Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 24590–24609.
DOI: 10.3934/math.20241198
Received: 28 June 2024
Revised: 30 July 2024
Accepted: 13 August 2024
Published: 22 August 2024

Research article

On fractional Bullen-type inequalities with applications

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Abstract: Integral inequalities in mathematical interpretations are a substantial and ongoing body of research. Because fractional calculus techniques are widely used in science, a lot of research has recently been done on them. A key concept in fractional calculus is the Caputo-Fabrizio fractional integral. In this work, we focus on using the Caputo-Fabrizio fractional integral operator to build a multi-parameter fractional integral identity. Using the obtained integral identity, certain generalized estimates of Bullen-type fractional inequalities have been generated. By establishing certain inequalities, this study advances the fields of fractional calculus and convex function research. Both graphical and numerical statistics are provided to show the correctness of our results. We finally provide applications to modified Bessel functions, $\mathfrak{h}$-divergence measures, and probability density functions.

Keywords: Bullen inequality; integral operator; modified Bessel functions; divergence measure; probability density function
Mathematics Subject Classification: 26D15, 26A33, 33C10, 94A17, 60E05

1. Introduction

Fractional calculus began with a legend in the 1800s there were two famous mathematicians, L' Hopital and Leibniz, who were discussing how to evaluate $\frac{d^n \mathfrak{f}}{dx^n}$ when $n = \frac{1}{2}$. In the 17th century, Leibniz published his book "Introductory Calculus", in which he talked about how to take derivatives of any function. After this brief discussion, the subject did not pick up much attention until 1819.

Therefore, there was another time point when another famous mathematician by the name of Lacroix wrote another book; the book was on fractional calculus, where he started to develop the formulation for evaluating these derivatives. More specifically, Lacroix developed the fractional formula $\frac{d^{\alpha}x^{m}}{dx^{\alpha}}$ for α and m being fractions. As a result, he found an answer to the famous question raised by L' Hopital and Leibniz, namely, what is the fractional derivative of a function of the order $\frac{1}{2}$. The discussion did not end there, although Lacroix has shown an initial way to evaluate fractional derivatives, which has some problems. To mitigate the problems, there was another mathematician by the name of Liouville who extended the Lacroix definition. Liouville developed the formula for $\frac{d^{\alpha}}{dx^{\alpha}}(\sum_{n=0}^{\infty} c_n \exp(a_n x))$ for $Re(a_n) > 0, c_n \in \mathbb{R}$, and α being a fraction. Liouville also developed the formula for $\frac{d^{\alpha}x^{m}}{dx^{\alpha}}$ for $m < 0$ and α being a fraction.

Fractional calculus has proven to be a potent and effective mathematical tool in recent years, helping to define the intricate dynamics of real-world issues from a variety of scientific and engineering disciplines [1–7]. Every traditional fractional differential operator has a distinct kernel and can be applied to certain problems. For example, the Caputo-Fabrizio fractional operator is used in the linear viscoelasticity framework. The most popular operator for computing a fractional-order integral among a number of operators is the Riemann-Liouville fractional integral. It is basically just a straightforward adaptation of the Cauchy formula from classical calculus for repeated integration. However, over the past half decade, a number of operators for fractional-order integrals and derivatives have been put out. These new operators are believed to arise because of the singularity in the kernel of the Riemann-Liouville integral at one endpoint of the integration interval $[0, T]$. It originates from the new fractional operator, in which the integral involves the non-singular kernel.

The main motivation of the Caputo-Fabrizio integral and derivative operator is that it is a generalization of classical integral and derivative. One of the characteristics that sets the operator apart from others is its kernel, which is essentially a real power transformed into an integral using the Laplace transform. As a result, finding an accurate answer to many issues is simple. An increasing number of mathematicians working in the applied sciences are using the Caputo-Fabrizio fractional integral operator to model their problems. For additional details, see [8–11]. The main benefit of the Caputo-Fabrizio integral operator is its ability to admit the same form for the boundary condition of fractional differential equations with Caputo-Fabrizio derivatives as it does for differential equations of integer order. For studying fractional differential equation solutions, fractional integral inequalities are crucial, particularly for determining the uniqueness of initial value problems. Using a function's convexity is one of the most effective techniques to establish integral inequalities. In fact, advances in the theory of convex functions are closely related to the development of mathematical inequalities. Convexity theory provides a powerful and efficient way to address a wide range of problems in different fields of pure and applied mathematics. The most well-known and fascinating outcome of the convex function is the Hermite-Hadamard integral inequality. The classical Hermite-Hadamard inequality, which provides us with an estimation of the mean value of a convex function $\mathfrak{f} : I \subseteq \mathbb{R} \to \mathbb{R}$ for $\mathfrak{a}_1, \mathfrak{a}_2 \in I$ with $\mathfrak{a}_1 < \mathfrak{a}_2$,

$$\mathfrak{f}\left(\frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\right) \leq \frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{f}(\mathfrak{x}) d\mathfrak{x} \leq \frac{\mathfrak{f}(\mathfrak{a}_1) + \mathfrak{f}(\mathfrak{a}_2)}{2}.$$

The geometrical relevance of this inequality led to its expansion, generalization, or improvement through the application of basic analytical procedures. Over the last few years, many mathematicians who have researched in this field have contributed to its development and made attempts to strengthen its modification in many ways [12–15].

Bullen [16] proved the inequality by giving the bound for the mean value of a convex function $\mathfrak{f} : I \subseteq \mathbb{R} \to \mathbb{R}$ for $\mathfrak{a}_1, \mathfrak{a}_2 \in I$ with $\mathfrak{a}_1 < \mathfrak{a}_2$,

$$\frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \mathfrak{f}(\mathfrak{x}) d\mathfrak{x} \leq \frac{1}{2}\left[\mathfrak{f}\left(\frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\right) + \frac{\mathfrak{f}(\mathfrak{a}_1) + \mathfrak{f}(\mathfrak{a}_2)}{2}\right].$$

We can observe that the right side of the Hermite-Hadamard inequality should be viewed as an extension of Bullen's inequality. Bullen's inequality holds a significant position in theory, as do other classical inequalities like Jensen, Ostrowski, and Hermite-Hadamard. Numerous fields, including numerical integration, midpoints, and trapezoidal quadrature rules, can benefit from its application. For more current findings about the extension and improvement of Bullen-type inequality, see [17–21].

The paper is organized in the following way: After this introduction in Section 2 we have discussed some basic related concepts, in Section 3 main results, in Section 4 numerically solved examples and their graph, in Section 5 applications to some extent, and in the last Section 6 conclusion of the whole paper.

2. Preliminaries

Some foundational ideas that are useful in understanding our main results are covered in this section.

Definition 1. *[22] Let* $f \in H^1(\mathfrak{m}_1, \mathfrak{m}_2)$, $\alpha \in [0, 1]$, *then the fractional integrals in the sense of Caputo and Fabrizio are defined by:*

$$({}^{CF}_{\mathfrak{m}_1+}I^{\alpha} f)(t) := \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} \int_{\mathfrak{m}_1}^{t} f(x)dx,$$

$$({}^{CF}_{\mathfrak{m}_2-}I^{\alpha} f)(t) := \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} \int_{t}^{\mathfrak{m}_2} f(x)dx,$$

provided that, $B(\alpha) > 0$ *is a normalization function satisfying* $B(0) = B(1) = 1$.

Theorem 1. *[23] Let* $f : [\mathfrak{m}_1, \mathfrak{m}_2] \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function on* $[\mathfrak{m}_1, \mathfrak{m}_2]$ *such that* $x_i \in [\mathfrak{m}_1, \mathfrak{m}_2]$, $\alpha_i \in [0, 1]$ *with* $\sum_{i=1}^{k} \alpha_i = 1$, $1 \leq i \leq k$, *then*

$$f\left(\mathfrak{m}_1 + \mathfrak{m}_2 - \sum_{i=1}^{k} \alpha_i x_i\right) \leq f(\mathfrak{m}_1) + f(\mathfrak{m}_2) - \sum_{i=1}^{k} \alpha_i f(x_i). \tag{2.1}$$

Proposition 1. *[24] Let* $f : [\mathfrak{m}_1, \mathfrak{m}_2] \subseteq \mathbb{R} \to \mathbb{R}^+$ *be a* $\log -$*convex function on* $[\mathfrak{m}_1, \mathfrak{m}_2]$ *such that* $x_i \in [\mathfrak{m}_1, \mathfrak{m}_2]$, $\alpha_i \in [0, 1]$ *with* $\sum_{i=1}^{k} \alpha_i = 1$, $1 \leq i \leq k$, *then Jensen-Mercer inequality is defined by:*

$$f\left(\mathfrak{m}_1 + \mathfrak{m}_2 - \sum_{i=1}^{n} \alpha_i x_i\right) \leq \frac{f(\mathfrak{m}_1) f(\mathfrak{m}_2)}{\prod_{i=1}^{k} f^{\alpha_i}(x_i)}. \tag{2.2}$$

Before going on, we make the following assumption:

$$\begin{aligned} I_{v,i}(\mathfrak{h}; \mathfrak{m}_1, \mathfrak{m}_2; \mathfrak{u}_1, \mathfrak{u}_2) \quad := \quad & \int_0^1 (\mathfrak{t} - \mathfrak{d})\mathfrak{h}'\Bigg((v - i - 1)\left\{\mathfrak{m}_1 - \mathfrak{t}\frac{\mathfrak{m}_1 + \mathfrak{m}_2}{2} - (1 - \mathfrak{t})\mathfrak{u}_1\right\} \\ & + i\left\{\mathfrak{m}_2 - \mathfrak{t}\mathfrak{u}_2 - \frac{(1 - \mathfrak{t})(\mathfrak{m}_1 + \mathfrak{m}_2)}{2}\right\} + w\Bigg) d\mathfrak{t}. \end{aligned} \tag{2.3}$$

3. Main results

Lemma 1. *Let* $\mathfrak{h} : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a differentiable function on* I° *(the interior of I), where* $\mathfrak{m}_1, \mathfrak{m}_2 \in I^\circ$ *with* $\mathfrak{m}_1 < \mathfrak{m}_2$, $v \in \mathbb{N}$; *let* $w \in [\mathfrak{u}_1, \mathfrak{u}_2]$; $\mathfrak{u}_1, \mathfrak{u}_2 \in [\mathfrak{m}_1, \mathfrak{m}_2]$ *such that* $\mathfrak{u}_1 \leq \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} \leq \mathfrak{u}_2$, $\varsigma \in (0, 1]$, $\mathfrak{d} \in [0, 1]$. *If* $\mathfrak{h}' \in L^1[\mathfrak{m}_1, \mathfrak{m}_2]$, *then*

$$\begin{aligned}
&J_v(\mathfrak{h}; \mathfrak{m}_1, \mathfrak{m}_2; \mathfrak{u}_1, \mathfrak{u}_2)\\
&:= \sum_{i=0}^{v-1}\Bigg[\frac{(1-v)(2\mathfrak{u}_1 - \mathfrak{m}_1 - \mathfrak{m}_2) + i(2\mathfrak{u}_1 + 2\mathfrak{u}_2 - 2\mathfrak{m}_1 - 2\mathfrak{m}_2)}{4} I_{v,i}(\mathfrak{h}; \mathfrak{m}_1, \mathfrak{m}_2; \mathfrak{u}_1, \mathfrak{u}_2)\\
&+\frac{(1-\varsigma)\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right)}{\varsigma[(1-v)(2\mathfrak{u}_1 - \mathfrak{m}_1 - \mathfrak{m}_2) + i(2\mathfrak{u}_1 + 2\mathfrak{u}_2 - 2\mathfrak{m}_1 - 2\mathfrak{m}_2)]}\Bigg]\\
&= \frac{1}{2}\sum_{i=0}^{v-1}\Bigg[(\mathfrak{d}-1)\mathfrak{h}\left(\frac{(v-1)(\mathfrak{m}_1 - \mathfrak{m}_2) + i(3\mathfrak{m}_2 - \mathfrak{m}_1 - 2\mathfrak{u}_2) + 2w}{2}\right)\\
&-\mathfrak{d}\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1 - \mathfrak{u}_1) + i(\mathfrak{m}_2 - 3\mathfrak{m}_1 + 2\mathfrak{u}_1) + 2w}{2}\right)\Bigg]\\
&+\frac{B(\varsigma)}{\varsigma}\sum_{i=0}^{v-1}\frac{{}^{CF}_{\frac{(v-1)(\mathfrak{m}_1-\mathfrak{m}_2)+i(3\mathfrak{m}_2-\mathfrak{m}_1-2\mathfrak{u}_2)+2w}{2}^+}I^{\varsigma}\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right)}{(1-v)(2\mathfrak{u}_1 - \mathfrak{m}_1 - \mathfrak{m}_2) + i(2\mathfrak{u}_1 + 2\mathfrak{u}_2 - 2\mathfrak{m}_1 - 2\mathfrak{m}_2)}.
\end{aligned} \tag{3.1}$$

