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Research article

A generalized Gronwall inequality via ψ-Hilfer proportional fractional operators and its applications to nonlocal Cauchy-type system

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Abstract: This paper establishes a novel generalized Gronwall inequality concerning the ψ-Hilfer proportional fractional operators. Before proving the main results, the solution of the linear nonlocal coupled ψ-Hilfer proportional Cauchy-type system with constant coefficients under the Mittag-Leffler kernel is created. The uniqueness result for the proposed coupled system is established using Banach's contraction mapping principle. Furthermore, a variety of the Mittag-Leffler-Ulam-Hyers stability of the solutions for the proposed coupled system is investigated. Finally, a numerical example is given to show the effectiveness and applicability of the obtained results, and graphical simulations in the case of linear systems are shown.

Keywords: generalized Gronwall inequality; existence and uniqueness; fixed point theorem; ψ-Hilfer proportional fractional operators; Mittag-Leffler-Ulam-Hyers stability
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1. Introduction

Fractional (non-integer) calculus is a strong tool for studying all kinds of complicated problems in applied science and engineering. Because nonlocal properties are suitable to specify memory phenomena in several real-world problems, fractional differential equations are required to explain intricate processes more precisely than differential equations (DEs) of integer order, see [1–4]. There are various distinct definitions of fractional integrals and derivatives in the literature. However, fractional operators in the context of Riemann-Liouville (RL) and Caputo senses are the most popularly used, see a variety of fractional operators, [5–8] and references therein. Recently, a new class of fractional derivative operators with proportional terms was developed and studied in [9, 10]. After that, Ahmed et al. [11] presented the idea of the Hilfer-type proportional fractional derivative operator (HPFDO), and Mallah et al. [12] proposed the ψ-Hilfer proportional fractional operator (ψ-HPFDO). For further information on the advantages of these fractional derivatives, refer to the following studies in regard to, *e.g.*, glass forming materials [13], advection-diffusion [14], Cauchy-problem [15], filtration processes [16, 17], and calcium diffusion [18].

Integral inequalities are applied in all kinds of mathematical areas, approximation theory, spectral analysis, statistical analysis, distribution theory, and so on. Due to the importance of integral inequalities and their applications in the qualitative theory of DEs, they are important in many disciplines of engineering and applied science [19]. The Gronwall inequality, proposed by Gronwall [20], is one of the most well-known integral inequalities. It has gained popularity because it is a useful instrument for analyzing the sufficient conditions of the qualitative and quantitative properties of solutions for nonlinear integral equations and DEs with integer-order and fractional-order by providing explicit bounds for solutions. There has been a steady increase in interest in this field of study to fulfill the demands of numerous applications of these properties. Many researchers have developed fractional Gronwall inequalities, employing a variety of methodologies to investigate and propose these inequalities. For more modern works on the Gronwall inequality with a variety of fractional operators, see [21–28]. Parallel to the increasing interest in the qualitative theory of DEs under various fractional operators, many researchers have discussed the existence and uniqueness of the solutions to these equations by using the well-known fixed point theorem. Meanwhile, the stability of Ulam-Hyer (UH) type is critical in the qualitative theory of linear and nonlinear DEs proposed by Ulam [29] and Hyers [30]. Rassias [31] extended the UH stability in 1978 to develop the stability of Ulam-Hyers-Rassias (UHR) type. The UH stability is the most effective stability approach analyzed. It is a form of data-dependent solution with applications in optimization, biology, economics, and other fields. Typically, we will study the existence and uniqueness of the solutions to the proposed problem in the first stage, followed by consideration of relevant differential inequalities. When the proposed problem under discussion has UH stability, there is a solution to the proposed problem so that both solutions are close enough. In addition, Wang et al. [32] proposed the fractional UH stability in 2011.

To understand our motivation to do this work, some inspirational works will be presented. In 2014, Wang and Li [33] introduced new ideas for Mittag-Leffler-Ulam (ML-U) stability of solutions for Cauchy-type problems under fractional Caputo derivatives using an extended Gronwall inequality, which differ from Ulam stability: Mittag-Leffler-Ulam-Hyers (ML-UH) stability, Mittag-Leffler-Ulam-Hyers-Rassias (ML-UHR) stability, and their generalization stabilities. Not long after that, in 2018, by applying fixed point techniques and the Mittag-Leffler (ML) function, Gao et al. [34] investigated the existence and uniqueness of solutions to nonlocal Cauchy-type problems under the Hilfer fractional derivative. In 2019, a generalized Gronwall inequality in the context of ψ-Hilfer fractional operators was introduced by Sousa and Oliveira [23]. It was applied to nonlinear ψ-Hilfer Cauchy-type problems. Later, Kucche et al. [35] offered a method for solving the ψ-Hilfer fractional Cauchy-type problems explicitly in terms of ML function. They proved the existence of the solutions for the proposed problem by using the Weissinger fixed point theory. By using an extended Gronwall inequality, Liu et al. [36] studied the existence, uniqueness, and ML-UH stability of the solutions to the ψ-Hilfer fractional delay differential equations. Alzabut et al. [24] investigated a new form for Gronwall inequality under the proportional fractional operators, and they proved the uniqueness of solution to proportional fractional delay DEs. In 2021, by using fixed point theory of Banach's and Schaefer's types, Almalahi and Panchal [37] established existence, uniqueness, and ML-U results for nonlocal problems of the form

$$\begin{cases} {}^{H}\mathfrak{D}_{a^+}^{\alpha,\beta;\psi}u(\tau) = \lambda u(\tau) + f(\tau, u(\tau)), \quad \alpha \in (1,2), \quad \beta \in [0,1], \quad \tau \in (a,b], \\ u(a) = 0, \quad u(b) = \sum_{i=1}^{m} \mu_i \mathcal{I}_{a^+}^{\delta;\psi} v(\xi_i), \quad \xi_i \in (a,b], \quad i = 1,2,\ldots,m, \quad m \in \mathbb{N}, \end{cases} \tag{1.1}$$

where ${}^{H}\mathfrak{D}_{a^+}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative of order α and type β, $\gamma = \alpha + \beta(2-\alpha)$, $\mathcal{I}_{a^+}^{\delta;\psi}$ is the ψ-fractional integral of order $\delta > 0$, $\mu_i \in \mathbb{R}$, and $i = 1,2,\ldots,m$, $\lambda < 0$, and $f \in C((a,b] \times \mathbb{R}, \mathbb{R})$. In 2023, Almalahi et al. [38] investigated sufficient conditions of the existence and uniqueness of the solutions for ψ-Hilfer fractional integro-differential equations by using Banach's and Schauder's fixed point thoerems with Hölder's inequality. They analyzed ML-U stability results by utilizing an extended Gronwall's inequality. Recently, Sudsutad et al. [28] constructed a generalized version for Gronwall inequality in the sense of the ψ-HPFDO. By using Banach's fixed point theorem, they proved a uniqueness result for nonlocal Cauchy-problems under the ψ-HPFDO, and a variety of ML-U stability results are analyzed.

A lot of works are interested in investigating fractional-order coupled systems, which arise in models for bioengineering [39], financial economics [40], dynamics [41], and so on. In 2023, Almalahi et al. [42] considered the existence and stability results for a ψ-Hilfer-type coupled system of sequential fractional differential equations, while the nonlinear coupled hybrid systems under the Hilfer fractional derivative operator (HFDO) were discussed in [43]. In 2022, Samadi et al. [44] studied a non-local coupled system of the HPFDO. In 2023, Ahmad and Aljoudi [45] investigated existence and uniqueness results for a coupled system of fractional differential equations with non-local integral conditions involving the Hilfer-Hadamard fractional derivative operator. For some features of the ψ-HPFDO, we refer the reader to the work [12]. It is worth noting that research on coupled systems of the ψ-HPFDO is limited and requires additional development. Motivated by previous modern works [26, 28, 33, 37], we discuss the following nonlinear coupled Cauchy-type system under nonlocal integral conditions involving the ψ-HPFDO as follows:

$$\begin{cases} {}^{H}_{\rho}\mathfrak{D}_{a^+}^{\alpha_1,\beta;\psi}u(\tau) = \lambda_1 u(\tau) + f(\tau, u(\tau), v(\tau)), \quad \tau \in (a,b], \quad 0 < a < b, \\ {}^{H}_{\rho}\mathfrak{D}_{a^+}^{\alpha_2,\beta;\psi}v(\tau) = \lambda_2 v(\tau) + g(\tau, u(\tau), v(\tau)), \quad \tau \in (a,b], \quad 0 < a < b, \\ u(a) = 0, \quad u(b) = \sum_{i=1}^{m} \theta_i {}_{\rho}\mathcal{I}_{a^+}^{\delta_i;\psi} v(\xi_i), \quad \xi_i \in (a,b], \quad i = 1,2,\ldots,m, \quad m \in \mathbb{N}, \\ v(a) = 0, \quad v(b) = \sum_{j=1}^{n} \mu_j {}_{\rho}\mathcal{I}_{a^+}^{\varrho_j;\psi} u(\eta_j), \quad \eta_j \in (a,b], \quad j = 1,2,\ldots,n, \quad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where ${}^{H}_{\rho}\mathfrak{D}_{a^+}^{\alpha_k,\beta;\psi}$ is the ψ-HPFDO of order $\alpha_k \in (1,2)$, $k = 1,2$, and type $\beta \in [0,1]$, $\rho \in (0,1]$, $\alpha_k \le \gamma_k := \alpha_k + (2-\alpha_k)\beta$, $\lambda_k < 0$, $k = 1,2$, and $f, g \in C([a,b] \times \mathbb{R}^2, \mathbb{R})$, ${}_{\rho}\mathcal{I}_{a^+}^{q;\psi}$ is the ψ-Hilfer proportional fractional integral operator (ψ-HPFIO) of order $q \in \{\delta_i, \varrho_j\} > 0$, $\theta_i, \mu_j \in \mathbb{R}$, $i = 1,2,\ldots,m$, and $j = 1,2,\ldots,n$.

The structure of this paper is as follows: Section 2 is separated into three subsections. The first part provides elementary ideas on the ψ-HPFDO and the ψ-HPFIO, some lemmas, and the basic properties of the ML functions that are used throughout this paper. Meanwhile, an extended Gronwall inequality in the context of the ψ-HPFIO is analyzed in the second part. Section 3 derives an analogous integral equation to the linearly linked ψ-Hilfer proportional system via constant coefficient in the context of the ML kernel. Section 3 is divided into two subsections. The uniqueness of the solutions to the proposed coupled system is established in the first part using Banach's fixed point theory, and a variety of ML-UH stability results are studied in the second part. However, we show numerical examples to present theoretically applicable results in Section 4, while the conclusion part is offered in the last section.

2. Preliminaries

2.1. The ψ-Hilfer proportional fractional operators (ψ-HPFOs)

Assume that $L^q(\mathcal{J},\mathbb{R})$ is a Banach space of all Lebesgue measurable $\theta:\mathcal{J}\to\mathbb{R}$ supplemented with the norm $\|\theta\|_{L^q(\mathcal{J})}<+\infty$, where $\mathcal{J}:=[a,b]$, and $\mathbb{U}=C(\mathcal{J},\mathbb{R})$ is the Banach space of a continuous function from $\mathcal{J}$ to $\mathbb{R}$ supplemented with $\|u\|=\sup_{\tau\in\mathcal{J}}\{|u(\tau)|\}$. Assume that g is an integrable function on $\mathcal{J}$ and $\psi\in C^1(\mathcal{J},\mathbb{R})$ is an increasing function with $\psi'(\tau)\neq 0$ for all $\tau\in\mathcal{J}$. For easily calculation, we provide the notation

$$\Psi_{\psi}^{\alpha-1}(\tau,s)=e^{\frac{\rho-1}{\rho}(\psi(\tau)-\psi(s))}(\psi(\tau)-\psi(s))^{\alpha-1}. \tag{2.1}$$

Definition 2.1. [9,10] *Let $\alpha>0$, $\rho\in(0,1]$. The ψ-RL-HPFIO of order α of g is given by*

$${}_{\rho}\mathcal{I}_{a^+}^{\alpha;\psi}g(\tau)=\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^{\tau}\Psi_{\psi}^{\alpha-1}(\tau,s)g(s)\psi'(s)ds,\quad \Gamma(\alpha)=\int_0^{\infty}s^{\alpha-1}e^{-s}ds,\ s>0. \tag{2.2}$$

Definition 2.2. [9,10] *Let $\alpha>0$, $\rho\in(0,1]$. The ψ-RL-HPFDO of order α of g is given by*

$${}_{\rho}^{RL}\mathfrak{D}_{a^+}^{\alpha;\psi}g(\tau)={}_{\rho}\mathfrak{D}^{n;\psi}\,{}_{\rho}\mathcal{I}_{a^+}^{n-\alpha;\psi}g(\tau)=\frac{{}_{\rho}\mathfrak{D}_{\tau}^{n;\psi}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^{\tau}\Psi_{\psi}^{n-\alpha-1}(\tau,s)g(s)\psi'(s)ds, \tag{2.3}$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of α, and ${}_{\rho}\mathfrak{D}^{n;\psi}=\underbrace{{}_{\rho}\mathfrak{D}^{\psi}\cdot{}_{\rho}\mathfrak{D}^{\psi}\cdots{}_{\rho}\mathfrak{D}^{\psi}}_{n\text{-times}}$ and

$${}_{\rho}\mathfrak{D}^{\psi}g(\tau)=(1-\rho)g(\tau)+\rho\frac{g'(\tau)}{\psi'(\tau)}. \tag{2.4}$$

Definition 2.3. [9,10] *Let $\alpha>0$, $\rho\in(0,1]$. The ψ-Caputo-PFDO of order α of g is given by*

$${}_{\rho}^{C}\mathfrak{D}_{a^+}^{\alpha;\psi}g(\tau)={}_{\rho}\mathcal{I}_{a^+}^{n-\alpha;\psi}\,{}_{\rho}\mathfrak{D}^{n;\psi}g(\tau)=\frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^{\tau}\Psi_{\psi}^{n-\alpha-1}(\tau,s)\,{}_{\rho}\mathfrak{D}^{n;\psi}g(s)\psi'(s)ds. \tag{2.5}$$

Definition 2.4. [12] *Let $\alpha\in(n-1,n)$, $n\in\mathbb{N}$, $\rho\in(0,1]$, $\beta\in[0,1]$, $g\in C^n(\mathcal{J},\mathbb{R})$, and $\psi\in C^n(\mathcal{J},\mathbb{R})$, for all $\tau\in\mathcal{J}$. The ψ-HPFDO of order α and type β of g is given by*

$${}_{\rho}^{H}\mathfrak{D}_{a^+}^{\alpha,\beta;\psi}g(\tau)={}_{\rho}\mathcal{I}_{a^+}^{\beta(n-\alpha);\psi}({}_{\rho}\mathfrak{D}^{n;\psi})\,{}_{\rho}\mathcal{I}_{a^+}^{(1-\beta)(n-\alpha);\psi}g(\tau)={}_{\rho}\mathcal{I}_{a^+}^{\beta(n-\alpha);\psi}\,{}_{\rho}\mathfrak{D}_{a^+}^{\gamma;\psi}g(\tau). \tag{2.6}$$

