AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 24576–24589.
DOI: 10.3934/math.20241197
Received: 06 June 2024
Revised: 31 July 2024
Accepted: 13 August 2024
Published: 21 August 2024

Research article

Fourth-order neutral dynamic equations oscillate on timescales with different arguments

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Abstract: The theory of neutral dynamic equations on timescales was based to unify the study of differential and difference equations. The article described several oscillating criteria that will be developed for fourth-order-neutral dynamic equations in the presence of various types of arguments on timescales. The goal was to establish all necessary conditions for the solutions of these models to be oscillatory. To construct observation values, ideas from [Y. Sui and Z. Han, Oscillation of second order neutral dynamic equations with deviating arguments on time scales, *Adv. Differ. Equ.*, 10 (2018)] were used. The research seeked to provide sufficient criteria that ensured the oscillation of solutions to these complex dynamic equations using a technique Riccati transformations generalized, emphasizing their importance in the study of oscillatory processes within various scientific and engineering contexts.

Keywords: timescales; oscillation; neutral; deviating arguments; nonlinear equations
Mathematics Subject Classification: 26E70, 34C10, 34K40

1. Introduction

The study of neutral dynamic systems that describe oscillatory processes is currently one of the recognized areas of modern nonlinear dynamics. Analysis of the connectivity of systems based on their time series is one of the current areas of modern science. The classical approach for this purpose is the Granger causality method [1]. The main idea of this method is to build a predictive model and if data from the first time series helps to more accurately predict the behavior of the second, then the first system is considered to influence the second. For a review of various variants of the Granger causality method and similar measures, see [2]. Hilger [3] proposed the timescale hypothesis to connect, broaden and summarize ideas from discrete math, quantum analytics, and persistent math to self-assured timescale analytics. Bohner and Peterson's books on timescale, or measure chain [4,5], summarize and compile a lot of timescale mathematics. The applied hypothesis of dynamic conditions associated with the study of oscillatory miracles in creativity, normal behavior, and sociologies includes the hypothesis of movements as a key component. An essential component of the connected hypothesis of energetic conditions associated with the inquiry of oscillatory wonders in innovation, normal science and social science is the notion of oscillations, see [6–8]. In recent years, the evolution of different energy situations over timescales has attracted a lot of interest.

Let $t \in \mathcal{I}_{t_0} = [t_0, +\infty) \cap \mathbb{T}$. We explore the dynamic equation of the type

$$\left(q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu}\right)^{\Delta} + \Phi\left(t, u^{\nu}(\upsilon(t))\right) = 0, \tag{1.1}$$

where

$$\varphi(t) = u(t) + p(t)u(\tau(t)),$$

and $\nu \geq 1$ is the ratio of two odd positive integers. The Eq (1.1) will be studied under the following assumptions:

(C_1) The function $\Phi \in C(\mathbb{R}, \mathbb{R})$ and there exists $r \in C^1(\mathcal{I}_{t_0}, [0, +\infty))$, such that

$$\frac{\Phi(t, u)}{u} \geq r(t), \forall t \in \mathcal{I}_{t_0} \text{ and } u \in \mathbb{R} - \{0\}.$$

(C_2) $\tau, \upsilon \in C^1_{rd}(\mathcal{I}_{t_0}, \mathcal{I}_{t_0}), p \in C^1(\mathcal{I}_{t_0}, [0, +\infty))$ such that

$$\tau(t) \leq t \leq \upsilon(t), \quad \tau^{\Delta}(t) > 0, \quad p^{\Delta}(t) \leq 0, \quad \text{for all } t \in \mathcal{I}_{t_0},$$

and

$$\lim_{t\to+\infty} \tau(t) = \lim_{t\to+\infty} \upsilon(t) = \infty.$$

By the solution of (1.1), we mean a nontrivial real-valued function

$$\varphi \in C^4\left(\mathcal{I}_{T_\varphi}, \mathbb{R}\right), \ T_\varphi \in \mathcal{I}_{s_0}.$$

Note that solutions which disappear in a neighborhood of infinity will be excluded from our consideration. A solution φ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is non-oscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory. We are interested in this type of research because there has recently been an increasing interest in obtaining suitable conditions for the oscillation and non-oscillation of combinations of various energy conditions on timescales. See, for example, the references [9–11].

J. Džurina et al. [9] studied oscillation for the second-order noncanonical delay differential equations

$$\left(r(t)(y'(t))^{\gamma}\right)' + q(t)\, y^{\gamma}(\tau(y)) = 0, \qquad t \in \mathcal{I}_{t_0},$$

under the condition

$$\int_{t_0}^{\infty} r^{-1/\gamma}(t)\, dt < \infty.$$

There have been any recent occurrences on the oscillatory of (1.1). The purpose of this research is to offer a few ambiguous criteria for this condition. Zhang et al. [10] studied a class of second-order nonlinear delay dynamic equations of neutral theory that was investigated by

$$\left[r(t)\left(\varphi^{\Delta}(t)\right)^{\nu}\right]^{\Delta} + q(t)\, \Phi(u(\delta(t))) = 0, \qquad t \in \mathcal{I}_{t_0},$$

where $\nu \geq 1$ is a ratio of odd integers and $\varphi(t) = u(t) - p(t)\, u(\tau(t))$.

