2 + C\|\partial_s z_k\|_{L^\infty(\Omega)}\langle|V_k|^2(1 + |z_k|^{q-2} + |V_k|^{q-2}), 1\rangle.$$

Combining (3.15), (3.29), and (3.31), we can infer that there exists a time $S' \le S$ such that, for sufficiently large k, we have

$$\frac{d}{ds}\mathcal{E}_{V_k}(s) + \varepsilon\mathcal{E}_{V_k}(s) \le C\|A_k(s)\|^2, \qquad \forall s \le S'.$$

Applying Gronwall's inequality, we obtain

$$\mathcal{E}_{V_k}(s) \le \mathcal{E}_{V_k}(s_k)e^{-\varepsilon(s-s_k)} + C\int_{s_k}^{s} e^{-\varepsilon(s-r)}\|A_k(r)\|^2 dr,$$

where constants C and ε are independent of k. Indeed, we have

$$\begin{aligned}
\|\partial_s V_k(s)\|^2 + \|V_k(s)\|^2_{\mathcal{H}^1} \le& C(1 + \|\xi_{v_k^{(k)}}(s_k)\|^2_{\mathscr{E}} + \|\xi_{z_k}(s_k)\|^2_{\mathscr{E}})e^{-\varepsilon(s-s_k)}+\\
&+ C\int_{s_k}^{s} e^{-\varepsilon(s-r)}\|A_k(r)\|^2 dr.
\end{aligned}\tag{3.32}$$

Now, passing to the limit $k \to \infty$ in (3.32), we obtain $\|\xi_V(s)\|_{\mathscr{E}} = 0$ for all $s \le S'$. The proof of Theorem 3.11 is now complete. □

Corollary 3.12. *Let the assumptions of Theorem 3.11 be satisfied. Then the uniform weak global attractor $\mathscr{A}_w$ for ES $\mathfrak{E}$ is in a more regular space: $\mathscr{A}_w \subset \mathscr{E}^1 \times C_b^2(\mathbb{R})$.*

Proof. Let $\Xi_u = (\xi_u, \sigma)$ be the complete trajectory of Eq (1.1), then $\Xi_v := (\xi_v, \eta)$ be the corresponding complete trajectory of Eq (2.3). Recall that there exists time S_0, such that $\xi_v(s) \in \mathscr{E}^1$ for all $s \leq S_0$. Due to Theorem 2.2, there is an extension $\bar{v}$ for $s \geq S_0$ such that $\bar{v}(s) = v(s)$ for $s \leq S_0$ and $\bar{v}(s)$ is a S–S solution of Eq (2.3) for all $s \in \mathbb{R}$. Indeed, we can conclude that $\xi_{\bar{v}}(s) \in \mathscr{E}^1$ for all $s \in \mathbb{R}$. We are now ready to prove that $\xi_{\bar{v}}(s) = \xi_v(s)$ for all $s \in \mathbb{R}$. Since $\Xi_v := (\xi_v, \eta) \in \bar{\mathfrak{E}}((-\infty, \infty))$, and applying Remark 3.9, we get

$$\partial_s^2 v_k^{(k)} - \Delta v_k^{(k)} + \eta_k(s)\partial_s v_k^{(k)} + \mathrm{P}_k g(v_k^{(k)}) = \mathrm{P}_k f, \quad \xi_{v_k}^{(k)}(s_k) = \mathrm{P}_k \xi_{v_k}(s_k), \tag{3.33}$$

where $s \geq s_k$ and $\lim_{k\to\infty} s_k = -\infty$. Obviously, $\bar{v}_k = \mathrm{P}_k \bar{v}$ satisfying

$$\partial_s^2 \bar{v}_k - \Delta \bar{v}_k + \eta(s)\partial_s \bar{v}_k + \mathrm{P}_k g(\bar{v}) = \mathrm{P}_k f, \quad \xi_{\bar{v}_k}(s_k) = \mathrm{P}_k \xi_{\bar{v}}(s_k). \tag{3.34}$$

Denote $W = v - \bar{v}$, $W_k = v_k^{(k)} - \bar{v}_k$, then combining (3.33) and (3.34) we know that W_k satisfies

$$\partial_s^2 W_k - \Delta W_k + \eta_k(s)\partial_s W_k + \mathrm{P}_k[g(v_k^{(k)}) - g(\bar{v}_k)] + L(-\Delta)^{-1} W_k = B_k, \tag{3.35}$$

where

$$B_k = \mathrm{P}_k[g(\bar{v}) - g(\bar{v}_k)] + [\eta - \eta_k]\partial_s \bar{v}_k + L(-\Delta)^{-1} W_k.$$