From Eq (2.6), we obtain the following two derivative types:

$${}_{\rho}^{H}\mathfrak{D}_{a^+}^{\alpha,\beta;\psi}g(\tau)=\begin{cases}{}_{\rho}\mathfrak{D}^{n;\psi}\,{}_{\rho}\mathcal{I}_{a^+}^{n-\alpha;\psi}g(\tau) & \text{if}\quad \beta=0,\\ {}_{\rho}\mathcal{I}_{a^+}^{n-\alpha;\psi}\,{}_{\rho}\mathfrak{D}^{n;\psi}g(\tau) & \text{if}\quad \beta=1.\end{cases} \tag{2.7}$$

Lemma 2.5. [9, 10] *Let $\rho \in (0,1]$ and $\alpha_i > 0$, $i = 1,2$. If g is continuous defined on $\mathcal{J}$, we have*

$$ {}_\rho\mathcal{I}_{a^+}^{\alpha_1;\psi}\,{}_\rho\mathcal{I}_{a^+}^{\alpha_2;\psi} g(\tau) = {}_\rho\mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\psi} g(\tau) = {}_\rho\mathcal{I}_{a^+}^{\alpha_2;\psi}\,{}_\rho\mathcal{I}_{a^+}^{\alpha_1;\psi} g(\tau). $$

Lemma 2.6. [9, 10] *Let $\rho \in (0,1]$ and $\alpha_i \in (n-1,n]$ with $0 < \alpha_2 < \alpha_1$, $n \in \mathbb{N}$, $i = 1,2$. Then,*

$$ {}_\rho\mathfrak{D}_{a^+}^{\alpha_2;\psi}\,{}_\rho\mathcal{I}_{a^+}^{\alpha_1;\psi} g(\tau) = {}_\rho\mathcal{I}_{a^+}^{\alpha_1-\alpha_2;\psi} g(\tau). $$

Lemma 2.7. [12] *Let $\rho \in (0,1]$, $\beta \in [0,1]$, and $\alpha \in (n-1,n)$, $\gamma \in (n-1,n)$, $\gamma := \alpha + \beta(n-\alpha)$, where $n \in \mathbb{N}$. If $g \in C_\gamma(\mathcal{J})$ and ${}_\rho\mathcal{I}_{a^+}^{n-\gamma,\psi} g \in C^n_{\gamma,\psi}(\mathcal{J})$, then we obtain*

$$ {}_\rho\mathcal{I}_{a^+}^{\alpha;\psi}\,{}^H_\rho\mathfrak{D}_{a^+}^{\alpha,\beta;\psi} g(\tau) = g(\tau) - \sum_{j=1}^{n} \frac{\Psi_\psi^{\gamma-j}(\tau,a)}{\rho^{\gamma-j}\Gamma(\gamma-j+1)}\left({}_\rho\mathcal{I}_{a^+}^{j-\gamma;\psi} g(a)\right). \tag{2.8} $$

Lemma 2.8. [28] *Let $\rho \in (0,1]$, $\beta \in [0,1]$, and $\alpha_i \in (n-1,n)$, $i = 1,2$ with $\alpha_2 \geq \alpha_1 + \beta(n-\alpha_1)$ where $n \in \mathbb{N}$. If $g \in C^n(\mathcal{J},\mathbb{R})$, then*

$$ {}^H_\rho\mathfrak{D}_{a^+}^{\alpha_1,\beta;\psi}\,{}_\rho\mathcal{I}_{a^+}^{\alpha_2,\psi} g(\tau) = {}_\rho\mathcal{I}_{a^+}^{\alpha_2-\alpha_1;\psi} g(\tau). \tag{2.9} $$

Proposition 2.9. [9, 10] *Let $\alpha \geq 0$, $\delta > 0$, and $\rho \in (0,1]$. Then we have*

$$ {}_\rho\mathcal{I}_{a^+}^{\alpha;\psi}\left[\Psi_\psi^{\delta-1}(\tau,a)\right] = \frac{\Gamma(\delta)}{\rho^\alpha\Gamma(\alpha+\delta)}\,\Psi_\psi^{\delta-1+\alpha}(\tau,a). \tag{2.10} $$

Proposition 2.10. [12] *Let $\alpha \in (n-1,n)$, $\beta \in [0,1]$, $\rho \in (0,1]$, $\gamma = \alpha + \beta(n-\alpha)$, $n = [\alpha]+1$. Then we have*

$$ {}^H_\rho\mathfrak{D}_{a^+}^{\alpha,\beta;\psi}\left[\Psi_\psi^{\delta-1}(\tau,a)\right] = \frac{\rho^\alpha\Gamma(\delta)}{\Gamma(\delta-\alpha)}\,\Psi_\psi^{\delta-1-\alpha}(\tau,a), \quad \delta > n \quad \delta \in \mathbb{R}. \tag{2.11} $$

Now, we give some details about the classical and generalized ML functions, $\mathbb{E}_\alpha$ and $\mathbb{E}_{\alpha,\beta}$, respectively, which are applied throughout this paper.

Lemma 2.11. ([46], Lemma 2, Page 1862). *Let $\alpha_i \in (0,1)$, $i = 1,2$. If the three functions $\mathbb{E}_{\alpha_1}$, $\mathbb{E}_{\alpha_1,\alpha_1}$, and $\mathbb{E}_{\alpha_1,\alpha_1+\alpha_2}$ are nonnegative functions, we obtain the following relations:*

$$ \mathbb{E}_{\alpha_1}(z) := \mathbb{E}_{\alpha_1,1}(z) \leq 1, \quad \mathbb{E}_{\alpha_1,\alpha_1}(z) \leq \frac{1}{\Gamma(\alpha_1)}, \quad \mathbb{E}_{\alpha_1,\alpha_1+\alpha_2}(z) \leq \frac{1}{\Gamma(\alpha_1+\alpha_2)}, \quad \forall z < 0, \tag{2.12} $$

with $\mathbb{E}_{\alpha_1}(0) = 1$, $\mathbb{E}_{\alpha_1,\alpha_1}(0) = 1/\Gamma(\alpha_1)$, $\mathbb{E}_{\alpha_1,\alpha_1+\alpha_2}(0) = 1/\Gamma(\alpha_1+\alpha_2)$, and

$$ \mathbb{E}_{\alpha_1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha_1+1)} \quad \text{and} \quad \mathbb{E}_{\alpha_1,\alpha_1+\alpha_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma((k+1)\alpha_1+\alpha_2)}, \quad z \in \mathbb{R},\ \alpha_1,\alpha_2 > 0. $$

In addition, for any $\lambda < 0$ and τ_1, $\tau_2 \in \mathcal{J}$, we get the following result:

$$ \mathbb{E}_{\alpha_1,\alpha_1+\alpha_2}(\lambda(\psi(\tau_2)-\psi(a))^{\alpha_1}) \to \mathbb{E}_{\alpha_1,\alpha_1+\alpha_2}(\lambda(\psi(\tau_1)-\psi(a))^{\alpha_1}) \quad \text{as} \quad \tau_2 \to \tau_1. $$

Lemma 2.12. [28] *Let α, δ, $\mu \in \mathbb{R}^+$, and $\lambda \in \mathbb{R}$. Then, we have the following properties:*

$$ \begin{aligned} {}^H_\rho\mathfrak{D}_{a^+}^{\alpha,\beta;\psi}\left[\Psi_\psi^{\delta-1}(\tau,a)\mathbb{E}_{\mu,\delta}(\lambda\rho^{-\mu}(\psi(\tau)-\psi(a))^\mu)\right] &= \rho^\alpha\Psi_\psi^{\delta-1-\alpha}(\tau,a)\,\mathbb{E}_{\mu,\delta-\alpha}(\lambda\rho^{-\mu}(\psi(\tau)-\psi(a))^\mu), \\ {}_\rho\mathcal{I}_{a^+}^{\alpha;\psi}\left[\Psi_\psi^{\delta-1}(\tau,a)\mathbb{E}_{\mu,\delta}(\lambda\rho^{-\mu}(\psi(\tau)-\psi(a))^\mu)\right] &= \frac{1}{\rho^\alpha}\Psi_\psi^{\delta-1+\alpha}(\tau,a)\mathbb{E}_{\mu,\delta+\alpha}(\lambda\rho^{-\mu}(\psi(\tau)-\psi(a))^\mu). \end{aligned} $$

Lemma 2.13. [28] *Let $\alpha > 0$, $\beta > 0$, $k > 0$, $\rho \in (0, 1]$, $\lambda \in \mathbb{R}$, and $g \in C(\mathcal{J}, \mathbb{R})$. Then*

$$\begin{aligned} & {}_{\rho}\mathcal{I}_{a^+}^{k;\psi}\left[\int_a^\tau \Psi_\psi^{\alpha-1}(\tau, s)\mathbb{E}_{\alpha,\alpha}(\lambda\rho^{-\alpha}(\psi(\tau) - \psi(s))^\alpha)g(s)\psi'(s)ds\right] \\ = & \frac{1}{\rho^k}\int_a^\tau \Psi_\psi^{\alpha+k-1}(\tau, s)\, \mathbb{E}_{\alpha,k+\alpha}(\lambda\rho^{-\alpha}(\psi(\tau) - \psi(s))^\alpha)g(s)\psi'(s)ds. \end{aligned} \tag{2.13}$$

Lemma 2.14. [28] *Let $(n + 1)\alpha > k > 0$, $\beta > 0$, $\rho \in (0, 1]$, $\lambda \in \mathbb{R}$, and $g \in C(\mathcal{J}, \mathbb{R})$. Then,*

$$\begin{aligned} & {}_{\rho}^{H}\mathfrak{D}_{a^+}^{k,\beta;\psi}\left[\int_a^\tau \Psi_\psi^{\alpha-1}(\tau, s)\mathbb{E}_{\alpha,\alpha}(\lambda\rho^{-\alpha}(\psi(\tau) - \psi(s))^\alpha)g(s)\psi'(s)ds\right] \\ = & \rho^k\int_a^\tau \Psi_\psi^{\alpha-k-1}(\tau, s)\mathbb{E}_{\alpha,\alpha-k}(\lambda\rho^{-\alpha}(\psi(\tau) - \psi(s))^\alpha)g(s)\psi'(s)ds. \end{aligned} \tag{2.14}$$

2.2. A generalized Gronwall inequality via ψ-HPFIO

Theorem 2.15. (*A generalized Gronwall's inequality via the ψ-HPFIO) Suppose that ρ, $\alpha_i \in \mathbb{R}^+$, $i = 1, 2$, and $\psi \in C^1(\mathcal{J}, \mathbb{R})$ is an increasing function so that $\psi'(\tau) \neq 0$, $\forall\tau \in \mathcal{J}$. Suppose the following properties:*

($\mathbb{P}_1$) *The functions $u(\tau)$, $v(\tau)$, and κ_i are nonnegative locally integrable on $\mathcal{J}$ for $i = 1, 2$;*
($\mathbb{P}_2$) *$\phi_i(\tau)$ are nonnegative, nondecreasing, and continuous functions defined on $\tau \in \mathcal{J}$, $i = 1, 2$, such that $\phi_i(\tau) \leq \phi_i^*$, where $\phi_i^* \in \mathbb{R}$ for $i = 1, 2$.*

If

$$u(\tau) \leq \kappa_1(\tau) + \rho^{\alpha_1}\Gamma(\alpha_1)\phi_1(\tau)\,{}_{\rho}\mathcal{I}_{a^+}^{\alpha_1;\psi}v(\tau), \tag{2.15}$$
$$v(\tau) \leq \kappa_2(\tau) + \rho^{\alpha_2}\Gamma(\alpha_2)\phi_2(\tau)\,{}_{\rho}\mathcal{I}_{a^+}^{\alpha_2;\psi}u(\tau), \tag{2.16}$$

then

$$\begin{aligned} u(\tau) \leq\ & \kappa_1(\tau) + \rho^{\alpha_1}\Gamma(\alpha_1)\phi_1(\tau)\,{}_{\rho}\mathcal{I}_{a^+}^{\alpha_1;\psi}\kappa_2(\tau) + \int_a^\tau \sum_{k=1}^{\infty}\frac{[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^k}{\Gamma(k(\alpha_1 + \alpha_2))}\Psi_\psi^{k(\alpha_1+\alpha_2)-1}(\tau, s) \\ & \times\Big(\kappa_1(s) + \rho^{\alpha_1}\Gamma(\alpha_1)\phi_1(s)\,{}_{\rho}\mathcal{I}_{a^+}^{\alpha_1;\psi}\kappa_2(s)\Big)\psi'(s)ds, \quad \tau \in \mathcal{J}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} v(\tau) \leq\ & \kappa_2(\tau) + \rho^{\alpha_2}\Gamma(\alpha_2)\phi_2(\tau)\,{}_{\rho}\mathcal{I}_{a^+}^{\alpha_2;\psi}\kappa_1(\tau) + \int_a^\tau \sum_{k=1}^{\infty}\frac{[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^k}{\Gamma(k(\alpha_1 + \alpha_2))}\Psi_\psi^{k(\alpha_1+\alpha_2)-1}(\tau, s) \\ & \times\Big(\kappa_2(s) + \rho^{\alpha_2}\Gamma(\alpha_2)\phi_2(s)\,{}_{\rho}\mathcal{I}_{a^+}^{\alpha_2;\psi}\kappa_1(s)\Big)\psi'(s)ds, \quad \tau \in \mathcal{J}. \end{aligned} \tag{2.18}$$

Proof. Define two operators

$$\mathcal{B}_1v(\tau) = \phi_1(\tau)\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau, s)v(s)\psi'(s)ds \quad \text{and} \quad \mathcal{B}_2u(\tau) = \phi_2(\tau)\int_a^\tau \Psi_\psi^{\alpha_2-1}(\tau, s)u(s)\psi'(s)ds.$$

From (2.15) and (2.16) imply that

$$u(\tau) \leq \kappa_1(\tau) + \mathcal{B}_1v(\tau) \quad \text{and} \quad v(\tau) \leq \kappa_2(\tau) + \mathcal{B}_2u(\tau). \tag{2.19}$$

By using (2.19) with the monotonic property of $\mathcal{B}_1$ and $\mathcal{B}_2$, we have

$$u(\tau) \leq \kappa_1(\tau) + \mathcal{B}_1 v(\tau) \leq \kappa_1(\tau) + \mathcal{B}_1\kappa_2(\tau) + \mathcal{B}_1\mathcal{B}_2\kappa_1(\tau) + \mathcal{B}_1\mathcal{B}_2\mathcal{B}_1\kappa_2(\tau) + (\mathcal{B}_1\mathcal{B}_2)^2 u(\tau).$$

By iterative procedure, for $m = 1, 2, \ldots$, we get

$$u(\tau) \leq \sum_{k=0}^{m-1}(\mathcal{B}_1\mathcal{B}_2)^k\kappa_1(\tau) + \sum_{k=0}^{m-1}(\mathcal{B}_1\mathcal{B}_2)^k\mathcal{B}_1\kappa_2(\tau) + (\mathcal{B}_1\mathcal{B}_2)^m u(t), \quad \tau \in \mathcal{J},$$

where $(\mathcal{B}_1\mathcal{B}_2)^0\kappa_1(\tau) = \kappa_1(\tau)$. Similarly, for $\tau \in \mathcal{J}$, one has

$$v(\tau) \leq \sum_{k=0}^{m-1}(\mathcal{B}_2\mathcal{B}_1)^k\kappa_2(\tau) + \sum_{k=0}^{m-1}(\mathcal{B}_2\mathcal{B}_1)^k\mathcal{B}_2\kappa_1(\tau) + (\mathcal{B}_2\mathcal{B}_1)^m v(t),$$

where $(\mathcal{B}_2\mathcal{B}_1)^0\kappa_2(\tau) = \kappa_2(\tau)$. Next, we will show that

$$(\mathcal{B}_1\mathcal{B}_2)^m u(\tau) \leq \int_a^\tau \frac{[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^m}{\Gamma(m(\alpha_1+\alpha_2))}\Psi_\psi^{m(\alpha_1+\alpha_2)-1}(\tau,s)u(s)\psi'(s)ds, \tag{2.20}$$

$$(\mathcal{B}_2\mathcal{B}_1)^m v(\tau) \leq \int_a^\tau \frac{[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^m}{\Gamma(m(\alpha_1+\alpha_2))}\Psi_\psi^{m(\alpha_1+\alpha_2)-1}(\tau,s)v(s)\psi'(s)ds, \tag{2.21}$$

and $(\mathcal{B}_1\mathcal{B}_2)^m u(\tau) \to 0$ and $(\mathcal{B}_2\mathcal{B}_1)^m v(\tau) \to 0$ as $m \to \infty$ for $t \in \mathcal{J}$.