Grace et al. [11] studied oscillation of fourth-order delay differential equations

$$\left(r_3(r_2(r_1 y')')'\right)'(t) + q(t)\, y(\tau(y)) = 0, \qquad t \in \mathcal{I}_{t_0},$$

under the assumption

$$\int_{t_0}^{\infty} r_i^{-1}(t)\, dt < \infty, \qquad \text{for } i \in \{1, 2, 3\}.$$

Some of the most common examples of calculus on timescales are differential calculus, difference calculus, and quantum calculus. Timescale dynamical equations have enormous potential for applications such as population dynamics. It can model insect populations that persist during a season, die in winter while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. There are applications of dynamical equations on timescales in quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. Notice that the model of problem (1.1) is more general than all the problems considered before. For example, if we consider problem (1), $\mathbb{T} = \mathbb{R}$, and $\Phi(t, u(t)) = q(t)\, u(t)$, we obtain the problem which is considered by a recent article by J. Džurina et al. in [9]; it discusses several possible applications.

2. Main results

In this section, we establish some criteria that guarantee that u of (1.1) oscillates on $\mathcal{I}_{t_0}$. Before stating the main results, we begin with the following lemma.

Lemma 2.1. *[12] Assume that u is an eventually positive solution of* (1.1). *Then, there are only the following three possible scenarios for* $\mathcal{I}_{t_1}$, *where* $t_1 \in \mathcal{I}_{t_0}$ *is sufficiently large:*

(1) $\varphi^{\Delta^3}(t) \geq 0, \varphi^{\Delta^2}(t) \leq 0, \varphi^{\Delta}(t) \geq 0,$
(2) $\varphi^{\Delta^3}(t) \leq 0, \varphi^{\Delta^2}(t) \geq 0, \varphi^{\Delta}(t) \geq 0,$
(3) $\varphi^{\Delta^3}(t) \leq 0, \varphi^{\Delta^2}(t) \geq 0, \varphi^{\Delta}(t) \leq 0.$

Here,

$$\eta_+(t) = \max\{\eta(t), 0\}, \quad \Lambda\eta(t) = \frac{\eta^\sigma(t)}{\eta(t)}, \quad \widetilde{\nu} = \frac{1}{\nu},$$

$$\pi_0(t) = \int_t^\infty q^{-\widetilde{\nu}}(s), \quad \pi_i(t) = \int_t^\infty \pi_{i-1}(s)\Delta s, \quad \text{for } i \in \{1,2\},$$

$$\xi(t) = \frac{1}{1+p(\upsilon(t))}\int_t^\infty q^{-\widetilde{\nu}}(v)\left(\int_v^\infty r(u)\Delta u\right)^{\widetilde{\nu}}\Delta v,$$

$$\beta(t) = \frac{q^{-\widetilde{\nu}}(t)}{1+p(\upsilon(t))}\left(\int_t^{+\infty} r(s)\Delta s\right)^{\widetilde{\nu}}, \quad P(t) = r(t)\left(1 - p(\upsilon(t))\frac{\pi_2(\upsilon(\tau(t)))}{\pi_2(\upsilon(t))}\right)^{\nu},$$

$$\phi(t) = P(t) + \frac{\pi_1(t)}{\pi_2^\sigma(t)\pi_2^\nu(t)} - \frac{\nu\pi_1(t)}{\pi_2(t)\pi_2^\nu(\sigma(t))}.$$

Theorem 2.1. *Assume that there exist three functions $\zeta_1, \zeta_2, \gamma \in C_{rd}^1(\mathcal{I}_{t_0}, (0,\infty))$ such that*

$$\int_{t_1}^\infty \zeta_1^\sigma(t)\xi(t) - \Lambda(t-t_1)\frac{\left(\zeta_1^\Delta(t)\right)^2}{4\zeta_1^\sigma(t)}\Delta t = \infty, \tag{2.1}$$

$$\int_{t_1}^\infty \gamma^\sigma(t)\beta(t) - \frac{\Lambda h_2(t,t_1)\left(\gamma^\Delta(t)\right)^2}{4(t-t_1)\gamma^\sigma(t)}\Delta t = \infty, \tag{2.2}$$

and

$$\int_{t_1}^\infty \zeta_2^\sigma(t)\phi(t) - \frac{\psi^{\nu+1}(t)\eta^\nu(t)}{(\nu+1)^{\nu+1}}\Delta t = \infty, \tag{2.3}$$

where

$$\psi(t) = \zeta_{2+}^\Delta(t) + (1+\nu)\zeta_2^\sigma(t)\frac{\pi_1(t)}{\pi_2^\sigma(t)}, \quad \eta(t) = \frac{\Lambda\pi_2(t)}{\zeta_2^\sigma(t)\pi_1(t)}.$$

