



Research article

Generalized variational inclusion: graph convergence and dynamical system approach

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Abstract: This work focused on the investigation of a generalized variation inclusion problem. The resolvent operator for generalized η -co-monotone mapping was structured, the Lipschitz constant was estimated and its relationship with the graph convergence was accomplished. An Ishikawa type iterative algorithm was designed by incorporating the resolvent operator and total asymptotically non-expansive mapping. By employing the novel implication of graph convergence and analyzing the convergence of the considered iterative method, the common solution of the generalized variational inclusion and the set of fixed points of a total asymptotically non-expansive mapping was obtained. Moreover, a generalized resolvent dynamical system was investigated. Some of its attributes were discussed and implemented to examine the considered generalized variation inclusion problem.

Keywords: η -co-monotone mapping; graph convergence; total asymptotically non-expansive mapping; resolvent dynamical system; generalized variational inclusion

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1. Introduction

The theory of variational inequalities (VIs) was initially originated from variational principles for investigating partial differential equations [31]. It is a dynamic tool for unifying and studying equilibrium problems. It has been recognized as a potential and compelling approach for exploring and analyzing nonlinear problems of science and engineering, complex boundary value problems, models of economics, and transportation and operations research by reformatting them as a VI. Since its rise, this theory has been augmented by diverse techniques and methodologies.

Among the succeeding expansions, one of the prominent, most fruitful and worthwhile generalizations of VI is variational inclusion. The variational inclusion problem plays a crucial role in the formulation of mathematical models of various real-life problems with practical implications across diverse disciplines. The monotone inclusion problem (MIP) is to discern an element $\theta^* \in \mathbb{H}$ so that

$$0 \in (\varphi + \psi)\theta^*, \quad (1.1)$$

where \mathbb{H} is a real Hilbert space, $\psi : \mathbb{H} \rightarrow \mathbb{H}$ is a single-valued monotone operator, and $\varphi : \mathbb{H} \rightarrow 2^{\mathbb{H}}$ is a maximal monotone operator. We indicate the solution set of the MIP (1.1) by $(\varphi + \psi)^{-1}(0)$. Variational inclusions have been implemented to tackle numerous equilibrium and optimization problems including image processing, image deblurring, convex minimization, DC programming, split feasibility, fixed point and VI problems; see [1,9,12,17,21,23,27–29,33]. Applicability and usefulness of variational inclusions have captivated the attentiveness of numerous scholars in a short span of time. As of now, a number of approaches have been carried out for figuring out the problem. One of the fundamental approaches to deal with these problems is to reduce the inclusion problem into an analogous fixed point problem by employing the technique of resolvent.

In recent times, for the sake of generalizing VIs and inclusions, the researchers have generalized the conception of monotone and accretive mappings such as m -accretive mappings [20] as an allied approach for maximal monotone, maximal η -monotone and η -subdifferential mappings. In this sequel, the concept of H -monotone mapping was incepted by Fang and Huang [15] in Hilbert spaces, and later they further coined an analogous notion in Banach spaces named H -accretive mappings [16]. In 2008, Zou and Huang [35] enriched the literature by defining the $H(\cdot, \cdot)$ -accretive operator in Banach spaces. Using these generalized monotone and accretive mappings, authors have examined numerous variational inclusions by implementing the resolvent operators. Moreover, the notion of the $H(\cdot, \cdot)$ -co-coercive mapping was set forth by Ahmad et al. [3]. This concept was further extended by defining $H(\cdot, \cdot)$ -co-monotone mapping [4], which is the combination of symmetric co-coercive and monotone mapping. An analogous conception was studied in Banach spaces and named as $H(\cdot, \cdot)$ -co-accretive mapping [5], which is the combination of symmetric co-coercive and accretive mapping. The researchers explored some properties of these operators and applied them to investigate a number of variational inclusions. Subsequently, a great deal of work has gone into examining variational inclusion problems involving generalized monotone and accretive mappings using the graph convergence. Li and Huang [22] brought the idea of graph convergence for $H(\cdot, \cdot)$ -accretive mappings and shown that it is homologous to the resolvent operator convergence. Further, Ahmad et al. [2] utilized the conception of graph convergence to examine a system of generalized variational inclusions involving $H(\cdot, \cdot)$ -co-accretive mapping. For a detailed literature on graph convergence, we refer to [6, 7, 10, 32].

Since the equilibrium point of the dynamical system leads to the solution of the corresponding VI and inclusion problem, dynamical systems represent cohesive, all-encompassing frameworks of VIs and inclusions as their equilibrium points serve as solutions to these problems. Thus, all the problems whose mathematical models can be solved using VIs can also be examined in the general framework of the dynamical systems. This characteristic has drawn the attention of researchers to study dynamical systems associated with VI and inclusion problems. One can transform the model of VI or inclusion problems into a fixed point problem by implementing the novel resolvent or projection operator, and such transformations allow us to suggest dynamical systems. Dynamical systems directly or indirectly appear in several useful areas encompassing celestial mechanics, environmental studies, financial

forecasting, modeling of neuroscience, etc., and allow us to describe the trajectories of physical process and real-world problems before achieving the steady state. For further applications of the dynamical systems, see [11, 13, 14, 18, 24–26].

Inspired and persuaded by the above stated work, in this study, we investigate a generalized variation inclusion problem. We define the resolvent operator for generalized η -co-monotone mapping and estimate its Lipschitz constant. Further, its relationship with the graph convergence is accomplished. An Ishikawa type iterative algorithm is structured and analyzed to obtain the common solution of the generalized variational inclusion and the set of fixed points of a total asymptotically non-expansive mapping by employing the novel implication of graph convergence. Moreover, we construct a generalized resolvent dynamical system associated to the generalized variational inclusion and discuss some of its attributes. Further, we investigated the considered generalized variation inclusion problem by implementing the generalized resolvent dynamical system. Also, the theoretical results are verified by illustrative examples.

2. Relevant concepts and auxiliary results

Now onward, \mathbb{H} is assumed to be a real Hilbert space endued with norm $\|\cdot\|$ which induces the metric d and inner product $\langle \cdot, \cdot \rangle$. The collection of all closed and bounded subsets of \mathbb{H} is signified as $CB(\mathbb{H})$.

Definition 2.1. Let $\eta : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ be a single-valued mapping. A mapping $\psi : \mathbb{H} \rightarrow \mathbb{H}$ is referred to as

(i) η -monotone if

$$\langle \psi(\theta) - \psi(\vartheta), \eta(\theta, \vartheta) \rangle \geq 0, \forall \theta, \vartheta \in \mathbb{H};$$

(ii) ρ -strongly η -monotone if $\exists \rho > 0$ so that

$$\langle \psi(\theta) - \psi(\vartheta), \eta(\theta, \vartheta) \rangle \geq \rho \|\theta - \vartheta\|^2, \forall \theta, \vartheta \in \mathbb{H};$$

(iii) ϖ -Lipschitz continuous if $\exists \varpi > 0$ so that

$$\|\psi(\theta) - \psi(\vartheta)\| \leq \varpi \|\theta - \vartheta\|, \forall \theta, \vartheta \in \mathbb{H};$$

(iv) ζ -expansive if $\exists \zeta > 0$ so that

$$\|\psi(\theta) - \psi(\vartheta)\| \geq \zeta \|\theta - \vartheta\|, \forall \theta, \vartheta \in \mathbb{H}.$$

The following lemma is a crucial instrument for carrying out the adopted scheme.

Lemma 2.1. [34] Let $\{p_k\}_{k=1}^{\infty}$ be a nonnegative real sequence and $\{q_k\}_{k=1}^{\infty}$ be a real sequence in $[0, 1]$ with $\sum_{k=0}^{\infty} q_k = \infty$ fulfilling the following inequality:

$$p_{k+1} \leq (1 - q_k)p_k + q_k\tau_k, \forall k \geq n_0,$$

where $\tau_k \geq 0, \forall k \geq 0$ and $\lim_{k \rightarrow \infty} \tau_k = 0$. Then $\lim_{k \rightarrow \infty} p_k = 0$.

2.1. Generalized η -co-monotone operator

Definition 2.2. Let $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $\psi, \varphi : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings. Then $\Phi(\cdot, \cdot)$ is known as

(i) m -mixed Lipschitz continuous with respect to ψ and φ if $\exists m > 0$ such that

$$\|\Phi(\psi(\theta), \varphi(\theta)) - \Phi(\psi(\vartheta), \varphi(\vartheta))\| \leq m\|\theta - \vartheta\|, \forall \theta, \vartheta \in \mathbb{H};$$

(ii) η -co-coercive with respect to ψ if $\exists \kappa' > 0$ such that

$$\langle \Phi(\psi(\theta), a) - \Phi(\psi(\vartheta), a), \eta(\theta, \vartheta) \rangle \geq \kappa' \|\psi(\theta) - \psi(\vartheta)\|^2, \forall \theta, \vartheta \in \mathbb{H};$$

(iii) relaxed η -co-coercive with respect to φ if $\exists \kappa'' > 0$ such that

$$\langle \Phi(b, \varphi(\theta)) - \Phi(b, \varphi(\vartheta)), \eta(\theta, \vartheta) \rangle \geq (-\kappa'') \|\varphi(\theta) - \varphi(\vartheta)\|^2, \forall \theta, \vartheta \in \mathbb{H};$$

(iv) symmetric η -co-coercive with respect to ψ and φ if $\Phi(\cdot, \cdot)$ satisfies (ii) and (iii).