Taking the multiplier $\partial_s W_k + \varepsilon W_k$ in (3.35) yields

$$\begin{aligned}&\frac{d}{ds}\mathcal{E}_{W_k}(s) + \varepsilon \mathcal{E}_{W_k}(s)\\ =&\Lambda_k(s) := 2\langle B_k(s), \partial_s W_k(s) + \varepsilon W_k(s)\rangle - Q_{W_k}(s) + \mathcal{G}_{W_k}(s),\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}\mathcal{E}_{W_k} =& \|\partial_s W_k\|^2 + \|W_k\|^2_{\mathcal{H}^1} + 2\varepsilon\langle \partial_s W_k, W_k\rangle\\ &+ L\|W_k\|^2_{\mathcal{H}^{-1}} + 2\langle G(W_k + \bar{v}_k) - G(\bar{v}_k) - g(\bar{v}_k)W_k, 1\rangle,\\ Q_{W_k} =& (2\varepsilon\eta_k(s) - 2\varepsilon^2)\langle \partial_s W_k, W_k\rangle + \varepsilon L\|W_k\|^2_{\mathcal{H}^{-1}} + \varepsilon\|W_k\|^2_{\mathcal{H}^1}\\ &+ (2\eta_k(s) - 3\varepsilon)\|\partial_s W_k\|^2,\\ \mathcal{G}_{W_k} =& 2\varepsilon[\langle G(W_k + \bar{v}_k) - G(\bar{v}_k) - g(\bar{v}_k)W_k, 1\rangle - \langle g(W_k + \bar{v}_k) - g(\bar{v}_k), W_k\rangle]+\\ &+ 2\langle g(W_k + \bar{v}_k) - g(\bar{v}_k) - g'(\bar{v}_k)W_k, \partial_t \bar{v}_k\rangle].\end{aligned}$$

Choosing L large enough and applying [20, Proposition 2.1], we discover that the right-hand side of (3.36) satisfies $\Lambda_k(s) \leq C\|B_k(s)\|^2$ and C is independent of k. Invoking Gronwall's inequality, we have the estimate

$$\mathcal{E}_{W_k}(s) \leq C(\mathcal{E}_{W_k}(s_k))e^{-\varepsilon(s-s_k)} + C\int_{s_k}^{s} e^{-\varepsilon(s-r)}\|B_k(r)\|^2 dr, \ \forall s \geq S_0. \tag{3.37}$$

Passing to the limit $k \to \infty$ and employing the convergence $B_k \to L(-\Delta)^{-1}W$ strongly in $C((-\infty, S_0], \mathcal{H}^0)$, $\|B_k\|_{C((-\infty,S_0],\mathcal{H}^0)} \leq C$ (C is independent of k), $\mathcal{E}_{W_k}$ is equivalent to $\|\xi_{W_k}\|^2_{\mathscr{E}}$ and the fact that $\bar{v}(s) = v(s)$ for $s \leq S_0$, we have

$$\|\xi_v(s) - \xi_{\bar{v}}(s)\|^2_{\mathscr{E}} \leq CL^2 \int_{S_0}^{s} e^{-\varepsilon(s-r)} \|(-\Delta)^{-1}(v(r) - \bar{v}(r)\|^2 dr, \ \forall s \geq S_0. \tag{3.38}$$

Invoking again Gronwall's inequality to relation (3.38) and noting $v(S_0) = \bar{v}(S_0)$, we derive that $v(s) = \bar{v}(s)$ for all $s \in \mathbb{R}$. □

Remark 3.13. *The proof of Corollary 3.12 indicates that for any $\Xi = (\xi_u, \sigma) \in \bar{\mathfrak{E}}((-\infty, \infty))$, then ξ_u is the S–S solution of Eq* (1.1)*, i.e., $\bar{\mathfrak{E}}((-\infty, \infty)) = \mathfrak{E}((-\infty, \infty))$. Moreover, we have $\xi_u(t) \in \mathscr{E}^1$ for all $t \in \mathbb{R}$.*

4. Strong global attractors

We introduce some definitions; see [2, 6, 11] for more details.

Definition 4.1. *Let $S(t)$ be a semigroup acting on a Banach space $\mathcal{Y}$. A set $\mathscr{A}_s \subset \mathcal{Y}$ is a (strong) global attractor of $S(t)$ if*

(1) The set $\mathscr{A}_s$ is compact in $\mathcal{Y}$;

(2) The set $\mathscr{A}_s$ is strictly invariant: $S(t)\mathscr{A}_s = \mathscr{A}_s$;

(3) It is an attracting set for the semigroup $S(t)$, i.e., for any bounded set $B \subset \mathcal{Y}$,

$$dist_{\mathcal{Y}}(S(t)B, \mathscr{A}_s) := \sup_{x \in B} \inf_{y \in \mathscr{A}_s} \|S(t)x - y\|_{\mathcal{Y}} \to 0, \quad as\ t \to \infty.$$

Definition 4.2. *A set $A \subset \mathcal{Y}$ is said to be uniformly (w.r.t. $\sigma \in \Sigma$) attracting for the family of processes $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$, if for any fixed $\tau \in \mathbb{R}$ and every bounded set $B \subset \mathcal{Y}$*

$$\lim_{t \to +\infty} \left(\sup_{\sigma \in \Sigma} dist_{\mathcal{Y}} \left(U_\sigma(t, \tau)B, A \right) \right) = 0.$$

A closed, uniformly attracting set $\mathscr{A}_s^{\Sigma}$ is said to be the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t, \tau)\}, \sigma \in \Sigma$, if it is contained in any closed uniformly attracting set (minimality property).

