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Research article

Supercommuting maps on unital algebras with idempotents

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Abstract: Let $\mathcal{A}$ be a unital algebra with nontrivial idempotents. We considered $\mathcal{A}$ as a superalgebra according to Ghahramani and Zadeh's method. We provided a description of supercommuting maps on $\mathcal{A}$. As a consequence, we gave a description of supercommuting maps on matrix algebras, which is different from the result on commuting maps of matrix algebras. Finally, we proved that every supercommuting map on triangular algebras is a commuting map.

Keywords: supercommuting map; commuting map; generalized matrix algebra; matrix algebra; triangular algebra
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1. Introduction

Let $\mathcal{A}$ be an associative algebra over R, a commutative ring with unity. By $Z(\mathcal{A})$, we denote the center of $\mathcal{A}$. Set $[x, y] = xy - yx$ and $x \circ y = xy + yx$.

A linear map f on $\mathcal{A}$ is called a *commuting map* if $[f(x), x] = 0$ for all $x \in \mathcal{A}$. It is clear that $f(x) = \lambda x + \tau(x)$ is a commuting map, where $\lambda \in Z(\mathcal{A})$ and $\tau : \mathcal{A} \to Z(\mathcal{A})$, which is said to be a *proper commuting map*.

In 1991, Brešar [1] proved that a commuting map on noncommutative Lie ideals of prime rings is always proper. In 1993, Breššar [2] discussed centralizing mappings and derivations in prime rings. In the same year, Breššar [3] discussed commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings. In 2020, Jia and Xiao [4] discussed commuting maps on certain incidence algebras. Results related to commuting maps are discussed in [5–7].

It should be mentioned that the study of commuting maps on rings initiated the theory of functional identities on rings (see [8] for details).

An associative algebra $\mathcal{A}$ is said to be a *superalgebra* if $\mathcal{A}$ is the direct sum of two R-submodules $\mathcal{A}_0$ and $\mathcal{A}_1$ such that $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ (modulo 2). We call $\mathcal{A}_0$ the even part and $\mathcal{A}_1$ the odd part of $\mathcal{A}$.

Elements in $\mathcal{H} = \mathcal{A}_0 \bigcup \mathcal{A}_1$ are called homogeneous, and we write $|a| = i$ to mean $a \in \mathcal{A}_i$. For $a, b \in \mathcal{H}$, the supercommutator of a and b is defined to be

$$[a, b]_s = ab - (-1)^{|a||b|} ba.$$

It is clear that $[a, b]_s = a \circ b$ if both a and b are odd, and $[a, b]_s = [a, b]$ if either a or b is even. The definition can be extended linearly to arbitrary $a, b \in \mathcal{A}$.

Let $\mathcal{A} = \mathcal{A}_0 \bigoplus \mathcal{A}_1$ be a superalgebra. A linear map $f : \mathcal{A} \to \mathcal{A}$ is said to be *supercommuting* if

$$[f(x), x]_s = 0$$

for all $x \in \mathcal{A}$.

In 2002, Beidar, Chen, Fong, and Ke [9] discussed graded polynomial identities with an antiautomorphism. In 2003, Beidar, Bresăr, and Chebotar [10] discussed Jordan superhomomorphisms on superalgebras. In 2008, Wang [11] discussed skew-supercommuting maps in superalgebras. In 2009, Wang [12] gave a description of supercentralizing superautomorphisms on prime superalgebras. In the same year, Lee and Wang [13] gave a description of supercommuting maps of prime superalgebras. In 2017, Fan and Dai [14] investigated Super-biderivations on Lie superalgebras. In 2019, Cheng and Sun [15] discussed Super-biderivations and linear supercommuting maps on the super-BMS3 algebras.

Let $\mathcal{A}$ be a unital algebra with an idempotent $e \neq 0, 1$. Let f denote the idempotent $1 - e$. In this case, $\mathcal{A}$ can be represented in the so-called Peirce decomposition form

$$\mathcal{A} = e\mathcal{A}e + e\mathcal{A}f + f\mathcal{A}e + f\mathcal{A}f,$$

where $e\mathcal{A}e$ and $f\mathcal{A}f$ are subalgebras with unitary elements e and f, respectively, $e\mathcal{A}f$ is an $(e\mathcal{A}e, f\mathcal{A}f)$-bimodule and $f\mathcal{A}e$ is an $(f\mathcal{A}f, e\mathcal{A}e)$-bimodule, which is said to be a *generalized matrix algebra* (see [16] for details).

For brevity, we set

$$A = e\mathcal{A}e, \quad M = e\mathcal{A}f, \quad N = f\mathcal{A}e, \quad B = f\mathcal{A}f.$$

It is clear that

$$\begin{cases} AM \subseteq M, MA = 0, AN = 0, NA \subseteq N, AB = 0, BA = 0, MN \subseteq A, \\ BM = 0, MB \subseteq M, BN \subseteq N, NB = 0, NM \subseteq B, MM = 0, NN = 0. \end{cases}$$

In 2012, Benkovič and Širovnik [17] defined the following useful condition:

$$\begin{cases} a \in A, \quad aM = 0 = Na \Rightarrow a = 0; \\ b \in B, \quad Mb = 0 = bN \Rightarrow b = 0. \end{cases} \tag{1.1}$$

Some examples of unital algebras with nontrivial idempotents having the property (1.1) are triangular algebras, matrix algebras, and prime (and hence in particular simple) algebras with nontrivial idempotents (see [17]).

In 2010, Xiao and Wei [16] initiated the study of commuting mappings of generalized matrix algebras, which generalized a typical result on commuting maps of triangular algebras (see [18] for details). In 2019, Li, Wei, and Fošner [19] discussed k-commuting mappings of generalized matrix algebras, which generalized a result on k-commuting maps of triangular algebras (see [20] for details). In 2002, Du and Wang [21] gave a description of Lie derivations of generalized matrix algebras. In 2018, Benkovič [22] discussed generalized Lie derivations of unital algebras with idempotents. Additional results on mappings of generalized matrix algebras can be found in [23–25].

In 2024, Ghahramani and Zadeh [26] considered $\mathcal{A}$ as a superalgebra by:

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1,$$

where

$$\mathcal{A}_0 = A + B \quad \text{and} \quad \mathcal{A}_1 = M + N.$$

They determined the class of generalized matrix algebras for which every Lie superderivation is proper (see [26, Theorem 5.1]. As a consequence, they gave some descriptions of Lie superderivations on both matrix algebras and triangular algebras (see [26] for details).

Recently, Chen [27] discussed Jordan superderivations of unital algebras with idempotents. As a consequence, she gave some descriptions of Jordan superderivations of matrix algebras and triangular algebras.

In the present paper, we give a description of supercommuting maps of unital algebras with idempotents. As a consequence, we give some descriptions of supercommuting maps of matrix algebras and triangular algebras.