Definition 2.3. Let $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$; $G, \psi, \varphi : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings and $\Psi : \mathbb{H} \rightarrow CB(\mathbb{H})$ be a set-valued mapping. Then, $\Phi(\cdot, \cdot)$ is known as mixed δ -strongly monotone with respect to G and Ψ , if for some $\omega \in \Psi(\theta), \bar{\omega} \in \Psi(\vartheta), \exists \delta > 0$ such that

$$\langle \Phi(\psi(\theta), \varphi(\theta)) - \Phi(\psi(\vartheta), \varphi(\vartheta)), \eta(G(\omega), G(\bar{\omega})) \rangle \geq \delta \|\theta - \vartheta\|^2, \forall \theta, \vartheta \in \mathbb{H}.$$

Definition 2.4. Let $\eta : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $f, g : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings. A set-valued mapping $M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ is known as

(i) τ' -strongly η -monotone with respect to f if $\exists \tau' > 0$ so that

$$\langle \mu - \nu, \eta(\theta, \vartheta) \rangle \geq \tau' \|\theta - \vartheta\|^2, \forall \theta, \vartheta, b \in \mathbb{H}, \mu \in M(f(\theta), b), \nu \in M(f(\vartheta), b);$$

(ii) τ'' -relaxed η -monotone with respect to g if $\exists \tau'' > 0$ so that

$$\langle \mu - \nu, \eta(\theta, \vartheta) \rangle \geq (-\tau'') \|\theta - \vartheta\|^2, \forall \theta, \vartheta, b \in \mathbb{H}, \mu \in M(b, g(\theta)), \nu \in M(b, g(\vartheta));$$

(iii) $M(\cdot, \cdot)$ is known as symmetric η -monotone with respect to f and g if $M(\cdot, \cdot)$ satisfies (i) and (ii).

Definition 2.5. Let $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $\psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings. A set-valued mapping $M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ is referred to as generalized η -co-monotone if $\Phi(\cdot, \cdot)$ is symmetric η -co-coercive with respect to ψ and φ , $M(\cdot, \cdot)$ is symmetric η -monotone with respect to f and g , and

$$[\Phi(\psi, \varphi) + \varrho M(f, g)](\mathbb{H}) = \mathbb{H}, \forall \varrho > 0. \quad (2.1)$$

Note 2.1. Now onward, M is generalized η -co-monotone means, $\Phi(\cdot, \cdot)$ is η -symmetric co-coercive with respect to ψ and φ with constants κ' and κ'' , respectively, and $M(\cdot, \cdot)$ is symmetric η -monotone with respect to f and g with constants τ' and τ'' , respectively, and satisfies (2.1).

Lemma 2.2. Let $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $\psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings. Let $M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ be a generalized η -co-monotone mapping. Let ψ be ζ -expansive and φ be l -Lipschitz continuous. Then, for all $\varrho > 0$, the mapping $[\Phi(\psi, \varphi) + \varrho M(f, g)]^{-1}$ is single-valued.

Definition 2.6. Let $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $\psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings. Let $M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ be a generalized η -co-monotone mapping. The resolvent $\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)} : \mathbb{H} \rightarrow \mathbb{H}$ is described as

$$\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) = [\Phi(\psi, \varphi) + \varrho M(f, g)]^{-1}(\theta), \forall \theta \in \mathbb{H}, \varrho > 0. \quad (2.2)$$

Proposition 2.1. Let $\eta : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ be a π -Lipschitz continuous mapping; $\Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $\psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings such that ψ is ς -expansive and φ is l -Lipschitz continuous. Let $M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ be a generalized η -co-monotone mapping. Then, $\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)} : \mathbb{H} \rightarrow \mathbb{H}$ is Ξ -Lipschitz continuous, where

$$\Xi = \frac{\pi}{[\varrho(\tau' - \tau'') + (\kappa' \varsigma^2 - \kappa'' l^2)]}.$$

Proof. For given $\theta, \vartheta \in \mathbb{H}$, it follows from (2.2) that

$$\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) = [\Phi(\psi, \varphi) + \varrho M(f, g)]^{-1}(\theta), \quad (2.3)$$

$$\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\vartheta) = [\Phi(\psi, \varphi) + \varrho M(f, g)]^{-1}(\vartheta). \quad (2.4)$$

From (2.3) and (2.4), one can write

$$\frac{1}{\varrho}(\theta - \Phi(\psi(\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta)), \varphi(\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta)))) \in M(f(\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta)), g(\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta))), \quad (2.5)$$

$$\frac{1}{\varrho}(\vartheta - \Phi(\psi(\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\vartheta)), \varphi(\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\vartheta)))) \in M(f(\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\vartheta)), g(\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\vartheta))). \quad (2.6)$$

For the sake of simplicity, we indicate $\Lambda(\theta) = \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta)$ and $\Lambda(\vartheta) = \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\vartheta)$, and since $M(\cdot, \cdot)$ is symmetric η -monotone, then

$$\begin{aligned} & \varrho(\tau' - \tau'') \|\Lambda(\theta) - \Lambda(\vartheta)\|^2 \\ & \leq \langle \theta - \Phi(\psi(\Lambda(\theta)), \varphi(\Lambda(\theta))) - (\vartheta - \Phi(\psi(\Lambda(\vartheta)), \varphi(\Lambda(\vartheta))), \eta(\Lambda(\theta), \Lambda(\vartheta)) \rangle \\ & \leq \langle \theta - \vartheta, \eta(\Lambda(\theta), \Lambda(\vartheta)) \rangle - \langle \Phi(\psi(\Lambda(\theta)), \varphi(\Lambda(\theta))) - \Phi(\psi(\Lambda(\vartheta)), \varphi(\Lambda(\vartheta))), \eta(\Lambda(\theta), \Lambda(\vartheta)) \rangle \\ & = \langle \theta - \vartheta, \eta(\Lambda(\theta), \Lambda(\vartheta)) \rangle - \langle \Phi(\psi(\Lambda(\theta)), \varphi(\Lambda(\theta))) - \Phi(\psi(\Lambda(\vartheta)), \varphi(\Lambda(\vartheta))), \eta(\Lambda(\theta), \Lambda(\vartheta)) \rangle \\ & \quad - \langle \Phi(\psi(\Lambda(\vartheta)), \varphi(\Lambda(\vartheta))) - \Phi(\psi(\Lambda(\theta)), \varphi(\Lambda(\theta))), \eta(\Lambda(\theta), \Lambda(\vartheta)) \rangle. \end{aligned}$$

Invoking symmetric η -co-coercivity of Φ , ς -expansiveness of ψ , l and π -Lipschitz continuities of φ and η , respectively, we attain

$$\varrho(\tau' - \tau'') \|\Lambda(\theta) - \Lambda(\vartheta)\|^2 \leq \langle \theta - \vartheta, \eta(\Lambda(\theta), \Lambda(\vartheta)) \rangle - (\kappa' \varsigma^2 - \kappa'' l^2) \|\Lambda(\theta) - \Lambda(\vartheta)\|^2, \text{ i.e.,}$$

$$[\varrho(\tau' - \tau'') + (\kappa' \varsigma^2 - \kappa'' l^2)] \|\Lambda(\theta) - \Lambda(\vartheta)\|^2 \leq \|\theta - \vartheta\| \|\eta(\Lambda(\theta), \Lambda(\vartheta))\| \leq \|\theta - \vartheta\| \pi \|\Lambda(\theta) - \Lambda(\vartheta)\|.$$

Thus, for all $\theta, \vartheta \in \mathbb{H}$, we obtain $\|\Lambda(\theta) - \Lambda(\vartheta)\| \leq \Xi \|\theta - \vartheta\|$, i.e.,

$$\|\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\vartheta)\| \leq \Xi \|\theta - \vartheta\|, \forall \theta, \vartheta \in \mathbb{H}, \quad (2.7)$$

where, $\Xi = \frac{\pi}{[\varrho(\tau' - \tau'') + (\kappa' \varsigma^2 - \kappa'' l^2)]}$. □

2.2. Graph convergence

Definition 2.7. The graph of a multivalued mapping $M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ is expressed as

$$\text{Graph}(M) = \{((\theta, \vartheta), \xi) : \xi \in M(\theta, \vartheta)\}.$$

Definition 2.8. Let $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $\psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings. For $n \geq 0$, let $M_n, M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ be generalized η -co-monotone mappings. Then, $\{M_n\}_{n=1}^\infty$ is known as graph convergent to M , indicated by $(M_n \xrightarrow{\mathbb{G}} M)$ if for each $(f(\theta), g(\theta), \xi) \in \text{Graph}(M)$, $\exists \{(f(\theta_n), g(\theta_n), \xi_n)\} \in \text{Graph}(M_n)$ so that

$$f(\theta_n) \rightarrow f(\theta), g(\theta_n) \rightarrow g(\theta) \text{ and } \xi_n \rightarrow \xi \text{ as } n \rightarrow \infty.$$