Note that (2.20) holds when $m = 1$ with the property that $\phi_2(\tau)$ is a nondecreasing function, $\phi_2(s) \leq \phi_2(\tau)$, for any $s \leq \tau$, that is,

$$\begin{aligned}
(\mathcal{B}_1\mathcal{B}_2)u(\tau) &= \phi_1(\tau)\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\left(\phi_2(s)\int_a^s \Psi_\psi^{\alpha_2-1}(s,r)u(r)\psi'(r)dr\right)\psi'(s)ds \\
&\leq \phi_1(\tau)\phi_2(\tau)\int_a^\tau\left(\int_r^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\Psi_\psi^{\alpha_2-1}(s,r)\psi'(s)ds\right)u(r)\psi'(r)dr \\
&\leq \phi_1(\tau)\phi_2(\tau)\int_a^\tau\left(\int_r^\tau e^{\frac{\rho-1}{\rho}(\psi(\tau)-\psi(s))}(\psi(\tau)-\psi(s))^{\alpha_1-1}\right. \\
&\quad \left.\times e^{\frac{\rho-1}{\rho}(\psi(s)-\psi(r))}(\psi(s)-\psi(r))^{\alpha_2-1}\psi'(s)ds\right)u(r)\psi'(r)dr \\
&= \phi_1(\tau)\phi_2(\tau)\int_a^\tau e^{\frac{\rho-1}{\rho}(\psi(\tau)-\psi(r))} \\
&\quad \times\left(\int_r^\tau (\psi(\tau)-\psi(s))^{\alpha_1-1}(\psi(s)-\psi(r))^{\alpha_2-1}\psi'(s)ds\right)u(r)\psi'(r)dr.
\end{aligned} \tag{2.22}$$

By modifying the variables $\psi(s) - \psi(r) = z(\psi(\tau) - \psi(r))$, inequality (2.22) reduces to

$$\begin{aligned}
(\mathcal{B}_1\mathcal{B}_2)u(\tau) &= \phi_1(\tau)\phi_2(\tau)\int_a^\tau e^{\frac{\rho-1}{\rho}(\psi(\tau)-\psi(r))}(\psi(\tau)-\psi(r))^{\alpha_1+\alpha_2-2} \\
&\quad \times\left(\int_r^\tau \left[1-\frac{\psi(s)-\psi(r)}{\psi(\tau)-\psi(r)}\right]^{\alpha_1-1}\left[\frac{\psi(s)-\psi(r)}{\psi(\tau)-\psi(r)}\right]^{\alpha_2-1}\psi'(s)ds\right)u(r)\psi'(r)dr \\
&= \phi_1(\tau)\phi_2(\tau)\int_a^\tau \Psi_\psi^{\alpha_1+\alpha_2-1}(\tau,s)\left(\int_0^1 [1-z]^{\alpha_1-1}z^{\alpha_2-1}dz\right)u(s)\psi'(s)ds \\
&= \int_a^\tau \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)}{\Gamma(\alpha_1+\alpha_2)}\Psi_\psi^{\alpha_1+\alpha_2-1}(\tau,s)u(s)\psi'(s)ds.
\end{aligned}$$

Let inequality (2.20) hold when $m = k$, for $k = 1, 2, \ldots$, that is,

$$(\mathcal{B}_1\mathcal{B}_2)^k u(\tau) \leq \int_a^\tau \frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^k}{\Gamma(k(\alpha_1+\alpha_2))}\Psi_\psi^{k(\alpha_1+\alpha_2)-1}(\tau, s)u(s)\psi'(s)ds.$$

For $m = k + 1$, by using mathematical induction, we have

$$\begin{aligned}
&(\mathcal{B}_1\mathcal{B}_2)^{k+1}u(\tau)\\
\leq\ & \mathcal{B}_1\mathcal{B}_2\left(\int_a^\tau \frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^k}{\Gamma(k(\alpha_1+\alpha_2))}\Psi_\psi^{k(\alpha_1+\alpha_2)-1}(\tau, s)u(s)\psi'(s)ds\right)\\
\leq\ & \int_a^\tau \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)}{\Gamma(\alpha_1+\alpha_2)}\Psi_\psi^{\alpha_1+\alpha_2-1}(\tau, s)\\
&\times\left(\int_a^s \frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(s)\phi_2(s)\right]^k}{\Gamma(k(\alpha_1+\alpha_2))}\Psi_\psi^{k(\alpha_1+\alpha_2)-1}(s, r)u(r)\psi'(r)dr\right)\psi'(s)ds\\
\leq\ & \frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^{k+1}}{\Gamma(\alpha_1+\alpha_2)\Gamma(k(\alpha_1+\alpha_2))}\int_a^\tau e^{\frac{\rho-1}{\rho}(\psi(\tau)-\psi(r))}\\
&\times\left(\int_r^\tau (\psi(\tau)-\psi(s))^{\alpha_1+\alpha_2-1}(\psi(s)-\psi(r))^{k(\alpha_1+\alpha_2)-1}\psi'(s)ds\right)u(r)\psi'(r)dr\\
=\ & \frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^{k+1}}{\Gamma(\alpha_1+\alpha_2)\Gamma(k(\alpha_1+\alpha_2))}\int_a^\tau e^{\frac{\rho-1}{\rho}(\psi(\tau)-\psi(r))}(\psi(\tau)-\psi(r))^{(k+1)(\alpha_1+\alpha_2)-2}\\
&\times\left(\int_r^\tau \left[1-\frac{\psi(s)-\psi(r)}{\psi(\tau)-\psi(r)}\right]^{\alpha_1+\alpha_2-1}\left[\frac{\psi(s)-\psi(r)}{\psi(\tau)-\psi(r)}\right]^{k(\alpha_1+\alpha_2)-1}\psi'(s)ds\right)u(r)\psi'(r)dr\\
=\ & \frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^{k+1}}{\Gamma(\alpha_1+\alpha_2)\Gamma(k(\alpha_1+\alpha_2))}\int_a^\tau e^{\frac{\rho-1}{\rho}(\psi(\tau)-\psi(r))}(\psi(\tau)-\psi(r))^{(k+1)(\alpha_1+\alpha_2)-1}\\
&\times\left(\int_0^1 [1-z]^{\alpha_1+\alpha_2-1}z^{k(\alpha_1+\alpha_2)-1}dz\right)u(r)\psi'(r)dr\\
=\ & \int_a^\tau \frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^{k+1}}{\Gamma((k+1)(\alpha_1+\alpha_2))}\Psi_\psi^{(k+1)(\alpha_1+\alpha_2)-1}(\tau, r)u(r)\psi'(r)dr.
\end{aligned}$$

In the same process of the proof of inequality (2.21) with the property $\phi_1(s) \leq \phi_1(\tau)$, for any $s \leq \tau$, we get

$$(\mathcal{B}_2\mathcal{B}_1)^{k+1}v(\tau) \leq \int_a^\tau \frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^{k+1}}{\Gamma((k+1)(\alpha_1+\alpha_2))}\Psi_\psi^{(k+1)(\alpha_1+\alpha_2)-1}(\tau, r)v(r)\psi'(r)dr.$$

Now we show that $(\mathcal{B}_1\mathcal{B}_2)^m u(\tau) \to 0$ and $(\mathcal{B}_2\mathcal{B}_1)^m v(\tau) \to 0$ as $m \to \infty$ for $t \in \mathcal{J}$. Since ϕ_1, $\phi_2 \in C(\mathcal{J}, \mathbb{R})$, then, by Weierstrass's theorem [47, 48], there are positive constants ϕ_1^* and ϕ_2^* such that $\phi_1(\tau) \leq \phi_1^*$ and $\phi_2(\tau) \leq \phi_2^*$ for any $t \in \mathcal{J}$, which yields that

$$(\mathcal{B}_1\mathcal{B}_2)^m u(\tau) \leq \int_a^\tau \frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1^*\phi_2^*\right]^m}{\Gamma(m(\alpha_1+\alpha_2))}\Psi_\psi^{m(\alpha_1+\alpha_2)-1}(\tau, s)u(s)\psi'(s)ds. \tag{2.23}$$

Since u is nonegative and locally integrable on $\mathcal{J}$, then u is bounded on $\mathcal{J}$. There exists a positive constant $\mathcal{M}_u$ such that $|u(\tau)| \leq \mathcal{M}_u$. Inequality (2.23) can be re-written as

$$(\mathcal{B}_1\mathcal{B}_2)^m u(\tau) \leq \frac{\mathcal{M}_u}{\Gamma(m(\alpha_1+\alpha_2)+1)}\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1^*\phi_2^*\Psi_\psi^{\alpha_1+\alpha_2}(b, a)\right]^m. \tag{2.24}$$

By applying Stirling's formula, $n! \sim \sqrt{2\pi n}(n/e)^n$, to (2.24), which implies that

$$(\mathcal{B}_1\mathcal{B}_2)^m u(\tau) \leq \frac{\mathcal{M}_u}{\sqrt{2\pi(\alpha_1+\alpha_2)}} \cdot \frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1^*\phi_2^*\Psi_\psi^{\alpha_1+\alpha_2}(b,a)\left(\frac{e}{\alpha_1+\alpha_2}\right)^{(\alpha_1+\alpha_2)}\right]^m}{m^{m(\alpha_1+\alpha_2)+\frac{1}{2}}}, \tag{2.25}$$

and this yields that $(\mathcal{B}_1\mathcal{B}_2)^m u(\tau) \to 0$ as $m \to \infty$. We obtain that

$$\begin{aligned} u(t) \leq\ & \kappa_1(\tau) + \phi_1(\tau)\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\kappa_2(s)\psi'(s)ds + \int_a^\tau \sum_{k=1}^{\infty} \frac{[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)]^k}{\Gamma(k(\alpha_1+\alpha_2))} \\ & \times\Psi_\psi^{k(\alpha_1+\alpha_2)-1}(\tau,s)\left(\kappa_1(s) + \phi_1(s)\int_a^s \Psi_\psi^{\alpha_1-1}(s,r)\kappa_2(r)\psi'(r)dr\right)\psi'(s)ds. \end{aligned}$$

Then, inequality (2.17) is obtained. Obtaining inequality (2.18) follows a similar process. □

If we take $\phi_i(\tau) \equiv b$ for $i = 1,2$ in Theorem 2.15, then we obtain the following result.

Corollary 2.16. *Assume that α_i, $i = 1,2$, ρ, and $\psi \in C^1(\mathcal{J},\mathbb{R})$ is an increasing function so that $\psi'(\tau) \neq 0$ for all $\tau \in \mathcal{J}$. Assume that the functions $u(\tau)$, $v(\tau)$, and $\kappa_i(\tau)$ are nonnegative locally integrable on $\mathcal{J}$, $\phi_i(\tau) \equiv b \geq 0$, $i = 1,2$. If*

$$u(\tau) \leq \kappa_1(\tau) + b\rho^{\alpha_1}\Gamma(\alpha_1)\,{}_\rho\mathcal{I}_{a^+}^{\alpha_1;\psi}v(\tau), \quad \tau \in \mathcal{J}, \tag{2.26}$$

$$v(\tau) \leq \kappa_2(\tau) + b\rho^{\alpha_2}\Gamma(\alpha_2)\,{}_\rho\mathcal{I}_{a^+}^{\alpha_2;\psi}u(\tau), \quad \tau \in \mathcal{J}, \tag{2.27}$$

then,

$$\begin{aligned} u(\tau) \leq\ & \kappa_1(\tau) + b\rho^{\alpha_1}\Gamma(\alpha_1)\,{}_\rho\mathcal{I}_{a^+}^{\alpha_1;\psi}\kappa_2(\tau) + \int_a^\tau \sum_{k=1}^{\infty} \frac{\left[b^2\Gamma(\alpha_1)\Gamma(\alpha_2)\right]^k}{\Gamma(k(\alpha_1+\alpha_2))}\Psi_\psi^{k(\alpha_1+\alpha_2)-1}(\tau,s) \\ & \times\left(\kappa_1(s) + b\rho^{\alpha_1}\Gamma(\alpha_1)\,{}_\rho\mathcal{I}_{a^+}^{\alpha_1;\psi}\kappa_2(s)\right)\psi'(s)ds, \quad \tau \in \mathcal{J}, \end{aligned} \tag{2.28}$$

$$\begin{aligned} v(\tau) \leq\ & \kappa_2(\tau) + b\rho^{\alpha_2}\Gamma(\alpha_2)\,{}_\rho\mathcal{I}_{a^+}^{\alpha_2;\psi}\kappa_1(\tau) + \int_a^\tau \sum_{k=1}^{\infty} \frac{\left[b^2\Gamma(\alpha_1)\Gamma(\alpha_2)\right]^k}{\Gamma(k(\alpha_1+\alpha_2))}\Psi_\psi^{k(\alpha_1+\alpha_2)-1}(\tau,s) \\ & \times\left(\kappa_2(s) + b\rho^{\alpha_2}\Gamma(\alpha_2)\,{}_\rho\mathcal{I}_{a^+}^{\alpha_2;\psi}\kappa_1(s)\right)\psi'(s)ds, \quad \tau \in \mathcal{J}. \end{aligned} \tag{2.29}$$

Corollary 2.17. *By the assumptions in Theorem 2.15, assume the functions $\kappa_i(\tau)$ are nondecreasing on $\tau \in \mathcal{J}$, $i = 1,2$. Then,*

$$u(\tau) \leq \left[\kappa_1(\tau) + \frac{\phi_1(\tau)\kappa_2(\tau)\Psi_\psi^{\alpha_1}(\tau,a)}{\alpha_1}\right]\mathbb{E}_{\alpha_1+\alpha_2}(\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}), \tag{2.30}$$

$$v(\tau) \leq \left[\kappa_2(\tau) + \frac{\phi_2(\tau)\kappa_1(\tau)\Psi_\psi^{\alpha_2}(b,a)}{\alpha_2}\right]\mathbb{E}_{\alpha_1+\alpha_2}(\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}). \tag{2.31}$$