Then every solution u of (1.1) *is oscillatory.*

Proof. Suppose, on the contrary, that u is a positive solution of (1.1) on $[t_0, +\infty)$. Then, there exists $t_1 \in \mathcal{I}_{t_0}$, such that $u(\tau(t)) > 0$, $u(\upsilon(t)) > 0$, and $u(t) > 0$, for all $t \geq t$. Assume that φ fulfills (2.1) in the claims of Lemma 2.1. Let

$$\Xi(t) = \zeta_1(t)\frac{\varphi^\Delta(t)}{\varphi(t)}, \qquad \text{for all } t \in \mathcal{I}_{t_1}.$$

Then, $\Xi(t) > 0$, for $t \in \mathcal{I}_{t_1}$ and

$$\Xi^\Delta(t) = \frac{\zeta_1^\Delta(t)}{\zeta_1(t)}\Xi(t) + \zeta_1^\sigma(t)\frac{\varphi^{\Delta^2}(t)}{\varphi(t)} - \zeta_1^\sigma(t)\frac{\left(\varphi^\Delta(t)\right)^2}{\varphi^\sigma(t)\varphi(t)}. \tag{2.4}$$

Since $\tau \in C^1(\mathcal{I}_{t_0}, \mathcal{I}_{t_0})$, then $u \circ \tau \in C^1(\mathcal{I}_{t_0}, \mathcal{I}_{t_0})$, and we have

$$(u\circ\tau)^\Delta(t) = \tau^\Delta(t)u^\Delta(\tau(t)), \qquad \text{for all } t \in \mathcal{I}_{t_1}.$$

Then

$$\varphi^{\Delta}(t) = u^{\Delta}(t) + p^{\sigma}(t)\,\tau^{\Delta}(t)\,u^{\Delta}(\tau(t)) + p^{\Delta}(t)\,u(\tau(t)) \geq 0,\ \forall t \in \mathcal{I}_{t_1}.$$

Since $p^{\Delta}(t) \leq 0$ and $\tau^{\Delta}(t) \geq 0$, for all $t \in \mathcal{I}_{t_1}$, then

$$u^{\Delta}(\tau(t))\,u^{\Delta}(t) \geq 0, \qquad \text{for all } t \geq \mathcal{I}_{t_1}.$$

Since τ is a bijective function, we obtain

$$u^{\Delta}(t) \geq 0, \qquad \text{for all } t \geq \mathcal{I}_{t_1}.$$

Integrate (1.1) from t to s, we get

$$-r(t)\left(\varphi^{\Delta^3}\right)^{\nu}(t) + r(t)\left(\varphi^{\Delta^3}\right)^{\nu}(s) = \int_s^t r(\tau)\,u^{\nu}(\upsilon(\tau))\,\Delta\tau.$$

When s tends to ∞ in the above inequality, we obtain

$$q(t)\left(\varphi^{\Delta^3}\right)^{\nu}(t) \geq \int_t^{\infty} r(\tau)\,u^{\nu}(\upsilon(\tau))\,\Delta\tau, \quad \text{for all } t \geq \mathcal{I}_{t_1},$$

which implies that

$$\varphi^{\Delta^3}(t) \geq u(\upsilon(t))\left(\frac{1}{q(t)}\int_t^{\infty} r(u)\,\Delta u\right)^{\widetilde{\nu}}, \quad \text{for all } t \geq \mathcal{I}_{t_1}.$$

Integrate from t to ∞ to get

$$\begin{aligned} -\varphi^{\Delta^2}(t) &\geq \int_t^{\infty} u(\upsilon(v))\left(\frac{1}{r(v)}\int_v^{\infty} q(u)\,\Delta u\right)^{\widetilde{\nu}}\Delta v \\ &\geq u(\upsilon(t))\int_t^{\infty}\left(\frac{1}{r(v)}\int_v^{\infty} q(u)\,\Delta u\right)^{\widetilde{\nu}}\Delta v. \end{aligned} \tag{2.5}$$

Since $u^{\Delta}(t) \geq 0$ and $\tau(t) \leq t$, for all $t \in \mathcal{I}_{t_1}$, then

$$u(t) \geq \frac{\varphi(t)}{1+p(t)}, \qquad \text{for all } t \geq \mathcal{I}_{t_1}. \tag{2.6}$$

Substituting (2.6) in (2.5), we get

$$\begin{aligned} -\varphi^{\Delta^2}(t) &\geq \frac{\varphi(\upsilon(t))}{1+p(\upsilon(t))}\int_t^{\infty}\left(\frac{1}{r(v)}\int_v^{\infty} q(u)\,\Delta u\right)^{\widetilde{\nu}}\Delta v \\ &\geq \frac{\varphi(t)}{1+p(\upsilon(t))}\int_t^{\infty}\left(\frac{1}{r(v)}\int_v^{\infty} q(u)\,\Delta u\right)^{\widetilde{\nu}}\Delta v = \xi(t)\,\varphi(t). \end{aligned} \tag{2.7}$$