The kernel $\mathcal{K}_\sigma$ consists of all bounded complete trajectories of the process $U_\sigma(t, \tau)$, i.e.,

$$\mathcal{K}_\sigma = \{u(\cdot) \mid \|u(t)\|_{\mathcal{Y}} \leqslant C_u, U_\sigma(t, \tau)u(\tau) = u(t) \quad \forall t \geq \tau,\ \tau \in \mathbb{R}\},$$

and $\mathcal{K}_\sigma(s)$ denotes the kernel section at a time moment $s \in \mathbb{R}$:

$$\mathcal{K}_\sigma(s) = \{u(s) \mid u(\cdot) \in \mathcal{K}_\sigma\}, \quad \mathcal{K}_\sigma(s) \subset \mathcal{Y}.$$

Theorem 4.3. *Under Assumption 1.1 assume that σ is translation compact in $C_b^2(\mathbb{R})$. Then the semigroup $\mathbb{S}$ defined in* (3.2) *possesses a strong global attractor $\mathscr{A}_s$ in $\mathscr{E}^1 \times C_b^2(\mathbb{R})$, which coincides with the uniform weak attractor $\mathscr{A}_w$ in Theorem 3.7 and satisfies the following properties:*

(i) $\Pi_1 \mathscr{A}_s = \mathscr{A}_s^\Sigma$ *is the uniform (w.r.t.* $\sigma \in \Sigma$*) attractor of the family of processes* $\{U_\sigma(t,\tau)\}$*,* $\sigma \in \Sigma$ *and* Π_1 *is projector from* $\mathscr{E} \times \Sigma$ *onto* $\mathscr{E}$*;*

(ii) The uniform attractor satisfies $\mathscr{A}_s^\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(0)$*, where* $\mathcal{K}_\sigma(0)$ *is the section at* $t = 0$ *of the kernel* $\mathcal{K}_\sigma$ *of the process* $\{U_\sigma(t,\tau)\}$ *with the symbol* $\sigma \in \Sigma$*.*

Proof. In order to apply [6, Theorem IV.5.1] and [18, Theorem 3.4], we have to check that the processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma, t\geq\tau}$ corresponding to the S–S solutions of Eq (1.1) be $(\mathscr{E}\times\Sigma, \mathscr{E})$-continuous and uniformly asymptotically compact.

Firstly, consider two S–S solutions ξ_{u_i} of Eq (1.1) with symbols σ_i and with initial values $\xi_{u_{i\tau}}$, then correspondingly, ξ_{v_i} are S–S solutions of Eq (2.3) with symbols η_i, $i = 1, 2$. Then $\xi_w = \xi_{v_1} - \xi_{v_2}$ satisfies the equation

$$\partial_s^2 w - \Delta w + \eta_1 \partial_s v_1 - \eta_2 \partial_s v_2 + g(v_1) - g(v_2) = 0. \tag{4.1}$$

Taking the scalar product of (4.1) with $\partial_s w$, we obtain

$$\frac{d}{ds}\|\xi_w(s)\|_{\mathscr{E}}^2 + (\eta_1 + \eta_2)\|\partial_s w\|^2 = 2\langle g(v_2) - g(v_1), \partial_s w\rangle + (\eta_1 - \eta_2)\langle \partial_s v_1 + \partial_s v_2, \partial_s w\rangle. \tag{4.2}$$

Thanks to Assumption 1.1 (**G**) and applying the elementary inequality, we have

$$2|\langle g(v_1) - g(v_2), \partial_s w\rangle| \leq C_g q(s)\|\xi_w\|_{\mathscr{E}}^2, \tag{4.3}$$

$$|(\eta_1(s) - \eta_2(s))\langle \partial_s v_1 + \partial_s v_2, \partial_s w\rangle| \leq \varepsilon\|\partial_s w\|^2 + C_\varepsilon|\eta_1(s) - \eta_2(s)|^2. \tag{4.4}$$

Where $q(s) = (1 + \|v_1(s)\|_{L^{12}(\Omega)}^4 + \|v_2(s)\|_{L^{12}(\Omega)}^4)$. Applying (4.3) and (4.4) in (4.2) and employing Gronwall's inequality, we obtain that

$$\|\xi_w(s)\|_{\mathscr{E}}^2 \leq e^{C_g \int_\mu^s q(r)dr}(\|\xi_w(\mu)\|_{\mathscr{E}}^2 + C_\varepsilon(s-\mu)\|\eta_1 - \eta_2\|_{C_b^1(\mathbb{R})}^2). \tag{4.5}$$

Then the $(\mathscr{E}\times\Sigma, \mathscr{E})$-continuity follows in a standard way from the energy inequality (4.5).