Proof. Applying (2.17) and (2.18) provided $\phi_i(\tau)$ are nondecreasing functions for any $\tau \in \mathcal{J}$ imply that $\kappa_i(s) \leq \kappa_i(\tau)$ for $i = 1,2$,

$$
\begin{aligned}
u(\tau) &\leq \kappa_1(\tau)+\rho^{\alpha_1}\Gamma(\alpha_1)\phi_1(\tau)_\rho\mathcal{I}_{a^+}^{\alpha_1;\psi}\kappa_2(\tau)+\int_a^\tau\sum_{k=1}^{\infty}\frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^k}{\Gamma(k(\alpha_1+\alpha_2))}\Psi_\psi^{k(\alpha_1+\alpha_2)-1}(\tau,s)\\
&\quad\times\left(\kappa_1(s)+\rho^{\alpha_1}\Gamma(\alpha_1)\phi_1(\tau)_\rho\mathcal{I}_{a^+}^{\alpha_1;\psi}\kappa_2(s)\right)\psi'(s)ds\\
&\leq \left[\kappa_1(\tau)+\frac{1}{\alpha_1}\phi_1(\tau)\kappa_2(\tau)\Psi_\psi^{\alpha_1}(\tau,a)\right]\left[1+\sum_{k=1}^{\infty}\frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^k}{\Gamma(k(\alpha_1+\alpha_2)+1)}\Psi_\psi^{k(\alpha_1+\alpha_2)}(\tau,a)\right]\\
&= \left[\kappa_1(\tau)+\frac{1}{\alpha_1}\phi_1(\tau)\kappa_2(\tau)\Psi_\psi^{\alpha_1}(\tau,a)\right]\sum_{k=0}^{\infty}\frac{\left[\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)\right]^k\Psi_\psi^{k(\alpha_1+\alpha_2)}(\tau,a)}{\Gamma(k(\alpha_1+\alpha_2)+1)}.
\end{aligned}
$$

Since $0<e^{\frac{\rho-1}{\rho}(\psi(\tau)-\psi(a))}\leq 1$, for all $a\leq\tau\leq b$, we have

$$
u(\tau)\leq\left[\kappa_1(\tau)+\frac{1}{\alpha_1}\phi_1(\tau)\kappa_2(\tau)\Psi_\psi^{\alpha_1}(b,a)\right]\mathbb{E}_{\alpha_1+\alpha_2}\left(\Gamma(\alpha_1)\Gamma(\alpha_2)\phi_1(\tau)\phi_2(\tau)(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}\right).
$$

Then, inequality (2.30) is obtained. In the case of inequality (2.31), obtaining it follows a similar process. □

2.3. The linear coupled ψ-Hilfer proportional system via constant coefficient

In 2023, Sudsutad et al. [28] presented the Cauchy-type problem with constant coefficient in the context of ψ-HPFDO

$$
\begin{cases}
{}^H_\rho\mathfrak{D}_{a^+}^{\alpha,\beta;\psi}u(\tau)=\lambda u(\tau)+h(\tau), \quad \alpha\in(n-1,n),\ \beta\in[0,1],\ \rho\in(0,1],\ \tau\in(a,b],\\
{}_\rho\mathcal{I}_{a^+}^{j-\gamma;\psi}u(a)=c_j,\ c_j\in\mathbb{R},\ j=1,\ldots,n,\ \alpha\leq\gamma=\alpha+(n-\alpha)\beta,\ \lambda<0,\ j-\gamma>0,
\end{cases}
\tag{2.32}
$$

which has an explicit solution in form of the ML function

$$
\begin{aligned}
u(\tau) &= \sum_{j=1}^{n}\frac{c_j}{\rho^{\gamma-j}}\Psi_\psi^{\gamma-j}(\tau,a)\,\mathbb{E}_{\alpha,\gamma-j+1}(\lambda\rho^{-\alpha}(\psi(\tau)-\psi(a))^{\alpha})\\
&\quad+\frac{1}{\rho^\alpha}\int_a^\tau\Psi_\psi^{\alpha-1}(\tau,s)\,\mathbb{E}_{\alpha,\alpha}(\lambda\rho^{-\alpha}(\psi(\tau)-\psi(s))^{\alpha})h(s)\psi'(s)ds.
\end{aligned}
\tag{2.33}
$$

Next, we demonstrate the following auxiliary lemma to determine the solution of the proposed coupled system (1.2).

Lemma 2.18. *Suppose that $\beta\in[0,1]$, $\rho\in(0,1]$, $h_i\in C(\mathcal{J},\mathbb{R})$, for $i=1,2$, and $\Theta_1\Theta_2\neq\Delta_1\Delta_2$. Then, the unique solution of the proposed problem*

$$
\begin{cases}
{}^H_\rho\mathfrak{D}_{a^+}^{\alpha_1,\beta;\psi}u(\tau)=\lambda_1u(\tau)+h_1(\tau), \quad \alpha_1,\in(1,2), \quad \gamma_1=\alpha_1+(2-\alpha_1)\beta,\ \tau\in(a,b],\\
{}^H_\rho\mathfrak{D}_{a^+}^{\alpha_2,\beta;\psi}v(\tau)=\lambda_2v(\tau)+h_2(\tau), \quad \alpha_2\in(1,2), \quad \gamma_2=\alpha_2+(2-\alpha_2)\beta,\ \tau\in(a,b],\\
u(a)=0, \qquad u(b)=\sum_{i=1}^{m}\theta_{i\rho}\mathcal{I}_{a^+}^{\delta_i;\psi}v(\xi_i), \quad \xi_i\in(a,b], \quad i=1,2,\ldots,m,\\
v(a)=0, \qquad v(b)=\sum_{j=1}^{n}\mu_{j\rho}\mathcal{I}_{a^+}^{\varrho_j;\psi}u(\eta_j), \quad \eta_j\in(a,b], \quad j=1,2,\ldots,n,
\end{cases}
\tag{2.34}
$$

can be presented under the equivalent integral equations as follows:

$$
\begin{aligned}
u(\tau) \;=\; & \left[\frac{\Psi_{\psi}^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}(\Theta_1\Theta_2-\Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_1,\gamma_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(a))^{\alpha_1})\right] \\
& \times\Bigg[\Theta_2\Bigg(-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_{\psi}^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds \\
& +\frac{1}{\rho^{\alpha_2}}\int_a^{b}\Psi_{\psi}^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds\Bigg) \\
& -\Delta_2\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_{\psi}^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds \\
& -\frac{1}{\rho^{\alpha_1}}\int_a^{b}\Psi_{\psi}^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds\Bigg)\Bigg] \\
& +\frac{1}{\rho^{\alpha_1}}\int_a^{\tau}\Psi_{\psi}^{\alpha_1-1}(\tau,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds, \qquad (2.35)
\end{aligned}
$$

$$
\begin{aligned}
v(\tau) \;=\; & \left[\frac{\Psi_{\psi}^{\gamma_2-1}(\tau,a)}{\rho^{\gamma_2-1}(\Theta_1\Theta_2-\Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_2,\gamma_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(a))^{\alpha_2})\right] \\
& \times\Bigg[\Delta_1\Bigg(-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_{\psi}^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds \\
& +\frac{1}{\rho^{\alpha_2}}\int_a^{b}\Psi_{\psi}^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds\Bigg) \\
& -\Theta_1\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_{\psi}^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds \\
& -\frac{1}{\rho^{\alpha_1}}\int_a^{b}\Psi_{\psi}^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds\Bigg)\Bigg] \\
& +\frac{1}{\rho^{\alpha_2}}\int_a^{\tau}\Psi_{\psi}^{\alpha_2-1}(\tau,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds, \qquad (2.36)
\end{aligned}
$$

where

$$
\Delta_k = \frac{\Psi_{\psi}^{\gamma_k-1}(b,a)}{\rho^{\gamma_k-1}}\,\mathbb{E}_{\alpha_k,\gamma_k}(\lambda_k\rho^{-\alpha_k}(\psi(b)-\psi(a))^{\alpha_k}), \quad k=1,2, \qquad (2.37)
$$

$$
\Theta_1 = \sum_{j=1}^{n}\frac{\mu_j\Psi_{\psi}^{\varrho_j+\gamma_1-1}(\eta_j,a)}{\rho^{\varrho_j+\gamma_1-1}}\,\mathbb{E}_{\alpha_1,\varrho_j+\gamma_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(a))^{\alpha_1}), \qquad (2.38)
$$

$$
\Theta_2 = \sum_{i=1}^{m}\frac{\theta_i\Psi_{\psi}^{\delta_i+\gamma_2-1}(\xi_i,a)}{\rho^{\delta_i+\gamma_2-1}}\,\mathbb{E}_{\alpha_2,\delta_i+\gamma_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(a))^{\alpha_2}). \qquad (2.39)
$$

Proof. By applying (2.33), the general solution for the proposed coupled system (2.34) has the form

$$
\begin{aligned}
u(\tau) &= \frac{c_1\Psi_\psi^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}}\,\mathbb{E}_{\alpha_1,\gamma_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(a))^{\alpha_1}) \\
&\quad+\frac{c_2\Psi_\psi^{\gamma_1-2}(\tau,a)}{\rho^{\gamma_1-2}}\,\mathbb{E}_{\alpha_1,\gamma_1-1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(a))^{\alpha_1}) \\
&\quad+\frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds, \qquad (2.40)\\
v(\tau) &= \frac{d_1\Psi_\psi^{\gamma_2-1}(\tau,a)}{\rho^{\gamma_2-1}}\,\mathbb{E}_{\alpha_2,\gamma_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(a))^{\alpha_2}) \\
&\quad+\frac{d_2\Psi_\psi^{\gamma_2-2}(\tau,a)}{\rho^{\gamma_2-2}}\,\mathbb{E}_{\alpha_2,\gamma_2-1}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(a))^{\alpha_2}) \\
&\quad+\frac{1}{\rho^{\alpha_2}}\int_a^\tau \Psi_\psi^{\alpha_2-1}(\tau,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds, \qquad (2.41)
\end{aligned}
$$

where $c_k = {}_\rho\mathcal{I}_{a^+}^{k-\gamma_1;\psi}u(a)\in\mathbb{R}$ and $d_k = {}_\rho\mathcal{I}_{a^+}^{k-\gamma_2;\psi}v(a)\in\mathbb{R}$ for $k=1,2$.

Setting $\tau = a$ to (2.40) and (2.41) with $\lim_{\tau\to a}(\psi(\tau)-\psi(a))^{\gamma_i-2}=\infty$, $i=1,2$, we obtain $c_2=d_2=0$. Setting $\tau=b$ to (2.40) and (2.41), we have

$$
\begin{aligned}
u(b) &= \frac{c_1\Psi_\psi^{\gamma_1-1}(b,a)}{\rho^{\gamma_1-1}}\,\mathbb{E}_{\alpha_1,\gamma_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(a))^{\alpha_1}) \\
&\quad+\frac{c_2\Psi_\psi^{\gamma_1-2}(b,a)}{\rho^{\gamma_1-2}}\,\mathbb{E}_{\alpha_1,\gamma_1-1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(a))^{\alpha_1}) \\
&\quad+\frac{1}{\rho^{\alpha_1}}\int_a^b \Psi_\psi^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds, \qquad (2.42)\\
v(b) &= \frac{d_1\Psi_\psi^{\gamma_2-1}(b,a)}{\rho^{\gamma_2-1}}\,\mathbb{E}_{\alpha_2,\gamma_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(a))^{\alpha_2}) \\
&\quad+\frac{d_2\Psi_\psi^{\gamma_2-2}(b,a)}{\rho^{\gamma_2-2}}\,\mathbb{E}_{\alpha_2,\gamma_2-1}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(a))^{\alpha_2}) \\
&\quad+\frac{1}{\rho^{\alpha_2}}\int_a^b \Psi_\psi^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds. \qquad (2.43)
\end{aligned}
$$

By applying Lemma 2.13 and Lemma 2.12 to (2.40) and (2.41), we obtain that

$$
\begin{aligned}
&\sum_{j=1}^{n}\mu_j{}_\rho\mathcal{I}_{a^+}^{\varrho_j;\psi}u(\eta_j) \\
&= \frac{c_1\mu_j\Psi_\psi^{\gamma_1-1+\varrho_j}(\eta_j,a)}{\rho^{\gamma_1-1+\varrho_j}}\,\mathbb{E}_{\alpha_1,\gamma_1+\varrho_j}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(a))^{\alpha_1}) \\
&\quad+\frac{c_2\mu_j\Psi_\psi^{\gamma_1-2+\varrho_j}(\eta_j,a)}{\rho^{\gamma_1-2+\varrho_j}}\,\mathbb{E}_{\alpha_1,\gamma_1-1+\varrho_j}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(a))^{\alpha_1}) \\
&\quad+\frac{\mu_j}{\rho^{\alpha_1+\varrho_j}}\int_a^{\eta_j} \Psi_\psi^{\alpha_1+\varrho_j-1}(\eta_j,s)\,\mathbb{E}_{\alpha_1,\alpha_1+\varrho_j}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds, \qquad (2.44)
\end{aligned}
$$

and

$$\begin{aligned}
&\sum_{i=1}^{m}\theta_{i\rho}\mathcal{I}_{a^+}^{\delta_i;\psi}v(\xi_i)\\
=\;&\frac{d_1\theta_i\Psi_{\psi}^{\gamma_2-1+\delta_i}(\xi_i,a)}{\rho^{\gamma_2-1+\delta_i}}\,\mathbb{E}_{\alpha_2,\gamma_2+\delta_i}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(a))^{\alpha_2})\\
&+\frac{d_2\theta_i\Psi_{\psi}^{\gamma_2-2+\delta_i}(\xi_i,a)}{\rho^{\gamma_2-2+\delta_i}}\,\mathbb{E}_{\alpha_2,\gamma_2-1+\delta_i}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(a))^{\alpha_2})\\
&+\frac{\theta_i}{\rho^{\alpha_2+\delta_i}}\int_a^{\xi_i}\Psi_{\psi}^{\alpha_2+\delta_i-1}(\xi_i,s)\,\mathbb{E}_{\alpha_2,\alpha_2+\delta_i}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds. \qquad (2.45)
\end{aligned}$$

From the nonlocal integral conditions with (2.42), (2.43), (2.44), and (2.45), we obtain the system

$$\begin{aligned}
\Delta_1c_1-\Theta_2d_1 &= \sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_{\psi}^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds\\
&\quad-\frac{1}{\rho^{\alpha_1}}\int_a^{b}\Psi_{\psi}^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds, \qquad (2.46)\\
\Theta_1c_1-\Delta_2d_1 &= -\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_{\psi}^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds\\
&\quad+\frac{1}{\rho^{\alpha_2}}\int_a^{b}\Psi_{\psi}^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds. \qquad (2.47)
\end{aligned}$$

As a result of the solution of Eqs (2.46) and (2.47), it follows that

$$\begin{aligned}
c_1 &= \frac{1}{\Theta_1\Theta_2-\Delta_1\Delta_2}\Bigg[\Theta_2\Bigg(\frac{1}{\rho^{\alpha_2}}\int_a^{b}\Psi_{\psi}^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds\\
&\quad-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_{\psi}^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds\Bigg)\\
&\quad-\Delta_2\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_{\psi}^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds\\
&\quad-\frac{1}{\rho^{\alpha_1}}\int_a^{b}\Psi_{\psi}^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds\Bigg)\Bigg],\\
d_1 &= \frac{1}{\Theta_1\Theta_2-\Delta_1\Delta_2}\Bigg[\Delta_1\Bigg(\frac{1}{\rho^{\alpha_2}}\int_a^{b}\Psi_{\psi}^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds\\
&\quad-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_{\psi}^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds\Bigg)\\
&\quad-\Theta_1\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_{\psi}^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})h_2(s)\psi'(s)ds\\
&\quad-\frac{1}{\rho^{\alpha_1}}\int_a^{b}\Psi_{\psi}^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})h_1(s)\psi'(s)ds\Bigg)\Bigg].
\end{aligned}$$

Then, after inserting c_i and d_i for $i = 1, 2$, into (2.40) and (2.41), we obtain (2.35) and (2.36), respectively.