We organize the paper as follows: In Section 2, we give preliminaries and the definition of proper supercommuting maps. In Section 3, we give a description of supercommuting maps of degree 0 on unital algebras. In Section 4, we give a description of supercommuting maps of degree 1 on unital algebras. In Section 5, we give the main result of the paper. In Section 6, we give a description of supercommuting maps on matrix algebras. As a consequence, we prove that every supercommuting map on matrix algebras over a 2-torsion free unital algebra is supercentral. In the last section, we prove that every supercommuting map on triangular algebras is a commuting map.

2. Preliminaries

Let $\mathcal{A} = \mathcal{A}_0 \bigoplus \mathcal{A}_1$ be a superalgebra. The supercenter of $\mathcal{A}$ is the set

$$Z(\mathcal{A})_s = \{a \in \mathcal{A} \mid [a, x]_s = 0 \quad \text{for all } x \in \mathcal{A}\}.$$

We set

$$Z(\mathcal{A})_0 = Z(\mathcal{A}) \cap \mathcal{A}_0 \quad \text{and} \quad Z(\mathcal{A})_1 = Z(\mathcal{A}) \cap \mathcal{A}_1.$$

It is easy to check that $Z(\mathcal{A}) = Z(\mathcal{A})_0 \bigoplus Z(\mathcal{A})_1$ is a graded subalgebra of $\mathcal{A}$ (see [4, Section 2] for details). It is clear that $Z(\mathcal{A})_0 \subseteq Z(\mathcal{A})_s$.

We begin with the following definition.

Definition 2.1. *Let $\mathcal{A} = \mathcal{A}_0 \bigoplus \mathcal{A}_1$ be a superalgebra. We call a linear map $f : \mathcal{A} \to \mathcal{A}$ a proper supercommuting map if*

$$f(x) = \lambda x + \tau(x)$$

for all $x \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})_0$ with $2\lambda\mathcal{A}_1^2 = \{0\}$, and $\tau : \mathcal{A} \to Z(\mathcal{A})_s$ is a linear map. In particular, if $f(x) \in Z(\mathcal{A})_s$ for all $x \in \mathcal{A}$, we call f a supercentral map.

The following result shows that a proper supercommuting map is a supercommuting map.

Lemma 2.1. *Let $\mathcal{A} = \mathcal{A}_0 \bigoplus \mathcal{A}_1$ be a superalgebra. Then*

$$f(x) = \lambda x + \tau(x)$$

for all $x \in \mathcal{A}$, is a supercommuting map of $\mathcal{A}$, where $\lambda \in Z(\mathcal{A})_0$ with $2\lambda\mathcal{A}_1^2 = \{0\}$, and $\tau : \mathcal{A} \to Z(\mathcal{A})_s$ is a linear map.

Proof. For any $x = x_0 + x_1 \in \mathcal{A}$ we get

$$\begin{aligned}
[f(x), x]_s &= [\lambda x + \tau(x), x]_s \\
&= \lambda[x, x]_s \\
&= \lambda[x_0 + x_1, x_0 + x_1]_s \\
&= \lambda[x_0, x_0] + \lambda[x_0, x_1] + \lambda[x_1, x_0] + \lambda[x_1, x_1]_s \\
&= \lambda[x_1, x_1]_s \\
&= 2\lambda x_1^2 \\
&= 0.
\end{aligned}$$

We obtain that f is a supercommuting map. □

Definition 2.2. *Let $\mathcal{A} = \mathcal{A}_0 \bigoplus \mathcal{A}_1$ be a superalgebra. A supercommuting map f on $\mathcal{A}$ is said to be a supercommuting map of degree 0 if $f(\mathcal{A}_0) \subseteq \mathcal{A}_0$ and $f(\mathcal{A}_1) \subseteq \mathcal{A}_1$. A supercommuting map f of $\mathcal{A}$ is said to be* a supercommuting map of degree 1 *if $f(\mathcal{A}_0) \subseteq \mathcal{A}_1$ and $f(\mathcal{A}_1) \subseteq \mathcal{A}_0$.*

The following result shows that a supercommuting map is the sum of a supercommuting map of degree 0 and a supercommuting map of degree 1.

Lemma 2.2. *Let $\mathcal{A} = \mathcal{A}_0 \bigoplus \mathcal{A}_1$ be a superalgebra. Let f be a supercommuting map of $\mathcal{A}$. Then*

$$f = f_0 + f_1$$

where f_0 is a supercommuting map of degree 0 on $\mathcal{A}$ and f_1 is a supercommuting map of degree 1 on $\mathcal{A}$.

Proof. For $i = 0$ or 1, let π_i be the canonical projection of $\mathcal{A}$. We set

$$f_0 = \pi_0 f \pi_0 + \pi_1 f \pi_1 \quad \text{and} \quad f_1 = \pi_0 f \pi_1 + \pi_1 f \pi_0.$$

It is easy to check that f_i is a linear map of $\mathcal{A}$ and $f = f_0 + f_1$, where $i = 0, 1$. Moreover, $f_0(\mathcal{A}_0) \subseteq \mathcal{A}_0$, $f_0(\mathcal{A}_1) \subseteq \mathcal{A}_1$, $f_1(\mathcal{A}_0) \subseteq \mathcal{A}_1$, and $f_1(\mathcal{A}_1) \subseteq \mathcal{A}_0$. We now claim that both f_0 and f_1 are supercommuting maps on $\mathcal{A}$.

For $i = 0, 1$, and any $x = x_0 + x_1 \in \mathcal{A}$, we have

$$\begin{aligned}
0 &= [f(x_i), x_i]_s \\
&= [f_0(x_i) + f_1(x_i), x_i]_s \\
&= [f_0(x_i), x_i]_s + [f_1(x_i), x_i]_s.
\end{aligned}$$

Since $[f_0(x_i), x_i]_s$ is even and $[f_1(x_i), x_i]_s$ is odd, we obtain that

$$[f_0(x_i), x_i]_s = 0 \quad \text{and} \quad [f_1(x_i), x_i]_s = 0. \tag{2.1}$$

It follows from (2.1) and the linearity of f_i, where $i = 0, 1$, that

$$[f_0(x_1), x_0]_s + [f_0(x_0), x_1]_s = 0 \tag{2.2}$$

and

$$[f_1(x_1), x_0]_s + [f_1(x_0), x_1]_s = 0. \tag{2.3}$$

For $i = 0, 1$, we get from (2.1), (2.2), and (2.3) that

$$\begin{aligned} [f_i(x), x]_s &= [f_i(x_0) + f_i(x_1), x_0 + x_1]_s \\ &= [f_i(x_0), x_0]_s + [f_i(x_0), x_1]_s + [f_i(x_1), x_0]_s + [f_i(x_1), x_1]_s \\ &= 0. \end{aligned}$$

This implies that f_i is a supercommuting map of degree i. The proof of the result is complete. □

Form now on we always assume that $\mathcal{A}$ is a unital algebra with nontrivial idempotents.

The following result is essentially the same as [26, Lemma 2.1].