Theorem 2.1. Let $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $\psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings such that $\Phi(\cdot, \cdot)$ is m -mixed Lipschitz continuous with respect to ψ and φ , and f, g are continuous mappings so that f is ζ -expansive. For $n \geq 0$, let $M_n, M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ be generalized η -co-monotone mappings. Then,

$$M_n \xrightarrow{\mathbb{G}} M \Leftrightarrow \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) \rightarrow \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta), \forall \theta \in \mathbb{H}, \varrho > 0.$$

Proof. For all $\theta \in \mathbb{H}$ and $\varrho > 0$, suppose that $\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) \rightarrow \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta)$. Assume that $(f(\theta), g(\theta), \xi) \in \text{Graph}(M)$, then

$$\theta = \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) + \varrho\xi]. \quad (2.8)$$

Letting

$$\theta_n = \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) + \varrho\xi], \quad (2.9)$$

which turns into

$$\Phi(\psi(\theta), \varphi(\theta)) + \varrho\xi \in [\Phi(\psi(\theta_n), \varphi(\theta_n)) + \varrho M(f(\theta_n), g(\theta_n))]. \quad (2.10)$$

For each $n \geq 0$, take $\xi_n \in M(f(\theta_n), g(\theta_n))$, then (2.10) yields

$$\Phi(\psi(\theta), \varphi(\theta)) + \varrho\xi = \Phi(\psi(\theta_n), \varphi(\theta_n)) + \varrho\xi_n. \quad (2.11)$$

Invoking the m -mixed Lipschitz continuity of Φ , it follows from (2.11) that

$$\|\varrho\xi_n - \varrho\xi\| = \|\Phi(\psi(\theta_n), \varphi(\theta_n)) - \Phi(\psi(\theta), \varphi(\theta))\| \leq (\rho' + \rho'')\|\theta_n - \theta\|. \quad (2.12)$$

Thus, $\|\xi_n - \xi\| \leq \frac{m}{\varrho}\|\theta_n - \theta\|$. Recalling the hypothesis $\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) \rightarrow \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta)$, it yields from (2.8) and (2.9) that $\|\theta_n - \theta\| \rightarrow 0$ and, hence, from (2.12), we acquire $\|\xi_n - \xi\| \rightarrow 0$ as $n \rightarrow \infty$. Accounting the continuity of f and g , we deduce $f(\theta_n) \rightarrow f(\theta)$ and $g(\theta_n) \rightarrow g(\theta)$, and so $M_n \xrightarrow{\mathbb{G}} M$.

On the contrary, assume that $M_n \xrightarrow{\mathbb{G}} M$ and choose an arbitrary but fixed $e \in \mathbb{H}$. Since $M(\cdot, \cdot)$ is a generalized η -co-monotone mapping, $\text{Range}[\Phi(\psi, \varphi) + \varrho M(f, g)] = \mathbb{H}$. Then, there exists $((f(\theta), g(\theta)), \xi) \in \text{Graph}(M)$ such that $e = \Phi(\psi(\theta), \varphi(\theta)) + \varrho\xi$. Since $(f(\theta), g(\theta), \xi) \in \text{Graph}(M)$ and suppose $(f(\theta_n), g(\theta_n), \xi_n) \in \text{Graph}(M_n)$,

$$\theta = \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) + \varrho\xi] \text{ and } \theta_n = \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta_n), \varphi(\theta_n)) + \varrho\xi_n].$$

Letting $e_n = \Phi(\psi(\theta_n), \varphi(\theta_n)) + \varrho\xi_n$, for all $n \geq 0$, adding the m -mixed Lipschitz continuity $\Phi(\cdot, \cdot)$ and making use of (2.7), we acquire

$$\begin{aligned}
 & \|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(e) - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(e)\| \\
 & \leq \|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(e) - \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(e_n)\| + \|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(e_n) - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(e)\| \\
 & \leq \Xi\|e_n - e\| + \|\theta_n - \theta\| \\
 & = \Xi\|\Phi(\psi(\theta_n), \varphi(\theta_n)) + \varrho\xi_n - [\Phi(\psi(\theta), \varphi(\theta)) + \varrho\xi]\| + \|\theta_n - \theta\| \\
 & \leq \Xi\|\Phi(\psi(\theta_n), \varphi(\theta_n)) - \Phi(\psi(\theta), \varphi(\theta))\| + \Xi\varrho\|\xi_n - \xi\| + \|\theta_n - \theta\| \\
 & \leq [1 + \Xi m]\|\theta_n - \theta\| + \Xi\varrho\|\xi_n - \xi\|.
 \end{aligned} \tag{2.13}$$

The ζ -expansiveness of f yields

$$\|f(\theta_n) - f(\theta)\| \geq \zeta\|\theta_n - \theta\| \geq 0. \tag{2.14}$$

Thus, we deduce from (2.14) and the Definition 2.8 that $\theta_n \rightarrow \theta$ and $\xi_n \rightarrow \xi$ as $n \rightarrow \infty$. Thus, from (2.13), we infer that $\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) \rightarrow \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta)$. \square

Example 2.1. Let $\mathbb{H} = \mathbb{R}^2$ with the usual inner product on \mathbb{R}^2 , i.e.,

$$\langle (\theta_1, \theta_2), (\vartheta_1, \vartheta_2) \rangle = \theta_1\vartheta_1 + \theta_2\vartheta_2, \forall (\theta_1, \theta_2), (\vartheta_1, \vartheta_2) \in \mathbb{R}^2.$$

Define $\psi, \varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $\eta, \Phi : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\begin{aligned}
 \psi(\theta_1, \theta_2) &= \left(\frac{\theta_1}{4}, \frac{\theta_2}{2}\right), \varphi(\theta_1, \theta_2) = \left(-\frac{\theta_1}{2}, -\frac{2\theta_2}{3}\right), \forall (\theta_1, \theta_2) \in \mathbb{R}^2, \\
 \Phi(\psi(\theta), \varphi(\theta)) &= \psi(\theta) + \varphi(\theta), \forall \theta \in \mathbb{R}^2, \eta(\theta, \vartheta) = \frac{\theta - \vartheta}{2}, \forall \theta, \vartheta \in \mathbb{R}^2.
 \end{aligned}$$

Then, for any fixed $\varsigma \in \mathbb{R}^2$, we find

$$\begin{aligned}
 \langle \Phi(\psi(\theta), \varsigma) - \Phi(\psi(\vartheta), \varsigma), \eta(\theta, \vartheta) \rangle &\geq 1\|\psi(\theta) - \psi(\vartheta)\|^2, \\
 \langle \Phi(\varsigma, \varphi(\theta)) - \Phi(\varsigma, \varphi(\vartheta)), \eta(\theta, \vartheta) \rangle &\geq (-1)\|\varphi(\theta) - \varphi(\vartheta)\|^2.
 \end{aligned}$$

Thus, $\Phi(\cdot, \cdot)$ is η -co-coercive with respect to ψ and relaxed η -co-coercive with respect to φ , hence $\Phi(\cdot, \cdot)$ is symmetric η -co-coercive. Next, we estimate the symmetric monotonicity of M . Define $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $M : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\begin{aligned}
 f(\theta_1, \theta_2) &= \left(\frac{\theta_1}{3} - \theta_2, \theta_1 + \frac{\theta_2}{2}\right), g(\theta_1, \theta_2) = \left(\frac{\theta_1}{3} + \frac{\theta_2}{2}, -\frac{\theta_1}{2} + \frac{\theta_2}{4}\right), \forall (\theta_1, \theta_2) \in \mathbb{R}^2, \\
 M(f(\theta), g(\theta)) &= f(\theta) - g(\theta), \forall \theta \in \mathbb{R}^2.
 \end{aligned}$$

Then for any fixed $\varpi \in \mathbb{R}^2$, we find

$$\begin{aligned}
 \langle M(f(\theta), \varpi) - M(f(\vartheta), \varpi), \eta(\theta, \vartheta) \rangle &\geq \frac{1}{6}\|\theta - \vartheta\|^2, \\
 \langle M(\varpi, g(\theta)) - M(\varpi, g(\vartheta)), \eta(\theta, \vartheta) \rangle &\geq -\frac{1}{6}\|\theta - \vartheta\|^2,
 \end{aligned}$$

i.e., $M(\cdot, \cdot)$ is η -monotone with respect to f and relaxed η -monotone with respect to g , hence $M(\cdot, \cdot)$ is symmetric η -monotone. Also, for any $\theta \in \mathbb{R}^2$ and $\varrho > 0$,

$$\begin{aligned} [\Phi(\psi, \varphi) + \varrho M(f, g)](\theta) &= \psi(\theta) + \varphi(\theta) + \varrho(f(\theta) - g(\theta)) \\ &= \left[-\frac{\theta_1}{4} - \frac{3\varrho}{2}\theta_2, \frac{3\varrho}{2}\theta_1 + \left(\frac{\varrho}{4} - \frac{1}{6}\right)\theta_2 \right], \end{aligned}$$

i.e., $[\Phi(\psi, \varphi) + \varrho M(f, g)](\mathbb{R}^2) = \mathbb{R}^2, \forall \varrho > 0$. Thus, $M(\cdot, \cdot)$ is a generalized η -co-monotone mapping. Further, we show that $M_n \xrightarrow{\mathbb{G}} M$. Let