On the other hand, direct computation easily confirms that the reverse scenario is correct. The proof is finished. □

3. Main results

In view of Lemma 2.18, we provide a fixed point problem related with $(u, v) = \mathcal{Q}(u, v)$. Define the operator $\mathcal{Q} : \mathbb{U} \times \mathbb{U} \to \mathbb{U} \times \mathbb{U}$, that is $\mathcal{Q}(u, v)(\tau) = (\mathcal{Q}_1(u, v)(\tau), \mathcal{Q}_2(u, v)(\tau))^{\mathcal{T}}$, where

$$
\begin{aligned}
&\mathcal{Q}_1(u,v)(\tau)\\
=\ &\left[\frac{\Psi_\psi^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}(\Theta_1\Theta_2-\Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_1,\gamma_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(a))^{\alpha_1})\right]\\
&\times\Bigg[\Theta_2\Bigg(-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})\mathbb{F}_{u,v}(s)\psi'(s)ds\\
&+\frac{1}{\rho^{\alpha_2}}\int_a^{b}\Psi_\psi^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})\mathbb{G}_{u,v}(s)\psi'(s)ds\Bigg)\\
&-\Delta_2\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})\mathbb{G}_{u,v}(s)\psi'(s)ds\\
&-\frac{1}{\rho^{\alpha_1}}\int_a^{b}\Psi_\psi^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})\mathbb{F}_{u,v}(s)\psi'(s)ds\Bigg)\Bigg]\\
&+\frac{1}{\rho^{\alpha_1}}\int_a^{\tau}\Psi_\psi^{\alpha_1-1}(\tau,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u,v}(s)\psi'(s)ds, \qquad (3.1)\\
&\mathcal{Q}_2(u,v)(\tau)\\
=\ &\left[\frac{\Psi_\psi^{\gamma_2-1}(\tau,a)}{\rho^{\gamma_2-1}(\Theta_1\Theta_2-\Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_2,\gamma_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(a))^{\alpha_2})\right]\\
&\times\Bigg[\Delta_1\Bigg(-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})\mathbb{F}_{u,v}(s)\psi'(s)ds\\
&+\frac{1}{\rho^{\alpha_2}}\int_a^{b}\Psi_\psi^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})\mathbb{G}_{u,v}(s)\psi'(s)ds\Bigg)\\
&-\Theta_1\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})\mathbb{G}_{u,v}(s)\psi'(s)ds\\
&-\frac{1}{\rho^{\alpha_1}}\int_a^{b}\Psi_\psi^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})\mathbb{F}_{u,v}(s)\psi'(s)ds\Bigg)\Bigg]\\
&+\frac{1}{\rho^{\alpha_2}}\int_a^{\tau}\Psi_\psi^{\alpha_2-1}(\tau,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})\mathbb{G}_{u,v}(s)\psi'(s)ds, \qquad (3.2)
\end{aligned}
$$

with $\mathbb{F}_{u,v}(\tau) = f(\tau, u(\tau), v(\tau))$ and $\mathbb{G}_{u,v}(\tau) = g(\tau, u(\tau), v(\tau))$. Accordingly, the product space $(\mathbb{U} \times \mathbb{U}, \|(u, v)\|)$ is a Banach space supplemented with the norm $\|(u, v)\| = \|u\| + \|v\|$ and the existence of fixed points of $\mathcal{Q}$ corresponding to the existence of the solutions for the proposed coupled system (1.2). For ease of computation, we give some symbols:

$$\omega_{1,k} = \frac{\Psi_{\psi}^{\alpha_k}(b,a)}{\rho^{\alpha_k}\Gamma(\alpha_k+1)}, \quad k=1,2, \tag{3.3}$$

$$\omega_{2,k} = \frac{\Psi_{\psi}^{\gamma_k-1}(b,a)}{\rho^{\gamma_k-1}\Gamma(\gamma_k)|\Theta_1\Theta_2-\Delta_1\Delta_2|}, \quad k=1,2, \tag{3.4}$$

$$\omega_3(\Xi,\Upsilon) = \frac{|\Xi|\Psi_{\psi}^{\alpha_1}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)} + \sum_{j=1}^{n}\frac{|\Upsilon||\mu_j|\Psi_{\psi}^{\varrho_j+\alpha_1}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)}, \quad \Xi,\Upsilon\in\{\Delta_k,\Theta_k\}, k=1,2, \tag{3.5}$$

$$\omega_4(\Xi,\Upsilon) = \frac{|\Xi|\Psi_{\psi}^{\alpha_2}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} + \sum_{i=1}^{m}\frac{|\Upsilon||\theta_i|\Psi_{\psi}^{\delta_i+\alpha_2}(\xi_i,a)}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)}, \quad \Xi,\Upsilon\in\{\Delta_k,\Theta_k\}, k=1,2, \tag{3.6}$$

$$\Omega_1 := \omega_{1,1}+\omega_{2,1}\omega_3(\Delta_2,\Theta_2)+\omega_{2,2}\omega_3(\Theta_1,\Delta_1), \tag{3.7}$$

$$\Omega_2 := \omega_{1,2}+\omega_{2,1}\omega_4(\Theta_2,\Delta_2)+\omega_{2,2}\omega_4(\Delta_1,\Theta_1). \tag{3.8}$$

3.1. Uniqueness result by Banach's fixed point theorem

By using Banach's fixed point theorem, the uniqueness result for the proposed coupled system (1.2) is proved in this subsection.

Lemma 3.1. [49] *Assume $\mathcal{B}$ is a non-empty closed subset of a Banach space $\mathbb{U}$. Then, any contraction mapping $\mathcal{Q}$ from $\mathcal{B}$ into itself has a unique fixed-point.*

Theorem 3.2. *Suppose $\psi\in C^2(\mathcal{J},\mathbb{R})$ where $\psi'(\tau)>0$, $\tau\in\mathcal{J}$, and f, $g\in C(\mathcal{J}\times\mathbb{R}^2,\mathbb{R})$, satisfying the following assumption:*

($\mathcal{A}_1$) *There are constants $\mathcal{L}>0$ and $\mathcal{K}>0$ so that*

$$|f(\tau,u_1,v_1)-f(\tau,u_2,v_2)|\le\mathcal{L}\left(|u_1-u_2|+|v_1-v_2|\right), \ \tau\in\mathcal{J}, \ u_i,v_i\in\mathbb{R}, \ i=1,2. \tag{3.9}$$

$$|g(\tau,u_1,v_1)-g(\tau,u_2,v_2)|\le\mathcal{K}\left(|u_1-u_2|+|v_1-v_2|\right), \ \tau\in\mathcal{J}, \ u_i,v_i\in\mathbb{R}, \ i=1,2. \tag{3.10}$$

Then, the proposed coupled system (1.2) *has a unique solution provided that*

$$\Omega_1\mathcal{L}+\Omega_2\mathcal{K}<1. \tag{3.11}$$

Proof. Define $\sup_{\tau\in\mathcal{J}}|\mathbb{F}_{0,0}(\tau)| := \mathcal{F}<\infty$, $\sup_{\tau\in\mathcal{J}}|\mathbb{G}_{0,0}(\tau)| := \mathcal{G}<\infty$, and a bounded, closed, and convex set $\mathbb{B}_{\mathcal{R}}\subset\mathbb{U}\times\mathbb{U}$ where $\mathbb{B}_{\mathcal{R}} := \{(u,v)\in\mathbb{U}\times\mathbb{U} : \|(u,v)\|\le\mathcal{R}\}$ such that the radius

$$\mathcal{R}\ge\frac{\Omega_1\mathcal{F}+\Omega_2\mathcal{G}}{1-[\Omega_1\mathcal{L}+\Omega_2\mathcal{K}]}>0.$$

Now, the process of the procedure is divided into two processes:

Step 1. We show that $\mathcal{Q}\mathbb{B}_{\mathcal{R}}\subseteq\mathbb{B}_{\mathcal{R}}$.

For any $(u,v)\in\mathbb{B}_{\mathcal{R}}$, $\tau\in\mathcal{J}$, we have

$$|\mathbb{F}_{u,v}(\tau)| \le |\mathbb{F}_{u,v}(\tau)-\mathbb{F}_{0,0}(\tau)|+|\mathbb{F}_{0,0}(\tau)|\le\mathcal{L}\left(\|u\|+\|v\|\right)+\mathcal{F}, \tag{3.12}$$

$$|\mathbb{G}_{u,v}(\tau)| \le |\mathbb{G}_{u,v}(\tau)-\mathbb{G}_{0,0}(\tau)|+|\mathbb{G}_{0,0}(\tau)|\le\mathcal{K}\left(\|u\|+\|v\|\right)+\mathcal{G}. \tag{3.13}$$

By applying properties in Lemma 2.9 and Lemma 2.11 with (3.12) and (3.13), we obtain

$$
\begin{aligned}
& |\mathcal{Q}_1(u,v)(\tau)| \\
\leq\ & \left[\frac{\Psi_\psi^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}|\Theta_1\Theta_2-\Delta_1\Delta_2|}\left|\mathbb{E}_{\alpha_1,\gamma_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(a))^{\alpha_1})\right|\right] \\
& \times\Bigg[|\Theta_2|\Bigg(\sum_{j=1}^{n}\frac{|\mu_j|}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s)\left|\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})\right||\mathbb{F}_{u,v}(s)|\psi'(s)ds \\
& +\frac{1}{\rho^{\alpha_2}}\int_a^{b}\Psi_\psi^{\alpha_2-1}(b,s)\left|\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})\right||\mathbb{G}_{u,v}(s)|\psi'(s)ds\Bigg) \\
& +|\Delta_2|\Bigg(\sum_{i=1}^{m}\frac{|\theta_i|}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s)\left|\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})\right||\mathbb{G}_{u,v}(s)|\psi'(s)ds \\
& +\frac{1}{\rho^{\alpha_1}}\int_a^{b}\Psi_\psi^{\alpha_1-1}(b,s)\left|\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})\right||\mathbb{F}_{u,v}(s)|\psi'(s)ds\Bigg)\Bigg] \\
& +\frac{1}{\rho^{\alpha_1}}\int_a^{\tau}\Psi_\psi^{\alpha_1-1}(\tau,s)\left|\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\right||\mathbb{F}_{u,v}(s)|\psi'(s)ds \\
\leq\ & \left[\frac{\Psi_\psi^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\right]\Bigg[|\Theta_2|\Bigg([\mathcal{L}(\|u\|+\|v\|)+\mathcal{F}]\sum_{j=1}^{n}\frac{|\mu_j|\Psi_\psi^{\varrho_j+\alpha_1}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)} \\
& +[\mathcal{K}(\|u\|+\|v\|)+\mathcal{G}]\frac{\Psi_\psi^{\alpha_2}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\Bigg)+|\Delta_2|\Bigg([\mathcal{K}(\|u\|+\|v\|)+\mathcal{G}]\sum_{i=1}^{m}\frac{|\theta_i|\Psi_\psi^{\delta_i+\alpha_2}(\xi_i,a)}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)} \\
& +[\mathcal{L}(\|u\|+\|v\|)+\mathcal{F}]\frac{\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\Bigg)\Bigg]+[\mathcal{L}(\|u\|+\|v\|)+\mathcal{F}]\frac{\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)} \\
\leq\ & \left[\frac{\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+\frac{\Psi_\psi^{\gamma_1-1}(b,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\left(\frac{|\Delta_2|\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+\sum_{j=1}^{n}\frac{|\Theta_2||\mu_j|\Psi_\psi^{\varrho_j+\alpha_1}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)}\right)\right] \\
& \times[\mathcal{L}(\|u\|+\|v\|)+\mathcal{F}]+\frac{\Psi_\psi^{\gamma_1-1}(b,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\Bigg(\frac{|\Theta_2|\Psi_\psi^{\alpha_2}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)} \\
& +\sum_{i=1}^{m}\frac{|\Delta_2||\theta_i|\Psi_\psi^{\delta_i+\alpha_2}(\xi_i,a)}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)}\Bigg)[\mathcal{K}(\|u\|+\|v\|)+\mathcal{G}] \\
=\ & [\omega_{1,1}+\omega_{2,1}\omega_3(\Delta_2,\Theta_2)][\mathcal{L}(\|u\|+\|v\|)+\mathcal{F}]+\omega_{2,1}\omega_4(\Theta_2,\Delta_2)[\mathcal{K}(\|u\|+\|v\|)+\mathcal{G}] \\
\leq\ & [\mathcal{L}(\omega_{1,1}+\omega_{2,1}\omega_3(\Delta_2,\Theta_2))+\mathcal{K}\omega_{2,1}\omega_4(\Theta_2,\Delta_2)]\mathcal{R} \\
& +(\omega_{1,1}+\omega_{2,1}\omega_3(\Delta_2,\Theta_2))\mathcal{F}+\omega_{2,1}\omega_4(\Theta_2,\Delta_2)\mathcal{G}.
\end{aligned}
$$

Similarly, we obtain the following result:

$$
\begin{aligned}
& |\mathcal{Q}_2(u,v)(\tau)| \\
\leq\ & \frac{\Psi_\psi^{\gamma_2-1}(b,a)}{\rho^{\gamma_2-1}\Gamma(\gamma_2)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\left(\frac{|\Theta_1|\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+\sum_{j=1}^{n}\frac{|\Delta_1||\mu_j|\Psi_\psi^{\varrho_j+\alpha_1}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)}\right)[\mathcal{L}(\|u\|+\|v\|)+\mathcal{F}]
\end{aligned}
$$

$$
\begin{aligned}
& +\left[\frac{\Psi_\psi^{\alpha_2}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}+\frac{\Psi_\psi^{\gamma_2-1}(b,a)}{\rho^{\gamma_2-1}\Gamma(\gamma_2)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\left(\frac{|\Delta_1|\Psi_\psi^{\alpha_2}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}+\sum_{i=1}^{m}\frac{|\Theta_1||\theta_i|\Psi_\psi^{\delta_i+\alpha_2}(\xi_i,a)}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)}\right)\right] \\
& \times[\mathcal{K}(\|u\|+\|v\|)+\mathcal{G}] \\
= \ & \omega_{2,2}\omega_3(\Theta_1,\Delta_1)[\mathcal{L}(\|u\|+\|v\|)+\mathcal{F}]+[\omega_{1,2}+\omega_{2,2}\omega_4(\Delta_1,\Theta_1)][\mathcal{K}(\|u\|+\|v\|)+\mathcal{G}] \\
\leq \ & [\mathcal{L}\omega_{2,2}\omega_3(\Theta_1,\Delta_1)+\mathcal{K}(\omega_{1,2}+\omega_{2,2}\omega_4(\Delta_1,\Theta_1))]\mathcal{R} \\
& +\omega_{2,2}\omega_3(\Theta_1,\Delta_1)\mathcal{F}+(\omega_{1,2}+\omega_{2,2}\omega_4(\Delta_1,\Theta_1))\mathcal{G}.
\end{aligned}
$$

Consequently, $\|Q(u,v)\| \leq [\Omega_1\mathcal{L}+\Omega_2\mathcal{K}]\mathcal{R}+\Omega_1\mathcal{F}+\Omega_2\mathcal{G} \leq \mathcal{R}$, which implies that $Q\mathbb{B}_{\mathcal{R}} \subseteq \mathbb{B}_{\mathcal{R}}$.

Step 2. We will show that Q is a contraction.