Lemma 2.3.

$$Z(\mathcal{A}) = Z(\mathcal{A})_s = \{X \in Z(\mathcal{A}_0) \mid [X, \mathcal{A}_1] = 0\}.$$

We define two natural projection $\pi_A : \mathcal{A} \to A$ and $\pi_B : \mathcal{A} \to B$ by

$$\pi_A(a + m + n + b) = a \quad \text{and} \quad \pi_B(a + m + n + b) = b.$$

The following result is essentially the same as [26, Lemma 2.3].

Lemma 2.4. *Let $\mathcal{A}$ be a unital algebra with nontrivial idempotents having the property (1.1). Then*

$$Z(\mathcal{A}) = Z(\mathcal{A})_s = \{X \in \mathcal{A}_0 \mid [X, \mathcal{A}_1] = 0\}.$$

Furthermore, $\pi_A(Z(\mathcal{A})) \subseteq Z(A)$, $\pi_B(Z(\mathcal{A})) \subseteq Z(B)$, and there exists a unique isomorphism φ from $\pi_A(Z(\mathcal{A}))$ to $\pi_B(Z(\mathcal{A}))$ such that $am = m\varphi(a)$, $mb = \varphi^{-1}(b)m$, $na = \varphi(a)n$, and $bn = n\varphi^{-1}(b)$ for all $m \in M, n \in N$.

3. Supercommuting maps of degree 0

We begin with the structure of supercommuting maps of degree 0.

Lemma 3.1. *Let f_0 be a supercommuting map of degree 0 on $\mathcal{A}$. Then*

$$f_0(a + m + n + b) = \alpha_1(a) + \alpha_4(b) + \alpha_1(1)m - m\beta_1(1) + n\alpha_1(1) - \beta_1(1)n + \beta_1(a) + \beta_4(b)$$

for all $a \in A$, $m \in M$, $n \in N$, and $b \in B$, where $\alpha_1 : A \to A$, $\alpha_4 : B \to Z(A)$, $\beta_1 : A \to Z(B)$, and $\beta_4 : B \to B$ are linear maps satisfying the following conditions:

(i) α_1 *and* β_4 *are commuting mappings of A and B, respectively. In particular,* $\alpha_1(1) \in Z(A), \beta_4(1) \in Z(B)$*;*

(ii) $\alpha_1(a)m - m\beta_1(a) = a(\alpha_1(1)m - m\beta_1(1)), \beta_1(a)n - n\alpha_1(a) = (n\alpha_1(1) - \beta_1(1)n)a$*;*

(iii) $\alpha_4(b)m - m\beta_4(b) = (m\beta_1(1) - \alpha_1(1)m)b, \beta_4(b)n - n\alpha_4(b) = b(n\alpha_1(1) - \beta_1(1)n)$*;*

(iv) $2\alpha_1(1)mn = 2m\beta_1(1)n$ *and* $2n\alpha_1(1)m = 2\beta_1(1)nm$*.*

Proof. Since $f_0(\mathcal{A}_0) \subseteq \mathcal{A}_0$ and $f_0(\mathcal{A}_1) \subseteq \mathcal{A}_1$, we can write

$$f_0(a + m + n + b) = \alpha_1(a) + \alpha_4(b) + \mu_2(m) + \mu_3(n) + \nu_2(m) + \nu_3(n) + \beta_1(a) + \beta_4(b) \tag{3.1}$$

for all $a \in A$, $m \in M$, $n \in N$, and $b \in B$, where $\alpha_1 : A \to A$, $\alpha_4 : B \to A$, $\mu_2 : M \to M$, $\mu_3 : N \to M$, $\nu_2 : M \to N$, $\nu_3 : N \to N$, $\beta_1 : A \to B$, and $\beta_4 : B \to B$ are linear maps.

Linearizing $[f_0(X), X]_s = 0$ leads to

$$[f_0(X), Y]_s + [f_0(Y), X]_s = 0 \tag{3.2}$$

for all $X, Y \in \mathcal{A}$. For any $m \in M$, taking $X = 1_A$ and $Y = m$ in (3.2) yields

$$[f_0(1_A), m] + [f_0(m), 1_A] = 0.$$

That is

$$[\alpha_1(1_A) + \beta_1(1_A), m] + [\mu_2(m) + \nu_2(m), 1_A] = 0.$$

This implies that $\nu_2(m) = 0$ and $\mu_2(m) = \alpha_1(1)m - m\beta_1(1)$ for all $m \in M$. Similarly, if we choose $X = 1_A$ and $Y = n$ in (3.2), then we arrive at $\mu_3(n) = 0$ and $\nu_3(n) = n\alpha_1(1) - \beta_1(1)n$ for all $n \in N$. Therefore (3.1) becomes

$$\begin{aligned} &f_0(a + m + n + b) \\ &\quad = \alpha_1(a) + \alpha_4(b) + \alpha_1(1)m - m\beta_1(1) + n\alpha_1(1) - \beta_1(1)n + \beta_1(a) + \beta_4(b) \end{aligned} \tag{3.3}$$

for all $a \in A$, $m \in M$, $n \in N$, and $b \in B$. For any $a \in A$ and $b \in B$, taking $X = a$ and $Y = b$ into (3.3) yields

$$[f_0(a), b] + [f_0(b), a] = 0.$$

That is

$$[\alpha_1(a) + \beta_1(a), b] + [\alpha_4(b) + \beta_4(b), a] = 0.$$

Then $[\alpha_4(b), a] = 0$ and $[\beta_1(a), b] = 0$ for all $a \in A$ and $b \in B$. This implies that $\alpha_4(B) \subseteq Z(A)$ and $\beta_1(A) \subseteq Z(B)$. By (3.3) we obtain

$$0 = [f_0(a + b), a + b] = [\alpha_1(a), a] + [\beta_4(b), b].$$

Then $[\alpha_1(a), a] = 0$ for all $a \in A$ and $[\beta_4(b), b] = 0$ for all $b \in B$. This implies that α_1 and β_4 are commuting mappings of A and B, respectively. It is easy to check that $\alpha_1(1) \in Z(A)$ and $\beta_4(1) \in Z(B)$. This proves the statement (i).

By (3.3) we get

$$[f_0(a), m + n] = \alpha_1(a)m - m\beta_1(a) + \beta_1(a)n - n\alpha_1(a)$$

and

$$[f_0(m+n), a] = a(m\beta_1(1) - \alpha_1(1)m) + (\beta_1(1)n - n\alpha_1(1)a.$$

Note that

$$[f_0(a), m+n] + [f_0(m+n), a] = 0.$$

The above three relations imply that $\alpha_1(a)m - m\beta_1(a) = a(\alpha_1(1)m - m\beta_1(1))$ and $\beta_1(a)n - n\alpha_1(a) = (n\alpha_1(1) - \beta_1(1)n)a$ for all $a \in A$, $m \in M$, and $n \in N$. This proves the statement (ii). Similarly, taking $X = b$ and $Y = m + n$ in (3.2) we can obtain that $\alpha_4(b)m - m\beta_4(b) = (m\beta_1(1) - \alpha_1(1)m)b$ and $\beta_4(b)n - n\alpha_4(b) = b(n\alpha_1(1) - \beta_1(1)n)$ for all $b \in B$, $m \in M$, and $n \in N$. This proves the statement (iii).