Proof. Integrating by parts the identity (2.3)

$$\begin{aligned}
&I_{v,i}(\mathfrak{h}; \mathfrak{m}_1, \mathfrak{m}_2; \mathfrak{u}_1, \mathfrak{u}_2)\\
&= (\mathfrak{t}-\mathfrak{d})\frac{\mathfrak{h}\left((v-i-1)\left\{\mathfrak{m}_1 - \mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} - (1-\mathfrak{t})\mathfrak{u}_1\right\} + i\left\{\mathfrak{m}_2 - \mathfrak{t}\mathfrak{u}_2 - \frac{(1-\mathfrak{t})(\mathfrak{m}_1+\mathfrak{m}_2)}{2}\right\} + w\right)}{(v-1)\left[\mathfrak{u}_1 - \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right] - i(\mathfrak{u}_1 + \mathfrak{u}_2 - \mathfrak{m}_1 - \mathfrak{m}_2)}\Bigg|_0^1\\
&-\int_0^1 \frac{\mathfrak{h}\left((v-i-1)\left\{\mathfrak{m}_1 - \mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} - (1-\mathfrak{t})\mathfrak{u}_1\right\} + i\left\{\mathfrak{m}_2 - \mathfrak{t}\mathfrak{u}_2 - \frac{(1-\mathfrak{t})(\mathfrak{m}_1+\mathfrak{m}_2)}{2}\right\} + w\right)}{(v-1)\left[\mathfrak{u}_1 - \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right] - i(\mathfrak{u}_1 + \mathfrak{u}_2 - \mathfrak{m}_1 - \mathfrak{m}_2)}d\mathfrak{t},
\end{aligned}$$

setting $z = (v-i-1)\left\{\mathfrak{m}_1 - \mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} - (1-\mathfrak{t})\mathfrak{u}_1\right\} + i\left\{\mathfrak{m}_2 - \mathfrak{t}\mathfrak{u}_2 - \frac{(1-\mathfrak{t})(\mathfrak{m}_1+\mathfrak{m}_2)}{2}\right\} + w$, so that $d\mathfrak{t} = \frac{dz}{(v-i-1)\left(\mathfrak{u}_1 - \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right) + i\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} - \mathfrak{u}_2\right)}$, and when $\mathfrak{t} = 0$, $z = (v-i-1)(\mathfrak{m}_1 - \mathfrak{u}_1) + i\left(\mathfrak{m}_2 - \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right) + w$, and when $t = 1$, $z = (v-i-1)\left(\mathfrak{m}_1 - \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right) + i(\mathfrak{m}_2 - \mathfrak{u}_2) + w$.

$$\begin{aligned}
&I_{v,i}(\mathfrak{h}; \mathfrak{m}_1, \mathfrak{m}_2; \mathfrak{u}_1, \mathfrak{u}_2)\\
&= \frac{2(1-\mathfrak{d})\mathfrak{h}\left(\frac{(v-1)(\mathfrak{m}_1-\mathfrak{m}_2)+i(3\mathfrak{m}_2-\mathfrak{m}_1-2\mathfrak{u}_2)+2w}{2}\right) + 2\mathfrak{d}\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right)}{(v-1)(2\mathfrak{u}_1 - \mathfrak{m}_1 - \mathfrak{m}_2) - i(2\mathfrak{u}_1 + 2\mathfrak{u}_2 - 2\mathfrak{m}_1 - 2\mathfrak{m}_2)}\\
&-\frac{4}{[(v-1)(2\mathfrak{u}_1 - \mathfrak{m}_1 - \mathfrak{m}_2) - i(2\mathfrak{u}_1 + 2\mathfrak{u}_2 - 2\mathfrak{m}_1 - 2\mathfrak{m}_2)]^2}\int_{\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}}^{\frac{(v-1)(\mathfrak{m}_1-\mathfrak{m}_2)+i(3\mathfrak{m}_2-\mathfrak{m}_1-2\mathfrak{u}_2)+2w}{2}} \mathfrak{h}(z)dz
\end{aligned}$$

$$\Rightarrow \frac{(1-v)(2\mathfrak{u}_1 - \mathfrak{m}_1 - \mathfrak{m}_2) + i(2\mathfrak{u}_1 + 2\mathfrak{u}_2 - 2\mathfrak{m}_1 - 2\mathfrak{m}_2)}{4} I_{v,i}(\mathfrak{h}; \mathfrak{m}_1, \mathfrak{m}_2; \mathfrak{u}_1, \mathfrak{u}_2)$$

$$
= \frac{(\mathfrak{d}-1)\mathfrak{h}\left(\frac{(v-1)(\mathfrak{m}_1-\mathfrak{m}_2)+i(3\mathfrak{m}_2-\mathfrak{m}_1-2\mathfrak{u}_2)+2w}{2}\right)-\mathfrak{d}\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right)}{2}
$$
$$
-\frac{1}{(1-v)(2\mathfrak{u}_1-\mathfrak{m}_1-\mathfrak{m}_2)+i(2\mathfrak{u}_1+2\mathfrak{u}_2-2\mathfrak{m}_1-2\mathfrak{m}_2)}\int_{\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}}^{\frac{(v-1)(\mathfrak{m}_1-\mathfrak{m}_2)+i(3\mathfrak{m}_2-\mathfrak{m}_1-2\mathfrak{u}_2)+2w}{2}}\mathfrak{h}(z)dz.
$$

Multiplying both sides by $\frac{\varsigma((1-v)(2\mathfrak{u}_1-\mathfrak{m}_1-\mathfrak{m}_2)+i(2\mathfrak{u}_1+2\mathfrak{u}_2-2\mathfrak{m}_1-2\mathfrak{m}_2))}{B(\varsigma)}$ and adding $\frac{1-\varsigma}{B(\varsigma)}\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right)$

$$
\frac{\varsigma\left[(1-v)(2\mathfrak{u}_1-\mathfrak{m}_1-\mathfrak{m}_2)+i(2\mathfrak{u}_1+2\mathfrak{u}_2-2\mathfrak{m}_1-2\mathfrak{m}_2)\right]^2}{4B(\varsigma)}I_{v,i}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{u}_1,\mathfrak{u}_2)
$$
$$
+\frac{1-\varsigma}{B(\varsigma)}\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right)
$$
$$
=\frac{\varsigma[(1-v)(2\mathfrak{u}_1-\mathfrak{m}_1-\mathfrak{m}_2)+i(2\mathfrak{u}_1+2\mathfrak{u}_2-2\mathfrak{m}_1-2\mathfrak{m}_2)]}{B(\varsigma)}\times
$$
$$
\frac{(\mathfrak{d}-1)\mathfrak{h}\left(\frac{(v-1)(\mathfrak{m}_1-\mathfrak{m}_2)+i(3\mathfrak{m}_2-\mathfrak{m}_1-2\mathfrak{u}_2)+2w}{2}\right)-\mathfrak{d}\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right)}{2}
$$
$$
+\frac{\varsigma}{B(\varsigma)}\int_{\frac{(v-1)(\mathfrak{m}_1-\mathfrak{m}_2)+i(3\mathfrak{m}_2-\mathfrak{m}_1-2\mathfrak{u}_2)+2w}{2}}^{\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}}\mathfrak{h}(z)dz+\frac{1-\varsigma}{B(\varsigma)}\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right).
$$

Now by the definition of Caputo-Fabrizio fractional operator

$$
\frac{(1-v)(2\mathfrak{u}_1-\mathfrak{m}_1-\mathfrak{m}_2)+i(2\mathfrak{u}_1+2\mathfrak{u}_2-2\mathfrak{m}_1-2\mathfrak{m}_2)}{4}I_{v,i}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{u}_1,\mathfrak{u}_2)
$$
$$
+\frac{(1-\varsigma)\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right)}{\varsigma[(1-v)(2\mathfrak{u}_1-\mathfrak{m}_1-\mathfrak{m}_2)+i(2\mathfrak{u}_1+2\mathfrak{u}_2-2\mathfrak{m}_1-2\mathfrak{m}_2)]}
$$
$$
=\frac{(\mathfrak{d}-1)\mathfrak{h}\left(\frac{(v-1)(\mathfrak{m}_1-\mathfrak{m}_2)+i(3\mathfrak{m}_2-\mathfrak{m}_1-2\mathfrak{u}_2)+2w}{2}\right)-\mathfrak{d}\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right)}{2}
$$
$$
+B(\varsigma)\frac{{}^{CF}_{\frac{(v-1)(\mathfrak{m}_1-\mathfrak{m}_2)+i(3\mathfrak{m}_2-\mathfrak{m}_1-2\mathfrak{u}_2)+2w}{2}+}I^{\varsigma}\mathfrak{h}\left(\frac{2(v-1)(\mathfrak{m}_1-\mathfrak{u}_1)+i(\mathfrak{m}_2-3\mathfrak{m}_1+2\mathfrak{u}_1)+2w}{2}\right)}{\varsigma[(1-v)(2\mathfrak{u}_1-\mathfrak{m}_1-\mathfrak{m}_2)+i(2\mathfrak{u}_1+2\mathfrak{u}_2-2\mathfrak{m}_1-2\mathfrak{m}_2)]},
$$

which completes the proof of (3.1). □

Remark 1. *In particular for $v=2$, identity* (3.1) *in Lemma 1 reduces to the following identity:*

$$
\frac{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}{4}I_{2,0}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{u}_1)+\frac{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}{4}I_{2,1}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{u}_2)
$$
$$
=(1-\mathfrak{d})\frac{\mathfrak{h}(\mathfrak{m}_2+w-\mathfrak{u}_2)+\mathfrak{h}\left(\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}\right)}{2}+\mathfrak{d}\frac{\mathfrak{h}(\mathfrak{m}_1+w-\mathfrak{u}_1)+\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2}
$$
$$
-\frac{B(\varsigma)}{\varsigma}\left\{\frac{{}^{CF}_{\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}+}I^{\varsigma}\mathfrak{h}(\mathfrak{m}_1-\mathfrak{u}_1+w)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}+\frac{{}^{CF}_{(w-\mathfrak{u}_2+\mathfrak{m}_2)+}I^{\varsigma}\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}\right\}+\frac{1-\varsigma}{\varsigma}\left[\frac{\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}+\frac{\mathfrak{h}(\mathfrak{m}_1+w-\mathfrak{u}_1)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}\right],
\tag{3.2}
$$

provided that

$$I_{2,0}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{u}_1) := \int_0^1 (\mathfrak{d}-\mathfrak{t})\mathfrak{h}'\left(\mathfrak{m}_1 + w - \mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} - (1-\mathfrak{t})\mathfrak{u}_1\right)d\mathfrak{t},$$

$$I_{2,1}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{u}_2) := \int_0^1 (\mathfrak{d}-\mathfrak{t})\mathfrak{h}'\left(\mathfrak{m}_2 + w - \mathfrak{t}\mathfrak{u}_2 - \frac{(1-\mathfrak{t})(\mathfrak{m}_1+\mathfrak{m}_2)}{2}\right)d\mathfrak{t}.$$