Substituting (2.7) in (2.4), we get

$$\Xi^{\Delta}(t) \leq -\zeta_1^{\sigma}(t)\,\xi(t) + \frac{\zeta_{1+}^{\Delta}(t)}{\zeta_1(t)}\Xi(t) - \zeta_1^{\sigma}(t)\,\frac{\left(\varphi^{\Delta}(t)\right)^2}{\varphi^{\sigma}(t)\,\varphi(t)}. \tag{2.8}$$

Since $\varphi^{\Delta^2}(t) \leq 0$, for all $t \in \mathcal{I}_{t_1}$, we have

$$\varphi(t) \geq \int_{t_1}^{t} \varphi^{\Delta}(s)\,\Delta s \geq \varphi^{\Delta}(t)(t-t_1), \quad \text{for all } t \in \mathcal{I}_{t_1}.$$

Therefore,

$$t \to \frac{\varphi}{t-t_1},$$

is nonincreasing on $\mathcal{I}_{t_1}$, then

$$\frac{\varphi^{\sigma}(t)}{\sigma(t)-t_1} \leq \frac{\varphi(t)}{t-t_1}, \forall t \in \mathcal{I}_{t_1}. \tag{2.9}$$

Substituting (2.9) in (2.8), we have

$$\Xi^{\Delta}(t) \leq -\zeta_1^{\sigma}(t)\xi(t) + \frac{\left[\zeta_1^{\Delta}(t)\right]_+}{\zeta_1(t)}u(t) - \Lambda(t-t_1)\Lambda\zeta_1(t)\frac{\Xi^2(t)}{\zeta_1(t)}.$$

As in [12], we obtain

$$\Xi^{\Delta}(t) \leq -\zeta_1^{\sigma}(t)\xi(t) + \frac{\Lambda(t-t_1)\left(\zeta_1^{\Delta}(t)\right)^2}{4\zeta_1^{\sigma}(t)}.$$

Integrating from t_1 to t, we have

$$\int_{t_1}^{t} \zeta_1^{\sigma}(s)\xi(s) - \frac{\Lambda(s-t_1)\left(\zeta_1^{\Delta}(s)\right)^2}{4\zeta_1^{\sigma}(s)}\Delta s \leq \Xi(t_2),$$

which contradicts (2.1).
Suppose that φ satisfies 2.1 of Lemma 2.1. Integrating (1.1) from t to ∞, we get

$$-q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu} \geq \int_{t}^{+\infty} r(s)\,u^{\nu}(\upsilon(s))\,\Delta s, \qquad \text{for all } t \in \mathcal{I}_{t_1}.$$

Since $\varphi^{\Delta}(t) \geq 0$, for all $t \in \mathcal{I}_{t_1}$, u is decreasing on $\mathcal{I}_{t_1}$, and by (2.6), we have

$$\begin{aligned} -\varphi^{\Delta^3}(t) &\geq u(\upsilon(t))\left(\frac{1}{q(t)}\int_{t}^{+\infty} r(s)\,\Delta s\right)^{\widetilde{\nu}} \\ &\geq \frac{\varphi(\upsilon(t))}{1+p(\upsilon(t))}\left(\frac{1}{q(t)}\int_{t}^{+\infty} r(s)\,\Delta s\right)^{\widetilde{\nu}} \\ &= \beta(t)\varphi(\upsilon(t)) \\ &\geq \beta(t)\varphi(t), \ \forall t \in \mathcal{I}_{t_1}. \end{aligned}$$

Let

$$\Theta(t) = \zeta_3(t)\frac{\varphi^{\Delta^2}(t)}{\varphi(t)}, \ \forall t \in \mathcal{I}_{t_1}.$$

Then,

$$\Theta^{\Delta}(t) = \frac{\zeta_3^{\Delta}(t)}{\zeta_2(t)}\Theta(t) + \zeta_3^{\sigma}(t)\frac{\varphi^{\Delta^3}(t)}{\varphi(t)} - \Lambda\zeta_3(t)\frac{\varphi^{\Delta}(t)}{\varphi^{\sigma}(t)}\Theta(t)$$

$$\leq \quad -\zeta_3^{\sigma}(t)\beta(t) + \frac{\left[\zeta_3^{\Delta}(t)\right]_+}{\zeta_3(t)}\Theta(t) - \Lambda\zeta_3(t)\frac{\varphi^{\Delta}(t)}{\varphi^{\sigma}(t)}\Theta(t). \tag{2.10}$$

Then, by $\varphi^{\Delta^3}(t) \leq 0,\ \forall t \in \mathcal{I}_{t_1}$, we have

$$\varphi^{\Delta}(t) \geq \int_{t_1}^{t} \varphi^{\Delta^2}(s)\,\Delta s \geq \varphi^{\Delta^2}(t)(t-t_1),\ \forall t \in \mathcal{I}_{t_1}. \tag{2.11}$$

Therefore, $\frac{\varphi^{\Delta}}{t-t_1}$ is a nonincreasing function $\mathcal{I}_{t_1}$, then

$$\begin{aligned} \varphi(t) &\geq \frac{\varphi^{\Delta}(t)}{t-t_1}\int_{t_1}^{t}(s-t_1)\,\Delta s \\ &\geq \frac{h_2(t,t_1)}{t-t_1}\varphi^{\Delta}(t), \quad \text{for all } t \in \mathcal{I}_{t_1}. \end{aligned}$$