$$\begin{aligned} f(\theta_n) &= \left(\frac{\theta_1}{3} - \theta_2 + \frac{2}{n}, \theta_1 + \frac{\theta_2}{2} + \frac{1}{n} \right), \\ g(\theta_n) &= \left(\frac{\theta_1}{3} + \frac{\theta_2}{2} + \frac{2}{n^2}, -\frac{\theta_1}{2} + \frac{\theta_2}{4} + \frac{1}{n^2} \right), \end{aligned}$$

and $\xi_n = M_n(f(\theta_n), g(\theta_n)) = f(\theta_n) - g(\theta_n) = \left(-\frac{3}{2}\theta_2 + \frac{2}{n} - \frac{2}{n^2}, \frac{3}{2}\theta_1 + \frac{1}{4}\theta_2 + \frac{1}{n} - \frac{1}{n^2} \right)$. One can observe that

$$\begin{aligned} \lim_{n \rightarrow \infty} f(\theta_n) &= \lim_{n \rightarrow \infty} \left(\frac{\theta_1}{3} - \theta_2 + \frac{2}{n}, \theta_1 + \frac{\theta_2}{2} + \frac{1}{n} \right) = \left(\frac{\theta_1}{3} - \theta_2, \theta_1 + \frac{\theta_2}{2} \right), \\ \lim_{n \rightarrow \infty} g(\theta_n) &= \lim_{n \rightarrow \infty} \left(\frac{\theta_1}{3} + \frac{\theta_2}{2} + \frac{2}{n^2}, -\frac{\theta_1}{2} + \frac{\theta_2}{4} + \frac{1}{n^2} \right) = \left(\frac{\theta_1}{3} + \frac{\theta_2}{2}, -\frac{\theta_1}{2} + \frac{\theta_2}{4} \right), \end{aligned}$$

and $\lim_{n \rightarrow \infty} \xi_n = \lim_{n \rightarrow \infty} \left(-\frac{3}{2}\theta_2 + \frac{2}{n} - \frac{2}{n^2}, \frac{3}{2}\theta_1 + \frac{1}{4}\theta_2 + \frac{1}{n} - \frac{1}{n^2} \right) = f(\theta) - g(\theta) = M(f(\theta), g(\theta)) = \xi$.

Thus, we acquire that $\lim_{n \rightarrow \infty} f(\theta_n) = f(\theta)$ and $\lim_{n \rightarrow \infty} g(\theta_n) = g(\theta)$ and $\lim_{n \rightarrow \infty} \xi_n = \xi$. Hence, $M_n \xrightarrow{\mathbb{G}} M$. Finally, it remains to manifest that $M_n \xrightarrow{\mathbb{G}} M \Leftrightarrow \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) \rightarrow \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta), \forall \theta \in \mathbb{H}, \varrho > 0$. Now, for $\varrho = 1$, the associated resolvent operators are estimated as:

$$\begin{aligned} \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) &= [\Phi(\psi, \varphi) + M_n(f, g)]^{-1}(\theta) \\ &= \left(-\frac{1}{4}\theta_1 - \frac{3}{2}\theta_2 + \frac{2}{n} - \frac{2}{n^2}, \frac{3}{2}\theta_1 + \frac{1}{12}\theta_2 + \frac{1}{n} - \frac{1}{n^2} \right)^{-1} \\ &= \frac{1}{107} \left(4\theta_1 + 72\theta_2 - \frac{80}{n} + \frac{80}{n^2}, -72\theta_1 - 12\theta_2 + \frac{156}{n} - \frac{156}{n^2} \right) \end{aligned}$$

and

$$\begin{aligned} \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) &= [\Phi(\psi, \varphi) + M(f, g)]^{-1}(\theta) = \left(-\frac{1}{4}\theta_1 - \frac{3}{2}\theta_2, \frac{3}{2}\theta_1 + \frac{1}{12}\theta_2 \right)^{-1} \\ &= \frac{1}{107} \left(4\theta_1 + 72\theta_2, -72\theta_1 - 12\theta_2 \right), \end{aligned}$$

which yields

$$\|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta)\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, we obtain

$$\|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta) \rightarrow \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}(\theta)\| \text{ as } M_n \xrightarrow{\mathbb{G}} M.$$

3. Problem formulation and convergence result

In this section, we employ a generalized η -co-monotone mapping for investigating a general variational inclusion (GVIP). We examine the problem of discerning $\theta \in \mathbb{H}$, $\omega \in \Psi(\theta)$ so that

$$0 \in G(\omega) + M(f(\theta), g(\theta)), \quad (3.1)$$

where $G, f, g : \mathbb{H} \rightarrow \mathbb{H}$ and $M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}; \Psi : \mathbb{H} \rightrightarrows CB(\mathbb{H})$ are single-valued and multivalued mappings, respectively. We signify the Problem (3.1) as GVIP and its solution set by $\Omega(\mathbb{H}, M, \Phi, G, \eta)$.

Lemma 3.1. *Let $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $G, \psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings and $\Psi : \mathbb{H} \rightrightarrows CB(\mathbb{H})$ a multivalued mapping. Let $M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ be a generalized η -co-monotone mapping. Then, (θ, ω) , where $\theta \in \mathbb{H}, \omega \in \Psi(\theta)$ solves GVIP (3.1) if, and only if,*

$$\theta = \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]. \quad (3.2)$$

Proof. One can obtain the conclusion immediately by implementing (2.2). \square

A mapping $F : \mathbb{H} \rightarrow \mathbb{H}$ is referred to as non-expansive (NM) if $\|F(\theta) - F(\vartheta)\| \leq \|\theta - \vartheta\|, \forall \theta, \vartheta \in \mathbb{H}$. In [19], the authors defined a generalized NM referred to as asymptotically nonexpansive (ANM) which properly includes the class of NM.

Definition 3.1. [19] *A mapping $F : \mathbb{H} \rightarrow \mathbb{H}$ is known as ANM if \exists is a sequence $\{\varepsilon_n\} \subseteq [1, \infty)$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 1$ and $\forall n \in \mathbb{N}$,*

$$\|F^n(\theta) - F^n(\vartheta)\| \leq \varepsilon_n \|\theta - \vartheta\|, \forall \theta, \vartheta \in \mathbb{H}.$$

In an attempt to obtain extension of NM and ANM, Sahu [30] introduced nearly asymptotically non-expansive mapping (NANM). The class of NANM is an intermediate class which contains the class of ANM and is contained in the class of mappings of asymptotically non-expansive type.

Definition 3.2. *A mapping $F : \mathbb{H} \rightarrow \mathbb{H}$ is known as NANM, if $\exists \{\varepsilon_n\} \subseteq [1, \infty)$ and $\{\nu_n\} \subseteq [0, \infty)$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 1, \lim_{n \rightarrow \infty} \nu_n = 0, \forall n \in \mathbb{N}$,*

$$\|F^n(\theta) - F^n(\vartheta)\| \leq \varepsilon_n \|\theta - \vartheta\| + \nu_n, \forall \theta, \vartheta \in \mathbb{H}.$$

Further, Alber et al. [8] made an attempt to unify some classes of generalized NMs by introducing total asymptotically non-expansive mapping (TANM).

Definition 3.3. *A mapping $F : \mathbb{H} \rightarrow \mathbb{H}$ is known as TANM if \exists nonnegative sequences of real numbers $\{\varepsilon_n\}, \{\nu_n\}$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0 = \lim_{n \rightarrow \infty} \nu_n$ and a strictly increasing continuous function $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\gamma(0) = 0$ and $\forall n \in \mathbb{N}$,*

$$\|F^n(\theta) - F^n(\vartheta)\| \leq \|\theta - \vartheta\| + \mu_n \gamma(\|\theta - \vartheta\|) + \nu_n, \forall \theta, \vartheta \in \mathbb{H}.$$

Let $F : \mathbb{H} \rightarrow \mathbb{H}$ be a TANM and presume that the mappings $\eta, \Phi, G, \psi, \varphi, f, g, \Psi$, and M are identical as in Lemma 3.1. Suppose that $\theta^* \in \text{Fix}(F) \cap \Omega(\mathbb{H}, M, \Phi, G, \eta)$, then from (3.2), one can achieve the following formulation:

$$\begin{aligned} \theta^* = F^n \theta^* &= \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta^*), \varphi(\theta^*)) - \varrho G(\omega^*)] \\ &= F^n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta^*), \varphi(\theta^*)) - \varrho G(\omega^*)], \end{aligned} \quad (3.3)$$

where $\varrho > 0$ and $\omega^* \in \Psi(\theta^*)$. By the virtue of formulation (3.3), we design the following Ishikawa type resolvent iterative scheme to explore a common element of $Fix(F)$ and $\Omega(\mathbb{H}, M, \Phi, G, \eta)$. Here, $Fix(F)$ indicates the set of fixed points of TANM F and $\Omega(\mathbb{H}, M, \Phi, G, \eta)$ indicates the solution set of GVIP (3.1).