Moreover, for $\mathfrak{u}_1 = \mathfrak{m}_1$, $\mathfrak{u}_2 = \mathfrak{m}_2$, $w = \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}$ *and* $\mathfrak{d} = \frac{1}{2}$, *it reduces to the following identity:*

$$\begin{aligned}\frac{\mathfrak{m}_2-\mathfrak{m}_1}{8}I(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2) &= \frac{1}{2}\left[\frac{\mathfrak{h}(\mathfrak{m}_1)+\mathfrak{h}(\mathfrak{m}_2)}{2} + \mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right] - \frac{B(\varsigma)}{\varsigma(\mathfrak{m}_2-\mathfrak{m}_1)} \\ &\times\left\{{}^{CF}_{\mathfrak{m}_1+}I^{\varsigma}\mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right) + {}^{CF}_{\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}+}I^{\varsigma}\mathfrak{h}(\mathfrak{m}_2)\right\} + \frac{1-\varsigma}{\varsigma}\frac{\mathfrak{h}(\mathfrak{m}_2)+\mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)}{\mathfrak{m}_2-\mathfrak{m}_1},\end{aligned} \tag{3.3}$$

$$I(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2) := \int_0^1 (1-2\mathfrak{t})\left\{\mathfrak{h}'\left(\mathfrak{t}\mathfrak{m}_1 + (1-\mathfrak{t})\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right) + \mathfrak{h}'\left(\mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} + (1-\mathfrak{t})\mathfrak{m}_2\right)\right\}d\mathfrak{t},$$

and further for $\varsigma = 1$, *it reduces to Lemma 2.1 of Xi and Qi [25].*

Theorem 2. *Let* $\mathfrak{h} : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a differentiable function on* I° *(the interior of* I*), where* $\mathfrak{m}_1, \mathfrak{m}_2 \in I^\circ$ *with* $\mathfrak{m}_1 < \mathfrak{m}_2$; *let* $w \in [\mathfrak{u}_1,\mathfrak{u}_2]$, $\mathfrak{u}_1,\mathfrak{u}_2 \in [\mathfrak{m}_1,\mathfrak{m}_2]$ *such that* $\mathfrak{u}_1 \le \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} \le \mathfrak{u}_2$, $\varsigma \in (0,1]$, $\mathfrak{d} \in [0,1]$. *If* $|\mathfrak{h}'|^{\mathfrak{a}}$ *is convex and* $\mathfrak{h}' \in L^1[\mathfrak{m}_1,\mathfrak{m}_2]$, $\mathfrak{a} \ge 1$, *then*

$$\begin{aligned}&\Bigg|(1-\mathfrak{d})\frac{\mathfrak{h}(\mathfrak{m}_2+w-\mathfrak{u}_2)+\mathfrak{h}\left(\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}\right)}{2} + \mathfrak{d}\frac{\mathfrak{h}(\mathfrak{m}_1+w-\mathfrak{u}_1)+\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2} \\ &-\frac{B(\varsigma)}{\varsigma}\left\{\frac{{}^{CF}_{\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}+}I^{\varsigma}\mathfrak{h}(\mathfrak{m}_1-\mathfrak{u}_1+w)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1} + \frac{{}^{CF}_{(w-\mathfrak{u}_2+\mathfrak{m}_2)+}I^{\varsigma}\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}\right\} + \frac{1-\varsigma}{\varsigma}\left[\frac{\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2} + \frac{\mathfrak{h}(\mathfrak{m}_1+w-\mathfrak{u}_1)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}\right]\Bigg| \\ &\le \mathfrak{d}^2\Bigg[\frac{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}{4}\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right) - (2-\mathfrak{d}+\mathfrak{a})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}} - \mathfrak{d}|\mathfrak{h}'(\mathfrak{u}_2)|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}} \\ &+\frac{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}{4}\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right) - (2-\mathfrak{d}+\mathfrak{a})|\mathfrak{h}'(\mathfrak{u}_1)|^{\mathfrak{a}} - \mathfrak{d}\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}}\Bigg] \\ &+(1-\mathfrak{d})^2\Bigg[\frac{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}{4}\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right) - (1+\mathfrak{d}+\mathfrak{a})|\mathfrak{h}'(\mathfrak{u}_2)|^{\mathfrak{a}} - (1-\mathfrak{d})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}} \\ &+\frac{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}{4}\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right) - (1+\mathfrak{d}+\mathfrak{a})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}} - (1-\mathfrak{d})|\mathfrak{h}'(\mathfrak{u}_1)|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}}\Bigg].\end{aligned} \tag{3.4}$$

Proof. For $\mathfrak{a} > 1$, by using the basic properties of modulus, Hölder integral inequality, convexity of $|\mathfrak{h}'|^{\mathfrak{a}}$, and relation (2.1) in Theorem 1 to identity defined by (3.2), we have

$$
\begin{aligned}
\left|I_{2,0}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{u}_1)\right| &= \left|\int_0^1 (\mathfrak{d}-\mathfrak{t})\mathfrak{h}'\left(\mathfrak{m}_1+w-\mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-(1-\mathfrak{t})\mathfrak{u}_1\right)d\mathfrak{t}\right| \\
&\le \mathfrak{d}^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_0^{\mathfrak{d}} (\mathfrak{d}-\mathfrak{t})^{\mathfrak{a}}\left|\mathfrak{h}'\left(\mathfrak{m}_1+w-\mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-(1-\mathfrak{t})\mathfrak{u}_1\right)\right|^{\mathfrak{a}}d\mathfrak{t}\right\}^{\frac{1}{\mathfrak{a}}} \\
&+(1-\mathfrak{d})^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_{\mathfrak{d}}^1 (\mathfrak{t}-\mathfrak{d})^{\mathfrak{a}}\left|\mathfrak{h}'\left(\mathfrak{m}_1+w-\mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-(1-\mathfrak{t})\mathfrak{u}_1\right)\right|^{\mathfrak{a}}d\mathfrak{t}\right\}^{\frac{1}{\mathfrak{a}}} \\
&\le \mathfrak{d}^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_0^{\mathfrak{d}} (\mathfrak{d}-\mathfrak{t})^{\mathfrak{a}}\left(|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}-\mathfrak{t}\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}-(1-\mathfrak{t})|\mathfrak{h}'(\mathfrak{u}_1)|^{\mathfrak{a}}\right)d\mathfrak{t}\right\}^{\frac{1}{\mathfrak{a}}} \\
&+(1-\mathfrak{d})^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_{\mathfrak{d}}^1 (\mathfrak{t}-\mathfrak{d})^{\mathfrak{a}}\left(|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}-\mathfrak{t}\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}-(1-\mathfrak{t})|\mathfrak{h}'(\mathfrak{u}_1)|^{\mathfrak{a}}\right)d\mathfrak{t}\right\}^{\frac{1}{\mathfrak{a}}} \\
&= \mathfrak{d}^2\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right)-(2-\mathfrak{d}+\mathfrak{a})|\mathfrak{h}'(\mathfrak{u}_1)|^{\mathfrak{a}}-\mathfrak{d}\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}} \\
&+(1-\mathfrak{d})^2\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right)-(1+\mathfrak{d}+\mathfrak{a})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}-(1-\mathfrak{d})|\mathfrak{h}'(\mathfrak{u}_1)|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}} \quad (3.5)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\left|I_{2,1}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{u}_2)\right| &= \left|\int_0^1 (\mathfrak{d}-\mathfrak{t})\mathfrak{h}'\left(\mathfrak{m}_2+w-(1-\mathfrak{t})\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-\mathfrak{t}\mathfrak{u}_2\right)d\mathfrak{t}\right| \\
&\le \mathfrak{d}^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_0^{\mathfrak{d}} (\mathfrak{d}-\mathfrak{t})^{\mathfrak{a}}\left|\mathfrak{h}'\left(\mathfrak{m}_2+w-(1-\mathfrak{t})\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-\mathfrak{t}\mathfrak{u}_2\right)\right|^{\mathfrak{a}}d\mathfrak{t}\right\}^{\frac{1}{\mathfrak{a}}} \\
&+(1-\mathfrak{d})^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_{\mathfrak{d}}^1 (\mathfrak{t}-\mathfrak{d})^{\mathfrak{a}}\left|\mathfrak{h}'\left(\mathfrak{m}_2+w-(1-\mathfrak{t})\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-\mathfrak{t}\mathfrak{u}_2\right)\right|^{\mathfrak{a}}d\mathfrak{t}\right\}^{\frac{1}{\mathfrak{a}}} \\
&\le \mathfrak{d}^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_0^{\mathfrak{d}} (\mathfrak{d}-\mathfrak{t})^{\mathfrak{a}}\left(|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}-(1-\mathfrak{t})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}-\mathfrak{t}|\mathfrak{h}'(\mathfrak{u}_2)|^{\mathfrak{a}}\right)d\mathfrak{t}\right\}^{\frac{1}{\mathfrak{a}}} \\
&+(1-\mathfrak{d})^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_{\mathfrak{d}}^1 (\mathfrak{t}-\mathfrak{d})^{\mathfrak{a}}\left(|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}-(1-\mathfrak{t})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}-\mathfrak{t}|\mathfrak{h}'(\mathfrak{u}_2)|^{\mathfrak{a}}\right)d\mathfrak{t}\right\}^{\frac{1}{\mathfrak{a}}} \\
&= \mathfrak{d}^2\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right)-(2-\mathfrak{d}+\mathfrak{a})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}-\mathfrak{d}|\mathfrak{h}'(\mathfrak{u}_2)|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}} \\
&+(1-\mathfrak{d})^2\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right)-(1+\mathfrak{d}+\mathfrak{a})|\mathfrak{h}'(\mathfrak{u}_2)|^{\mathfrak{a}}-(1-\mathfrak{d})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}} \quad (3.6)
\end{aligned}
$$