Thus $\frac{\varphi}{h_2(.,t_1)}$is a nonincreasing function $\mathcal{I}_{t_1}$, and we get

$$\frac{\varphi^{\sigma}(t)}{h_2^{\sigma}(t,t_1)} \leq \frac{\varphi(t)}{h_2(t,t_1)}, \quad \text{for all } t \in \mathcal{I}_{t_1}. \tag{2.12}$$

Substituting (2.12) in (2.10), we get

$$\Theta^{\Delta}(t) \leq -\zeta_3^{\sigma}(t)\beta(t) + \frac{\zeta_{3+}^{\Delta}(t)}{\zeta_3(t)}\Theta(t) - (t-t_1)\frac{\Lambda\zeta_3(t)}{\zeta_3(t)\Lambda h_2(t,t_1)}\Theta^2(t).$$

As in [12], we obtain

$$\Theta^{\Delta}(t) \leq -\zeta_3^{\sigma}(t)\beta(t) + \frac{\Lambda h_2(t,t_1)\left(\zeta_3^{\Delta}(t)\right)^2}{4(t-t_1)\zeta_3^{\sigma}(t)}.$$

Integrate from t_1 to t, we have

$$\int_{t_1}^{t} \zeta_3^{\sigma}(s)\beta(s) - \frac{\Lambda h_2(s,t_1)\left(\zeta_3^{\Delta}(s)\right)^2}{4(s-t_1)\zeta_3^{\sigma}(s)}\Delta s \leq \Theta(t_1),$$

which contradicts (2.2).

Suppose that u satisfies claim 2.1 of Lemma 2.1. By the definition of φ, we have $\varphi(t) \geq u(t)$, and from (1.1), we have

$$\left(q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu}\right)^{\Delta} = -\Phi(t, u^{\nu}(\tau(t))) \leq -r(t)u^{\nu}(\upsilon(t)) \leq 0,\ \forall t \in \mathcal{I}_{t_0}. \tag{2.13}$$

Thus,

$$t \to q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu},$$

is decreasing on $\mathcal{I}_{t_1}$, then, for any $s \geq \mathcal{I}_{t_1}$, we have

$$q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu} \geq p(s)\left(\varphi^{\Delta^3}(s)\right)^{\nu}, \quad \text{for all } t \in \mathcal{I}_{t_0}.$$

Then,

$$\varphi^{\Delta^3}(s) \leq \left(\frac{q(t)}{q(s)}\right)^{\widetilde{\nu}} \varphi^{\Delta^3}(t), \quad \text{for all } s, t \in \mathcal{I}_{t_0}.$$

Integrating from t to v,

$$\varphi^{\Delta^2}(v) - \varphi^{\Delta^2}(t) \leq q^{\widetilde{\nu}}(t)\,\varphi^{\Delta^3}(t) \int_t^v q^{-\widetilde{\nu}}(s)\,\Delta s.$$

When v tends to ∞ in the above inequality, we obtain

$$\begin{aligned} \varphi^{\Delta^2}(t) &\geq -q^{\widetilde{\nu}}(t)\left(\int_t^\infty q^{-\widetilde{\nu}}(s)\,\Delta s\right)\varphi^{\Delta^3}(t) \\ &= -q^{\widetilde{\nu}}(t)\,\pi_0(t)\,\varphi^{\Delta^3}(t), \end{aligned} \tag{2.14}$$

thus

$$\left(\frac{\varphi^{\Delta^2}(t)}{\pi_0(t)}\right)^{\Delta} = \frac{\varphi^{\Delta^3}(t)\,\pi_0(t) + q^{-\widetilde{\nu}}(t)\,\varphi^{\Delta^2}(t)}{\pi_0(t)\,\pi_0^{\sigma}(t)} \geq 0.$$

Therefore,

$$t \to \frac{\varphi^{\Delta^2}}{\pi_0},$$

is an increasing function on $\mathcal{I}_{t_1}$. Thus, we get

$$\begin{aligned} -\varphi^{\Delta}(t) &\geq \int_t^\infty \pi_0(s)\,\frac{\varphi^{\Delta^2}(s)}{\pi_0(s)}\Delta s \\ &\geq \frac{\varphi^{\Delta^2}(t)}{\pi_0(t)} \int_t^\infty \pi_0(s)\,\Delta s \\ &= \frac{\pi_1(t)}{\pi_0(t)}\varphi^{\Delta^2}(t). \end{aligned} \tag{2.15}$$

Then,

$$\left(\frac{\varphi^{\Delta}(t)}{\pi_1(t)}\right)^{\Delta} = \frac{\varphi^{\Delta^2}(t)\,\pi_1(t) + \pi_0(t)\,\varphi^{\Delta}(t)}{\pi_1(t)\,\pi_1^{\sigma}(t)} \leq 0.$$