Algorithm 3.1. Let $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $G, \psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings. Let $\Psi : \mathbb{H} \rightrightarrows CB(\mathbb{H})$ be a multivalued mapping; $M_n, M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ be generalized η -co-monotone mappings, and $F : \mathbb{H} \rightarrow \mathbb{H}$ be a TANM. For initial points $\theta_0, \vartheta_0 \in \mathbb{H}, \omega_0 \in \Psi(\theta_0)$, estimate the sequences $\{\theta_n\}, \{\omega_n\}$ by the following procedure:

$$\theta_{n+1} = (1 - \alpha_n)\theta_n + \alpha_n F^n \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n)], \quad (3.4)$$

$$\vartheta_n = (1 - \beta_n)\theta_n + \beta_n F^n \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta_n), \varphi(\theta_n)) - \varrho G(\omega_n)], \quad (3.5)$$

for $n = 0, 1, 2, \dots, \omega_n \in \Psi(\theta_n), \bar{\omega}_n \in \Psi(\vartheta_n), 0 \leq \alpha_n, \beta_n \leq 1, \sum_{n=0}^{\infty} \alpha_n = \infty$, and $\varrho > 0$.

Theorem 3.1. Let $\eta : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ be a π -Lipschitz continuous mapping; $\Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $\psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ be the single-valued mappings such that $\Phi(\cdot, \cdot)$ is m -mixed Lipschitz continuous with respect to ψ and φ and ι -mixed strongly monotone with respect to G and Ψ , G is t -Lipschitz continuous, Ψ is \mathcal{D} -Lipschitz continuous with constant r , ψ is ς -expansive, and φ is l -Lipschitz continuous. Let $M_n, M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ be generalized η -co-monotone mappings. Let $F : \mathbb{H} \rightarrow \mathbb{H}$ be a $(\{\mu_n\}, \{\nu_n\}, \phi)$ -TANM so that $Fix(F) \cap \Omega(\mathbb{H}, M, \Phi, G, \eta) \neq \emptyset$. If $\varrho > 0$ obeys the following relation:

$$\sqrt{m^2 - 2\varrho\iota + \varrho^2 t^2 r^2} < \frac{[\varrho(\tau' - \tau'') + (\kappa' s^2 - \kappa'' l^2)]}{\pi}. \quad (3.6)$$

(i) Then $\Omega(\mathbb{H}, M, \Phi, G, \eta)$ is singleton.

(ii) If $M_n \xrightarrow{\rightarrow} M$, then the sequence $\{\theta_n\}$ induced by (3.4)-(3.5) converges strongly to $\theta \in Fix(F) \cap \Omega(\mathbb{H}, M, \Phi, G, \eta)$.

Proof. (i) Define $\mathcal{G} : \mathbb{H} \rightarrow \mathbb{H}$ as

$$\mathcal{G}(\theta) = \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)], \quad \forall \theta \in \mathbb{H}. \quad (3.7)$$

By making use of Proposition 2.1 and (3.7), for all $\theta, \vartheta \in \mathbb{H}$, we acquire

$$\begin{aligned} \|\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\| &= \|\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)] \\ &\quad - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta), \varphi(\vartheta)) - \varrho G(\bar{\omega})]\| \\ &\leq \Xi \|\Phi(\psi(\theta), \varphi(\theta)) - \Phi(\psi(\vartheta), \varphi(\vartheta)) - \varrho(G(\omega) - \varrho G(\bar{\omega}))\|. \end{aligned} \quad (3.8)$$

Utilizing the m -mixed Lipschitz continuity and ι -mixed strong monotonicity of Φ and Lipschitz continuities of G and Ψ , we acquire

$$\begin{aligned} &\|\Phi(\psi(\theta), \varphi(\theta)) - \Phi(\psi(\vartheta), \varphi(\vartheta)) - \varrho(G(\omega) - \varrho G(\bar{\omega}))\|^2 \\ &= \|\Phi(\psi(\theta), \varphi(\theta)) - \Phi(\psi(\vartheta), \varphi(\vartheta))\|^2 - 2\varrho \langle \Phi(\psi(\theta), \varphi(\theta)) - \Phi(\psi(\vartheta), \\ &\quad \varphi(\vartheta)), G(\omega) - G(\bar{\omega}) \rangle + \varrho^2 \|G(\omega) - G(\bar{\omega})\|^2 \\ &\leq (m^2 - 2\varrho\iota + \varrho^2 t^2 r^2) \|\theta - \vartheta\|^2. \end{aligned} \quad (3.9)$$

After simplification, the above inequality turns into

$$\|\Phi(\psi(\theta), \varphi(\theta)) - \Phi(\psi(\vartheta), \varphi(\vartheta)) - \varrho(G(\omega) - \varrho G(\bar{\omega}))\| \leq \sqrt{m^2 - 2\varrho\iota + \varrho^2 t^2 r^2} \|\theta - \vartheta\|. \quad (3.10)$$

Thus, (3.10) and (3.8) yield

$$\|\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\| \leq \Theta \|\theta - \vartheta\|, \quad (3.11)$$

where $\Theta = \Xi \sqrt{m^2 - 2\varrho\iota + \varrho^2 t^2 r^2}$ and $\Xi = \frac{\pi}{[\varrho(\tau' - \tau'') + (\kappa' \zeta^2 - \kappa'' l^2)]}$. Taking the premise (3.6) into consideration, we see that $0 \leq \Theta < 1$. Thus, \mathcal{G} being a contraction mapping owns a unique fixed point, i.e., \exists a unique $\theta \in \mathbb{H}$ so that $\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)] = \theta$. Consequently, Lemma 3.1 guarantees that $\Omega(\mathbb{H}, M, \Phi, G, \eta)$ is singleton.

(ii) By the assumption that $\emptyset \neq \text{Fix}(F) \cap \Omega(\mathbb{H}, M, \Phi, G, \eta)$ and in (i), we confirmed that $\Omega(\mathbb{H}, M, \Phi, G, \eta)$ is singleton. Suppose that $\Omega(\mathbb{H}, M, \Phi, G, \eta) = \{\theta\}$, then we deduce that $\theta \in \text{Fix}(F)$ and consequently, by (3.3), one can express

$$\begin{aligned} \theta &= (1 - \alpha_n)\theta + \alpha_n F^n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta), \varphi(\vartheta)) - \varrho G(\bar{\omega})] \\ &= (1 - \beta_n)\theta + \beta_n F^n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]. \end{aligned} \quad (3.12)$$

Utilizing the Proposition 2.1, we get

$$\begin{aligned} &\|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n)] - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\| \\ &\leq \|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n)] - \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\| \\ &\quad + \|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)] - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\| \\ &\leq \Xi \|\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n) - [\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\| + \|\Sigma_n\| \\ &= \Xi \|\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \Phi(\psi(\theta), \varphi(\theta)) - \varrho[G(\bar{\omega}_n) - G(\omega)]\| + \|\Sigma_n\|, \end{aligned} \quad (3.13)$$

where $\Sigma_n = \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)] - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]$. Utilizing the Lipschitz continuities of Ψ and G , we acquire

$$\|G(\bar{\omega}_n) - G(\omega)\| \leq t \|\bar{\omega}_n - \omega\| \leq t \mathcal{D}(\Psi(\bar{\omega}_n), \Psi(\omega)) \leq tr \|\vartheta_n - \theta\|. \quad (3.14)$$

Also, utilizing m -mixed Lipschitz continuity of Φ regarding ψ and φ , ι -mixed strong monotonicity with respect to G and Ψ and combining (3.14), we acquire

$$\begin{aligned} &\|\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \Phi(\psi(\theta), \varphi(\theta)) - \varrho[G(\bar{\omega}_n) - G(\omega)]\|^2 \\ &= \|\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \Phi(\psi(\theta), \varphi(\theta))\|^2 - 2\varrho \langle \Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) \\ &\quad - \Phi(\psi(\theta), \varphi(\theta)), G(\bar{\omega}_n) - G(\omega) \rangle + \varrho^2 \|G(\bar{\omega}_n) - G(\omega)\|^2 \\ &\leq m^2 \|\vartheta_n - \theta\|^2 - 2\varrho\iota \|\vartheta_n - \theta\|^2 + \varrho^2 t^2 r^2 \|\vartheta_n - \theta\|^2 \\ &= (m^2 - 2\varrho\iota + \varrho^2 t^2 r^2) \|\vartheta_n - \theta\|^2, \end{aligned}$$

which yields

$$\begin{aligned} &\|\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \Phi(\psi(\theta), \varphi(\theta)) - \varrho[G(\bar{\omega}_n) - G(\omega)]\| \\ &\leq \sqrt{m^2 - 2\varrho\iota + \varrho^2 t^2 r^2} \|\vartheta_n - \theta\|. \end{aligned} \quad (3.15)$$

After substituting (3.15) into (3.13), we obtain

$$\|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n)] - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\| \leq \Theta \|\vartheta_n - \theta\| + \|\Sigma_n\|, \quad (3.16)$$

where $\Theta = \Xi \sqrt{m^2 - 2\varrho u + \varrho^2 t^2 r^2}$. Now, recalling that F is $(\{\mu_n\}, \{\nu_n\}, \phi)$ -total asymptotically non-expansive and applying (3.4) and (3.12), we get

$$\begin{aligned} \|\theta_{n+1} - \theta\| &= \|(1 - \alpha_n)\theta_n + \alpha_n F^n \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n)] \\ &\quad - [(1 - \alpha_n)\theta + \alpha_n F^n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]]\| \\ &\leq (1 - \alpha_n)\|\theta_n - \theta\| + \alpha_n \|F^n \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n)] \\ &\quad - F^n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\| \\ &\leq (1 - \alpha_n)\|\theta_n - \theta\| + \alpha_n \left[\|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n)] \right. \\ &\quad \left. - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\| \right. \\ &\quad \left. + \mu_n \phi[\|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n)] \right. \\ &\quad \left. - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\|] + \nu_n \right]. \end{aligned} \quad (3.17)$$