Multiplying (3.5) and (3.6) by, respectively, $\frac{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}{4}$ and $\frac{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}{4}$, then addition yields

$$\Bigg|(1-\mathfrak{d})\frac{\mathfrak{h}\left(\mathfrak{m}_2+w-\mathfrak{u}_2\right)+\mathfrak{h}\left(\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}\right)}{2}+\mathfrak{d}\frac{\mathfrak{h}(\mathfrak{m}_1+w-\mathfrak{u}_1)+\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2}$$
$$-\frac{B(\varsigma)}{\varsigma}\left\{\frac{{}^{CF}_{\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}+}I^{\varsigma}\mathfrak{h}\left(\mathfrak{m}_1-\mathfrak{u}_1+w\right)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}+\frac{{}^{CF}_{(w-\mathfrak{u}_2+\mathfrak{m}_2)+}I^{\varsigma}\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}\right\}+\frac{1-\varsigma}{\varsigma}\left[\frac{\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}+\frac{\mathfrak{h}\left(\mathfrak{m}_1+w-\mathfrak{u}_1\right)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}\right]\Bigg|$$
$$\leq \mathfrak{d}^2\left[\frac{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}{4}\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right)-(2-\mathfrak{d}+\mathfrak{a})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}-\mathfrak{d}\,|\mathfrak{h}'(\mathfrak{u}_2)|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}}\right.$$
$$\left.+\frac{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}{4}\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right)-(2-\mathfrak{d}+\mathfrak{a})\,|\mathfrak{h}'\left(\mathfrak{u}_1\right)|^{\mathfrak{a}}-\mathfrak{d}\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}}\right]$$
$$+(1-\mathfrak{d})^2\left[\frac{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}{4}\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right)-(1+\mathfrak{d}+\mathfrak{a})\,|\mathfrak{h}'\left(\mathfrak{u}_2\right)|^{\mathfrak{a}}-(1-\mathfrak{d})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}}\right.$$
$$\left.+\frac{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}{4}\left\{\frac{(\mathfrak{a}+2)\left(|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}}+|\mathfrak{h}'(w)|^{\mathfrak{a}}\right)-(1+\mathfrak{d}+\mathfrak{a})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|^{\mathfrak{a}}-(1-\mathfrak{d})\,|\mathfrak{h}'(\mathfrak{u}_1)|^{\mathfrak{a}}}{(\mathfrak{a}+1)(\mathfrak{a}+2)}\right\}^{\frac{1}{\mathfrak{a}}}\right]. \quad (3.7)$$

For $\mathfrak{a}=1$, by using basic properties of modulus, convexity of $|\mathfrak{h}'|$, and relation (2.1) in Theorem 1 to identity defined by (3.2), we have

$$\begin{aligned}\left|I_{2,0}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{u}_1)\right| &= \left|\int_0^1(\mathfrak{d}-\mathfrak{t})\mathfrak{h}'\left(\mathfrak{m}_1+w-\mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-(1-\mathfrak{t})\mathfrak{u}_1\right)d\mathfrak{t}\right|\\
&\leq \int_0^{\mathfrak{d}}(\mathfrak{d}-\mathfrak{t})^{\mathfrak{a}}\left|\mathfrak{h}'\left(\mathfrak{m}_1+w-\mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-(1-\mathfrak{t})\mathfrak{u}_1\right)\right|d\mathfrak{t}\\
&+\int_{\mathfrak{d}}^1(\mathfrak{t}-\mathfrak{d})\left|\mathfrak{h}'\left(\mathfrak{m}_1+w-\mathfrak{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-(1-\mathfrak{t})\mathfrak{u}_1\right)\right|d\mathfrak{t}\\
&\leq \int_0^{\mathfrak{d}}(\mathfrak{d}-\mathfrak{t})\Big(|\mathfrak{h}'(\mathfrak{m}_1)|+|\mathfrak{h}'(w)|-\mathfrak{t}\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|-(1-\mathfrak{t})\,|\mathfrak{h}'(\mathfrak{u}_1)|\Big)d\mathfrak{t}\\
&+\int_{\mathfrak{d}}^1(\mathfrak{t}-\mathfrak{d})\Big(|\mathfrak{h}'(\mathfrak{m}_1)|+|\mathfrak{h}'(w)|-\mathfrak{t}\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|-(1-\mathfrak{t})\,|\mathfrak{h}'(\mathfrak{u}_1)|\Big)d\mathfrak{t}\\
&= \mathfrak{d}^2\frac{(3\left(|\mathfrak{h}'(\mathfrak{m}_1)|+|\mathfrak{h}'(w)|\right)-(3-\mathfrak{d})\,|\mathfrak{h}'\left(\mathfrak{u}_1\right)|-\mathfrak{d}\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|}{6}\\
&+(1-\mathfrak{d})^2\frac{3\left(|\mathfrak{h}'(\mathfrak{m}_1)|+|\mathfrak{h}'(w)|\right)-(2+\mathfrak{d})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|-(1-\mathfrak{d})\,|\mathfrak{h}'(\mathfrak{u}_1)|}{6}. \quad (3.8)\end{aligned}$$

Similarly

$$\begin{aligned}\left|I_{2,1}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{u}_2)\right| &= \left|\int_0^1 (\mathfrak{d}-\mathrm{t})\mathfrak{h}'\left(\mathfrak{m}_2+w-(1-\mathrm{t})\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-\mathrm{t}\mathfrak{u}_2\right)dt\right| \\ &\le \int_0^{\mathfrak{d}} (\mathfrak{d}-\mathrm{t})\left|\mathfrak{h}'\left(\mathfrak{m}_2+w-(1-\mathrm{t})\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-\mathrm{t}\mathfrak{u}_2\right)\right|d\mathrm{t} \\ &+ \int_{\mathfrak{d}}^1 (\mathrm{t}-\mathfrak{d})\left|\mathfrak{h}'\left(\mathfrak{m}_2+w-(1-\mathrm{t})\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-\mathrm{t}\mathfrak{u}_2\right)\right|d\mathrm{t} \\ &\le \int_0^{\mathfrak{d}} (\mathfrak{d}-\mathrm{t})\left(|\mathfrak{h}'(\mathfrak{m}_2)|+|\mathfrak{h}'(w)|-(1-\mathrm{t})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|-\mathrm{t}\,|\mathfrak{h}'(\mathfrak{u}_2)|\right)d\mathrm{t} \\ &+ \int_{\mathfrak{d}}^1 (\mathrm{t}-\mathfrak{d})\left(|\mathfrak{h}'(\mathfrak{m}_2)|+|\mathfrak{h}'(w)|-(1-\mathrm{t})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|-\mathrm{t}\,|\mathfrak{h}'(\mathfrak{u}_2)|\right)d\mathrm{t} \\ &= \mathfrak{d}^2\frac{3\left(|\mathfrak{h}'(\mathfrak{m}_2)|+|\mathfrak{h}'(w)|\right)-(3-\mathfrak{d})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|-\mathfrak{d}\,|\mathfrak{h}'(\mathfrak{u}_2)|}{6} \\ &+(1-\mathfrak{d})^2\frac{3\left(|\mathfrak{h}'(\mathfrak{m}_2)|+|\mathfrak{h}'(w)|\right)-(2+\mathfrak{d})\,|\mathfrak{h}'(\mathfrak{u}_2)|-(1-\mathfrak{d})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|}{6}. \end{aligned} \tag{3.9}$$

Multiplying (3.8) and (3.9) by, respectively, $\frac{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}{4}$ and $\frac{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}{4}$, then addition yields

$$\begin{aligned}&\Bigg|(1-\mathfrak{d})\frac{\mathfrak{h}\left(\mathfrak{m}_2+w-\mathfrak{u}_2\right)+\mathfrak{h}\left(\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}\right)}{2}+\mathfrak{d}\frac{\mathfrak{h}(\mathfrak{m}_1+w-\mathfrak{u}_1)+\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2} \\ &-\frac{B(\varsigma)}{\varsigma}\left\{\frac{{}^{CF}_{\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}+}I^{\varsigma}\mathfrak{h}\left(\mathfrak{m}_1-\mathfrak{u}_1+w\right)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}+\frac{{}^{CF}_{(w-\mathfrak{u}_2+\mathfrak{m}_2)+}I^{\varsigma}\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}\right\}+\frac{1-\varsigma}{\varsigma}\left[\frac{\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}+\frac{\mathfrak{h}\left(\mathfrak{m}_1+w-\mathfrak{u}_1\right)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}\right]\Bigg| \\ &\le \mathfrak{d}^2\Bigg\{(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2)\frac{3\left(|\mathfrak{h}'(\mathfrak{m}_2)|+|\mathfrak{h}'(w)|\right)-(3-\mathfrak{d})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|-\mathfrak{d}\,|\mathfrak{h}'(\mathfrak{u}_2)|}{24} \\ &+(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1)\frac{3\left(|\mathfrak{h}'(\mathfrak{m}_1)|+|\mathfrak{h}'(w)|\right)-(3-\mathfrak{d})\,|\mathfrak{h}'(\mathfrak{u}_1)|-\mathfrak{d}\left|\mathfrak{h}'(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2})\right|}{24}\Bigg\} \\ &+(1-\mathfrak{d})^2\Bigg\{(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2)\frac{3\left(|\mathfrak{h}'(\mathfrak{m}_2)|+|\mathfrak{h}'(w)|\right)-(2+\mathfrak{d})\,|\mathfrak{h}'(\mathfrak{u}_2)|-(1-\mathfrak{d})\left|\mathfrak{h}'(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2})\right|}{24} \\ &+(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1)\frac{3\left(|\mathfrak{h}'(\mathfrak{m}_1)|+|\mathfrak{h}'(w)|\right)-(2+\mathfrak{d})\left|\mathfrak{h}'\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|-(1-\mathfrak{d})\,|\mathfrak{h}'(\mathfrak{u}_1)|}{24}\Bigg\}. \end{aligned} \tag{3.10}$$

A combination of (3.7) and (3.10), yields the desired result (3.4). This completes the desired result. □

Theorem 3. *Let* $\mathfrak{h}: I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a differentiable function on* I° *(the interior of* I*), where* $\mathfrak{m}_1, \mathfrak{m}_2 \in I^\circ$ *with* $\mathfrak{m}_1 < \mathfrak{m}_2$*; let* $w \in [\mathfrak{m}_1, \mathfrak{m}_2]$, $\varsigma \in (0,1]$, $\mathfrak{d} \in [0,1]$. *If* $|\mathfrak{h}'|^{\mathfrak{a}}$ *is* log-*convex and* $\mathfrak{h}' \in L^1[\mathfrak{m}_1, \mathfrak{m}_2]$, $\mathfrak{a} \ge 1$, *then*

$$\left|(1-\mathfrak{d})\mathfrak{h}\left(\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}\right)+\mathfrak{d}\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)+\frac{2(1-\varsigma)}{\varsigma(\mathfrak{m}_2-\mathfrak{m}_1)}\left\{\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)+\mathfrak{h}\left(w\right)\right\}\right.$$

$$+\mathfrak{h}(w)-\frac{2B(\varsigma)}{\varsigma(\mathfrak{m}_2-\mathfrak{m}_1)}\left\{{}^{CF}_{\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}+}I^{\varsigma}\mathfrak{h}\left(w\right)+{}^{CF}_{w+}I^{\varsigma}\mathfrak{h}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)\right\}\Bigg|$$

$$\leq\frac{(1+\sqrt[\mathfrak{a}]{\alpha})(\mathfrak{m}_2-\mathfrak{m}_1)\left|\mathfrak{h}'(w)\right|\left\{\left(\frac{\mathfrak{d}^2}{2}\right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}(h_1(\mathfrak{d},\alpha))^{\frac{1}{\mathfrak{a}}}+\left(\frac{(1-\mathfrak{d})^2}{2}\right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}(h_2(\mathfrak{d},\alpha))^{\frac{1}{\mathfrak{a}}}\right\}}{2\sqrt[\mathfrak{a}]{\alpha}}, \tag{3.11}$$

provided that $\alpha=\left|\frac{\mathfrak{h}'(\mathfrak{m}_1)}{\mathfrak{h}'(\mathfrak{m}_2)}\right|^{\frac{\mathfrak{a}}{2}}$,

$$h_1(\mathfrak{d},\alpha):=\begin{cases}-\frac{\mathfrak{d}}{\ln\alpha}+\frac{\alpha^{\mathfrak{d}}-1}{(\ln\alpha)^2}, & \alpha\neq 1;\\ \frac{\mathfrak{d}^2}{2}, & \alpha=1.\end{cases},\quad h_2(\mathfrak{d},\alpha):=\begin{cases}\frac{\alpha(1-\mathfrak{d})}{\ln\alpha}+\frac{\alpha^{\mathfrak{d}}-\alpha}{(\ln\alpha)^2}, & \alpha\neq 1;\\ \frac{(1-\mathfrak{d})^2}{2}, & \alpha=1.\end{cases}$$