Secondly, we intend to verify the uniform asymptotic compactness of the processes $\{\widetilde{U}_\eta(s,\mu)\}_{\eta\in\widetilde{\Sigma}, s\geq\mu}$ corresponding to the S–S solutions of Eq (2.3), where $\widetilde{\Sigma} = [T_h(\eta), h \in \mathbb{R}]_{C_b^1(\mathbb{R})}$. Let $\{\eta_n\} \subset \widetilde{\Sigma}$, $\{-\mu_n\} \subset (-\infty, 0]$, $\mu_n \to \infty$ as $n \to \infty$. ξ_{μ_n} belongs to a bounded subset in $\mathscr{E}$. Since σ is translation compact in $C_b^2(\mathbb{R})$, without loss of generality, we may assume that $\eta_n \to \eta$ $(n \to \infty)$ in $C_b^1(\mathbb{R})$, and $\xi_{\mu_n} \rightharpoonup \xi_\mu$ weakly in $\mathscr{E}$ as $n \to \infty$. Denote $\xi_{v_n}(s) = \widetilde{U}_{\eta_n}(s,\mu_n)\xi_{\mu_n}$ the corresponding solutions, then v_n solves

$$\partial_s^2 v_n - \Delta v_n + \eta_n(s)\partial_s v_n + g(v_n) = f, \quad s \geq \mu_n \text{ and } \xi_{v_n}(\mu_n) = \xi_{\mu_n}. \tag{4.6}$$

Taking the multiplier $\partial_s v_n + \varepsilon v_n$ with $0 < \varepsilon \ll 1$ in Eq (4.6), we derive the following energy type identity:

$$\frac{d}{ds}\mathcal{E}_{v_n}(s) + \varrho\mathcal{E}_{v_n}(s) + \mathcal{Q}_{v_n}(s) + \mathcal{G}_{v_n}(s) + \mathcal{F}_{v_n}(s) = 0, \tag{4.7}$$

where

$$\mathcal{E}_{v_n} = \|\partial_s v_n\|^2 + \|v_n\|_{\mathcal{H}^1}^2 + 2\varepsilon\langle \partial_s v_n, v_n\rangle + 2\langle G(v_n), 1\rangle - 2\langle f, v_n\rangle,$$

$$\mathcal{Q}_{v_n} = (2\eta_n - 2\varepsilon - \varrho)\|\partial_s v_n\|^2 + (2\varepsilon - \varrho)\|v_n\|^2_{\mathcal{H}^1} + 2\varepsilon(\eta_n - \varrho)\langle \partial_s v_n, v_n\rangle,$$
$$\mathcal{G}_{v_n} = 2[\varepsilon\langle g(v_n), v_n\rangle - \varrho\langle G(v_n), 1\rangle], \quad \mathcal{F}_{v_n} = 2(\varrho - \varepsilon)\langle f, v_n\rangle.$$

Now, integrate Eq (4.7) with respect to $s \in [-\mu_n, 0]$ to deduce that

$$\mathcal{E}_{v_n}(0) + \int_{-\mu_n}^{0} e^{\varrho r}(\mathcal{Q}_{v_n}(r) + \mathcal{G}_{v_n}(r) + \mathcal{F}_{v_n}(r))dr = \mathcal{E}_{v_n}(0)e^{-\varrho\mu_n}. \tag{4.8}$$

Our intention now is to pass to the limit $n \to \infty$ in equality (4.8). To do this, we remind that ξ_{v_n} is uniformly bounded in $C((-\mu_n, \infty], \mathscr{E})$ and $\eta_n \to \eta$ $(n \to \infty)$ in $C^1_b(\mathbb{R})$, then we get that

$$\Xi_n = (\xi_{v_n}, \eta_n) \rightharpoonup \Xi = (\xi_v, \eta), \quad \text{in } C(\mathbb{R}, \mathbb{X}_w)$$

and $\Xi = (\xi_v, \eta) \in \bar{\mathfrak{E}}((-\infty, \infty)) = \mathfrak{E}((-\infty, \infty))$ by recalling Corollary 3.12. In addition, we also know that ξ_v is an S–S solution and $\xi_{v_n}(0) \rightharpoonup \xi_v(0)$ weakly in $\mathscr{E}$. Applying the compact embedding $C_{loc}((-\infty, 0], \mathscr{E}) \subset\subset C_{loc}((-\infty, 0], \mathcal{H})$, we can get that $v_n \to v$ strongly in $C_{loc}((-\infty, 0], \mathcal{H})$, including almost everywhere. On the other hand, from the assumption (1.3), we can choose $\varrho = \frac{\varepsilon}{4}$, which guarantees that $\mathcal{G}_{v_n}(s) \geq -\kappa_2|\Omega|$ and choose $0 < \varepsilon \leq \varepsilon_0$ small enough such that the quadratic form $\mathcal{Q}_{v_n}$ is positive definite and satisfying $C_1\|\xi_v\|^2_{\mathscr{E}} \leq \mathcal{Q}_v \leq C_2\|\xi_v\|^2_{\mathscr{E}}$. Now, using the Fatou lemma, we conclude that

$$\begin{aligned} 0 = \liminf_{n\to\infty}\Big(\mathcal{E}_{v_n}(0) + \int_{-\mu_n}^{0} e^{\varrho r}(\mathcal{Q}_{v_n}(r) + \mathcal{G}_{v_n}(r) + \mathcal{F}_{v_n}(r))dr\Big) \\ \geq \mathcal{E}_v(0) + \int_{-\infty}^{0} e^{\varrho r}(\mathcal{Q}_v(r) + \mathcal{G}_v(r) + \mathcal{F}_v(r))dr. \end{aligned} \tag{4.9}$$