Now, let $(u_1,v_1),(u_2,v_2) \in \mathbb{U}\times\mathbb{U}$ and for all $\tau \in \mathcal{J}$, we have

$$
\begin{aligned}
|\mathbb{F}_{u_1,v_1}(s)-\mathbb{F}_{u_2,v_2}(s)| &\leq \mathcal{L}(\|u_1-u_2\|+\|v_1-v_2\|), \quad (3.14)\\
|\mathbb{G}_{u_1,v_1}(s)-\mathbb{G}_{u_2,v_2}(s)| &\leq \mathcal{K}(\|u_1-u_2\|+\|v_1-v_2\|). \quad (3.15)
\end{aligned}
$$

By applying Lemma 2.9 and Lemma 2.11 with (3.14) and (3.15) again, one has

$$
\begin{aligned}
& |Q_1(u_1,v_1)(\tau)-Q_1(u_2,v_2)(\tau)| \\
\leq \ & \left[\frac{\Psi_\psi^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\right] \\
& \times\Bigg[|\Theta_2|\Bigg(\sum_{j=1}^{n}\frac{|\mu_j|}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)}\int_a^{\eta_j}\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s)|\mathbb{F}_{u_1,v_1}(s)-\mathbb{F}_{u_2,v_2}(s)|\psi'(s)ds \\
& +\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{b}\Psi_\psi^{\alpha_2-1}(b,s)|\mathbb{G}_{u_1,v_1}(s)-\mathbb{G}_{u_2,v_2}(s)|\psi'(s)ds\Bigg) \\
& +|\Delta_2|\Bigg(\sum_{i=1}^{m}\frac{|\theta_i|}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)}\int_a^{\xi_i}\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s)|\mathbb{G}_{u_1,v_1}(s)-\mathbb{G}_{u_2,v_2}(s)|\psi'(s)ds \\
& +\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\int_a^{b}\Psi_\psi^{\alpha_1-1}(b,s)|\mathbb{F}_{u_1,v_1}(s)-\mathbb{F}_{u_2,v_2}(s)|\psi'(s)ds\Bigg)\Bigg] \\
& +\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\int_a^{\tau}\Psi_\psi^{\alpha_1-1}(\tau,s)|\mathbb{F}_{u_1,v_1}(s)-\mathbb{F}_{u_2,v_2}(s)|\psi'(s)ds \\
\leq \ & \left[\frac{\Psi_\psi^{\gamma_1-1}(b,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\right]\Bigg[|\Theta_2|\Bigg(\sum_{j=1}^{n}\frac{|\mu_j|\Psi_\psi^{\varrho_j+\alpha_1}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)}\mathcal{L}\Big[\|u_1-u_2\|+\|v_1-v_2\|\Big] \\
& +\frac{\Psi_\psi^{\alpha_2}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\mathcal{K}\Big[\|u_1-u_2\|+\|v_1-v_2\|\Big]\Bigg) \\
& +|\Delta_2|\Bigg(\sum_{i=1}^{m}\frac{|\theta_i|\Psi_\psi^{\delta_i+\alpha_2}(\xi_i,a)}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)}\mathcal{K}\Big[\|u_1-u_2\|+\|v_1-v_2\|\Big] \\
& +\frac{\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\mathcal{L}\Big[\|u_1-u_2\|+\|v_1-v_2\|\Big]\Bigg)\Bigg]+\frac{\Psi_\psi^{\alpha_1}(b,s)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\mathcal{L}\Big[\|u_1-u_2\|+\|v_1-v_2\|\Big] \\
= \ & \left[\frac{\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+\frac{\Psi_\psi^{\gamma_1-1}(b,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\left(\frac{|\Delta_2|\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+\sum_{j=1}^{n}\frac{|\Theta_2||\mu_j|\Psi_\psi^{\varrho_j+\alpha_1}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times\mathcal{L}\Big[\|u_1-u_2\|+\|v_1-v_2\|\Big]+\frac{\Psi_\psi^{\gamma_1-1}(b,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\Bigg(\frac{|\Theta_2|\Psi_\psi^{\alpha_2}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\\
&+\sum_{i=1}^{m}\frac{|\Delta_2||\theta_i|\Psi_\psi^{\delta_i+\alpha_2}(\xi_i,a)}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)}\Bigg)\mathcal{K}\Big[\|u_1-u_2\|+\|v_1-v_2\|\Big]\\
=\;&[\omega_{1,1}+\omega_{2,1}\omega_3(\Delta_2,\Theta_2)]\mathcal{L}[\|u_1-u_2\|+\|v_1-v_2\|]\\
&+\omega_{2,1}\omega_4(\Theta_2,\Delta_2)\mathcal{K}[\|u_1-u_2\|+\|v_1-v_2\|]\\
\le\;&[(\omega_{1,1}+\omega_{2,1}\omega_3(\Delta_2,\Theta_2))\mathcal{L}+\omega_{2,1}\omega_4(\Theta_2,\Delta_2)\mathcal{K}]\mathcal{R}. \qquad (3.16)
\end{aligned}
$$

On the other hand, we get

$$
\begin{aligned}
&|\mathcal{Q}_2(u_1,v_1)(\tau)-\mathcal{Q}_2(u_2,v_2)(\tau)|\\
=\;&\omega_{2,2}\omega_3(\Theta_1,\Delta_1)\mathcal{L}[\|u_1-u_2\|+\|v_1-v_2\|]\\
&+[\omega_{1,2}+\omega_{2,2}\omega_4(\Delta_1,\Theta_1)]\mathcal{K}[\|u_1-u_2\|+\|v_1-v_2\|]\\
\le\;&[\omega_{2,2}\omega_3(\Theta_1,\Delta_1)\mathcal{L}+(\omega_{1,2}+\omega_{2,2}\omega_4(\Delta_1,\Theta_1))\mathcal{K}]\mathcal{R}. \qquad (3.17)
\end{aligned}
$$

From (3.16) and (3.17), one has $\|\mathcal{Q}(u_1,v_1)-\mathcal{Q}(u_2,v_2)\|\le[\Omega_1\mathcal{L}+\Omega_2\mathcal{K}]\mathcal{R}$. Since $\Omega_1\mathcal{L}+\Omega_2\mathcal{K}<1$, then, $\mathcal{Q}$ is a contraction mapping. Then, by Lemma 3.1, $\mathcal{Q}$ has unique fixed point, which is the unique solution of the proposed coupled system (1.2). □

3.2. The Mittag-Leffler-Ulam stability and its generalization

This section proves the ML-U stability for the proposed coupled system (1.2) by means of the integral representation of its solution provided by $u(\tau)=\mathcal{Q}_1(u,v)(\tau)$ and $v(\tau)=\mathcal{Q}_2(u,v)(\tau)$, where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined by (3.1) and (3.2), respectively. Next, we give some definitions of ML-UH stability for the proposed coupled system (1.2). Let $\lambda<0$ and $f,g\in C(\mathcal{J}\times\mathbb{R}^2,\mathbb{R})$.

Definition 3.3. *The proposed coupled system* (1.2) *is called ML-UH stable, if there exist* $\mathfrak{E}_f>0$, $\mathfrak{E}_g>0$ *so that, for every* $\epsilon>0$, *and for any solution* $(u^*,v^*)\in\mathbb{U}\times\mathbb{U}$ *of*

$$
\begin{aligned}
\Big\|{}^H_\rho\mathfrak{D}_{a^+}^{\alpha_1,\beta;\psi}u^*(\tau)-\lambda_1u^*(\tau)-\mathbb{F}_{u^*,v^*}(\tau)\Big\| &\le \epsilon,\quad \tau\in\mathcal{J}, \qquad (3.18)\\
\Big\|{}^H_\rho\mathfrak{D}_{a^+}^{\alpha_2,\beta;\psi}v^*(\tau)-\lambda_2v^*(\tau)-\mathbb{G}_{u^*,v^*}(\tau)\Big\| &\le \epsilon,\quad \tau\in\mathcal{J}, \qquad (3.19)
\end{aligned}
$$

there is a solution $(u,v)\in\mathbb{U}\times\mathbb{V}$ *of the proposed coupled system* (1.2) *satisfying*

$$
\|(u,v)-(u^*,v^*)\|\le(\mathfrak{E}_f+\mathfrak{E}_g)\epsilon\mathbb{E}_{\alpha_1+\alpha_2}(\kappa_{f,g}(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}),\quad \kappa_{f,g}\ge0,\quad \tau\in\mathcal{J}. \qquad (3.20)
$$

Definition 3.4. *The proposed coupled system* (1.2) *is called generalized ML-UH stable if there are two functions* χ_f, $\chi_g\in C(\mathbb{R}^+,\mathbb{R}^+)$ *with* $\chi_f(0)=0=\chi_g(0)$, *so that for every* $\epsilon>0$, *and for any* $(u^*,v^*)\in\mathbb{U}\times\mathbb{U}$ *of*

$$
\begin{aligned}
\Big\|{}^H_\rho\mathfrak{D}_{a^+}^{\alpha_1,\beta;\psi}u^*(\tau)-\lambda_1u^*(\tau)-\mathbb{F}_{u^*,v^*}(\tau)\Big\| &\le \chi_f(\tau),\quad \tau\in\mathcal{J}, \qquad (3.21)\\
\Big\|{}^H_\rho\mathfrak{D}_{a^+}^{\alpha_2,\beta;\psi}v^*(\tau)-\lambda_2v^*(\tau)-\mathbb{G}_{u^*,v^*}(\tau)\Big\| &\le \chi_g(\tau),\quad \tau\in\mathcal{J}, \qquad (3.22)
\end{aligned}
$$

there is a solution $(u,v)\in\mathbb{U}\times\mathbb{U}$ *of the proposed coupled system* (1.2) *satisfying*

$$
\|(u,v)-(u^*,v^*)\|\le(\chi_f(\epsilon)+\chi_g(\epsilon))\mathbb{E}_{\alpha_1+\alpha_2}(\kappa_{f,g}(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}),\quad \kappa_{f,g}\ge0,\quad \tau\in\mathcal{J}. \qquad (3.23)
$$

Definition 3.5. *The proposed coupled system* (1.2) *is called ML-UHR stable with respect to another function* $\Phi(\tau)$ *if there exist* $\mathfrak{E}_{f_\Phi} > 0$, $\mathfrak{E}_{g_\Phi} > 0$ *so that for every* $\epsilon > 0$, *and for any solution* $(u^*, v^*) \in \mathbb{U}\times\mathbb{U}$ *of*

$$\begin{aligned}\left\|{}^{H}_{\rho}\mathfrak{D}^{\alpha_1,\beta;\psi}_{a^+}u^*(\tau) - \lambda_1 u^*(\tau) - \mathbb{F}_{u^*,v^*}(\tau)\right\| &\leq \epsilon\Phi(\tau), \quad \tau \in \mathcal{J}, &(3.24)\\ \left\|{}^{H}_{\rho}\mathfrak{D}^{\alpha_2,\beta;\psi}_{a^+}v^*(\tau) - \lambda_2 v^*(\tau) - \mathbb{G}_{u^*,v^*}(\tau)\right\| &\leq \epsilon\Phi(\tau), \quad \tau \in \mathcal{J}, &(3.25)\end{aligned}$$

there is a solution $(u, v) \in \mathbb{U}\times\mathbb{V}$ *of the proposed coupled system* (1.2) *satisfying*

$$\|(u, v) - (u^*, v^*)\| \leq (\mathfrak{E}_{f_\Phi} + \mathfrak{E}_{g_\Phi})\epsilon\Phi(\tau)\mathbb{E}_{\alpha_1+\alpha_2}(\kappa_{f_\Phi,g_\Phi}(\psi(\tau) - \psi(a))^{\alpha_1+\alpha_2}), \quad \tau \in \mathcal{J}, \tag{3.26}$$

where $\kappa_{f_\Phi,g_\Phi} \geq 0$.

Definition 3.6. *The proposed coupled system* (1.2) *is called generalized ML-UHR stable with respect to another function* $\Phi(\tau)$ *so that for every* $\epsilon_i > 0$, $i = 1, 2$, *and for any solution* $(u^*, v^*) \in \mathbb{U}\times\mathbb{U}$ *of*

$$\begin{aligned}\left\|{}^{H}_{\rho}\mathfrak{D}^{\alpha_1,\beta;\psi}_{a^+}u^*(\tau) - \lambda_1 u^*(\tau) - \mathbb{F}_{u^*,v^*}(\tau)\right\| &\leq \Phi(\tau), \quad \tau \in \mathcal{J}, &(3.27)\\ \left\|{}^{H}_{\rho}\mathfrak{D}^{\alpha_2,\beta;\psi}_{a^+}v^*(\tau) - \lambda_2 v^*(\tau) - \mathbb{G}_{u^*,v^*}(\tau)\right\| &\leq \Phi(\tau), \quad \tau \in \mathcal{J}, &(3.28)\end{aligned}$$

there is a solution $(u, v) \in \mathbb{U}\times\mathbb{V}$ *of the proposed coupled system* (1.2) *satisfying*

$$\|(u, v) - (u^*, v^*)\| \leq (\mathfrak{E}_{f_\Phi} + \mathfrak{E}_{g_\Phi})\Phi(\tau)\mathbb{E}_{\alpha_1+\alpha_2}(\kappa_{f_\Phi,g_\Phi}(\psi(\tau) - \psi(a))^{\alpha_1+\alpha_2}), \tag{3.29}$$

for any $\tau \in \mathcal{J}$, $\mathfrak{E}_{f_\Phi}$, $\mathfrak{E}_{g_\Phi} > 0$, *and* $\kappa_{f_\Phi,g_\Phi} \geq 0$.

Remark 3.7. *Let* $(u^*, v^*) \in \mathbb{U}\times\mathbb{U}$ *be the solution of* (3.18) *and* (3.19) *if and only if there exist* $w_{u^*,v^*} \in \mathbb{U}\times\mathbb{U}$ *(depends on* (u^*, v^*)*) so that*

(i) $|w_{u^*,v^*}(\tau)| \leq \epsilon\mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(\tau) - \psi(a))^{\alpha_1+\alpha_2})$, $\epsilon = \max\{\epsilon_1, \epsilon_2\}$, $\bar{\lambda}^* = \max\{|\lambda_1|, |\lambda_2|\}$, $\tau \in \mathcal{J}$.
(ii) ${}^{H}_{\rho}\mathfrak{D}^{\alpha_1,\beta;\psi}_{a^+}u^*(\tau) = \lambda_1 u^*(\tau) + f(\tau, u^*(\tau), v^*(\tau)) + w_{u^*,v^*}(\tau)$ *and* ${}^{H}_{\rho}\mathfrak{D}^{\alpha_2,\beta;\psi}_{a^+}v^*(\tau) = \lambda_2 v^*(\tau) + g(\tau, u^*(\tau), v^*(\tau)) + w_{u^*,v^*}(\tau)$, $\tau \in \mathcal{J}$.

Remark 3.8. *Let* $(u^*, v^*) \in \mathbb{U}\times\mathbb{U}$ *be the solution of* (3.24) *and* (3.25) *if and only if there exist* $z_{u^*,v^*} \in \mathbb{U}\times\mathbb{U}$ *(depends on* (u^*, v^*)*) so that*

(i) $|z_{u^*,v^*}(\tau)| \leq \epsilon\Phi(\tau)\mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(\tau) - \psi(a))^{\alpha_1+\alpha_2})$, $\epsilon = \max\{\epsilon_1, \epsilon_2\}$, $\bar{\lambda}^* = \max\{|\lambda_1|, |\lambda_2|\}$, $\tau \in \mathcal{J}$.
(ii) ${}^{H}_{\rho}\mathfrak{D}^{\alpha_1,\beta;\psi}_{a^+}u^*(\tau) = \lambda_1 u^*(\tau) + f(\tau, u^*(\tau), v^*(\tau)) + w_{u^*,v^*}(\tau)$ *and* ${}^{H}_{\rho}\mathfrak{D}^{\alpha_2,\beta;\psi}_{a^+}v^*(\tau) = \lambda_2 v^*(\tau) + g(\tau, u^*(\tau), v^*(\tau)) + z_{u^*,v^*}(\tau)$, $\tau \in \mathcal{J}$.