Thus,

$$t \to \frac{\varphi^{\Delta}}{\pi_1},$$

is a nonincreasing function on $\mathcal{I}_{t_1}$, and we have

$$\begin{aligned} -\varphi(t) &\leq \int_t^\infty \varphi^{\Delta}(t)\,\Delta s \\ &\leq \frac{\varphi^{\Delta}(t)}{\pi_1(t)} \int_t^\infty \pi_1(s)\,\Delta s = \frac{\pi_2(t)}{\pi_1(t)}\varphi^{\Delta}(t), \end{aligned} \tag{2.16}$$

then

$$\left(\frac{\varphi(t)}{\pi_2(t)}\right)^{\Delta} = \frac{\varphi^{\Delta}(t)\,\pi_2(t) + \pi_1(t)\,\varphi(t)}{\pi_2(t)\,\pi_2^{\sigma}(t)} \geq 0. \tag{2.17}$$

Thus,

$$t \to \frac{\varphi}{\pi_2},$$

is an increasing function on $\mathcal{I}_{t_1}$, and we have

$$\varphi(\tau(t)) \leq \frac{\pi_2(\tau(t))}{\pi_2(t)}\varphi(t), \quad \text{for all } t \in \mathcal{I}_{t_1}. \tag{2.18}$$

Moreover, by the definition of φ and (2.18), we have

$$\begin{aligned} u(t) &= \varphi(t) - p(t)\, u(\tau(t)) \\ &\geq \varphi(t) - p(t)\, \varphi(\tau(t)) \\ &\geq \left[1 - p(t)\frac{\pi_2(\tau(t))}{\pi_2(t)}\right]\varphi(t). \end{aligned} \tag{2.19}$$

From (2.19), and (2.13), we get

$$\begin{aligned} \left(q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu}\right)^{\Delta} &\leq -r(t)\left[1 - p(\upsilon(t))\frac{\pi_2(\upsilon(\tau(t)))}{\pi_2(\upsilon(t))}\right]^{\nu}\varphi^{\nu}(\upsilon(t)) \\ &= -P(t)\,\varphi^{\nu}(\upsilon(t)). \end{aligned} \tag{2.20}$$

From (2.14)–(2.16), we obtain

$$\varphi(t) \geq -\pi_2(t)\, q^{\widetilde{\nu}}(t)\, \varphi^{\Delta^3}(t), \quad \text{for all } t \in \mathcal{I}_{t_1}. \tag{2.21}$$

Let

$$\omega(t) = \zeta_2(t)\left[\frac{q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu}}{\varphi^{\nu}(t)} + \pi_2^{-\nu}(t)\right], \quad \text{for all } t \in \mathcal{I}_{t_1}.$$

From (2.21), we have $\omega(t) \geq 0$, for all $t \in \mathcal{I}_{t_1}$ and

$$\begin{aligned} \omega^{\Delta}(t) &= \frac{\zeta_2^{\Delta}(t)}{\zeta_2(t)}\omega(t) + \zeta_2^{\sigma}(t)\frac{\left[q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu}\right]^{\Delta}}{\varphi^{\nu}(t)} \\ &\quad -\zeta_2^{\sigma}(t)\frac{q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu}(\varphi^{\nu}(t))^{\Delta}}{\varphi^{\nu}(t)\,\varphi^{\nu}(\sigma(t))} - \zeta_2^{\sigma}(t)\frac{\left(\pi_2^{\nu}(t)\right)^{\Delta}}{\pi_2^{\nu}(t)\,\pi_2^{\nu}(\sigma(t))}. \end{aligned} \tag{2.22}$$

By Pöotzsche's chain rule [4, Theorem 1.90], we get

$$\begin{aligned} (\varphi^{\nu}(t))^{\Delta} &= \nu\varphi(t)^{\Delta}\int_0^1 (h\varphi(t) + (1-h)\,\varphi^{\sigma}(t))^{\nu-1}\,\Delta h \\ &\leq \nu\varphi(t)^{\Delta}\,\varphi^{\nu-1}(t), \end{aligned} \tag{2.23}$$

and

$$-\left(\pi_2^{\nu}(t)\right)^{\Delta} \leq \nu\pi_1(t)\,\pi_2^{\nu-1}(t), \quad \text{for all } t \in \mathcal{I}_{t_1}. \tag{2.24}$$

Substituting (2.20), (2.23) , and (2.24) in (2.22), we have

$$\begin{aligned}\omega^{\Delta}(t) &= \frac{\zeta_2^{\Delta}(t)}{\zeta_2(t)}\omega(t) - \zeta_2^{\sigma}(t) P(t) \frac{\varphi^{\nu}(\upsilon(t))}{\varphi^{\nu}(t)} \\ &\quad -\nu\zeta_2^{\sigma}(t) \frac{p(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu}\varphi^{\Delta}(t)}{\varphi(t)\varphi^{\nu}(\sigma(t))} + \frac{\nu\zeta_2^{\sigma}(t)\pi_1(t)}{\pi_2(t)\pi_2^{\nu}(\sigma(t))}.\end{aligned} \tag{2.25}$$