Proof. By power mean inequality and log −convexity of $|\mathfrak{h}'|^{\mathfrak{a}}$ to identity defined by (3.2), we have

$$\begin{aligned}
&\left|I_{2,0}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{m}_1)\right|\\
&=\left|\int_0^1(\mathfrak{d}-\mathrm{t})\mathfrak{h}'\left(\mathfrak{m}_1+w-\mathrm{t}\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}-(1-\mathrm{t})\mathfrak{m}_1\right)dt\right|\\
&\leq\int_0^{\mathfrak{d}}(\mathfrak{d}-\mathrm{t})\left|\mathfrak{h}'\left(\mathfrak{m}_1+w-\frac{2-\mathrm{t}}{2}\mathfrak{m}_1-\frac{\mathrm{t}}{2}\mathfrak{m}_2\right)\right|dt\\
&+\int_{\mathfrak{d}}^1(\mathrm{t}-\mathfrak{d})\left|\mathfrak{h}'\left(\mathfrak{m}_1+w-\frac{2-\mathrm{t}}{2}\mathfrak{m}_1-\frac{\mathrm{t}}{2}\mathfrak{m}_2\right)\right|dt\\
&\leq\left\{\int_0^{\mathfrak{d}}(\mathfrak{d}-\mathrm{t})dt\right\}^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_0^{\mathfrak{d}}(\mathfrak{d}-\mathrm{t})\left|\mathfrak{h}'\left(\mathfrak{m}_1+w-\frac{2-\mathrm{t}}{2}\mathfrak{m}_1-\frac{\mathrm{t}}{2}\mathfrak{m}_2\right)\right|^{\mathfrak{a}}dt\right\}^{\frac{1}{\mathfrak{a}}}\\
&+\left\{\int_{\mathfrak{d}}^1(\mathrm{t}-\mathfrak{d})dt\right\}^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_{\mathfrak{d}}^1(\mathrm{t}-\mathfrak{d})\left|\mathfrak{h}'\left(\mathfrak{m}_1+w-\frac{2-\mathrm{t}}{2}\mathfrak{m}_1-\frac{\mathrm{t}}{2}\mathfrak{m}_2\right)\right|^{\mathfrak{a}}dt\right\}^{\frac{1}{\mathfrak{a}}}\\
&\leq\left(\frac{\mathfrak{d}^2}{2}\right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_0^{\mathfrak{d}}\frac{(\mathfrak{d}-\mathrm{t})|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}}|\mathfrak{h}'(w)|^{\mathfrak{a}}}{|\mathfrak{h}'(\mathfrak{m}_1)|^{\frac{\mathfrak{a}(2-\mathrm{t})}{2}}|\mathfrak{h}'(\mathfrak{m}_2)|^{\frac{\mathfrak{a}\mathrm{t}}{2}}}dt\right\}^{\frac{1}{\mathfrak{a}}}+\left(\frac{(1-\mathfrak{d})^2}{2}\right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_{\mathfrak{d}}^1\frac{(\mathrm{t}-\mathfrak{d})|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}}|\mathfrak{h}'(w)|^{\mathfrak{a}}}{|\mathfrak{h}'(\mathfrak{m}_1)|^{\frac{\mathfrak{a}(2-\mathrm{t})}{2}}|\mathfrak{h}'(\mathfrak{m}_2)|^{\frac{\mathfrak{a}\mathrm{t}}{2}}}dt\right\}^{\frac{1}{\mathfrak{a}}}\\
&=\left(\frac{\mathfrak{d}^2}{2}\right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}|\mathfrak{h}'(w)|\left\{\int_0^{\mathfrak{d}}(\mathfrak{d}-\mathrm{t})\left|\frac{\mathfrak{h}'(\mathfrak{m}_1)}{\mathfrak{h}'(\mathfrak{m}_2)}\right|^{\frac{\mathfrak{a}\mathrm{t}}{2}}dt\right\}^{\frac{1}{\mathfrak{a}}}+\left(\frac{(1-\mathfrak{d})^2}{2}\right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}|\mathfrak{h}'(w)|\left\{\int_{\mathfrak{d}}^1(\mathrm{t}-\mathfrak{d})\left|\frac{\mathfrak{h}'(\mathfrak{m}_1)}{\mathfrak{h}'(\mathfrak{m}_2)}\right|^{\frac{\mathfrak{a}\mathrm{t}}{2}}dt\right\}^{\frac{1}{\mathfrak{a}}}\\
&=|\mathfrak{h}'(w)|\left[\left(\frac{\mathfrak{d}^2}{2}\right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_0^{\mathfrak{d}}(\mathfrak{d}-\mathrm{t})\alpha^{\mathrm{t}}dt\right\}^{\frac{1}{\mathfrak{a}}}+\left(\frac{(1-\mathfrak{d})^2}{2}\right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}\left\{\int_{\mathfrak{d}}^1(\mathrm{t}-\mathfrak{d})\alpha^{\mathrm{t}}dt\right\}^{\frac{1}{\mathfrak{a}}}\right]\\
&=|\mathfrak{h}'(w)|\left\{\left(\frac{\mathfrak{d}^2}{2}\right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}(h_1(\mathfrak{d},\alpha))^{\frac{1}{\mathfrak{a}}}+\left(\frac{(1-\mathfrak{d})^2}{2}\right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}}(h_2(\mathfrak{d},\alpha))^{\frac{1}{\mathfrak{a}}}\right\}.
\end{aligned}\tag{3.12}$$

Similarly

$$\left|I_{2,1}(\mathfrak{h};\mathfrak{m}_1,\mathfrak{m}_2;\mathfrak{m}_2)\right|=\left|\int_0^1(\mathfrak{d}-\mathrm{t})\mathfrak{h}'\left(\mathfrak{m}_2+w-\mathrm{t}\mathfrak{m}_2-\frac{(\mathfrak{m}_1+\mathfrak{m}_2)(1-\mathrm{t})}{2}\right)dt\right|$$

$$\leq\int_0^{\mathfrak{d}}(\mathfrak{d}-\mathrm{t})\left|\mathfrak{h}'\left(\mathfrak{m}_2+w-\frac{1+\mathrm{t}}{2}\mathfrak{m}_2-\frac{1-\mathrm{t}}{2}\mathfrak{m}_1\right)\right|dt+\int_{\mathfrak{d}}^1(\mathrm{t}-\mathfrak{d})\left|\mathfrak{h}'\left(\mathfrak{m}_2+w-\frac{1+\mathrm{t}}{2}\mathfrak{m}_2-\frac{1-\mathrm{t}}{2}\mathfrak{m}_1\right)\right|dt$$

$$\leq \left\{ \int_0^{\mathfrak{d}} (\mathfrak{d}-\mathfrak{t})d\mathfrak{t} \right\}^{\frac{\mathfrak{a}-1}{\mathfrak{a}}} \left\{ \int_0^{\mathfrak{d}} (\mathfrak{d}-\mathfrak{t}) \left| \mathfrak{h}' \left(\mathfrak{m}_2 + w - \frac{1+\mathfrak{t}}{2}\mathfrak{m}_2 - \frac{1-\mathfrak{t}}{2}\mathfrak{m}_1 \right) \right|^{\mathfrak{a}} d\mathfrak{t} \right\}^{\frac{1}{\mathfrak{a}}}$$

$$+ \left\{ \int_{\mathfrak{d}}^1 (\mathfrak{t}-\mathfrak{d})d\mathfrak{t} \right\}^{\frac{\mathfrak{a}-1}{\mathfrak{a}}} \left\{ \int_{\mathfrak{d}}^1 (\mathfrak{t}-\mathfrak{d}) \left| \mathfrak{h}' \left(\mathfrak{m}_2 + w - \frac{1+\mathfrak{t}}{2}\mathfrak{m}_2 - \frac{1-\mathfrak{t}}{2}\mathfrak{m}_1 \right) \right|^{\mathfrak{a}} d\mathfrak{t} \right\}^{\frac{1}{\mathfrak{a}}}$$

$$\leq \left(\frac{\mathfrak{d}^2}{2} \right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}} \left\{ \int_0^{\mathfrak{d}} \frac{(\mathfrak{d}-\mathfrak{t}) |\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}} |\mathfrak{h}'(w)|^{\mathfrak{a}}}{|\mathfrak{h}'(\mathfrak{m}_1)|^{\frac{\mathfrak{a}(1-\mathfrak{t})}{2}} |\mathfrak{h}'(\mathfrak{m}_2)|^{\frac{\mathfrak{a}(1+\mathfrak{t})}{2}}} d\mathfrak{t} \right\}^{\frac{1}{\mathfrak{a}}} + \left(\frac{(1-\mathfrak{d})^2}{2} \right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}} \left\{ \int_{\mathfrak{d}}^1 \frac{(\mathfrak{t}-\mathfrak{d}) |\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}} |\mathfrak{h}'(w)|^{\mathfrak{a}}}{|\mathfrak{h}'(\mathfrak{m}_1)|^{\frac{\mathfrak{a}(1-\mathfrak{t})}{2}} |\mathfrak{h}'(\mathfrak{m}_2)|^{\frac{\mathfrak{a}(1+\mathfrak{t})}{2}}} d\mathfrak{t} \right\}^{\frac{1}{\mathfrak{a}}}$$

$$= \left(\frac{\mathfrak{d}^2}{2} \right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}} |\mathfrak{h}'(w)| \left| \frac{\mathfrak{h}'(\mathfrak{m}_2)}{\mathfrak{h}'(\mathfrak{m}_1)} \right|^{\frac{1}{2}} \left\{ \int_0^{\mathfrak{d}} (\mathfrak{d}-\mathfrak{t}) \left| \frac{\mathfrak{h}'(\mathfrak{m}_1)}{\mathfrak{h}'(\mathfrak{m}_2)} \right|^{\frac{\mathfrak{a}\mathfrak{t}}{2}} d\mathfrak{t} \right\}^{\frac{1}{\mathfrak{a}}} + \left(\frac{(1-\mathfrak{d})^2}{2} \right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}} |\mathfrak{h}'(w)| \left| \frac{\mathfrak{h}'(\mathfrak{m}_2)}{\mathfrak{h}'(\mathfrak{m}_1)} \right|^{\frac{1}{2}} \left\{ \int_{\mathfrak{d}}^1 (\mathfrak{t}-\mathfrak{d}) \left| \frac{\mathfrak{h}'(\mathfrak{m}_1)}{\mathfrak{h}'(\mathfrak{m}_2)} \right|^{\frac{\mathfrak{a}\mathfrak{t}}{2}} d\mathfrak{t} \right\}^{\frac{1}{\mathfrak{a}}}$$

$$= \frac{|\mathfrak{h}'(w)|}{\sqrt[\mathfrak{a}]{\alpha}} \left[\left(\frac{\mathfrak{d}^2}{2} \right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}} \left\{ \int_0^{\mathfrak{d}} (\mathfrak{d}-\mathfrak{t}) \alpha^{\mathfrak{t}} d\mathfrak{t} \right\}^{\frac{1}{\mathfrak{a}}} + \left(\frac{(1-\mathfrak{d})^2}{2} \right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}} \left\{ \int_{\mathfrak{d}}^1 (\mathfrak{t}-\mathfrak{d}) \alpha^{\mathfrak{t}} d\mathfrak{t} \right\}^{\frac{1}{\mathfrak{a}}} \right]$$

$$= \frac{|\mathfrak{h}'(w)|}{\sqrt[\mathfrak{a}]{\alpha}} \left\{ \left(\frac{\mathfrak{d}^2}{2} \right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}} (h_1(\mathfrak{d},\alpha))^{\frac{1}{\mathfrak{a}}} + \left(\frac{(1-\mathfrak{d})^2}{2} \right)^{\frac{\mathfrak{a}-1}{\mathfrak{a}}} (h_2(\mathfrak{d},\alpha))^{\frac{1}{\mathfrak{a}}} \right\}. \tag{3.13}$$

Multiplying both (3.12) and (3.13) by $\frac{\mathfrak{m}_2-\mathfrak{m}_1}{4}$, yields the desired result. □

An observation about the equality of the functional value of the the mean position and mean position of the functional values comes to mind, that is, for a real valued function $\mathfrak{h} : [\mathfrak{m}_1, \mathfrak{m}_2] \subseteq \mathbb{R} \to \mathbb{R}$

$$\mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} \right) = \frac{\mathfrak{h}(\mathfrak{m}_1)+\mathfrak{h}(\mathfrak{m}_2)}{2}. \tag{3.14}$$

The affirmative answer about the validity of (3.14) was given by Xi and Qi [25] by the function $\mathfrak{h}(\mathfrak{t}) = \pm\frac{\mathfrak{t}^3-9\mathfrak{t}^2+27\mathfrak{t}}{3}$, $\mathfrak{t} \in [1,5]$.