According to Theorem 3.11, v is an S–S solution with more regularity in $\mathscr{E}^1$, and obviously v satisfies the energy equality. Then, by repeating the derivation of (4.8), for solution v, we obtain the energy equality

$$\mathcal{E}_v(0) + \int_{-\infty}^{0} e^{\varrho r}(\mathcal{Q}_v(r) + \mathcal{G}_v(r) + \mathcal{F}_v(r))dr = 0. \tag{4.10}$$

Combining (4.9) and (4.10), we ascertain

$$\liminf_{n\to\infty} \mathcal{E}_{v_n}(0) = \mathcal{E}_v(0). \tag{4.11}$$

Applying the Fatou lemma and weak lower semi-continuous of the norm, we find that

$$\liminf_{n\to\infty}\langle G(v_n(0)), 1\rangle \geq \langle G(v(0)), 1\rangle, \quad \liminf_{n\to\infty}\|\xi_{v_n}(0)\|^2_{\mathscr{E}} \geq \|\xi_v(0)\|^2_{\mathscr{E}}. \tag{4.12}$$

Obviously, energy equality is true only when inequalities (4.12) are also equalities. Recalling $\xi_{v_n}(0) \rightharpoonup \xi_v(0)$, we obtain $U_{\eta_n}(0, \mu_n)\xi_{\mu_n} = \xi_{v_n}(0) \to \xi_v(0)$ strongly in $\mathscr{E}$. Finally, we get the uniformly asymptotically compact of the processes $\{U_\sigma(t, \tau)\}_{\sigma\in\Sigma, t\geq\tau}$ and the theorem is proved. □

Theorem 4.4. *Under Assumption 1.1 assume that σ is translation compact in $C^2_b(\mathbb{R})$. Then the global attractor $\mathscr{A}_s$ of the solution semigroup $\mathbb{S}(t)$ defined in* (3.2) *is a bounded set in $\mathscr{E}^1 \times C^2_b(\mathbb{R})$.*

Proof. For any initial data $(\xi_{u_0}, \sigma_0) \in \mathscr{A}_s$, we will prove that

$$\|\xi_u(t)\|^2_{\mathscr{E}^1} \le e^{-\frac{\lambda}{8}t} Q(\|\xi_{u_0}\|^2_{\mathscr{E}^1} + \|f\|^2) + C_{\|f\|^2, \mathscr{A}_s}, \quad \forall t \ge 0, \tag{4.13}$$

where the positive constant λ and monotone increasing function $Q(\cdot)$ are independent of u and t. We proceed in three steps:

Step 1. Claim #1: Denote the restriction of the trajectory in $\mathfrak{E}((-\infty,\infty))$ to the time interval $t \in [0,1]$ as

$$\mathcal{F} := \{u|_{t\in[0,1]}, \Xi = (\xi_u, \sigma) \in \mathfrak{E}((-\infty,\infty))\}.$$

Then $\mathcal{F}$ is a compact set of $L^4(0,1;L^{12}(\Omega))$:

$$\mathcal{F} \subset\subset L^4(0,1;L^{12}(\Omega)). \tag{4.14}$$

Proof of claim. First note that the attractor $\mathscr{A}_s$ is compact in $\mathscr{E}\times C^2_b(\mathbb{R})$, then there exists $T = T(\mathscr{A}_s) > 0$ such that

$$\|u\|_{L^4(0,T;L^{12}(\Omega))} \le C$$

for any S–S solution $u(t)$ with $\xi_u(0) \in \Pi_1\mathscr{A}_s$, where C may depend on $\mathscr{A}_s$, but is independent of u. Indeed, we obtain

$$\|u\|_{L^4(0,T;L^{12}(\Omega))} \le C, \qquad \forall\ \Xi = (\xi_u, \sigma) \in \mathfrak{E}((-\infty,\infty)). \tag{4.15}$$

Using $\mathfrak{E}((-\infty,\infty))$ is invariant with respect to time shifts, for any $\Xi = (\xi_u,\sigma) \in \mathfrak{E}((-\infty,\infty))$ we have

$$\sup_{t\in\mathbb{R}} \|u\|_{L^4(t,t+1;L^{12}(\Omega))} \le C_0 \tag{4.16}$$

for some positive constant C_0, independent of u. Now let ξ_{v_i} are two S–S solutions of Eq (2.3) with symbols η_i, $i = 1,2$, and denote $\xi_w = \xi_{v_1} - \xi_{v_2}$, then recalling (4.5) and applying (4.15) or (4.16) to find that

$$\|\xi_w(s)\|^2_{\mathscr{E}} \le Ce^{Ks}(\|\xi_w(0)\|^2_{\mathscr{E}} + C\|\eta_1 - \eta_2\|^2_{C^1_b(\mathbb{R})}), \quad \forall s \in [0,1], \tag{4.17}$$

where the constants C and K are independent of $\xi_{v_i}(0)$, $i = 1,2$. Then, applying (1.2) and (4.17), we have

$$\begin{aligned}\|g(v_1) - g(v_2)\|_{L^1(0,1;L^2(\Omega))} &\le C\int_0^1 (1 + \|v_1\|^4_{L^{12}(\Omega)} + \|v_2\|^4_{L^{12}(\Omega)})\|v_1 - v_2\|_{L^6(\Omega)} ds \\ &\le C(\|\xi_w(0)\|_{\mathscr{E}} + C\|\eta_1 - \eta_2\|_{C^1_b(\mathbb{R})}).\end{aligned} \tag{4.18}$$