Since $[f_0(m+n), m+n]_s = 0$ for all $m \in M$, $n \in N$, we get from (3.3) that

$$[a(m\beta_1(1) - \alpha_1(1)m) + a_1(1)n - n\alpha_1(1))a, m+n]_s = 0.$$

That is

$$\begin{aligned}(a(m\beta_1(1) - \alpha_1(1)m) + a_1(1)n - n\alpha_1(1))a)(m+n) \\ + (m+n)(a(m\beta_1(1) - \alpha_1(1)m) + a_1(1)n - n\alpha_1(1))a) = 0\end{aligned}$$

for all $m \in M$ and $n \in N$. This implies that

$$2\alpha_1(1)mn = 2m\beta_1(1)n \quad \text{and} \quad 2n\alpha_1(1)m = 2\beta_1(1)nm$$

for all $m \in M$, $n \in N$. This proves the statement (iv). We complete the proof of the result. □

The idea of proving the following result is taken from [18, Lemma 1].

Lemma 3.2. *Let $\mathcal{A}$ be a unital algebra with nontrivial idempotents having the property (1.1). Let f_0 be a supercommuting map of degree 0 on $\mathcal{A}$. With notations as above, then $\beta_1^{-1}(\pi_B(Z(\mathcal{A})))$ and $\alpha_4^{-1}(\pi_A(Z(\mathcal{A})))$ are ideals of A and B, respectively. Furthermore, $[A, A] \subseteq \beta_1^{-1}(\pi_B(Z(\mathcal{A})))$ and $[B, B] \subseteq \alpha_4^{-1}(\pi_A(Z(\mathcal{A})))$.*

Proof. We prove the part of the statement related to β_1. The part related to α_4 can be proved analogously. For any $a, a' \in A$, $m \in M$, and $n \in N$ we get from Lemma 3.1(ii) that

$$a'a(\alpha_1(1)m - m\beta_1(1)) = \alpha_1(a'a)m - m\beta_1(a'a) \tag{3.4}$$
$$a'a(\alpha_1(1)m - m\beta_1(1)) = a'(\alpha_1(a)m - m\beta_1(a)) \tag{3.5}$$
$$aa'(\alpha_1(1)m - m\beta_1(1)) = \alpha_1(aa')m - m\beta_1(aa') \tag{3.6}$$
$$a(\alpha_1(1)a'm - a'm\beta_1(1)) = \alpha_1(a)a'm - a'm\beta_1(a). \tag{3.7}$$

From (3.4) and (3.5), we have

$$\alpha_1(a'a)m - m\beta_1(a'a) - a'(\alpha_1(a)m - m\beta_1(a)) = 0, \tag{3.8}$$

and from (3.6) and (3.7), we have

$$\alpha_1(aa')m - m\beta_1(aa') - \alpha_1(a)a'm + a'm\beta_1(a) = 0. \tag{3.9}$$

Taking the difference of (3.8) and (3.9), we have

$$(\alpha_1([a,a']) - [\alpha_1(a), a'])m = m\beta_1([a,a']). \tag{3.10}$$

For any $a, a' \in A$ and $n \in N$, we get from Lemma 3.1(ii) that

$$\begin{aligned}
(n\alpha_1(1) - \beta_1(1)n)a'a &= \beta_1(a'a)n - n\alpha_1(a'a) & (3.11)\\
(n\alpha_1(1) - \beta_1(1)n)a'a &= (\beta_1(a')n - n\alpha_1(a'))a & (3.12)\\
(n\alpha_1(1) - \beta_1(1)n)aa' &= \beta_1(aa')n - n\alpha_1(aa') & (3.13)\\
(na\alpha_1(1) - \beta_1(1)na)a' &= \beta_1(a')na - na\alpha_1(a'). & (3.14)
\end{aligned}$$

From (3.11) and (3.12), we have

$$\beta_1(a'a)n - n\alpha_1(a'a) - \beta_1(a')na + n\alpha_1(a')a = 0, \tag{3.15}$$

and from (3.13) and (3.14), we have

$$\beta_1(aa')n - n\alpha_1(aa') - \beta_1(a')na + na\alpha_1(a') = 0. \tag{3.16}$$

Taking the difference of (3.15) and (3.16), we have

$$\beta_1([a,a'])n = n(\alpha_1(([a,a']) - [a, \alpha_1(a')]). \tag{3.17}$$

Since α_1 is commuting map of A, we get that $[a, \alpha_1(a')] = [\alpha_1(a), a']$. Thus, we get from (3.17) that

$$\beta_1([a,a'])n = n(\alpha_1(([a,a']) - [\alpha_1(a), a']). \tag{3.18}$$

In view of Lemma 2.4 we get from both (3.10) and (3.18) that $\beta_1([a,a']) \in \pi_B(Z(\mathcal{A}))$. Hence $[A,A] \subseteq \beta_1^{-1}(\pi_B(Z(\mathcal{A})))$.

Suppose that $a \in \beta_1^{-1}(\pi_B(Z(\mathcal{A})))$. From both (3.8) and (3.15) we have

$$\begin{aligned}
m\beta_1(a'a) &= (\alpha_1(a'a) - a'\alpha_1(a) + a'\varphi^{-1}(\beta_1(a)))m;\\
\beta_1(a'a)n &= n(\alpha_1(a'a) - a'\alpha_1(a) + a'\varphi^{-1}(\beta_1(a))).
\end{aligned}$$

By Lemma 2.4 we get that $\beta_1(a'a) \in \pi_B(Z(\mathcal{A})))$. Hence $a'a \in \beta_1^{-1}(\pi_B(Z(\mathcal{A})))$. Similarly, from both (3.9) and (3.16) we have

$$\begin{aligned}
m\beta_1(aa') &= (\alpha_1(aa') - \alpha_1(a)a' + a'\varphi^{-1}(\beta_1(a)))m;\\
\beta_1(aa')n &= n(\alpha_1(aa') - \alpha_1(a)a' + a'\varphi^{-1}(\beta_1(a))).
\end{aligned}$$

By Lemma 2.4 we get that $\beta_1(aa') \in \pi_B(Z(\mathcal{A})))$. Hence $aa' \in \beta_1^{-1}(\pi_B(Z(\mathcal{A})))$. As a result, $\beta_1^{-1}(\pi_B(Z(\mathcal{A})))$ is an ideal of A containing $[A,A]$. This proves the result. □

Now we obtain necessary and sufficient conditions for a supercommuting map of degree 0 on $\mathcal{A}$ to be proper. The idea of proving the following result is taken from [18, Theorem 1].