Lemma 3.9. *Let* $\alpha_i \in (1, 2)$, $i = 1, 2$, $\beta \in [0, 1]$, *and* $\rho \in (0, 1]$. *If* $(u^*, v^*) \in \mathbb{U}\times\mathbb{U}$ *satisfies inequalities* (3.18) *and* (3.19), *then* (u^*, v^*) *satisfies the inequalities*

$$\begin{aligned}\left|u^*(\tau) - \mathcal{M}_{u^*,v^*}(\tau) - \frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi^{\alpha_1-1}_\psi(\tau, s)\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau) - \psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds\right| &\leq \Lambda_1\epsilon,\\ \left|v^*(\tau) - \mathcal{N}_{u^*,v^*}(\tau) - \frac{1}{\rho^{\alpha_2}}\int_a^\tau \Psi^{\alpha_2-1}_\psi(\tau, s)\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau) - \psi(s))^{\alpha_2})\mathbb{G}_{u^*,v^*}(s)\psi'(s)ds\right| &\leq \Lambda_2\epsilon,\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_{u^*,v^*}(\tau) &= \left[\frac{\Psi_{\psi}^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}(\Theta_1\Theta_2-\Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_1,\gamma_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(a))^{\alpha_1})\right]\\
&\times\Bigg[\Theta_2\Bigg(\frac{1}{\rho^{\alpha_2}}\int_a^b \Psi_{\psi}^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})\mathbb{G}_{u^*,v^*}(s)\psi'(s)ds\\
&-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_{\psi}^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds\Bigg)\\
&-\Delta_2\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_{\psi}^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})\mathbb{G}_{u^*,v^*}(s)\psi'(s)ds\\
&-\frac{1}{\rho^{\alpha_1}}\int_a^b\Psi_{\psi}^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds\Bigg)\Bigg], \qquad (3.30)\\
\mathcal{N}_{u^*,v^*}(\tau) &= \left[\frac{\Psi_{\psi}^{\gamma_2-1}(\tau,a)}{\rho^{\gamma_2-1}(\Theta_1\Theta_2-\Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_2,\gamma_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(a))^{\alpha_2})\right]\\
&\times\Bigg[\Delta_1\Bigg(\frac{1}{\rho^{\alpha_2}}\int_a^b \Psi_{\psi}^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})\mathbb{G}_{u^*,v^*}(s)\psi'(s)ds\\
&-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_{\psi}^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds\Bigg)\\
&-\Theta_1\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_{\psi}^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})\mathbb{G}_{u^*,v^*}(s)\psi'(s)ds\\
&-\frac{1}{\rho^{\alpha_1}}\int_a^b\Psi_{\psi}^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds\Bigg)\Bigg], \qquad (3.31)
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_k(\Xi,\Upsilon) &= \omega_{2,k}\Bigg[|\Xi|\Bigg(\sum_{j=1}^{n}\frac{|\mu_j|\Psi_{\psi}^{\varrho_j+\alpha_1-1}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1-2}\Gamma(\varrho_j+\alpha_1)}\mathbb{E}_{\alpha_1+\alpha_2,2}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(\eta_j)-\psi(a))^{\alpha_1+\alpha_2})\\
&+\frac{\Psi_{\psi}^{\alpha_2}(b,a)}{\rho^{\alpha_2}}\mathbb{E}_{\alpha_1+\alpha_2,\alpha_2+1}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(b)-\psi(a))^{\alpha_1+\alpha_2})\Bigg)\\
&+|\Upsilon|\Bigg(\sum_{i=1}^{m}\frac{|\theta_i|\Psi_{\psi}^{\delta_i+\alpha_2}(\xi_i,a)}{\rho^{\delta_i+\alpha_2-2}\Gamma(\delta_i+\alpha_2)}\mathbb{E}_{\alpha_1+\alpha_2,2}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(\xi_i)-\psi(a))^{\alpha_1+\alpha_2})\\
&+\frac{\Psi_{\psi}^{\alpha_1}(b,a)}{\rho^{\alpha_1}}\mathbb{E}_{\alpha_1+\alpha_2,\alpha_1+1}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(b)-\psi(a))^{\alpha_1+\alpha_2})\Bigg)\Bigg]\\
&+\frac{\Psi_{\psi}^{\alpha_k}(b,a)}{\rho^{\alpha_k}}\mathbb{E}_{\alpha_1+\alpha_2,\alpha_k+1}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}), \quad k=1,2. \qquad (3.32)
\end{aligned}$$

Proof. Assume (u^*, v^*) is the solution of (3.18) and (3.19). By using property (*ii*) of Remark 3.7, we get

$$\begin{cases} {}^{H}_{\rho}\mathfrak{D}^{\alpha_1,\beta;\psi}_{a^+}u^*(\tau) = \lambda_1 u^*(\tau) + \mathbb{F}_{u^*,v^*}(\tau) + w_{u^*,v^*}(\tau), \quad \tau \in (a,b], \\ {}^{H}_{\rho}\mathfrak{D}^{\alpha_2,\beta;\psi}_{a^+}v^*(\tau) = \lambda_2 v^*(\tau) + \mathbb{G}_{u^*,v^*}(\tau) + w_{u^*,v^*}(\tau), \quad \tau \in (a,b], \\ u^*(a) = 0, \quad u^*(b) = \sum_{i=1}^{m} \theta_{i\rho}\mathcal{I}^{\delta_i;\psi}_{a^+}v^*(\xi_i), \quad \xi_i \in (a,b], \quad i = 1,2,\ldots,m, \\ v^*(a) = 0, \quad v^*(b) = \sum_{j=1}^{n} \mu_{j\rho}\mathcal{I}^{\varrho_j;\psi}_{a^+}u^*(\eta_j), \quad \eta_j \in (a,b], \quad j = 1,2,\ldots,n. \end{cases} \tag{3.33}$$

In view of Lemma 2.18, the solution of the proposed system (3.33) is

$$\begin{aligned} u^*(\tau) \;=\;& \mathcal{M}_{u^*,v^*}(\tau) + \frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds \\ &+\left[\frac{\Psi_\psi^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}(\Theta_1\Theta_2-\Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_1,\gamma_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(a))^{\alpha_1})\right] \\ &\times\Bigg[\Theta_2\Bigg(\frac{1}{\rho^{\alpha_2}}\int_a^b \Psi_\psi^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})w_{u^*,v^*}(s)\psi'(s)ds \\ &-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})w_{u^*,v^*}(s)\psi'(s)ds\Bigg) \\ &-\Delta_2\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})w_{u^*,v^*}(s)\psi'(s)ds \\ &-\frac{1}{\rho^{\alpha_1}}\int_a^b \Psi_\psi^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})w_{u^*,v^*}(s)\psi'(s)ds\Bigg)\Bigg] \\ &+\frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})w_{u^*,v^*}(s)\psi'(s)ds, \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} v^*(\tau) \;=\;& \mathcal{N}_{u^*,v^*}(\tau) + \frac{1}{\rho^{\alpha_2}}\int_a^\tau \Psi_\psi^{\alpha_2-1}(\tau,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})\mathbb{G}_{u^*,v^*}(s)\psi'(s)ds \\ &+\left[\frac{\Psi_\psi^{\gamma_2-1}(\tau,a)}{\rho^{\gamma_2-1}(\Theta_1\Theta_2-\Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_2,\gamma_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(a))^{\alpha_2})\right] \\ &\times\Bigg[\Delta_1\Bigg(\frac{1}{\rho^{\alpha_2}}\int_a^b \Psi_\psi^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})w_{u^*,v^*}(s)\psi'(s)ds \\ &-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})w_{u^*,v^*}(s)\psi'(s)ds\Bigg) \\ &-\Theta_1\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})w_{u^*,v^*}(s)\psi'(s)ds \\ &-\frac{1}{\rho^{\alpha_1}}\int_a^b \Psi_\psi^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})w_{u^*,v^*}(s)\psi'(s)ds\Bigg)\Bigg] \\ &+\frac{1}{\rho^{\alpha_2}}\int_a^\tau \Psi_\psi^{\alpha_2-1}(\tau,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})w_{u^*,v^*}(s)\psi'(s)ds, \end{aligned} \tag{3.35}$$

where $\mathcal{M}_{u^*,v^*}(\tau)$ and $\mathcal{N}_{u^*,v^*}(\tau)$ are provided by (3.30) and (3.31). Now, we apply property (i) of Remark 3.7, Lemma 2.9, Lemma 2.11, and Lemma 2.12 with (3.34), which implies that

$$
\begin{aligned}
& \left| u^*(\tau) - \mathcal{M}_{u^*,v^*}(\tau) - \frac{1}{\rho^{\alpha_1}} \int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\, \mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1 \rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1}) \mathbb{F}_{u^*,v^*}(s)\psi'(s)ds \right| \\
\leq\ & \left[\frac{\Psi_\psi^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|} \right] \Bigg[|\Theta_2| \Bigg(\sum_{j=1}^n \frac{|\mu_j|}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)} \\
& \times \int_a^{\eta_j} \Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s)|w_{u^*,v^*}(s)|\psi'(s)ds + \frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)} \int_a^b \Psi_\psi^{\alpha_2-1}(b,s)|w_{u^*,v^*}(s)|\psi'(s)ds \Bigg) \\
& + |\Delta_2| \Bigg(\sum_{i=1}^m \frac{|\theta_i|}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)} \int_a^{\xi_i} \Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s)|w_{u^*,v^*}(s)|\psi'(s)ds \\
& + \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \int_a^b \Psi_\psi^{\alpha_1-1}(b,s)|w_{u^*,v^*}(s)|\psi'(s)ds \Bigg) \Bigg] + \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)|w_{u^*,v^*}(s)|\psi'(s)ds \\
\leq\ & \epsilon \Bigg\{ \left[\frac{\Psi_\psi^{\gamma_1-1}(b,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|} \right] \Bigg[|\Theta_2| \Bigg(\sum_{j=1}^n \frac{|\mu_j|}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)} \\
& \times \int_a^{\eta_j} \Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s) \mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^* \rho^{-(\alpha_1+\alpha_2)}(\psi(s)-\psi(a))^{\alpha_1+\alpha_2})\psi'(s)ds \\
& + \frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)} \int_a^b \Psi_\psi^{\alpha_2-1}(b,s) \mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^* \rho^{-(\alpha_1+\alpha_2)}(\psi(s)-\psi(a))^{\alpha_1+\alpha_2})\psi'(s)ds \Bigg) \\
& + |\Delta_2| \Bigg(\sum_{i=1}^m \frac{|\theta_i|}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)} \int_a^{\xi_i} \Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s) \mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^* \rho^{-(\alpha_1+\alpha_2)}(\psi(s)-\psi(a))^{\alpha_1+\alpha_2})\psi'(s)ds \\
& + \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \int_a^b \Psi_\psi^{\alpha_1-1}(b,s) \mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^* \rho^{-(\alpha_1+\alpha_2)}(\psi(s)-\psi(a))^{\alpha_1+\alpha_2})\psi'(s)ds \Bigg) \Bigg] \\
& + \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)} \int_a^b \Psi_\psi^{\alpha_1-1}(b,s) \mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^* \rho^{-(\alpha_1+\alpha_2)}(\psi(s)-\psi(a))^{\alpha_1+\alpha_2})\psi'(s)ds \Bigg\} \\
\leq\ & \epsilon \Bigg\{ \Bigg[|\Theta_2| \Bigg(\sum_{j=1}^n \frac{|\mu_j|\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1-2}\Gamma(\varrho_j+\alpha_1)} \mathbb{E}_{\alpha_1+\alpha_2,2}(\bar{\lambda}^* \rho^{-(\alpha_1+\alpha_2)}(\psi(\eta_j)-\psi(a))^{\alpha_1+\alpha_2}) \\
& + \frac{\Psi_\psi^{\alpha_2}(b,a)}{\rho^{\alpha_2}} \mathbb{E}_{\alpha_1+\alpha_2,\alpha_2+1}(\bar{\lambda}^* \rho^{-(\alpha_1+\alpha_2)}(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}) \Bigg) \\
& + |\Delta_2| \Bigg(\sum_{i=1}^m \frac{|\theta_i|\Psi_\psi^{\delta_i+\alpha_2}(\xi_i,a)}{\rho^{\delta_i+\alpha_2-2}\Gamma(\delta_i+\alpha_2)} \mathbb{E}_{\alpha_1+\alpha_2,2}(\bar{\lambda}^* \rho^{-(\alpha_1+\alpha_2)}(\psi(\xi_i)-\psi(a))^{\alpha_1+\alpha_2}) \\
& + \frac{\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_1}} \mathbb{E}_{\alpha_1+\alpha_2,\alpha_1+1}(\bar{\lambda}^* \rho^{-(\alpha_1+\alpha_2)}(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}) \Bigg) \Bigg] \left[\frac{\Psi_\psi^{\gamma_1-1}(b,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|} \right] \\
& + \frac{\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_1}} \mathbb{E}_{\alpha_1+\alpha_2,\alpha_1+1}(\bar{\lambda}^* \rho^{-(\alpha_1+\alpha_2)}(\psi(b)-\psi(a))^{\alpha_1+\alpha_2}) \Bigg\} \\
=\ & \Lambda_1(\Theta_2,\Delta_2)\epsilon.
\end{aligned}
$$

In the same process, we obtain

$$\begin{aligned}&\left|v^*(\tau)-\mathcal{N}_{u^*,v^*}(\tau)-\frac{1}{\rho^{\alpha_2}}\int_a^\tau \Psi_\psi^{\alpha_2-1}(\tau,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})\mathbb{G}_{u^*,v^*}(s)\psi'(s)ds\right|\\ \le\;& \Lambda_2(\Delta_1,\Theta_1)\epsilon.\end{aligned}$$

The proof is done. □

Theorem 3.10. *Under the assumptions in Theorem 3.2, the proposed coupled system* (1.2) *is ML-UH stable and therefore generalized ML-UH stable on* $\mathcal{J}$.

Proof. Let $\epsilon > 0$ and $(u^*, v^*) \in \mathbb{U}\times\mathbb{U}$ be the function satisfying (3.18) and (3.19). Assume that $(u, v) \in \mathbb{U}\times\mathbb{U}$ is the unique solution of the proposed coupled system (1.2). Then, by Lemma 2.18, we get

$$u(\tau)=\mathcal{M}_{u,v}(\tau)+\frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u,v}(s)\psi'(s)ds, \tag{3.36}$$

$$v(\tau)=\mathcal{N}_{u,v}(\tau)+\frac{1}{\rho^{\alpha_2}}\int_a^\tau \Psi_\psi^{\alpha_2-1}(\tau,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})\mathbb{G}_{u,v}(s)\psi'(s)ds. \tag{3.37}$$

Clearly, $u(a) = u^*(a)$, $u(b) = u^*(b)$, ${}_\rho\mathcal{I}_{a^+}^{\varrho_j;\psi}u(\eta_j) = {}_\rho\mathcal{I}_{a^+}^{\varrho_j;\psi}u^*(\eta_j)$ $v(a) = v^*(a)$, $v(b) = v^*(b)$, and ${}_\rho\mathcal{I}_{a^+}^{\delta_i;\psi}v(\xi_i) = {}_\rho\mathcal{I}_{a^+}^{\delta_i;\psi}v^*(\xi_i)$, $i = 1, 2\ldots, m$, and $j = 1, 2, \ldots, n$, which yields that $\mathcal{M}_{u,v}(\tau) = \mathcal{M}_{u^*,v^*}(\tau)$ and $\mathcal{N}_{u,v}(\tau) = \mathcal{N}_{u^*,v^*}(\tau)$. We obtain that $|\mathcal{M}_{u,v}(\tau) - \mathcal{M}_{u^*,v^*}(\tau)| = 0$.