Applying Strichartz estimates for Eq (4.1) and recalling (4.17) and (4.18) gives us

$$\|v_1 - v_2\|_{L^4(0,1;L^{12}(\Omega))} \le C(\|\xi_w(0)\|_{\mathscr{E}} + \|\eta_1 - \eta_2\|_{C^1_b(\mathbb{R})}). \tag{4.19}$$

Assertion (4.14) is now a consequence of (4.19) and the general fact that $\mathscr{A}_s$ is compact in $\mathscr{E} \times C^2_b(\mathbb{R})$.

Step 2. Claim #2: For any $\varepsilon > 0$ and any $\Xi = (\xi_u, \sigma) \in \mathfrak{E}((-\infty, \infty))$, we can split the solution into two parts $u = \bar{u} + \tilde{u}$, where

$$\sup_{t\geq 0} \|\tilde{u}\|_{L^4(t,t+1;L^{12}(\Omega))} \leq \varepsilon \quad \text{and} \quad \|\bar{u}(t)\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^2)} \leq C_\varepsilon. \tag{4.20}$$

Here, the constant C_ε depends on ε, but is independent of u.
Proof of claim. In fact, $\forall \varepsilon > 0$, there exists a finite ε-net $\{(y_i, \sigma_i)\}_{i=1}^m \subset \mathcal{D}$ satisfying

$$\mathcal{D} \subset \bigcup_{1\leq i\leq m} B_{L^4(0,1;L^{12}(\Omega))\times C_b^2(\mathbb{R})}\left((y_i, \sigma_i), \frac{\varepsilon}{4}\right), \tag{4.21}$$

where $\mathcal{D} := \{(u, \sigma)|_{t\in[0,1]}, \Xi = (\xi_u, \sigma) \in \mathfrak{E}((-\infty, \infty))\}$ and $B_X(x_0, r)$ denotes the r-ball centered on x_0 in the space X. By (4.16), we also have

$$\sup_{1\leq i\leq m} \|y_i\|_{L^4(0,1;L^{12}(\Omega))} \leq C_0.$$

Then approximate y_i by a smoother function $\tilde{y}_i$ such that

$$\|\tilde{y}_i - y_i\|_{L^4(0,1;L^{12}(\Omega))} \leq \frac{\varepsilon}{4} \quad \text{and} \quad \|\tilde{y}_i\|_{C(0,1;\mathcal{H}^2)} \leq C_\varepsilon, \tag{4.22}$$

where $i = 1, 2, \cdots, m$ and the constant C_ε is independent of y_i. Combining (4.21) and (4.22), we otain

$$\mathcal{D} \subset \bigcup_{1\leq i\leq m} B_{L^4(0,1;L^{12}(\Omega))\times C_b^2(\mathbb{R})}\left((\tilde{y}_i, \sigma_i), \frac{\varepsilon}{2}\right). \tag{4.23}$$

For every $\Xi = (\xi_u, \sigma) \in \mathfrak{E}((-\infty, \infty))$, we observe that

$$(u, \sigma)|_{t\in[n,n+1]} = \mathbb{S}(n)(u, \sigma)|_{t\in[0,1]} \in \mathcal{D}. \tag{4.24}$$

So in view of (4.23) and (4.24), there exists $(\tilde{y}_{i_n}, \sigma_{i_n})$ such that

$$(u, \sigma)|_{t\in[n,n+1]} \in B_{L^4(0,1;L^{12}(\Omega))\times C_b^2(\mathbb{R})}\left((\tilde{y}_{i_n}, \sigma_{i_n}), \frac{\varepsilon}{2}\right). \tag{4.25}$$

Define the function $\tilde{u}(t)$ as

$$\bar{u}(t) = \tilde{y}_{i_n}(t - n), \quad \text{if } t \in [n, n+1),\ \forall n \in \mathbb{N},$$

and the function $\tilde{u}(t) = u(t) - \bar{u}(t)$. Then

$$\|\bar{u}(t)\|_{L^\infty(\mathbb{R}^+;\mathcal{H}^2)} \leq \sup_{n\in\mathbb{N}} \|\tilde{y}_{i_n}\|_{C(n,n+1;\mathcal{H}^2)} \leq C_\varepsilon. \tag{4.26}$$