Lemma 3.3. *Let $\mathcal{A}$ be a unital algebra with nontrivial idempotents having the property (1.1). Let f_0 be a supercommuting map of degree 0 on $\mathcal{A}$ such that*

$$f_0(a+m+n+b) = \alpha_1(a)+\alpha_4(b)+\alpha_1(1)m - m\beta_1(1)+n\alpha_1(1)-\beta_1(1)n+\beta_1(a)+\beta_4(b).$$

Then, the following three conditions are equivalent:

(i) f_0 *is proper;*
(ii) $\beta_1(A)\subseteq \pi_B(Z(\mathcal{A}))$ *and* $\alpha_4(B)\subseteq \pi_A(Z(\mathcal{A}))$*;*
(iii) $\alpha_1(1)\in \pi_A(Z(\mathcal{A}))$ *and* $\beta_1(1)\in\pi_B(Z(\mathcal{A}))$.

Proof. (ii)⇒ (iii). $\beta_1(1)\in\beta_1(A)\subseteq\pi_B(Z(\mathcal{A}))$. Taking $b=1$ in Lemma 3.1(iii), we get

$$\alpha_1(1)m = m(\beta_4(1)+\beta_1(1)-\varphi(\alpha_4(1)))$$
$$n\alpha_1(1) = (\beta_4(1)+\beta_1(1)-\varphi(\alpha_4(1)))n$$

for all $m\in M$ and $n\in N$. By Lemma 2.4 we get that $\alpha_1(1)\in\pi_A(Z(\mathcal{A}))$.

(iii) ⇒ (ii). Since $\beta_1(1)\in\pi_B(Z(\mathcal{A}))$, the ideal $\beta_1^{-1}(\pi_B(Z(\mathcal{A})))$ of A contains 1. Hence $A = \beta_1^{-1}(\pi_B(Z(\mathcal{A})))$. We have that $\beta_1(A)\subseteq\pi_B(Z(\mathcal{A}))$. By Lemma 3.1(iii), we have $\alpha_4(b)m - m\beta_4(b) = (m\beta_1(1)-\alpha_1(1)m)b$, which implies

$$\alpha_4(b)m = m(\beta_4(b)+\beta_1(1)b-\varphi(\alpha_1(1))b) \tag{3.19}$$

for all $m\in M$ and $b\in B$. By Lemma 3.1(iii) again, we have $\beta_4(b)n - n\alpha_4(b) = b(n\alpha_1(1)-\beta_1(1)n)$, which implies

$$n\alpha_4(b) = (\beta_4(b)+\beta_1(1)b-\varphi(\alpha_1(1))b)n \tag{3.20}$$

for all $n\in N$ and $b\in B$. In view of Lemma 2.4 we get from both (3.19) and (3.20) that $\alpha_4(b)\in\pi_A(Z(\mathcal{A}))$ for all $b\in B$.

(iii) ⇒ (i). We set

$$\tau(X) = f_0(X)-\lambda X$$

for all $X\in\mathcal{A}$, where $\lambda = \alpha_1(1)-\varphi^{-1}(\beta_1(1))+\varphi(\alpha_1(1))-\beta_1(1)\in Z(\mathcal{A})$. We claim that $\tau(\mathcal{A})\subseteq Z(\mathcal{A})$. Indeed, we have

$$\begin{aligned}\tau(a+m+n+b) &= f_0(a+m+n+b)-\lambda(a+m+n+b)\\ &= (\alpha_1(a)+\alpha_4(b)+\alpha_1(1)m-m\beta_1(1)+n\alpha_1(1)-\beta_1(1)n+\beta_1(a)+\beta_4(b)\\ &\quad -(\alpha_1(1)-\varphi^{-1}(\beta_1(1))+\varphi(\alpha_1(1))-\beta_1(1)))(a+m+n+b)\\ &= \alpha_1(a)-(\alpha_1(1)-\varphi^{-1}(\beta_1(1)))+\beta_1(a)+\alpha_4(b)+\beta_4(b)-(\varphi(\alpha_1(1))-\beta_1(1))b.\end{aligned}$$

By Lemma 3.1(ii) we have

$$(\alpha_1(a)-(\alpha_1(1)-\varphi^{-1}(\beta_1(1)))a)m = m\beta_1(a)$$
$$n(\alpha_1(a)-(\alpha_1(1)-\varphi^{-1}(\beta_1(1)))a) = \beta_1(a)n$$

for all $a\in A$, $m\in M$, and $n\in N$. By Lemma 2.4 we get that

$$\alpha_1(a)-(\alpha_1(1)-\varphi^{-1}(\beta_1(1)))a+\beta_1(a)\in Z(\mathcal{A})$$

for all $a \in A$. Similarly, we get from Lemma 3.1(iii) that

$$\alpha_4(b)m = m(\beta_4(b) - (\varphi(\alpha_1(1)) - \beta_1(1))b);$$
$$n\alpha_4(b) = (\beta_4(b) - (\varphi(\alpha_1(1)) - \beta_1(1))b)n$$

for all $m \in M$, $n \in N$, and $b \in B$. By Lemma 2.4 we get that

$$\alpha_4(b) + \beta_4(b) - (\varphi(\alpha_1(1)) - \beta_1(1))b \in Z(\mathcal{A})$$

for all $b \in B$. We obtain that $\tau(a+m+n+b) \in Z(\mathcal{A})$ for all $a \in A$, $m \in M$, $n \in N$, and $b \in B$ as desired. We next claim that $2\lambda\mathcal{A}_1^2 = \{0\}$.

For any $m \in M$ and $n \in N$, by Lemma 3.1(iv) we have

$$\begin{aligned} 2\pi_A(\lambda)mn &= 2(\alpha_1(1) - \varphi^{-1}(\beta_1(1)))mn \\ &= 2\alpha_1(1)mn - 2m\beta_1(1)n = 0 \end{aligned}$$

and

$$\begin{aligned} 2\pi_B(\lambda)nm &= 2(\varphi(\alpha_1(1)) - \beta_1(1))nm \\ &= 2n\alpha_1(1)m - 2\beta_1(1)nm = 0. \end{aligned}$$

It follows that

$$\begin{aligned} 2\lambda(m+n)(m'+n') &= 2\lambda(mn' + nm') \\ &= 2\pi_A(\lambda)mn' + 2\pi_B(\lambda)nm' \\ &= 0 \end{aligned}$$

for all $m, m' \in M$, $n, n' \in N$. This implies that $2\lambda\mathcal{A}_1^2 = \{0\}$.