Corollary 1. *Let* $\mathfrak{h} : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a differentiable function on* I° *(the interior of* I*), where* $\mathfrak{m}_1, \mathfrak{m}_2 \in I^\circ$ *with* $\mathfrak{m}_1 < \mathfrak{m}_2$. *If* $|\mathfrak{h}'|^{\mathfrak{a}}$ *is convex and* $\mathfrak{h}' \in L^1[\mathfrak{m}_1, \mathfrak{m}_2]$, $\mathfrak{a} \geq 1$, *then*

$$\left| \frac{1}{2} \left\{ \frac{\mathfrak{h}(\mathfrak{m}_1)+\mathfrak{h}(\mathfrak{m}_2)}{2} + \mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} \right) \right\} + \frac{(1-\varsigma)\left\{ \mathfrak{h}(\mathfrak{m}_2) + \mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} \right) \right\}}{\varsigma(\mathfrak{m}_2-\mathfrak{m}_1)} - \frac{B(\varsigma)\left\{ {}^{CF}_{\mathfrak{m}_1+}I^{\varsigma}\mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} \right) + {}^{CF}_{\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}+}I^{\varsigma}\mathfrak{h}(\mathfrak{m}_2) \right\}}{\varsigma(\mathfrak{m}_2-\mathfrak{m}_1)} \right|$$

$$\leq \frac{\mathfrak{m}_2-\mathfrak{m}_1}{\sqrt[\mathfrak{a}]{4^{2\mathfrak{a}+1}(\mathfrak{a}+1)(\mathfrak{a}+2)}} \left(\sqrt[\mathfrak{a}]{(2\mathfrak{a}+5)|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}} + (2\mathfrak{a}+3)|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}} + \sqrt[\mathfrak{a}]{|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}} + (4\mathfrak{a}+7)|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}} \right.$$

$$\left. + \sqrt[\mathfrak{a}]{(4\mathfrak{a}+7)|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}} + |\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}} + \sqrt[\mathfrak{a}]{(2\mathfrak{a}+3)|\mathfrak{h}'(\mathfrak{m}_1)|^{\mathfrak{a}} + (2\mathfrak{a}+5)|\mathfrak{h}'(\mathfrak{m}_2)|^{\mathfrak{a}}} \right). \tag{3.15}$$

Proof. The proof directly follows by setting $\mathfrak{u}_1 = \mathfrak{m}_1$, $\mathfrak{u}_2 = \mathfrak{m}_2$, $\mathfrak{d} = \frac{1}{2}$, $w = \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}$ in Theorem 2. □

Corollary 2. *Let* $\mathfrak{h} : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a differentiable function on* I° *(the interior of* I*), where* $\mathfrak{m}_1, \mathfrak{m}_2 \in I^\circ$ *with* $\mathfrak{m}_1 < \mathfrak{m}_2$. *If* $|\mathfrak{h}'|^{\mathfrak{a}}$ *is* log *−convex and* $\mathfrak{h}' \in L^1[\mathfrak{m}_1, \mathfrak{m}_2]$, $\mathfrak{a} \geq 1$, *then*

$$\left| \frac{1}{2} \left\{ \frac{\mathfrak{h}(\mathfrak{m}_1)+\mathfrak{h}(\mathfrak{m}_2)}{2} + \mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} \right) \right\} + \frac{(1-\varsigma)\left\{ \mathfrak{h}(\mathfrak{m}_2) + \mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} \right) \right\}}{\varsigma(\mathfrak{m}_2-\mathfrak{m}_1)} - \frac{B(\varsigma)\left\{ {}^{CF}_{\mathfrak{m}_1+}I^{\varsigma}\mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2} \right) + {}^{CF}_{\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}+}I^{\varsigma}\mathfrak{h}(\mathfrak{m}_2) \right\}}{\varsigma(\mathfrak{m}_2-\mathfrak{m}_1)} \right|$$

$$\leq \frac{(1+\sqrt[\alpha]{\alpha})(\mathfrak{m}_2-\mathfrak{m}_1)\sqrt{|\mathfrak{h}'(\mathfrak{m}_1)||\mathfrak{h}'(\mathfrak{m}_2)|}\left\{\sqrt[\alpha]{h_1\left(\frac{1}{2},\alpha\right)}+\sqrt[\alpha]{h_2\left(\frac{1}{2},\alpha\right)}\right\}}{2^{\frac{5\alpha-3}{\alpha}}\sqrt[\alpha]{\alpha}}. \tag{3.16}$$

Proof. The proof directly follows by setting $\mathfrak{u}_1 = \mathfrak{m}_1$, $\mathfrak{u}_2 = \mathfrak{m}_2$, $\mathfrak{d} = \frac{1}{2}$, $w = \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}$ in Theorem 3. □

Remark 2. *For $\varsigma = 1$, Corollaries 1 and 2 coincides with Theorems 3.2 and 3.7 of Xi and Qi [25] respectively.*

In particular, under the relation (3.14), the left sides in (3.15) and (3.16) can be replaced by the relations either (3.17) or (3.18) to get trapezoidal type inequality or midpoint type inequality

$$\left|\frac{\mathfrak{h}(\mathfrak{m}_1)+\mathfrak{h}(\mathfrak{m}_2)}{2}+\frac{(1-\varsigma)\left\{\mathfrak{h}(\mathfrak{m}_2)+\mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right\}-B(\varsigma)\left\{{}^{CF}_{\mathfrak{m}_1+}I^{\varsigma}\mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)+{}^{CF}_{\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}+}I^{\varsigma}\mathfrak{h}(\mathfrak{m}_2)\right\}}{\varsigma(\mathfrak{m}_2-\mathfrak{m}_1)}\right|, \tag{3.17}$$

$$\left|\mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)+\frac{(1-\varsigma)\left\{\mathfrak{h}(\mathfrak{m}_2)+\mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right\}-B(\varsigma)\left\{{}^{CF}_{\mathfrak{m}_1+}I^{\varsigma}\mathfrak{h}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)+{}^{CF}_{\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}+}I^{\varsigma}\mathfrak{h}(\mathfrak{m}_2)\right\}}{\varsigma(\mathfrak{m}_2-\mathfrak{m}_1)}\right|. \tag{3.18}$$

4. Analysis in numbers and graphics

In order to better grasp the theoretical results, we go over the numerical and graphical analysis of our main results in this part. Tables and figures in each example are unrelated to one another. Both sets of statistics were selected at random. The table and graphic in each case demonstrate that the inequality's left-hand side is less than or equal to its right-hand side, according to the corresponding theorem.

Example 1. *Let $\mathfrak{h}(\mathfrak{t}) = \frac{2}{5}\sqrt{\mathfrak{t}^5}$ be such that $\mathfrak{t} \in [0,\infty)$ and $\varsigma = \mathfrak{a} = 1$. In Table 1, we compute the values from result (3.4) of Theorem 2. Furthermore, the validity of result (3.4) of Theorem 2 is graphically shown in Figure 1 by considering $\mathfrak{h}(\mathfrak{t})$ with the following values: $\mathfrak{m}_1 = 3$, $\mathfrak{u}_1 = 5$, $w = 18$, $\mathfrak{u}_2 = 20$, $20 \leq \mathfrak{m}_2 \leq 30$, $0 \leq \mathfrak{d} \leq 1$, $\mathfrak{a} = 7$.*

Table 1. Comparison of values in result of Theorem 2.

$\mathfrak{m}_1$	$\mathfrak{u}_1$	w	$\mathfrak{u}_2$	$\mathfrak{m}_2$	$\mathfrak{d}$	LHS of (3.4)	RHS of (3.4)
5	6	15	15	16	0	123.6568	127.9318
23	33	33	44	50	0.2	339.7169	401.0339
11	11	47	75	100	0.4	208.3972	2.5144e+03
63	80	90	100	129	0.6	826.1879	1.8423e+03
2	3	30	40	60	0.8	1.0376e+03	1.1879e+03
101	102	106	107	111	0.99	1.3199e+03	1.3204e+03
20	30	40	75	75	1	3.6029e+03	3.7572e+03

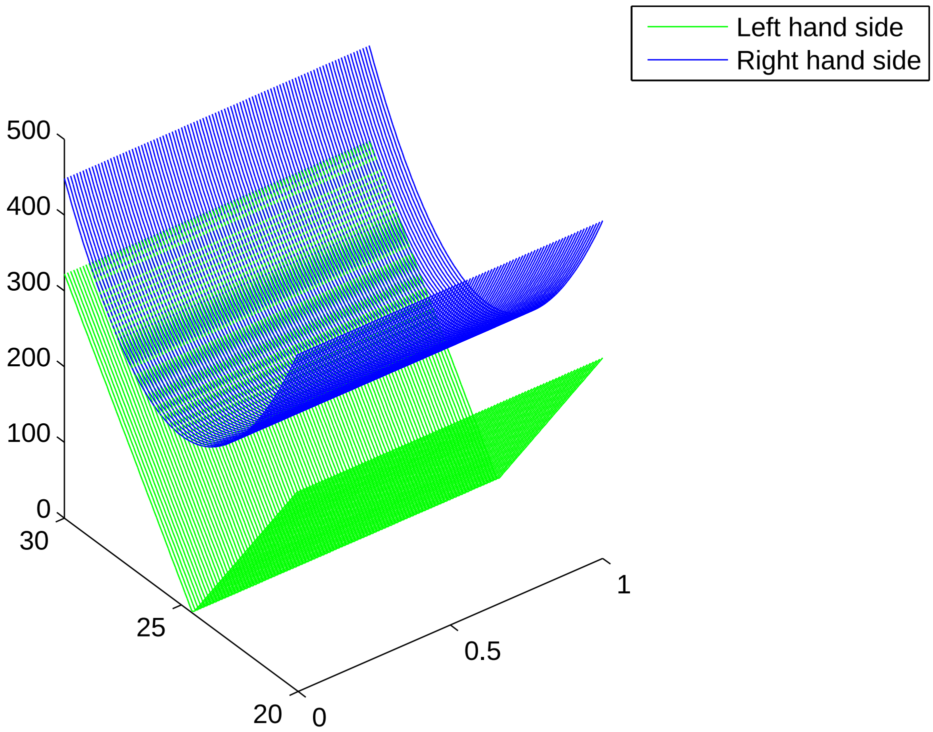

Figure 1. Validity of inequality (3.4) in Theorem 3.