By substituting (3.16) in (3.17), we acquire

$$\begin{aligned} \|\theta_{n+1} - \theta\| &\leq (1 - \alpha_n)\|\theta_n - \theta\| + \alpha_n [(\Theta \|\vartheta_n - \theta\| + \|\Sigma_n\|) \\ &\quad + \mu_n \phi(\Theta \|\vartheta_n - \theta\| + \|\Sigma_n\|) + \nu_n]. \end{aligned} \quad (3.18)$$

Following the same steps and employing the same facts as in (3.17), it follows from (3.5) and (3.12) that

$$\begin{aligned} \|\vartheta_n - \theta\| &= \|(1 - \beta_n)\theta_n + \beta_n F^n \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta_n), \varphi(\theta_n)) - \varrho G(\omega_n)] \\ &\quad - [(1 - \beta_n)\theta + \beta_n F^n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]]\| \\ &\leq (1 - \beta_n)\|\theta_n - \theta\| + \beta_n \|F^n \mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta_n), \varphi(\theta_n)) - \varrho G(\omega_n)] \\ &\quad - F^n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\| \\ &\leq (1 - \beta_n)\|\theta_n - \theta\| + \beta_n \left[\|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta_n), \varphi(\theta_n)) - \varrho G(\omega_n)] \right. \\ &\quad \left. - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\| \right. \\ &\quad \left. + \mu_n \phi[\|\mathbb{R}_{\varrho, M_n(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta_n), \varphi(\theta_n)) - \varrho G(\omega_n)] \right. \\ &\quad \left. - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]\|] + \nu_n \right] \\ &\leq (1 - \beta_n)\|\theta_n - \theta\| + \beta_n [(\Theta \|\theta_n - \theta\| + \|\Sigma_n\|) \\ &\quad + \mu_n \phi(\Theta \|\theta_n - \theta\| + \|\Sigma_n\|) + \nu_n]. \end{aligned} \quad (3.19)$$

By the hypothesis $M_n \xrightarrow{\mathbb{G}} M$, we obtain $\Sigma_n \rightarrow 0$, thus from (3.18) and (3.19), we deduce that

$$\|\theta_{n+1} - \theta\| \leq (1 - \alpha_n)\|\theta_n - \theta\| + \alpha_n [\Theta \|\vartheta_n - \theta\| + \mu_n \phi(\Theta \|\vartheta_n - \theta\|) + \nu_n]. \quad (3.20)$$

$$\|\vartheta_n - \theta\| \leq (1 - \beta_n)\|\theta_n - \theta\| + \beta_n[\Theta\|\theta_n - \theta\| + \mu_n\phi(\Theta\|\theta_n - \theta\|) + \nu_n]. \quad (3.21)$$

Substituting (3.21) into (3.20), we acquire

$$\begin{aligned} \|\theta_{n+1} - \theta\| &\leq (1 - \alpha_n)\|\theta_n - \theta\| + \alpha_n[\Theta\{(1 - \beta_n)\|\theta_n - \theta\| + \beta_n[\Theta\|\theta_n - \theta\| \\ &\quad + \mu_n\phi(\Theta\|\theta_n - \theta\|) + \nu_n]\} + \mu_n\phi(\Theta\{(1 - \beta_n)\|\theta_n - \theta\| + \beta_n[\Theta\|\theta_n - \theta\| \\ &\quad + \mu_n\phi(\Theta\|\theta_n - \theta\|) + \nu_n]\}) + \nu_n] \\ &= [(1 - \alpha_n) + \alpha_n\Theta[1 - \beta_n(1 - \Theta)]]\|\theta_n - \theta\| + \alpha_n[\beta_n\Theta\{\mu_n\phi(\Theta\|\theta_n - \theta\|) + \nu_n\} \\ &\quad + \mu_n\phi\{\Theta[1 - \beta_n(1 - \Theta)]\|\theta_n - \theta\| + \mu_n\phi(\Theta\|\theta_n - \theta\|) + \nu_n\} + \nu_n] \\ &\leq [1 - \alpha_n(1 - \Theta)]\|\theta_n - \theta\| \\ &\quad + \alpha_n(1 - \Theta)\frac{[\beta_n\Theta\{\mu_n\phi(\Psi_n) + \nu_n\} + \mu_n\phi\{\Theta[\Gamma_n + \mu_n\phi(\Psi_n) + \nu_n]\} + \nu_n]}{(1 - \Theta)}, \end{aligned} \quad (3.22)$$

where $\Psi_n = \Theta\|\theta_n - \theta\|$ and $\Gamma_n = [1 - \beta_n(1 - \Theta)]\|\theta_n - \theta\|$. For each $n \geq n_0$, setting $p_n = \|\theta_n - \theta\|$, $q_n = \alpha_n(1 - \Theta)$ and $\tau_n = \frac{\beta_n\Theta[\mu_n\phi(\Psi_n) + \nu_n] + \mu_n\phi\{\Theta[\Gamma_n + \mu_n\phi(\Psi_n) + \nu_n]\} + \nu_n}{(1 - \Theta)}$. Clearly, $\sum_{n=0}^{\infty} q_n = \infty$ because of $\sum_{n=0}^{\infty} \alpha_n = \infty$. In fact, $\mu_n, \nu_n \rightarrow 0$ as $n \rightarrow \infty$ yields $\tau_n = 0$. Thus, we deduce from Lemma 2.1 that $\lim_{n \rightarrow \infty} p_n = 0$ and, hence, $\lim_{n \rightarrow \infty} \theta_n = \theta$. \square

Example 3.1. Let $\mathbb{H} = [0, \infty)$ with inner product $\langle \theta, \vartheta \rangle = \theta\vartheta$ and norm $|\cdot|$. Define $F(\theta) = \sin \theta$, $\forall \theta \in [0, \infty)$. Then, clearly $0 \in \text{Fix}(F)$ for all $\theta, \vartheta \in [0, \infty)$, and we express

$$\begin{aligned} \|F(\theta) - F(\vartheta)\| &= \|\sin \theta - \sin \vartheta\| \\ &= \left\| 2 \sin \frac{(\theta - \vartheta)}{2} \cdot \cos \frac{(\theta + \vartheta)}{2} \right\| \\ &\leq \|\theta - \vartheta\|. \end{aligned}$$

Thus, F is non-expansive, hence F is TANM with $\mu_n = \frac{1}{n^2}$ and $\nu_n = \frac{1}{n^3}$, $\forall n \geq 1$. Define the mappings $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $G, \psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ by

$$\begin{aligned} \eta(\theta, \vartheta) &= \frac{\theta - \vartheta}{4}, \Phi(\psi(\theta), \varphi(\theta)) = \psi(\theta) + \varphi(\theta), G(\theta) = \frac{\theta - 1}{2}, \\ \psi(\theta) &= \frac{3\theta + 1}{4}, \varphi(\theta) = \frac{-\theta - 1}{4}, f(\theta) = 4\theta + 1, g(\theta) = -2\theta + \frac{1}{2}. \end{aligned}$$

It can be easily observed that η, G , and φ are Lipschitz continuous with constants $\pi = \frac{1}{4}$, $t = \frac{1}{2}$, and $\varphi = \frac{1}{4}$, respectively, and ψ is $\frac{3}{4}$ -expansive. Also,

$$\begin{aligned} \|\Phi(\psi(\theta), \varphi(\theta)) - \Phi(\psi(\vartheta), \varphi(\vartheta))\| &\leq \frac{1}{2}\|\theta - \vartheta\|, \forall \theta, \vartheta \in \mathbb{H}, \\ \langle \Phi(\psi(\theta), \varphi(\theta)) - \Phi(\psi(\vartheta), \varphi(\vartheta)), \eta(\theta, \vartheta) \rangle &\geq \frac{1}{8}\|\theta - \vartheta\|^2, \forall \theta, \vartheta \in \mathbb{H}, \\ \langle \Phi(\psi(\theta), k) - \Phi(\psi(\vartheta), k), \eta(\theta, \vartheta) \rangle &\geq \frac{1}{3}\|\psi(\theta) - \psi(\vartheta)\|^2, \forall \theta, \vartheta, k \in \mathbb{H}, \\ \langle \Phi(k, \varphi(\theta)) - \Phi(k, \varphi(\vartheta)), \eta(\theta, \vartheta) \rangle &\geq (-1)\|\varphi(\theta) - \varphi(\vartheta)\|^2, \forall \theta, \vartheta, k \in \mathbb{H}, \end{aligned}$$

i.e., $\Phi(\cdot, \cdot)$ is 1/2-mixed Lipschitz continuous, 1/8-mixed strongly monotone, and symmetric η -co-coercive with respect to ψ and φ with constants 1/3 and 1, respectively. Define $M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ and $\Psi : \mathbb{H} \rightarrow CB(\mathbb{H})$ by $M(f(\theta), g(\theta)) = \frac{f(\theta) + g(\theta)}{3}$ and $\Psi(\theta) = \left\{ \frac{\theta}{5} + 1 \right\}$. Then, Ψ is 1/5-Lipschitz continuous and