(i) $\Rightarrow$ (iii). Suppose that $f_0(X) = \lambda X + \tau(X)$ for all $X \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})$ with $2\lambda\mathcal{A}_1^2 = \{0\}$ and $\tau : \mathcal{A} \to Z(\mathcal{A})$ is a linear map. For any $m \in M$ and $n \in N$, we have

$$f_0(m+n) = (\pi_A(\lambda) + \pi_B(\lambda))(m+n) + \tau(m+n).$$

By Lemma 3.1 we get that

$$\alpha_1(1)m - m\beta_1(1) + n\alpha_1(1) - \beta_1(1)n = \pi_A(\lambda)m + \pi_B(\lambda)n + \tau(m+n).$$

We get from the last relation that

$$\alpha_1(1)m - m\beta_1(1) = \pi_A(\lambda)m;$$
$$n\alpha_1(1) - \beta_1(1)n = \pi_B(\lambda)n.$$

This implies that

$$(\alpha_1(1) - \pi_A(\lambda))m = m\beta_1(1);$$
$$n(\alpha_1(1) - \pi_A(\lambda)) = \beta_1(1)n.$$

By Lemma 2.4 we get that $\alpha_1(1) - \pi_A(\lambda) \in \pi_A(Z(\mathcal{A}))$ and $\beta_1(1) \in \pi_B(Z(\mathcal{A}))$. Hence, $\alpha_1(1) \in \pi_A(Z(\mathcal{A}))$ and $\beta_1(1) \in \pi_B(Z(\mathcal{A}))$ as desired. The proof of the result is complete. $\square$

We now give sufficient conditions for every supercommuting map of degree 0 on $\mathcal{A}$ to be proper. The idea of proving the following result is taken from [18, Theorem 2].

Theorem 3.1. *Let $\mathcal{A}$ be a unital algebra with nontrivial idempotents having the property (1.1). Suppose that the following two conditions are satisfied:*

(i) $Z(B) = \pi_B(Z(\mathcal{A}))$, *or* $A = [A, A]$*;*
(ii) $Z(A) = \pi_A(Z(\mathcal{A}))$, *or* $B = [B, B]$.

Then every supercommuting map of degree 0 *on $\mathcal{A}$ is proper.*

Proof. Let f_0 be a supercommuting mapping of degree 0 on $\mathcal{A}$. With notations as above, we note that $\alpha_4(B) \subseteq Z(A)$ and $\beta_1(A) \subseteq Z(B)$. By the condition (i) we note that either $Z(B) = \pi_B(Z(\mathcal{A}))$ or $A = [A, A]$. Suppose first that $Z(B) = \pi_B(Z(\mathcal{A}))$. We get that $\beta_1(A) \subseteq \pi_B(Z(\mathcal{A}))$. Suppose next that $A = [A, A]$. In view of Lemma 3.2 we note that $[A, A] \subseteq \beta_1^{-1}(\pi_B(Z(\mathcal{A})))$. This implies that $\beta_1(A) \subseteq \pi_B(Z(\mathcal{A}))$.

By the condition (ii) we note that either $Z(A) = \pi_A(Z(\mathcal{A}))$ or $B = [B, B]$. Suppose first that $Z(A) = \pi_A(Z(\mathcal{A}))$. We get that $\alpha_4(B) \subseteq \pi_A(Z(\mathcal{A}))$. Suppose next that $B = [B, B]$. In view of Lemma 3.2 we note that $[B, B] \subseteq \alpha_4^{-1}(\pi_A(Z(\mathcal{A})))$. We obtain that $\alpha_4(B) \subseteq \pi_A(Z(\mathcal{A}))$. By Lemma 3.3 we obtain that f_0 is proper. This proves the result. ◻

4. Supercommuting maps of degree 1

We first give the structure of supercommuting map of degree 1 on $\mathcal{A}$.

Lemma 4.1. *Let $\mathcal{A}$ be a unital algebra with nontrivial idempotents. Let f_1 be a commuting mapping of degree* 1 *on $\mathcal{A}$. Then f_1 is of the form*

$$f_1(a + m + n + b) = \alpha_2(m) + \alpha_3(n) + \beta_2(m) + \beta_3(n)$$

for all $a \in A$, $m \in M$, $n \in N$, and $b \in B$, where $\alpha_2 : M \to Z(A)$, $\alpha_3 : N \to Z(A)$, $\beta_2 : M \to Z(B)$, and $\beta_3 : N \to Z(B)$ are linear maps satisfying the following conditions:

(i) $(\alpha_2(m) + \alpha_3(n))m = m(\beta_2(m) + \beta_3(n))$*;*
(ii) $n(\alpha_2(m) + \alpha_3(n)) = (\beta_2(m) + \beta_3(n))n$

for all $m \in M$, $n \in N$.

Proof. Note that $f_1(\mathcal{A}_0) \subseteq \mathcal{A}_1$ and $f_1(\mathcal{A}_1) \subseteq \mathcal{A}_0$. So f_1 is of the form

$$f_1(a + m + n + b) = \alpha_2(m) + \alpha_3(n) + \mu_1(a) + \mu_4(b) + \nu_1(a) + \nu_4(b) + \beta_2(m) + \beta_3(m) \tag{4.1}$$

for all $a \in A$, $m \in M$, $n \in N$, and $b \in B$, where $\alpha_2 : M \to A$, $\alpha_3 : N \to A$, $\mu_1 : A \to M$, $\mu_4 : B \to M$, $\nu_1 : A \to N$, $\nu_4 : B \to N$, $\beta_2 : M \to B$, and $\beta_3 : N \to B$ are linear maps. Note that

$$[f_1(1_A), 1_A] = 0.$$

We get that

$$[\mu_1(1) + \nu_1(1), 1_A] = 0.$$

This implies that $\mu_1(1) = 0 = \nu_1(1)$. Linearizing $[f_1(X), X]_s = 0$ leads to

$$[f_1(X), Y]_s + [f_1(Y), X]_s = 0 \tag{4.2}$$

for all $X, Y \in \mathcal{A}$. For any $a \in A$ and $b \in B$, taking $X = a + b$ and $Y = 1_A$ into (4.2) yields

$$[f_1(a + b), 1_A] + [f_1(1_A), a + b] = 0.$$

That is

$$[\mu_1(a) + \mu_4(b) + \nu_1(a) + \nu_4(b), 1_A] + [0, a + b] = 0.$$

This implies that $\mu_1(a)+\mu_4(b) = 0$ and $\nu_1(a)+\nu_4(b) = 0$ for all $a \in A$ and $b \in B$. We get that $\mu_1 = \mu_4 = 0$ and $\nu_1 = \nu_4 = 0$. Thus, the relation (4.1) becomes

$$f_1(a + m + n + b) = \alpha_2(m) + \alpha_3(n) + \beta_2(m) + \beta_3(m) \tag{4.3}$$

for all $a \in A$, $m \in M$, $n \in N$, and $b \in B$. For any $a \in A$ and $b \in B$, $m \in M$, and $n \in N$, taking $X = a + b$ and $Y = m + n$ in (4.3) yields

$$[f_1(a + b), m + n] + [f_1(m + n), a + b] = 0.$$

It follows from (4.3) that

$$[\alpha_2(m) + \alpha_3(n) + \beta_2(m) + \beta_3(n), a + b] = 0.$$

This implies that $[\alpha_2(m) + \alpha_3(n), a] = 0$ and $[\beta_2(m) + \beta_3(n), b] = 0$ for all $a \in A$ and $b \in B$, $m \in M$, and $n \in N$. We get that $[\alpha_2(m), a] = 0$, $[\alpha_3(n), a] = 0$, $[\beta_2(m), b] = 0$, and $[\beta_3(n), b] = 0$ for all $a \in A$ and $b \in B$, $m \in M$, and $n \in N$. That is, $\alpha_2(m), \alpha_3(n) \in Z(A)$ and $\beta_2(m), \beta_3(n) \in Z(B)$ for all $m \in M$ and $n \in N$.