For any $t \geq 0$, $[t, t+1] \subset [n, n+2)$ for some n, then combining (4.24) and (4.25) leads us to the estimate

$$\begin{aligned}
\|\tilde{u}\|_{L^4(t,t+1;L^{12}(\Omega))} &\leq \|(u - \bar{u}, \sigma - \sigma_{i_n})\|_{L^4(n,n+1;L^{12}(\Omega))\times C_b^2(\mathbb{R})} \\
&\qquad + \|(u - \bar{u}, \sigma - \sigma_{i_{n+1}})\|_{L^4(n+1,n+2;L^{12}(\Omega))\times C_b^2(\mathbb{R})} \\
&= \|\mathbb{S}(n)(u, \sigma) - (\tilde{y}_{i_n}, \sigma_{i_n})\|_{L^4(0,1;L^{12}(\Omega))\times C_b^2(\mathbb{R})}
\end{aligned}$$

$$+\|\mathbb{S}(n+1)(u,\sigma)-(\tilde{y}_{i_{n+1}},\sigma_{i_{n+1}})\|_{L^4(0,1;L^{12}(\Omega))\times C_b^2(\mathbb{R})}\leq\varepsilon. \tag{4.27}$$

Hence, (4.26) and (4.27) imply the stated assertion (4.20) easily.

Step 3. The following estimates will be deduced by a formal argument, which can be justified by using Faedo–Galerkin method. Differentiating Eq (2.3) and setting $\theta(s)=\partial_s v$, we see that

$$\partial_s^2\theta-\Delta\theta+\eta(s)\partial_s\theta+\eta'(s)\theta+g'(v)\theta=0 \tag{4.28}$$

with the initial condition

$$\xi_\theta(0)=(\partial_s v(0),\partial_s^2 v(0))=(v_1,\Delta v_0-g(v_0)-\eta(0)v_1+f)\in\mathscr{E}. \tag{4.29}$$

Taking the multiplier $\partial_s\theta+\lambda\theta$ in (4.28), we can discover

$$\frac{d}{ds}\mathcal{E}_\theta(s)+\mathcal{Q}_\theta(s)+\mathcal{G}_\theta(s)=0,$$

where

$$\begin{aligned}
&\mathcal{E}_\theta(s)=\|\partial_s\theta\|^2+\|\nabla\theta\|^2+\lambda\langle\partial_s\theta,\theta\rangle,\\
&\mathcal{Q}_\theta(s)=(2\eta(s)-\lambda)\|\partial_s\theta\|^2+\lambda\|\nabla\theta\|^2+(2\eta'(s)+\lambda\eta(s))\langle\theta,\partial_s\theta\rangle+\lambda\eta'(s)\|\theta\|^2,\\
&\mathcal{G}_\theta(s)=\lambda\langle g'(v),\theta^2\rangle+2\langle g'(v)\theta,\partial_s\theta\rangle.
\end{aligned}$$

Choosing λ small enough such that

$$\mathcal{E}_\theta\cong\|\xi_\theta\|_{\mathscr{E}}^2,\quad\text{and}\quad\frac{d}{ds}\mathcal{E}_\theta(s)+\frac{\lambda}{2}\mathcal{E}_\theta(s)\leq C_{\lambda,\|\eta\|_{C_b^1(\mathbb{R})}}\|\theta(s)\|^2-2\langle g'(v)\theta,\partial_s\theta\rangle. \tag{4.30}$$

We employ the decomposition (4.20) and then let

$$\bar{v}(x,s)=\bar{u}(x,\phi^{-1}(s)),\quad\tilde{v}(x,s)=\tilde{u}(x,\phi^{-1}(s)),$$

to discover

$$\begin{aligned}
&|\langle g'(v)\theta,\partial_s\theta\rangle|\leq|\langle(g'(\tilde{v}+\bar{v})-g'(\bar{v}))\theta,\partial_s\theta\rangle|+|\langle g'(\bar{v})\theta,\partial_s\theta\rangle|\\
\leq&C\langle(1+|\bar{v}|^3+|\tilde{v}|^3)|\tilde{v}|,|\theta||\partial_s\theta|\rangle+\|g'(\bar{v})\|_{L^\infty}\|\theta\|\|\partial_s\theta\|\\
\leq&C(1+\|\tilde{v}\|_{L^{12}(\Omega)}^3+\|\bar{v}\|_{L^{12}(\Omega)}^3)\|\tilde{v}\|_{L^{12}(\Omega)}\|\theta\|_{L^6(\Omega)}\|\partial_s\theta\|+C(1+\|\bar{v}\|_{\mathcal{H}^2}^4)\|\partial_s v\|\|\partial_s\theta\|\\
\leq&C(1+\|\tilde{v}\|_{L^{12}(\Omega)}^3+\|\bar{v}\|_{L^{12}(\Omega)}^3)\|\tilde{v}\|_{L^{12}(\Omega)}\|\xi_\theta\|_{\mathscr{E}}^2+\frac{\lambda}{4}\|\partial_s\theta\|^2+C_{\lambda,\mathscr{A}_s,\|\eta\|_{C_b^1(\mathbb{R})}}\|\partial_s v\|^2\\
\leq&l_\varepsilon(s)\|\xi_\theta\|_{\mathscr{E}}^2+C_{\lambda,\mathscr{A}_s,\|\eta\|_{C_b^1(\mathbb{R})}}\|\partial_s v\|^2+\frac{\lambda}{4}\|\partial_s\theta\|^2,
\end{aligned} \tag{4.31}$$