Since $\varphi^{\Delta}(t) < 0$ and $\left(\pi_2^{-1}(t)\varphi(t)\right)^{\Delta} > 0$, for all $t \in \mathcal{I}_{t_1}$, we obtain

$$\frac{\varphi^{\nu}(\upsilon(t))}{\varphi^{\nu}(t)} \geq 1 \quad \text{and} \quad \varphi^{\sigma}(t) \geq \frac{\pi_2^{\sigma}(t)}{\pi_2(t)}\varphi(t), \quad \text{for all } t \in \mathcal{I}_{t_1}. \tag{2.26}$$

By (2.14) and (2.15), we get

$$\varphi^{\Delta}(t) \leq \pi_1(t) p^{\widetilde{\nu}}(t)\varphi^{\Delta^3}(t), \quad \text{for all } t \in \mathcal{I}_{t_1}. \tag{2.27}$$

Substituting (2.26) and (2.27) in (2.25), we have

$$\begin{aligned}\omega^{\Delta}(t) &= \frac{\zeta_2^{\Delta}(t)}{\zeta_2(t)}\omega(t) - \zeta_2^{\sigma}(t) P(t) \\ &\quad -\nu\zeta_2^{\sigma}(t)\frac{\pi_1(t)}{T\pi_2(t)}\left[\frac{\omega(t)}{\zeta_2(t)} - \frac{1}{\pi_2^{\nu}(t)}\right]^{1+\widetilde{\nu}} + \frac{\nu\zeta_2^{\sigma}(t)\pi_1(t)}{\pi_2(t)\pi_2^{\nu}(\sigma(t))}.\end{aligned}$$

We conclude that

$$\left[\frac{\omega(t)}{\zeta_2(t)} - \frac{1}{\pi_2^{\nu}(t)}\right]^{1+\widetilde{\nu}} \geq \left(\frac{\omega(t)}{\zeta_2(t)}\right)^{1+\widetilde{\nu}} - (1+\widetilde{\nu})\frac{\omega(t)}{\pi_2(t)\zeta_2(t)} + \widetilde{\nu}\frac{1}{\pi_2^{\nu+1}(t)}.$$

Thus,

$$\begin{aligned}\omega^{\Delta}(t) &= -\left[\zeta_2^{\sigma}(t) P(t) + \frac{\zeta_2^{\sigma}(t)\pi_1(t)}{\pi_2^{\sigma}(t)\pi_2^{\nu}(t)} - \frac{\nu\zeta_2^{\sigma}(t)\pi_1(t)}{\pi_2(t)\pi_2^{\nu}(\sigma(t))}\right] \\ &\quad + \left[\frac{\zeta_{2+}^{\Delta}(t)}{\zeta_2(t)} + (1+\nu)\Lambda\zeta_2(t)\frac{\pi_1(t)}{\pi_2^{\sigma}(t)}\right]\omega(t) - \nu\frac{\Lambda\zeta_2(t)\pi_1(t)}{\zeta_2^{\widetilde{\nu}}(t)\Lambda\pi_2(t)}(\omega(t))^{1+\widetilde{\nu}}.\end{aligned}$$

Then,

$$\begin{aligned}\omega^{\Delta}(t) &= -\zeta_2^{\sigma}(t)\left[P(t) + \frac{\pi_1(t)}{\pi_2^{\sigma}(t)\pi_2^{\nu}(t)} - \frac{\nu\pi_1(t)}{\pi_2(t)\pi_2^{\nu}(\sigma(t))}\right] + \\ &\quad \frac{1}{(\nu+1)^{\nu+1}}\left[\zeta_{2+}^{\Delta}(t) + (1+\nu)\zeta_2^{\sigma}(t)\frac{\pi_1(t)}{\pi_2^{\sigma}(t)}\right]^{\nu+1}\left[\frac{\Lambda\pi_2(t)}{\zeta_2^{\sigma}(t)\pi_1(t)}\right]^{\nu} \\ &= -\zeta_2^{\sigma}(t)\phi(t) + \frac{1}{(\nu+1)^{\nu+1}}\psi^{\nu+1}(t)\eta^{\nu}(t).\end{aligned}$$

Integrating from t_1 to t, we have

$$\int_{t_1}^{t} \zeta_2^{\sigma}(s)\phi(s) - \frac{\psi^{\nu+1}(s)\eta^{\nu}(s)}{(\nu+1)^{\nu+1}}\Delta s \leq \omega(t_1),$$

which contradicts (2.3). □

Corollary 2.1. *If*

$$\int_{t_1}^{\infty} \xi(t)\,\Delta t = \int_{t_1}^{\infty} \phi(t)\,\Delta t = \int_{t_1}^{\infty} \beta(t)\,\Delta t = \infty, \tag{2.28}$$

then every solution u of (1.1) *is oscillatory.*

Proof. Similar to the proof of Theorem 2.1, we put

$$\zeta_1(t) = \zeta_2(t) = \gamma(t) = 1,$$

in Eqs (2.1)–(2.3), and we find (2.28) □

Theorem 2.2. *Assume that there exist three functions $\zeta \in C_{rd}^1(\mathcal{I}_{t_0}, (0,\infty))$ such that* (2.1) *holds and*

$$\pi_0(t_0) = \infty.$$

Then, every solution u of (1.1) *is oscillatory.*

Proof. Suppose, on the contrary, that u is a positive solution of (1.1) on $\mathcal{I}_{t_0}$. Then, there exists $t_1 \in \mathcal{I}_{t_0}$, such that

$$u(\tau(t)) > 0, \quad u(\upsilon(t)) > 0 \quad \text{and} \quad u(t) > 0, \quad \text{for all } t \in \mathcal{I}_{t_1}.$$