For any $m \in M$ and $n \in N$, we have that

$$[f_1(m + n), m + n] = 0.$$

It follows from (4.3) that

$$[\alpha_2(m) + \alpha_3(n) + \beta_2(m) + \beta_3(n), m + n] = 0.$$

This implies that $(\alpha_2(m) + \alpha_3(n))m = m(\beta_2(m) + \beta_3(n))$ and $n(\alpha_2(m) + \alpha_3(n)) = (\beta_2(m) + \beta_3(n))n$ for all $m \in M$ and $n \in N$. We complete the proof of the result. □

We now give a sufficient condition for every supercommuting map of degree 1 on $\mathcal{A}$ to be supercentral.

Theorem 4.1. *Let f_1 be a supercommuting mapping of degree 1 on $\mathcal{A}$. Suppose that there exists $Y_1 \in \mathcal{A}_1$ such that*

$$Z(\mathcal{A}) = \{X \in Z(\mathcal{A}_0) \mid [X, Y_1] = 0\}.$$

Then $f_1(\mathcal{A}) \subseteq Z(\mathcal{A})$.

Proof. We set

$$Y_1 = m_0 + n_0.$$

By Lemma 4.1 we note that

$$f_1(a + m + n + b) = \alpha_2(m) + \alpha_3(n) + \beta_2(m) + \beta_3(m) \tag{4.4}$$

for all $a \in A$, $m \in M$, $n \in N$, and $b \in B$, where $\alpha_2 : M \to Z(A)$, $\alpha_3 : N \to Z(A)$, $\beta_2 : M \to Z(B)$, and $\beta_3 : N \to Z(B)$. Moreover,

$$(\alpha_2(m) + \alpha_3(n))m = m(\beta_2(m) + \beta_3(n))$$
$$n(\alpha_2(m) + \alpha_3(n)) = (\beta_2(m) + \beta_3(n))n$$

for all $m \in M$ and $n \in N$. We set

$$\delta(m, n) = \alpha_2(m) + \alpha_3(n) + \beta_2(m) + \beta_3(n)$$

for all $m \in M$ and $n \in N$. It is easy to check that

$$Z(\mathcal{A}_0) = Z(A) + Z(B).$$

It follows that $\delta(m, n) \in Z(\mathcal{A}_0)$ for all $m \in M$, $n \in N$. Since

$$[f_1(a + m + n + b), a + m + n + b]_s = 0$$

for all $a \in A$, $m \in M$, $n \in N$, and $b \in B$ we get from (4.4) that

$$[\delta(m, n), m + n] = 0$$

for all $m \in M$ and $n \in N$. In particular, $[\delta(m_0, n_0), Y_1] = 0$. By assumption we have that $\delta(m_0, n_0) \in Z(\mathcal{A})$. Note that

$$[\delta(m + m_0, n + n_0), m + m_0 + n + n_0] = 0 \tag{4.5}$$

for all $m \in M$ and $n \in N$. It is clear that

$$\delta(m + m_0, n + n_0) = \delta(m, n) + \delta(m_0, n_0)$$

for all $m \in M$ and $n \in N$. We get from (4.5) that

$$[\delta(m, n), m + n] + [\delta(m, n), m_0 + n_0] + [\delta(m_0, n_0), m + n] + [\delta(m_0, n_0), m_0 + n_0] = 0$$

for all $m \in M$ and $n \in N$. This implies that

$$[\delta(m, n), m_0 + n_0] = 0$$

for all $m \in M$ and $n \in N$. By assumption again we obtain that $\delta(m, n) \in Z(\mathcal{A})$ for all $m \in M$ and $n \in N$. It follows from (4.4) that $f_1(\mathcal{A}) \subseteq Z(\mathcal{A})$. This proves the result. □

5. The main results

We are in a position to give the main result of the paper.

Theorem 5.1. *Let $\mathcal{A}$ be a unital algebra with nontrivial idempotents satisfying (1.1). Suppose that*

(i) $Z(B) = \pi_B(Z(\mathcal{A}))$*, or* $A = [A, A]$*;*
(ii) $Z(A) = \pi_A(Z(\mathcal{A}))$*, or* $B = [B, B]$*;*
(iii) there exists $Y_1 \in \mathcal{A}_1$ *such that*

$$Z(\mathcal{A}) = \{X \in Z(\mathcal{A}_0) \mid [X, Y_1] = 0\}.$$

Then, every supercommuting map of $\mathcal{A}$ is proper.

Proof. Let f is a supercommuting map of $\mathcal{A}$. By Lemma 2.2 we have that $f = f_0 + f_1$, where f_i is a supercommuting map of degree i on $\mathcal{A}$, $i = 0, 1$. By Theorem 3.1 we have that

$$f_0(X) = \lambda X + \tau_0(X)$$

for all $X \in \mathcal{A}$, where $\lambda \in Z(\mathcal{A})$ with $2\lambda\mathcal{A}_1^2 = \{0\}$ and $\tau_0 : \mathcal{A} \to Z(\mathcal{A})$ is a linear map. By Theorem 4.1 we have that $f_1(\mathcal{A}) \subseteq Z(\mathcal{A})$. We set $\tau = \tau_0 + f_1$. Then

$$f(X) = \lambda X + \tau(X)$$

for all $X \in \mathcal{A}$. This proves the result. □

6. Supercommuting maps of matrix algebras

Let S be a unital algebra over R. Let $M_n(S)$ be the set of all $n \times n$ matrices over S, where $n \geq 2$. We can view $M_n(S)$ as a unital algebra with nontrivial idempotents:

$$M_n(S) = A + M + N + B,$$

where $A = M_s(S)$, $M = M_{s,t}(S)$, $N = M_{t,s}(S)$, $B = M_t(S)$, where $s+t = n$. Thus, $M_n(S)$ is a generalized matrix algebra (see [16] for details). We set

$$M_n(S)_0 = A + B \quad \text{and} \quad M_n(S)_1 = M + N.$$

It is easy to check that $M_n(S) = M_n(S)_0 \bigoplus M_n(S)_1$ is a superalgebra (see [26] for details).

Applying Theorem 3.1, we give a description of supercommuting maps of matrix algebras, which is different from the result on commuting maps of matrix algebras (see [16, Corollary 4.1] for details).