$$\begin{aligned} \langle M(f(\theta), k) - M(f(\vartheta), k), \eta(\theta, \vartheta) \rangle &\geq 1/3 \|\theta - \vartheta\|^2, \forall \theta, \vartheta, k \in \mathbb{H}, \\ \langle M(k, g(\theta)) - M(k, g(\vartheta)), \eta(\theta, \vartheta) \rangle &\geq (-1/6) \|\theta - \vartheta\|^2, \forall \theta, \vartheta, k \in \mathbb{H}, \end{aligned}$$

i.e., $M(\cdot, \cdot)$ is symmetric η -monotone with respect to f and g with constants 1/3 and 1/6, respectively. Thus, $M(\cdot, \cdot)$ is a generalized η -co-monotone mapping. Also, $\mathcal{G}(\theta) = \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)] = \frac{5}{2}\theta$ and the estimated constants satisfy (3.6), that is, $\sqrt{m^2 - 2\varrho l + \varrho^2 t^2 r^2} < \frac{[\varrho(\tau' - \tau'') + (\kappa' \varsigma^2 - \kappa'' l^2)]}{\pi}$. Therefore, $\theta^* = 0 \in \mathbb{H}$ is a unique fixed point of \mathcal{G} . Thus, we have $\theta^* = 0 \in \text{Fix}(F) \cap \Omega(\mathbb{H}, M, \Phi, G, \eta)$. Next, we compute the sequence $\{\theta_n\}$ by employing Algorithm 3.1. Let $\alpha_n = \frac{n}{2n+1}$ and $\beta_n = \frac{1}{n+1}$. Then

$$\begin{aligned} \theta_{n+1} &= (1 - \alpha_n)\theta_n + \alpha_n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n)], \\ \theta_{n+1} &= \left(\frac{n+1}{2n+1}\right)\theta_n + \left(\frac{3n}{5(2n+1)}\right)\vartheta_n, \\ \vartheta_n &= (1 - \beta_n)\theta_n + \beta_n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta_n), \varphi(\theta_n)) - \varrho G(\omega_n)], \\ \vartheta_n &= \left(\frac{n}{n+1}\right)\theta_n + \left(\frac{3}{5(n+1)}\right)\theta_n. \end{aligned}$$

For different initial points: $\theta_0 = -10.5$ and $\theta_0 = 1.5$, the sequence $\theta_n \rightarrow 0$ and $0 \in \text{Fix}(F) \cap \Omega(\mathbb{H}, M, \Phi, G, \eta)$ and the convergence behavior of $\{\theta_n\}$ is shown in Figure 1.

Example 3.2. Let $\mathbb{H} = \mathbb{R}$ with inner product $\langle \theta, \vartheta \rangle = \theta \cdot \vartheta$ and norm $|\cdot|$. Define $F(\theta) = \sin \theta, \forall \theta \in \mathbb{R}$. Then, clearly F is TANM with $\mu_n = \frac{1}{n^2}$ and $\nu_n = \frac{1}{n^3}, \forall n \geq 1$ and $0 \in \text{Fix}(F)$. Define the mappings $\eta, \Phi : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$ and $G, \psi, \varphi, f, g : \mathbb{H} \rightarrow \mathbb{H}$ by

$$\begin{aligned} \eta(\theta, \vartheta) &= \frac{\theta - \vartheta}{3}, \Phi(\psi(\theta), \varphi(\theta)) = \frac{\psi(\theta) + \varphi(\theta)}{2}, G(\theta) = \frac{\theta}{3}, \\ \psi(\theta) &= \frac{\theta}{2}, \varphi(\theta) = \frac{-\theta}{4}, f(\theta) = 2\theta, g(\theta) = -\theta. \end{aligned}$$

Then η, G , and φ are Lipschitz continuous with constants $\pi = t = \frac{1}{3}$ and $l = \frac{1}{4}$, respectively, and ψ is $\frac{1}{2}$ -expansive. Also, $\Phi(\cdot, \cdot)$ is 1/8-mixed Lipschitz continuous, 1/16-mixed strongly monotone, and symmetric η -co-coercive with respect to ψ and φ with constants 1/3 and 2/3, respectively. Define $M : \mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ and $\Psi : \mathbb{H} \rightarrow CB(\mathbb{H})$ by $M(f(\theta), g(\theta)) = \frac{f(\theta) + g(\theta)}{3}$ and $\Psi(\theta) = \{\theta\}$. Then, Ψ is 1-Lipschitz continuous and $M(\cdot, \cdot)$ is symmetric η -monotone with respect to f and g with constants 2/9 and 1/9, respectively. Thus, $M(\cdot, \cdot)$ is a generalized η -co-monotone mapping. Also, for $\varrho = 1$, the estimated constants satisfy (3.6), that is, $\sqrt{m^2 - 2\varrho l + \varrho^2 t^2 r^2} < \frac{[\varrho(\tau' - \tau'') + (\kappa' \varsigma^2 - \kappa'' l^2)]}{\pi}$ and $\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)] = -\frac{5}{11}\theta$. Therefore, $0 \in \mathbb{R}$ is a unique fixed point of $\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]$. Thus, we have $0 \in \text{Fix}(F) \cap \Omega(\mathbb{H}, M, \Phi, G, \eta)$. Now for $\alpha_n = \frac{n}{2n+1}$ and $\beta_n = \frac{1}{n+1}$, we compute the sequence $\{\theta_n\}$ by employing Algorithm 3.1 as under:

$$\begin{aligned}\theta_{n+1} &= (1 - \alpha_n)\theta_n + \alpha_n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\vartheta_n), \varphi(\vartheta_n)) - \varrho G(\bar{\omega}_n)], \\ \theta_{n+1} &= \left(\frac{n+1}{2n+1}\right)\theta_n - \left(\frac{5n}{11(2n+1)}\right)\vartheta_n, \\ \vartheta_n &= (1 - \beta_n)\theta_n + \beta_n \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta_n), \varphi(\theta_n)) - \varrho G(\omega_n)], \\ \vartheta_n &= \left(\frac{n}{n+1}\right)\theta_n - \left(\frac{5}{11(n+1)}\right)\theta_n.\end{aligned}$$

For different initial points: $\theta_0 = 6$, and $\theta_0 = 1$, the sequence $\theta_n \rightarrow 0$ and $0 \in \text{Fix}(F) \cap \Omega(\mathbb{H}, M, \Phi, G, \eta)$ and the convergence is shown by the graph (Figure 1) below.

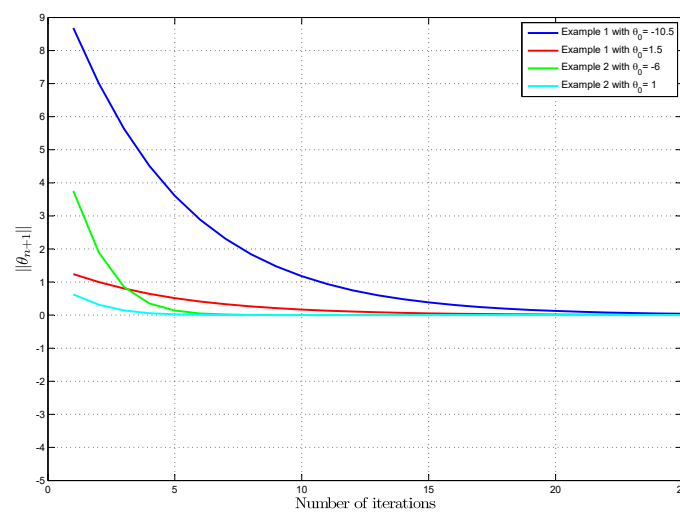


Figure 1. Convergence behavior of $\{\theta_n\}$ for Example 3.1 and Example 3.2 with initial values $\theta_0 = -10.5$, $\theta_0 = 1.5$, and $\theta_0 = -6$, $\theta_0 = 1$, respectively.

4. Generalized resolvent dynamical system

Herein, we employ the technique of the dynamical system to explore the solution of GVIP (3.1). By utilizing Lemma 3.1, the generalized resolvent dynamical system (GRDS) that we examine is as under:

$$\frac{d\theta}{ds} = \xi \{ \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta, \varphi\theta) - \varrho G(\omega)] - \theta \}, \theta(s_0) = \theta_0 \in \mathbb{H}, \quad (4.1)$$

where $\theta \in \mathbb{H}$, $\omega \in \Psi(\theta)$, and $\xi > 0$ is a parameter.

Definition 4.1. [24] It is stated that the GRDS (4.1) converges to the solution set $\Omega(\mathbb{H}, M, \Phi, G, \eta)$ of GVIP (3.1) if the trajectory of the dynamical system, irrespective of the initial point, satisfies

$$\lim_{s \rightarrow \infty} \text{dist}(\theta(s), \Omega) = 0,$$

where

$$\text{dist}(\theta(s), \Omega) = \inf_{\vartheta \in \Omega} \|\theta - \vartheta\|.$$

If θ^* is a unique point of Ω , then $\lim_{t \rightarrow \infty} \theta(s) = \theta^*$.

Definition 4.2. [25] The dynamical system is referred to as globally exponentially stable with degree k at θ^* if the trajectory of the dynamical system, irrespective of the initial point, satisfies

$$\|\theta(s) - \theta^*\| \leq c_0 \|\theta(s_0) - \theta^*\| \exp(-k(s - s_0)), s \geq s_0,$$

where positive constants k and c_0 do not depend on the initial point.