where $l_\varepsilon(s)=C(1+\|\tilde{v}\|_{L^{12}(\Omega)}^3+\|\bar{v}\|_{L^{12}(\Omega)}^3)\|\tilde{v}\|_{L^{12}(\Omega)}$. Owing to (4.16) and (4.20), we conclude

$$\begin{aligned}
&\int_s^{s+1}l_\varepsilon(r)dr\leq C\left(\int_s^{s+1}(1+\|\bar{v}\|_{L^{12}(\Omega)}^3+\|\tilde{v}\|_{L^{12}(\Omega)}^3)^{\frac{4}{3}}dr\right)^{\frac{3}{4}}\left(\int_s^{s+1}\|\tilde{v}\|_{L^{12}(\Omega)}^4dr\right)^{\frac{1}{4}}\\
\leq&C\left(1+\|\tilde{v}\|_{L^4(s,s+1;L^{12}(\Omega))}^3+\|\bar{v}\|_{L^4(s,s+1;L^{12}(\Omega))}^3\right)\|\tilde{v}\|_{L^4(s,s+1;L^{12}(\Omega))}\leq C\varepsilon
\end{aligned} \tag{4.32}$$

for some positive constant C independent of ε. Combining now (4.30) and (4.31) and employing Gronwall's inequality, we deduce

$$\|\xi_\theta(s)\|_{\mathscr{E}}^2 \leq e^{-\int_0^s (\frac{\lambda}{4}-l_\varepsilon(r))dr} Q(\|\xi_\theta(0)\|_{\mathscr{E}}^2) + C\int_0^s e^{-\int_0^r (\frac{\lambda}{4}-l_\varepsilon(\mu))d\mu} \|\xi_v(r)\|_{\mathscr{E}}^2 dr \tag{4.33}$$

for some monotone function $Q(\cdot)$ and positive constant λ, which are independent of ε and v. We estimate using (4.32) and (4.33):

$$\|\xi_\theta(s)\|_{\mathscr{E}}^2 \leq e^{-\frac{\lambda}{8}s} Q(\|\xi_\theta(0)\|_{\mathscr{E}}^2) + C\|\xi_v\|_{C(\mathbb{R}_+;\mathscr{E})}^2 \leq e^{-\frac{\lambda}{8}s} Q(\|\xi_\theta(0)\|_{\mathscr{E}}^2) + C_{\|f\|^2,\mathscr{A}_s}. \tag{4.34}$$

Recalling now (4.29), we see that in fact

$$\|\xi_\theta(0)\|_{\mathscr{E}}^2 \leq C(\|\xi_v(0)\|_{\mathscr{E}^1}^2 + \|f\|^2).$$

Inserting this estimate into (4.34), we discover that

$$\|\xi_\theta(s)\|_{\mathscr{E}}^2 \leq e^{-\frac{\lambda}{8}s} Q(\|\xi_v(0)\|_{\mathscr{E}^1}^2 + \|f\|^2) + C_{\|f\|^2,\mathscr{A}_s}. \tag{4.35}$$

Recalling (2.3) and employing (1.2), we deduce that

$$\|v(s)\|_{\mathcal{H}^2}^2 \leq C(\|f\|^2 + \|\xi_\theta(s)\|^2). \tag{4.36}$$

Combining (4.35) and (4.36) and remembering (2.5), we derive the estimate (4.13). The estimate, together with the invariance of $\mathscr{A}_s$, completes the proof. □

Corollary 4.5. *Under the assumptions of Theorem 4.4, the family of processes* $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ *corresponding to Eq* (1.1) *has a compact uniform (w.r.t.*$\sigma \in \Sigma$*) attractor* $\mathscr{A}_s^\Sigma$ *which is bounded in the phase space* $\mathscr{E}^1$.

Indeed, applying Theorems 4.3 and 4.4, we can state the result on the boundedness of the strong uniform attractor $\mathscr{A}_s^\Sigma$ in $\mathscr{E}^1$.

5. Conclusions

We have investigated the dynamical behavior of a wave equation with time-dependent coefficients and quintic nonlinearity on a bounded domain, and established results on the existence and smoothness of a uniform attractor $\mathscr{A}_s^\Sigma$ in natural energy spaces $\mathscr{E}$.

There is still much work that needs to be done in this field. For example, the continuity of pullback attractors for Eq (1.1) with cubic nonlinearity g was studied by Aragão et al. in [1]. Hence, a natural question is: Is it possible to otain similar results for Eq (1.1) when the non-linearity g is assumed to have a sub-quintic or quintic rate? As we have already mentioned, the key difficulty in this problem is establishing the so-called ETS estimate (2.2). Nevertheless, we conjecture it is true, at least in the sub-quintic case. On the other hand, up to the moment, we do not know how to establish an ETS estimate in the quintic case, and this can be regarded as an open problem.

Author contributions

Feng Zhou: Conceptualisation, writing-original draft, formal analysis, project administration, writing-review and editing; Hongfang Li: Supervision and editiong; Kaixuan Zhu: Editing and project administration; Xin Li: Review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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