Example 2. *Let* $\mathfrak{h}(\mathfrak{t}) = \exp \mathfrak{t}$ *be such that* $\mathfrak{t} \in (0, \infty)$ *and* $\varsigma = 1$. *In Table 2, we compute the values from result* (3.11) *of Theorem* 3. *Furthermore, the validity of result* (3.11) *of Theorem* 3 *is graphically shown in Figure 2 by considering* $\mathfrak{h}(\mathfrak{t})$ *with the following values:* $\mathfrak{m}_1 = 9$, $9 \leq w \leq 12$, $\mathfrak{m}_2 = 12$, $\mathfrak{a} = 3$, $0 \leq \mathfrak{d} \leq 1$.

Table 2. Comparison of values in result of Theorem 3

$\mathfrak{m}_1$	w	$\mathfrak{m}_2$	$\mathfrak{a}$	$\mathfrak{d}$	LHS of (3.11)	RHS of (3.11)
1	4	7	2	0	307.3219	3.9033e+03
12	12	30	11	0.2	1.1739e+08	1.8195e+12
21	40	40	7	0.3	6.1262e+20	1.1768e+25
7	10	11	3	0.5	2.5007e+04	2.1551e+05
30	31	52	4	0.8	1.2333e+18	1.4996e+23
22	29	43	5	0.99	1.2775e+17	1.2082e+22
99	150	171	6	1	5.8417e+80	1.9028e+97

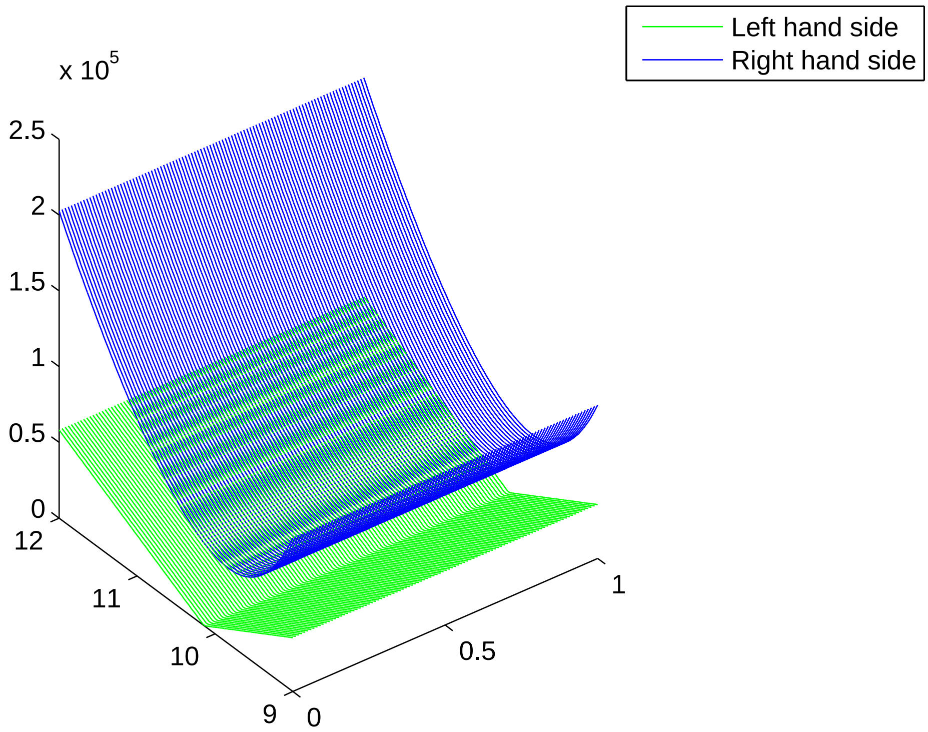

Figure 2. Validity of inequality (3.11) in Theorem 3.

5. Applications

5.1. Modified Bessel functions

The modified Bessel functions of first and second kind are defined, respectively by Watson [26]

$$I_\rho(\xi) = \sum_{n=0}^{\infty} \frac{\left(\frac{\xi}{2}\right)^{\rho+2n}}{n!\Gamma(\rho+n+1)}; \quad K_\rho(\xi) = \frac{\pi}{2}\frac{I_{-\rho}(\xi) - I_\rho(\xi)}{\sin\pi\rho}.$$

Watson also defined the functions $J_\rho, L_\rho : \mathbb{R} \to [1, \infty)$ by

$$J_\rho(\xi) = \Gamma(\rho+1)\left(\frac{\xi}{2}\right)^{-\rho} I_\rho(\xi); \;\; L_\rho(\xi) = \Gamma(\rho+1)\left(\frac{\xi}{2}\right)^{-\rho} K_\rho(\xi) \;\forall\, \xi \in \mathbb{R},\; \rho > -1,$$

differentiating with respect to ξ twice yields:
$J'_\rho(\xi) = \frac{\xi J_{\rho+1}(\xi)}{2(\rho+1)}$; $J''_\rho(\xi) = \frac{\xi^2 J_{\rho+2}(\xi)+2(\rho+2)J_{\rho+1}(\xi)}{4(\rho+1)(\rho+2)}$ and $L'_\rho(\xi) = \frac{\xi L_{\rho+1}(\xi)}{2(\rho+1)}$, $L''_\rho(\xi) = \frac{\xi^2 L_{\rho+2}(\xi)+2(\rho+2)L_{\rho+1}(\xi)}{4(\rho+1)(\rho+2)}$. Convexities of $J_\rho(\xi)$ and $L_\rho(\xi)$ directly follows from here. We incorporate this function as a result.

Proposition 2. *For* $\mathfrak{h}(\mathfrak{t}) = J'_\rho(\mathfrak{t})$; $\mathfrak{a} = 1$ *in Theorem 2, we have*

$$\Bigg|(1-\mathfrak{d})\frac{2(\mathfrak{m}_2 + w - \mathfrak{u}_2)J_{\rho+1}(\mathfrak{m}_2 + w - \mathfrak{u}_2) + (\mathfrak{m}_1 - \mathfrak{m}_2 + 2w)J_{\rho+1}\left(\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}\right)}{8(\rho+1)}$$

$$+\mathfrak{d}\frac{2(\mathfrak{m}_1+w-\mathfrak{u}_1)J_{\rho+1}(\mathfrak{m}_1+w-\mathfrak{u}_1)+(\mathfrak{m}_2-\mathfrak{m}_1+2w)J_{\rho+1}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{8(\rho+1)}$$
$$\left.+\frac{J_\rho(\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2})-J_\rho(\mathfrak{m}_1+w-\mathfrak{u}_1)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}+\frac{J_\rho(\mathfrak{m}_2+w-\mathfrak{u}_2)-J_\rho(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2})}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}\right|$$
$$\leq\frac{\left(2\mathfrak{d}^2-2\mathfrak{d}+1\right)(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1)}{32(\rho+1)(\rho+2)}\left(\mathfrak{m}_1^2J_{\rho+2}(\mathfrak{m}_1)+2(\rho+2)J_{\rho+1}(\mathfrak{m}_1)\right)$$
$$+\frac{\left(2\mathfrak{d}^2-2\mathfrak{d}+1\right)(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2)}{32(\rho+1)(\rho+2)}\left(\mathfrak{m}_2^2J_{\rho+2}(\mathfrak{m}_2)+2(\rho+2)J_{\rho+1}(\mathfrak{m}_2)\right)$$
$$+\frac{\left(2\mathfrak{d}^2-2\mathfrak{d}+1\right)(\mathfrak{u}_2-\mathfrak{u}_1)}{16(\rho+1)(\rho+2)}\left(w^2J_{\rho+2}(w)+2(\rho+2)J_{\rho+1}(w)\right)$$
$$+\frac{\left(2\mathfrak{d}^3-6\mathfrak{d}^2+3\mathfrak{d}-1\right)(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1)}{96(\rho+1)(\rho+2)}\left(\mathfrak{u}_1^2J_{\rho+2}(\mathfrak{u}_1)+2(\rho+2)J_{\rho+1}(\mathfrak{u}_1)\right)$$
$$+\frac{\left(-2\mathfrak{d}^3+3\mathfrak{d}-2\right)(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2)}{96(\rho+1)(\rho+2)}\left(\mathfrak{u}_2^2J_{\rho+2}(\mathfrak{u}_2)+2(\rho+2)J_{\rho+1}(\mathfrak{u}_2)\right)$$
$$+\frac{\left(2\mathfrak{d}^3-6\mathfrak{d}^2+3\mathfrak{d}-1\right)(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2)-\left(2\mathfrak{d}^3-3\mathfrak{d}+2\right)(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1)}{384(\rho+1)(\rho+2)}$$
$$\times\left((\mathfrak{m}_1+\mathfrak{m}_2)^2J_{\rho+2}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)+8(\rho+2)J_{\rho+1}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right).$$

Proposition 3. *For* $\mathfrak{h}(\mathfrak{t})=L'_\rho(\mathfrak{t})$; $\mathfrak{a}=1$ *in Theorem 2, we have*

$$\left|(1-\mathfrak{d})\frac{2(\mathfrak{m}_2+w-\mathfrak{u}_2)L_{\rho+1}(\mathfrak{m}_2+w-\mathfrak{u}_2)+(\mathfrak{m}_1-\mathfrak{m}_2+2w)L_{\rho+1}\left(\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}\right)}{8(\rho+1)}\right.$$
$$+\mathfrak{d}\frac{2(\mathfrak{m}_1+w-\mathfrak{u}_1)L_{\rho+1}(\mathfrak{m}_1+w-\mathfrak{u}_1)+(\mathfrak{m}_2-\mathfrak{m}_1+2w)L_{\rho+1}\left(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{8(\rho+1)}$$
$$\left.+\frac{L_\rho(\frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2})-L_\rho(\mathfrak{m}_1+w-\mathfrak{u}_1)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}+\frac{L_\rho(\mathfrak{m}_2+w-\mathfrak{u}_2)-L_\rho(\frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2})}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}\right|$$
$$\leq\frac{\left(2\mathfrak{d}^2-2\mathfrak{d}+1\right)(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1)}{32(\rho+1)(\rho+2)}\left(\mathfrak{m}_1^2L_{\rho+2}(\mathfrak{m}_1)+2(\rho+2)L_{\rho+1}(\mathfrak{m}_1)\right)$$
$$+\frac{\left(2\mathfrak{d}^2-2\mathfrak{d}+1\right)(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2)}{32(\rho+1)(\rho+2)}\left(\mathfrak{m}_2^2L_{\rho+2}(\mathfrak{m}_2)+2(\rho+2)L_{\rho+1}(\mathfrak{m}_2)\right)$$
$$+\frac{\left(2\mathfrak{d}^2-2\mathfrak{d}+1\right)(\mathfrak{u}_2-\mathfrak{u}_1)}{16(\rho+1)(\rho+2)}\left(w^2L_{\rho+2}(w)+2(\rho+2)L_{\rho+1}(w)\right)$$
$$+\frac{\left(2\mathfrak{d}^3-6\mathfrak{d}^2+3\mathfrak{d}-1\right)(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1)}{96(\rho+1)(\rho+2)}\left(\mathfrak{u}_1^2L_{\rho+2}(\mathfrak{u}_1)+2(\rho+2)L_{\rho+1}(\mathfrak{u}_1)\right)$$

$$+\frac{\left(-2\mathfrak{d}^3+3\mathfrak{d}-2\right)(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2)}{96(\rho+1)(\rho+2)}\left(\mathfrak{u}_2^2L_{\rho+2}(\mathfrak{u}_2)+2(\rho+2)L_{\rho+1}(\mathfrak{u}_2)\right)$$
$$+\frac{\left(2\mathfrak{d}^3-6\mathfrak{d}^2+3\mathfrak{d}-1\right)(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2)-\left(2\mathfrak{d}^3-3\mathfrak{d}+2\right)(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1)}{384(\rho+1)(\rho+2)}$$
$$\times\left((\mathfrak{m}_1+\mathfrak{m}_2)^2L_{\rho+2}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)+8(\rho+2)L_{\rho+1}\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right).$$