Lemma 4.1. [26] Let $\tilde{\theta}$ and $\tilde{\vartheta}$ be real-valued nonnegative continuous functions with domain $\{s : s \geq s_0\}$ and let $\alpha(s) = \alpha_0(|s - s_0|)$ where α_0 is a monotonic increasing function. If for all $s \geq s_0$,

$$\tilde{\theta}(s) \leq \alpha(s) + \int_{s_0}^s \tilde{\theta}(t) \tilde{\vartheta}(t) dt,$$

then

$$\tilde{\theta}(s) \leq \alpha(s) \cdot \exp\left(\int_{s_0}^s \tilde{\vartheta}(t) dt\right).$$

Next, by utilizing Lemma 4.1 and Theorem 3.1, we investigate the unique solution of GRDS (4.1).

Theorem 4.1. Assume that the Theorem 3.1 holds. Then, for each $\theta_0 \in \mathbb{H}$ with $\omega_0 \in \Psi(\theta_0)$, there exists a unique continuous solution $\theta(s)$ with $\theta(s_0) = \theta_0$ of GRDS (4.1) over $[s_0, \infty)$.

Proof. Define

$$\mathcal{G}(\theta) = \xi \{\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta, \varphi\theta) - \varrho G(\omega)] - \theta\}, \forall \theta \in \mathbb{H}.$$

Invoking the arguments as for (3.8), we obtain

$$\begin{aligned} \|\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\| &= \xi \|\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta, \varphi\theta) - \varrho G(\omega)] - \theta \\ &\quad - \{\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\vartheta, \varphi\vartheta) - \varrho G(\bar{\omega})] - \vartheta\}\| \\ &\leq \xi \Xi \|\Phi(\psi\theta, \varphi\theta) - \Phi(\psi\vartheta, \varphi\vartheta) - \varrho(G(\omega) - G(\bar{\omega}))\| \\ &\quad + \xi \|\theta - \vartheta\|. \end{aligned} \tag{4.2}$$

Invoking the arguments as employed to (3.9), we obtain

$$\|\Phi(\psi\theta, \varphi\theta) - \Phi(\psi\vartheta, \varphi\vartheta) - \varrho(G(\omega) - G(\bar{\omega}))\| \leq \sqrt{m^2 - 2\varrho\iota + \varrho^2 t^2 r^2} \|\theta - \vartheta\|. \tag{4.3}$$

(4.2) and (4.3) together yields

$$\|\mathcal{G}(\theta) - \mathcal{G}(\vartheta)\| \leq \xi(1 + \Theta) \|\theta - \vartheta\|, \tag{4.4}$$

which proves that \mathcal{G} is locally Lipschitz continuous in \mathbb{H} . Thus, for each $\theta_0 \in \mathbb{H}$, \exists a unique continuous solution $\theta(s)$ of GRDS (4.1) with $\theta(s_0) = \theta_0$ in the interval $s_0 \leq s < \mathcal{S}$. Let the maximal interval of its existence be $[s_0, \mathcal{S})$. Next, we substantiate that $\mathcal{S} = \infty$. Now, for any $\theta \in \mathbb{H}$, $\omega \in \Psi(\theta)$, we have

$$\begin{aligned} \|\mathcal{G}(\theta)\| &= \xi \|\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta, \varphi\theta) - \varrho G(\omega)] - \theta\| \\ &\leq \xi \|\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta, \varphi\theta) - \varrho G(\omega)] - \theta^*\| + \xi \|\theta - \theta^*\| \\ &= \xi \|\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta, \varphi\theta) - \varrho G(\omega)] \\ &\quad - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta^*, \varphi\theta^*) - \varrho G(\omega^*)]\| + \xi \|\theta - \theta^*\| \\ &\leq \xi \Xi \|\Phi(\psi\theta, \varphi\theta) - \Phi(\psi\theta^*, \varphi\theta^*) - \varrho(G(\omega) - G(\omega^*))\| + \xi \|\theta - \theta^*\| \\ &\leq \xi(1 + \Theta) \|\theta - \theta^*\| \\ &\leq \xi(1 + \Theta) \|\theta\| + \xi(1 + \Theta) \|\theta^*\|. \end{aligned} \tag{4.5}$$

Employing the integral on (4.5) over $[s_0, s]$ and utilizing Lemma 4.1, we get

$$\begin{aligned} \|\theta(s)\| &\leq \|\theta_0\| + \int_{s_0}^s \|\mathcal{G}(\theta(\tau))\| d\tau \\ &\leq \{\|\theta_0\| + k_5(s - s_0)\} + k_6 \int_{s_0}^s \|\theta(\tau)\| d\tau, \forall s \in [s_0, \mathcal{S}] \\ &\leq \{\|\theta_0\| + k_5(s - s_0)\} \exp\{k_6(s - s_0)\}, \forall s \in [s_0, \mathcal{S}], \end{aligned} \quad (4.6)$$

where $k_5 = \xi(1 + \Theta)\|\theta^*\|$ and $k_6 = \xi(1 + \Theta)$. Hence the solution is bounded on $[s_0, \mathcal{S}]$, so $\mathcal{S} = \infty$. \square

In the next theorem, we shall examine GVIP (3.1) by the convergence of the trajectory of the solution of considered GRDS (4.1).

Theorem 4.2. *Assume Theorem 3.1 is true. Then, GRDS (4.1) converges globally exponentially to the unique solution $\theta^* \in \Omega(\mathbb{H}, M, \Phi, G, \eta)$.*

Proof. It is evident from the Theorem 4.1 that GRDS (4.1) owns a unique solution. Assume that $\theta(s) = (s, s_0; \theta_0)$ is a solution of GRDS (4.1) with $\theta(s_0) = \theta_0$. Define the Lyapunov function \mathcal{L} on \mathbb{H} by

$$\mathcal{L}(\theta) = \frac{1}{2} \|\theta - \theta^*\|^2, \forall \theta \in \mathbb{H}. \quad (4.7)$$

We obtained the relation $\theta = \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi(\theta), \varphi(\theta)) - \varrho G(\omega)]$ from the Lemma 3.1 and, utilizing (3.8)–(3.11), we acquire

$$\begin{aligned} \frac{d\mathcal{L}}{ds} &= \left\langle \theta(s) - \theta^*, \frac{d\theta}{ds} \right\rangle \\ &= \xi \langle \theta(s) - \theta^*, \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta, \varphi\theta) - \varrho G(\omega)] - \theta \rangle \\ &= -\xi \langle \theta(s) - \theta^*, \theta(s) - \theta^* \rangle + \xi \langle \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta, \varphi\theta) - \varrho G(\omega)] - \theta^* \rangle \\ &\leq -\xi \|\theta(s) - \theta^*\|^2 + \xi \langle \theta(s) - \theta^*, \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta, \varphi\theta) - \varrho G(\omega)] \\ &\quad - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta^*, \varphi\theta^*) - \varrho G(\omega^*)] \rangle \\ &\leq -\xi \|\theta(s) - \theta^*\|^2 + \xi \|\theta(s) - \theta^*\| \|\mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta, \varphi\theta) - \varrho G(\omega)] \\ &\quad - \mathbb{R}_{\varrho, M(\cdot, \cdot)}^{\eta, \Phi(\cdot, \cdot)}[\Phi(\psi\theta^*, \varphi\theta^*) - \varrho G(\omega^*)]\| \\ &\leq -\xi \|\theta(s) - \theta^*\|^2 + \xi \Theta \|\theta(s) - \theta^*\|^2, \end{aligned}$$

which yields

$$\frac{d}{ds} \frac{1}{2} \|\theta - \theta^*\|^2 = -\xi(1 - \Theta) \|\theta(s) - \theta^*\|^2, \quad (4.8)$$

where $\Theta = \Xi \sqrt{m^2 - 2\varrho t + \varrho^2 t^2 r^2}$ and $\Xi = \frac{\pi}{[\varrho(\tau' - \tau'') + (\kappa' \zeta^2 - \kappa'' l^2)]}$. Thus, we acquire

$$\|\theta - \theta^*\| \leq \|\theta_0 - \theta^*\| e^{-\xi(1-\Theta)(s-s_0)}. \quad (4.9)$$

From (3.6), we know that $1 - \Theta > 0$. As a result, the trajectory of the solution of GRDS (4.1) converges globally exponentially to the unique solution of GVIP (3.1). \square

5. Conclusions

In this work, we investigate a generalized variation inclusion problem. The resolvent operator for generalized η -co-monotone mapping is structured, the Lipschitz constant is estimated and its relationship with the graph convergence is accomplished. An Ishikawa type iterative algorithm is designed and employed to explore the common solution of the generalized variational inclusion and the set of fixed points of a TANM by using the novel implication of graph convergence. Moreover, a generalized resolvent dynamical system is considered and implemented to examine the considered generalized variation inclusion problem.

Author contributions

DF: funding, writing review and editing, supervision; MD: conceptualization, writing review and editing; MA: conceptualization, writing original draft preparation, writing review and editing, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used AI tools in the creation of this article.

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Conflict of interest

Authors declare no conflicts of interest in this paper.

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