From (1.1), we have that $q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu}$ is decreasing for all $t \in \mathcal{I}_{t_1}$. We claim that $q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu} > 0$, for all $t \in \mathcal{I}_{t_1}$. If not, then there exist a $t_2 \in \mathcal{I}_{t_1}$ such that

$$q(t)\left(\varphi^{\Delta^3}(t)\right)^{\nu} \le -m, \qquad \text{for all } \mathcal{I}_{t_2},$$

where $m > 0$. Integrating from t_2 to t, we obtain

$$\varphi^{\Delta^2}(t) \le -m^{\widetilde{\nu}}\pi_0(t_2), \qquad \text{for all } \mathcal{I}_{t_2}.$$

This gives

$$\lim_{t\to-\infty} \varphi^{\Delta^2}(t) = -\infty,$$

then

$$\lim_{t\to-\infty} \varphi(t) = -\infty,$$

which is a contradiction. □

As some applications of the main results, we present the following example.

Example 2.1. *Consider a fourth-order half-linear delay dynamic equation*

$$\left[u(t) + u\left(\frac{t}{2}\right)\right]^{(4)} + \frac{1}{t^3}u(2t) = 0, \qquad \textit{for all } t \ge 1. \tag{2.29}$$

Here, $\mathbb{T} = \mathbb{R}$, $\nu = 1$, $q(t) = p(t) = 1$, $r(t) = t^{-3}$, $\tau(t) = \frac{t}{2}$, and $\upsilon(t) = 2t$.
Set $\zeta(t) = 1$, then

$$\xi(t) = \frac{1}{4t}, \qquad \pi_0(t_0) = \infty.$$

Thus, (2.1) *holds. By Theorem 2.2, Eq* (2.29) *is oscillatory.*

Remark 2.1. *If fourth-order hybrid nonlinear functional dynamic equations with damping on timescale are considered*

$$\left(\frac{q(t)\left(\left[u(t)+p(t)u(\tau(t))\right]^{\Delta^3}\right)^{\nu}}{h(t,u(t))}\right)^{\Delta}+\Phi\left(t,u^{\nu}(\upsilon(t))\right)=0, \forall t\in\mathcal{I}_{t_0}, \tag{2.30}$$

on an arbitrary timescale $\mathbb{T}$ *with* $\sup\mathbb{T}=\infty$*, with the function*

$$h:\mathcal{I}_{t_0}\times\mathbb{R}\to\mathbb{R},$$

such that

$$h\in C\left(\mathcal{I}_{t_0}\times\mathbb{R},\mathbb{R}\right),\ u\Phi(t,u)>0,,\forall\ (t,u)\in\mathcal{I}_{t_0}\times\mathbb{R}-\{0\},$$

and there exists $\psi\in C\left(\mathcal{I}_{t_0},[0,+\infty)\right)$*, so that*

$$h(t,u)\geq\psi(t),\forall\ (t,u)\in\mathcal{I}_{t_0}\times\mathbb{R}-\{0\}.$$

Thus, (1.1) *becomes a special case of* (2.30) *in a case* $h\equiv 1$. *Using the same method presented in this research, we can obtain some oscillation criteria for* (2.30)*. It means obtaining generalizations of Theorems 2.1 and 2.2.*

3. Conclusions

The investigation of sufficient circumstances ensures oscillation of all fourth-order neutral functional differential equation delay solutions. However, this problem remains largely open for future research. We used the generalized Riccati transformations to present a few new theorems for the oscillation of (1.1). Our results here support some well-known results that have recently been published in the literature. A certain number of results reported in the literature are supplemented by the results provided; for more details, see [13–15].

Furthermore, our results can be extended to study a class of systems of higher order hybrid advanced differential equations, for example,

$$\left(q(t)\left(\frac{\varphi^{\Delta^3}(t)}{F(t,u(t))}\right)^{\nu}\right)^{\Delta}+\Phi\left(t,u^{\nu}(\upsilon(t))\right)=0, \tag{3.1}$$

where

$$\varphi(t)=u(t)+p(t)u(\tau(t)),$$

and $\nu\geq 1$ is the ratio of two odd positive integers, see [16–19]. It will be our next work.

Author contributions

Abdelkader Moumen: Methodology, Formal analysis; Amin Benaissa Cherif: Writing-original draft preparation; Fatima Zohra Ladrani: Writing-original draft preparation; Keltoum Bouhali: Supervision; Mohamed Bouye: Writing-review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

Acknowledgments

The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through Large group research project under grant number RGP2/281/45.

Conflict of interest

The authors declare that there is no conflict of interest.

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