5.2. $\mathfrak{h}$-divergence measures

Let the set ϕ and the σ finite measure μ be given, and let the set of all probability densities on μ be defined on $\Omega := \{\chi|\chi:\phi\to\mathbb{R},\chi(\varpi)>0,\int_\phi\chi(\varpi)d\mu(\varpi)=1\}$. Let $\mathfrak{h}:\mathbb{R}^+\to\mathbb{R}$ be given mapping and consider $D_\mathfrak{h}(\chi,\psi)$ defined by:

$$D_\mathfrak{h}(\chi,\psi):=\int_\phi\chi(\varpi)\mathfrak{h}\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)d\mu(\varpi),\ \ \chi,\psi\in\Omega. \tag{5.1}$$

If $\mathfrak{h}$ is convex, then (5.1) is called Csisźar $\mathfrak{h}$-divergence. Consider the following Hermite-Hadamard (HH) divergence:

$$D^\mathfrak{h}_{HH}(\chi,\psi):=\int_\phi\chi(\varpi)\frac{\int_1^{\frac{\psi(\varpi)}{\chi(\varpi)}}\mathfrak{h}(\mathfrak{t})d\mathfrak{t}}{\frac{\psi(\varpi)}{\chi(\varpi)}-1}d\mu(\varpi),\ \ \chi,\psi\in\Omega, \tag{5.2}$$

where $\mathfrak{h}$ is convex on $\mathbb{R}^+$ with $\mathfrak{h}(1)=0$. Consider $D_v(\chi,\psi)$ defined by:

$$D_v(\chi,\psi)=\int_\phi|\chi(\varpi)-\psi(\varpi)|d\mu(\varpi), \tag{5.3}$$

so-called variation distance. Note that $D^\mathfrak{h}_{HH}(\chi,\psi)\geq 0$ with equality holds if and only if $\chi=\psi$.

Proposition 4. *Let $\mathfrak{h}:I\subseteq\mathbb{R}^+\to\mathbb{R}$ be a differentiable function on I°, interior of I, $\mathfrak{m}_1,\mathfrak{m}_2\in I^\circ$ such that $|\mathfrak{h}'|$ is convex and $\mathfrak{h}(1)=0$, then*

$$\left|\frac{2D_\mathfrak{h}\left(\chi,\frac{\psi+\chi}{2}\right)+D_\mathfrak{h}(\chi,\psi)}{4}-D^\mathfrak{h}_{HH}(\chi,\psi)\right|$$
$$\leq\frac{|\mathfrak{h}'(1)|\,D_v(\chi,\psi)}{32}+\int_\phi\frac{|\psi(\varpi)-\chi(\varpi)|\left\{\left|\mathfrak{h}'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|+2\left|\mathfrak{h}'\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right)\right|\right\}}{32}d\mu(\varpi). \tag{5.4}$$

Proof. Let $\Phi_1:=\{\varpi\in\phi:\psi(\varpi)>\chi(\varpi)\}$; $\Phi_2:=\{\varpi\in\phi:\psi(\varpi)<\chi(\varpi)\}$ and $\Phi_3:=\{\varpi\in\phi:\psi(\varpi)=\chi(\varpi)\}$. Obviously, if $\varpi\in\Phi_3$, then equality holds in (5.4). Now, if $\varpi\in\Phi_1$, then for $\mathfrak{u}_1=\mathfrak{m}_1$, $w=\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}$; $\mathfrak{m}_1=\mathfrak{a}=1$; $\mathfrak{u}_2=\mathfrak{m}_2=\frac{\psi(\varpi)}{\chi(\varpi)}$; $\mathfrak{d}=\frac{1}{2}$ in Theorem 2, multiplying both sides by the obtained result by $\chi(\varpi)$ and integrating over Φ_1, we have

$$\left|\frac{1}{2}\int_{\Phi_1}\chi(\varpi)\mathfrak{h}\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right)d\mu(\varpi)+\frac{1}{4}\int_{\Phi_1}\chi(\varpi)\mathfrak{h}\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)d\mu(\varpi)-\int_{\Phi_1}\frac{\chi(\varpi)\int_1^{\frac{\psi(\varpi)}{\chi(\varpi)}}\mathfrak{h}(\mathfrak{t})d\mathfrak{t}}{\frac{\psi(\varpi)}{\chi(\varpi)}-1}d\mu(\varpi)\right|$$

$$\leq \int_{\Phi_1} \frac{\psi(\varpi)-\chi(\varpi)}{32}\left\{|\mathfrak{h}'(1)|+\left|\mathfrak{h}'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|+2\left|\mathfrak{h}'\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right)\right|\right\} d\mu(\varpi). \tag{5.5}$$

Similarly, if $\varpi \in \Phi_2$, then for $\mathfrak{u}_1 = \mathfrak{m}_1 = \frac{\psi(\varpi)}{\chi(\varpi)}$, $w = \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}$; $\mathfrak{a} = 1$; $\mathfrak{u}_2 = \mathfrak{m}_2 = 1$; $\mathfrak{d} = \frac{1}{2}$ in Theorem 2, multiplying both sides by the obtained result by $\chi(\varpi)$ and integrating over Φ_2, we have

$$\left|\frac{1}{2}\int_{\Phi_2} \chi(\varpi)\mathfrak{h}\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right) d\mu(\varpi)+\frac{1}{4}\int_{\Phi_2} \chi(\varpi)\mathfrak{h}\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right) d\mu(\varpi)-\int_{\Phi_2} \frac{\chi(\varpi)\int_1^{\frac{\psi(\varpi)}{\chi(\varpi)}} \mathfrak{h}(\mathfrak{t})d\mathfrak{t}}{\frac{\psi(\varpi)}{\chi(\varpi)}-1} d\mu(\varpi)\right|$$
$$\leq \int_{\Phi_2} \frac{\chi(\varpi)-\psi(\varpi)}{32}\left\{|\mathfrak{h}'(1)|+\left|\mathfrak{h}'\left(\frac{\psi(\varpi)}{\chi(\varpi)}\right)\right|+2\left|\mathfrak{h}'\left(\frac{\psi(\varpi)+\chi(\varpi)}{2\chi(\varpi)}\right)\right|\right\} d\mu(\varpi). \tag{5.6}$$

Adding inequalities (5.5) and (5.6) and utilizing triangular inequality, we obtain the desired result (5.4). □

5.3. *Probability density functions*

Let $f : [\mathfrak{m}_1, \mathfrak{m}_2] \to [0, 1]$ be the probability density function of m continuous random variable X with the cumulative distribution function, F, given by:

$$F(\varrho) = Pr(X \leq \varrho) = \int_{\mathfrak{m}_1}^{\varrho} f(\mathfrak{t})d\mathfrak{t} \ \text{ and } E(X) = \int_{\mathfrak{m}_1}^{\mathfrak{m}_2} \mathfrak{t}dF(\mathfrak{t}) = \mathfrak{m}_2 - \int_{\mathfrak{m}_1}^{\mathfrak{m}_2} F(\mathfrak{t})d\mathfrak{t}.$$

Then, from Theorem 2 for $\mathfrak{a} = 1$, we have the following result:

$$\left|\frac{(1-\mathfrak{d})\left[Pr\left(X \leq \mathfrak{m}_2+w-\mathfrak{u}_2\right)+Pr\left(X \leq \frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}\right)\right]}{2}+\frac{\mathfrak{d}\left[Pr\left(X \leq \mathfrak{m}_1+w-\mathfrak{u}_1\right)+Pr\left(X \leq \frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)\right]}{2}\right.$$
$$\left.-\frac{Pr\left(X \leq \mathfrak{m}_1+w-\mathfrak{u}_1\right)-Pr\left(X \leq \frac{\mathfrak{m}_1-\mathfrak{m}_2+2w}{2}\right)}{\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1}+\frac{Pr\left(X \leq \mathfrak{m}_2+w-\mathfrak{u}_2\right)-Pr\left(X \leq \frac{\mathfrak{m}_2-\mathfrak{m}_1+2w}{2}\right)}{2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2}\right|$$
$$\leq \frac{\left(2\mathfrak{d}^2-2\mathfrak{d}+1\right)\left\{\left(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1\right)|f(\mathfrak{m}_1)|+\left(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2\right)|f(\mathfrak{m}_2)|+2\left(\mathfrak{u}_2-\mathfrak{u}_1\right)|f(w)|\right\}}{8}$$
$$+\frac{\left(2\mathfrak{d}^3-6\mathfrak{d}^2+3\mathfrak{d}-1\right)\left(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1\right)|f(\mathfrak{u}_1)|+\left(-2\mathfrak{d}^3+3\mathfrak{d}-2\right)\left(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2\right)|f(\mathfrak{u}_2)|}{24}$$
$$+\frac{\left(2\mathfrak{d}^3-6\mathfrak{d}^2+3\mathfrak{d}-1\right)(2\mathfrak{u}_2-\mathfrak{m}_1-\mathfrak{m}_2)-\left(2\mathfrak{d}^3-3\mathfrak{d}+2\right)\left(\mathfrak{m}_1+\mathfrak{m}_2-2\mathfrak{u}_1\right)}{24}\left|f\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|. \tag{5.7}$$

In particular, for $\mathfrak{u}_1 = \mathfrak{m}_1$, $\mathfrak{u}_2 = \mathfrak{m}_2$, $\mathfrak{d} = \frac{1}{2}$ and $w = \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}$, (5.7) reduces to

$$\left|\frac{Pr\left(X \leq \mathfrak{m}_1\right)+Pr\left(X \leq \mathfrak{m}_2\right)+2Pr\left(X \leq \frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)}{4}-\frac{\mathfrak{m}_2-E(X)}{\mathfrak{m}_2-\mathfrak{m}_1}\right| \leq \frac{(\mathfrak{m}_2-\mathfrak{m}_1)\left(|f(\mathfrak{m}_1)|+|f(\mathfrak{m}_2)|+2\left|f\left(\frac{\mathfrak{m}_1+\mathfrak{m}_2}{2}\right)\right|\right)}{32}.$$

6. Concluding remarks

By constructing a multi-parameter fractional integral identity in the form of the Caputo-Fabrizio fractional integral operator, we have generated some new generalized estimates for fractional Bullen-type inequalities by using convexity, log-convexity, Hölder inequality, and power mean inequality. We have also included numerical and graphical examples to demonstrate the correctness of the generated results. Additionally, modified Bessel functions, $\mathfrak{h}$-divergence measures, and probability density functions are given as implementations of the resulting conclusions. It is anticipated that the paper's findings will pique readers's interest.

Author contributions

Sabir Hussain and Jongsuk Ro: Conceptualization, formal analysis; Sobia Rafeeq and Sabir Hussain: Methodology, writing-original draft preparation, validation; Sobia Rafeeq: Software, investigation; Jongsuk Ro: Resources; Sobia Rafeeq, Sabir Hussain and Jongsuk Ro: Writing-review and editing; Sobia Rafeeq and Jongsuk Ro: Visualization. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. NRF-2022R1A2C2004874). This work was also supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. NRF-2022R1A2C2004874).

Conflict of interest

The authors declare no conflict of interest.

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