Theorem 6.1. *Let S be a unital algebra. Let $M_n(S)$ be the $n \times n$ matrix algebra. Then every supercommuting map on $M_n(R)$ is the form $X \to \lambda X + \tau(X)$, where $\lambda \in Z(S)$ with $2\lambda = 0$, and $\tau : M_n(S) \to Z(S) \cdot 1$ is a linear map.*

Proof. Note that $M_n(S)$ satisfies the condition (1.1) (see [17, Section 1]). It is easy to check that $Z(M_n(S)) = Z(S)\cdot 1$. This implies that the conditions (i) and (ii) in Theorem 5.1 are satisfied. We now claim that the condition (iii) in Theorem 5.1 is satisfied.

We set

$$\mathcal{W} = \{X \in Z(\mathcal{A}_0) \mid [X, e_{1,s+1}] = 0\}.$$

It is clear that $Z(S)\cdot 1 \subseteq \mathcal{W}$. Note that

$$Z(\mathcal{A}_0) = Z(A) + Z(B).$$

For any $X \in \mathcal{W}$, we may assume that

$$X = \lambda_1 \cdot 1_A + \lambda_2 \cdot 1_B,$$

where $\lambda_1, \lambda_2 \in Z(S)$. We have that

$$[\lambda_1 \cdot 1_A + \lambda_2 \cdot 1_B, e_{1,s+1}] = 0.$$

This implies that $(\lambda_1 - \lambda_2)e_{1,s+1} = 0$. Hence, $\lambda_1 = \lambda_2$. It follows that $X \in Z(S)\cdot 1$. Thus, $\mathcal{W} \subseteq Z(S)\cdot 1$. Hence, $\mathcal{W} = Z(S)\cdot 1$, as desired. We obtain that $M_n(S)$ satisfies all conditions in Theorem 5.1.

Let f be a supercommuting map of $M_n(S)$. By Theorem 5.1 we have that there exist $\lambda \in Z(S)$ with $2\lambda M_n(S)_1^2 = 0$, and a supercentral map $\tau : M_n(S) \to Z(S)\cdot 1$ such that

$$f(X) = \lambda X + \tau(X)$$

for all $X \in M_n(S)$. It suffices to prove that $2\lambda = 0$. It is clear that $e_{1,s+1} + e_{s+1,1} \in M_n(S)_1$. It follows that

$$2\lambda(e_{11} + e_{s+1,s+1}) = 2\lambda(e_{1,s+1} + e_{s+1,1})^2 = 0.$$

Hence, $2\lambda = 0$. This proves the result. □

As a consequence of theorem 6.1 we have the following interesting result.

Corollary 6.1. *Let S be a 2-torsion free unital algebra. Let $M_n(S)$ be the $n \times n$ matrix algebra with $n \geq 2$. Then every supercommuting map on $M_n(R)$ is supercentral.*

Proof. Let f is a supercommuting map on $M_n(S)$. By Theorem 6.1 we have that $f(X) = \lambda X + \tau(X)$, where $\lambda \in Z(S)$ with $2\lambda = 0$, and $\tau : M_n(S) \to Z(S)\cdot 1$. Since S is 2-torsion free we get that $\lambda = 0$. This implies that $f(M_n(S)) \subseteq Z(S)\cdot 1$. This proves the result. □

We conclude the section with an example, which implies that a commuting map is not a supercommuting map in general.

Example 6.1. *Let S be a 2-torsion free unital algebra. Let $M_n(S)$ be the $n \times n$ matrix algebra with $n \geq 2$. We defined a linear map $f : M_n(S) \to M_n(S)$ by*

$$f(x) = x$$

for all $x \in M_n(S)$. Then f is a commuting map on $M_n(S)$, but f is not a supercommuting map on $M_n(S)$.

Proof. It is clear that f is a commuting map. Assume that f is a supercommuting map. By Corollary 6.1 we get that $f(x) = x \in Z(S)\cdot 1$ for all $x \in M_n(S)$. In particular, we have that

$$f(e_{12}) = e_{12} \in Z(S)\cdot 1,$$

which is a contradiction. Hence, f is not a supercommuting map. □

7. Supercommuting maps of triangular algebras

Let $\mathcal{A}$ be a unital algebra with a nontrivial idempotent e satisfying (1.1). If $f\mathcal{A}e = 0$, we have that

$$\mathcal{A} = e\mathcal{A}e + e\mathcal{A}f + f\mathcal{A}f,$$

where M is a faithful (A, B)-bimodule. In this case, $\mathcal{A}$ is said to be a *triangular algebra* (see [18,28] for details). Upper triangular matrix algebras and nest algebras are the two usual examples of triangular algebras (see [18] for details).

In 2001, Cheung [18] initiated the study of commuting maps on triangular algebra. He determined the class of triangular algebras for which every commuting map is proper (see [18, Theorem 8]. In 2003, Cheung [28] gave a description of Lie derivations of triangular algebras. In 2012, Du and Wang [20] discussed k-commuting maps of triangular algebras, where $k \geq 1$.

We set

$$\mathcal{A}_0 = e\mathcal{A}e + f\mathcal{A}f \quad \text{and} \quad \mathcal{A}_1 = e\mathcal{A}f.$$

Note that $\mathcal{A} = \mathcal{A}_0 \bigoplus \mathcal{A}_1$ is a superalgebra (see [26] for details).

We begin with the following result, which shows that a supercommutator is a commutator in triangular algebras.

Proposition 7.1. *Let $\mathcal{A}$ be a triangular algebra. Then $[x, y]_s = [x, y]$ for all $x, y \in \mathcal{A}$.*

Proof. For any $a, a' \in e\mathcal{A}e$, $m, m' \in e\mathcal{A}f$, and $b, b' \in f\mathcal{A}f$, we note that

$$[m, m']_s = 0 = [m, m'].$$

We get that

$$\begin{aligned}
[a + m + b, a' + m' + b']_s &= [a + b, a' + b'] + [a + b, m'] + [m, a' + b'] + [m, m']_s \\
&= [a + b, a' + b'] + [a + b, m'] + [m, a' + b'] + [m, m'] \\
&= [a + b, a' + b' + m'] + [m, a' + b' + m'] \\
&= [a + b + m, a' + b' + m'].
\end{aligned}$$

We obtain that $[x, y]_s = [x, y]$ for all $x, y \in \mathcal{A}$. This proves the result. □

As a consequence of Proposition 7.1, we have the following interesting result.

Corollary 7.1. *Every supercommuting map is the same as a commuting map on triangular algebras.*

8. Conclusions

Let $\mathcal{A}$ be a unital algebra with nontrivial idempotents. We consider $\mathcal{A}$ as a generalized matrix algebra according to Ghahramani and Zadeh's method. We give a description of supercommuting maps on generalized matrix algebras. As a consequence, we give a description of supercommuting maps on matrix algebras, which is different from the result on commuting maps of matrix algebras. We finally prove that every supercommuting map is the same as a commuting map on triangular algebras.

Author contributions

Yingyu Luo: Writing-riginal draft, Funding acquisition; Yu Wang: Validation, Writing-review & editing.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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