By applying Lemma 3.9 with $|x - y| \le |x| + |y|$, for any $x, y \in \mathbb{R}$, $\tau \in \mathcal{J}$, it follows that

$$\begin{aligned}&|u^*(\tau)-u(\tau)|\\ \le\;&\left|u^*(\tau)-\mathcal{M}_{u,v}(\tau)-\frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u,v}(s)\psi'(s)ds\right|\\ \le\;&\left|u^*(\tau)-\mathcal{M}_{u^*,v^*}(\tau)-\frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds\right|\\ &+\frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\left|\mathbb{F}_{u^*,v^*}(s)-\mathbb{F}_{u,v}(s)\right|\psi'(s)ds\\ &+|\mathcal{M}_{u,v}(\tau)-\mathcal{M}_{u^*,v^*}(\tau)|\\ \le\;&\frac{\mathcal{L}}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})[|u^*(s)-u(s)|+|v^*(s)-v(s)|]\psi'(s)ds\\ &+\Lambda_1(\Theta_2,\Delta_2)\epsilon.\end{aligned}$$

Then,

$$\|u^*-u\|\le\frac{\Lambda_1(\Theta_2,\Delta_2)\rho^{\alpha_1}\Gamma(\alpha_1+1)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_\psi^{\alpha_1}(b,a)}\epsilon+\frac{\rho^{\alpha_1}\Gamma(\alpha_1+1)\mathcal{L}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_\psi^{\alpha_1}(b,a)}{}_\rho\mathcal{I}_{a^+}^{\alpha_1;\psi}[\|v^*-v\|]. \tag{3.38}$$

Using the same procedure, we have

$$\|v^*-v\|\le\frac{\Lambda_2(\Delta_1,\Theta_1)\rho^{\alpha_2}\Gamma(\alpha_2+1)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_\psi^{\alpha_2}(b,a)}\epsilon+\frac{\rho^{\alpha_2}\Gamma(\alpha_2+1)\mathcal{K}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_\psi^{\alpha_2}(b,a)}{}_\rho\mathcal{I}_{a^+}^{\alpha_2;\psi}[\|u^*-u\|]. \tag{3.39}$$

Utilizing Theorem 2.15 and Corollary 2.17 to (3.38) and (3.39) implies that

$$\begin{aligned}\|u^*-u\| &\le \left[\frac{\epsilon}{\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_{\psi}^{\alpha_1}(b,a)}\left(\Lambda_1(\Theta_2,\Delta_2)\rho^{\alpha_1}\Gamma(\alpha_1+1)\right.\right.\\ &\quad\left.\left.+\frac{\mathcal{L}\Lambda_2(\Delta_1,\Theta_1)\rho^{\alpha_2}\Gamma(\alpha_2+1)\Psi_{\psi}^{\alpha_1}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_{\psi}^{\alpha_2}(b,a)}\right)\right]\\ &\quad\times\mathbb{E}_{\alpha_1+\alpha_2}\left(\frac{\mathcal{L}\mathcal{K}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}}{[\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_{\psi}^{\alpha_1}(b,a)][\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_{\psi}^{\alpha_2}(b,a)]}\right),\end{aligned}$$

and

$$\begin{aligned}\|v^*-v\| &\le \left[\frac{\epsilon}{\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_{\psi}^{\alpha_2}(b,a)}\left(\Lambda_2(\Delta_1,\Theta_1)\rho^{\alpha_2}\Gamma(\alpha_2+1)\right.\right.\\ &\quad\left.\left.+\frac{\mathcal{K}\Lambda_1(\Theta_2,\Delta_2)\rho^{\alpha_1}\Gamma(\alpha_1+1)\Psi_{\psi}^{\alpha_2}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_{\psi}^{\alpha_1}(b,a)}\right)\right]\\ &\quad\times\mathbb{E}_{\alpha_1+\alpha_2}\left(\frac{\mathcal{L}\mathcal{K}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}}{[\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_{\psi}^{\alpha_1}(b,a)][\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_{\psi}^{\alpha_2}(b,a)]}\right).\end{aligned}$$

Then,

$$\|(u,v)-(u^*,v^*)\| \le (\mathfrak{E}_f+\mathfrak{E}_g)\epsilon\mathbb{E}_{\alpha_1+\alpha_2}(\kappa_{f,g}(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}),$$

where

$$\begin{aligned}\mathfrak{E}_f &:= \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_{\psi}^{\alpha_1}(b,a)}\left(\Lambda_1(\Theta_2,\Delta_2)\rho^{\alpha_1}\Gamma(\alpha_1+1)\right.\\ &\quad\left.+\frac{\mathcal{L}\Lambda_2(\Delta_1,\Theta_1)\rho^{\alpha_2}\Gamma(\alpha_2+1)\Psi_{\psi}^{\alpha_1}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_{\psi}^{\alpha_2}(b,a)}\right),\\ \mathfrak{E}_g &:= \frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_{\psi}^{\alpha_2}(b,a)}\left(\Lambda_2(\Delta_1,\Theta_1)\rho^{\alpha_2}\Gamma(\alpha_2+1)\right.\\ &\quad\left.+\frac{\mathcal{K}\Lambda_1(\Theta_2,\Delta_2)\rho^{\alpha_1}\Gamma(\alpha_1+1)\Psi_{\psi}^{\alpha_2}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_{\psi}^{\alpha_1}(b,a)}\right),\\ \kappa_{f,g} &:= \frac{\mathcal{L}\mathcal{K}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{[\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_{\psi}^{\alpha_1}(b,a)][\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_{\psi}^{\alpha_2}(b,a)]}.\end{aligned}$$

Therefore, the proposed coupled system (1.2) is ML-UH stable. In addition, by setting $\chi_f(\epsilon)=\mathfrak{E}_f\epsilon$ and $\chi_g(\epsilon)=\mathfrak{E}_g\epsilon$ with $\chi_f(0)=0$ and $\chi_g(0)=0$,

$$\|(u,v)-(u^*,v^*)\| \le (\chi_f(\epsilon)+\chi_g(\epsilon))\mathbb{E}_{\alpha_1+\alpha_2}(\kappa_{f,g}(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}).$$

Hence, the proposed coupled system (1.2) is generalized ML-UH stable. The proof is done. □

Next, we define the following assumption:

($\mathcal{A}_2$) Assume $\Phi \in C(\mathcal{J}, \mathbb{R})$ is a nondecreasing function. There is a constant $\mathfrak{E}_\Phi \in \mathbb{R}^+$ so that for every $\tau \in \mathcal{J}$,

$$\int_a^\tau \Psi_\psi^{\alpha_1+\alpha_2-1}(\tau, s)\mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(\tau)-\psi(s))^{\alpha_1+\alpha_2})\Phi(s)\psi'(s)ds \le \mathfrak{E}_\Phi\Phi(\tau). \tag{3.40}$$

Lemma 3.11. *Let $\alpha_i \in (1,2)$, $i = 1,2$, $\beta \in [0,1]$, and $\rho \in (0,1]$. If $(u^*, v^*) \in \mathbb{U}\times\mathbb{U}$ satisfies* (3.24) *and* (3.25)*, then (u^*, v^*) satisfies the following inequalities:*

$$\begin{aligned}
&\left|u^*(\tau) - \mathcal{M}_{u^*,v^*}(\tau) - \frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds\right| \\
\le\ & \sigma_1(\Theta_2, \Delta_2)\mathfrak{E}_\Phi\epsilon\Phi(\tau), \\
&\left|v^*(\tau) - \mathcal{N}_{u^*,v^*}(\tau) - \frac{1}{\rho^{\alpha_2}}\int_a^\tau \Psi_\psi^{\alpha_2-1}(\tau,s)\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})\mathbb{G}_{u^*,v^*}(s)\psi'(s)ds\right| \\
\le\ & \sigma_2(\Delta_1, \Theta_1)\mathfrak{E}_\Phi\epsilon\Phi(\tau),
\end{aligned}$$

where $\mathcal{M}_{u^*,v^*}(\tau)$ and $\mathcal{N}_{u^*,v^*}(\tau)$ are given by (3.30) *and* (3.31)*, respectively.*

$$\begin{aligned}
\sigma_k(\Xi, \Upsilon) &= \frac{1}{\rho^{\alpha_k}\Gamma(\alpha_k)} + \left[\frac{\Psi_\psi^{\gamma_k-1}(b,a)}{\rho^{\gamma_k-1}\Gamma(\gamma_k)|\Theta_1\Theta_2 - \Delta_1\Delta_2|}\right]\Bigg[|\Xi|\left(\sum_{j=1}^n \frac{|\mu_j|\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)} + \frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\right) \\
&\quad + |\Upsilon|\left(\sum_{i=1}^m \frac{|\theta_i|\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,a)}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)} + \frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\right)\Bigg], \quad \Xi, \Upsilon \in \Theta_k, \Delta_k, k = 1,2,
\end{aligned} \tag{3.41}$$

where Δ_i and Θ_i are given by (2.37)–(2.39)*, $i = 1,2$, respectively.*

Proof. Suppose (u^*, v^*) is the solution of (3.24) and (3.25). By using (*ii*) in Remark 3.8, we get

$$\begin{cases}
{}^H_\rho\mathfrak{D}_{a^+}^{\alpha_1,\beta;\psi} u^*(\tau) = \lambda_1 u^*(\tau) + \mathbb{F}_{u^*,v^*}(\tau) + z_{u^*,v^*}(\tau), \quad \tau \in (a,b], \\
{}^H_\rho\mathfrak{D}_{a^+}^{\alpha_2,\beta;\psi} v^*(\tau) = \lambda_2 v^*(\tau) + \mathbb{G}_{u^*,v^*}(\tau) + z_{u^*,v^*}(\tau), \quad \tau \in (a,b], \\
u^*(a) = 0, \quad u^*(b) = \sum_{i=1}^m \theta_{i\rho}\mathcal{I}_{a^+}^{\delta_i;\psi} v^*(\xi_i), \quad \xi_i \in (a,b], \quad i = 1,2,\dots,m, \\
v^*(a) = 0, \quad v^*(b) = \sum_{j=1}^n \mu_{j\rho}\mathcal{I}_{a^+}^{\varrho_j;\psi} u^*(\eta_j), \quad \eta_j \in (a,b], \quad j = 1,2,\dots,n.
\end{cases} \tag{3.42}$$

In view of Lemma 2.18, the solution of the proposed system (3.42) is

$$\begin{aligned}
u^*(\tau) &= \mathcal{M}_{u^*,v^*}(\tau) + \frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds \\
&\quad + \left[\frac{\Psi_\psi^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}(\Theta_1\Theta_2 - \Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_1,\gamma_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(a))^{\alpha_1})\right] \\
&\quad \times\Bigg[\Theta_2\Bigg(\frac{1}{\rho^{\alpha_2}}\int_a^b \Psi_\psi^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})z_{u^*,v^*}(s)\psi'(s)ds \\
&\quad - \sum_{j=1}^n \frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j} \Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})z_{u^*,v^*}(s)\psi'(s)ds\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&-\Delta_2\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})z_{u^*,v^*}(s)\psi'(s)ds\\
&-\frac{1}{\rho^{\alpha_1}}\int_a^{b}\Psi_\psi^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})z_{u^*,v^*}(s)\psi'(s)ds\Bigg)\Bigg]\\
&+\frac{1}{\rho^{\alpha_1}}\int_a^{\tau}\Psi_\psi^{\alpha_1-1}(\tau,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})z_{u^*,v^*}(s)\psi'(s)ds, \qquad (3.43)
\end{aligned}
$$

and

$$
\begin{aligned}
v^*(\tau) \;=\;& \mathcal{N}_{u^*,v^*}(\tau)+\frac{1}{\rho^{\alpha_2}}\int_a^{\tau}\Psi_\psi^{\alpha_2-1}(\tau,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})\mathbb{G}_{u^*,v^*}(s)\psi'(s)ds\\
&+\Bigg[\frac{\Psi_\psi^{\gamma_2-1}(\tau,a)}{\rho^{\gamma_2-1}(\Theta_1\Theta_2-\Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_2,\gamma_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(a))^{\alpha_2})\Bigg]\\
&\times\Bigg[\Delta_1\Bigg(\frac{1}{\rho^{\alpha_2}}\int_a^{b}\Psi_\psi^{\alpha_2-1}(b,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(s))^{\alpha_2})z_{u^*,v^*}(s)\psi'(s)ds\\
&-\sum_{j=1}^{n}\frac{\mu_j}{\rho^{\varrho_j+\alpha_1}}\int_a^{\eta_j}\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(s))^{\alpha_1})z_{u^*,v^*}(s)\psi'(s)ds\Bigg)\\
&-\Theta_1\Bigg(\sum_{i=1}^{m}\frac{\theta_i}{\rho^{\delta_i+\alpha_2}}\int_a^{\xi_i}\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(s))^{\alpha_2})z_{u^*,v^*}(s)\psi'(s)ds\\
&-\frac{1}{\rho^{\alpha_1}}\int_a^{b}\Psi_\psi^{\alpha_1-1}(b,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(s))^{\alpha_1})z_{u^*,v^*}(s)\psi'(s)ds\Bigg)\Bigg]\\
&+\frac{1}{\rho^{\alpha_2}}\int_a^{\tau}\Psi_\psi^{\alpha_2-1}(\tau,s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})z_{u^*,v^*}(s)\psi'(s)ds, \qquad (3.44)
\end{aligned}
$$

where $\mathcal{M}_{u^*,v^*}(\tau)$ and $\mathcal{N}_{u^*,v^*}(\tau)$ are provided by (3.30) and (3.31). Now, we apply the property (i) of Remark 3.8, Lemma 2.9, Lemma 2.11, and Lemma 2.12 with (3.34), we obtain

$$
\begin{aligned}
&\left|u^*(\tau)-\mathcal{M}_{u^*,v^*}(\tau)-\frac{1}{\rho^{\alpha_1}}\int_a^{\tau}\Psi_\psi^{\alpha_1-1}(\tau,s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds\right|\\
\le\;& \Bigg[\frac{\Psi_\psi^{\gamma_1-1}(\tau,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\Bigg]\Bigg[|\Theta_2|\Bigg(\sum_{j=1}^{n}\frac{|\mu_j|}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)}\int_a^{\eta_j}\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,s)|z_{u^*,v^*}(s)|\psi'(s)ds\\
&+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^{b}\Psi_\psi^{\alpha_2-1}(b,s)|z_{u^*,v^*}(s)|\psi'(s)ds\Bigg)\\
&+|\Delta_2|\Bigg(\sum_{i=1}^{m}\frac{|\theta_i|}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)}\int_a^{\xi_i}\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,s)|z_{u^*,v^*}(s)|\psi'(s)ds\\
&+\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\int_a^{b}\Psi_\psi^{\alpha_1-1}(b,s)|z_{u^*,v^*}(s)|\psi'(s)ds\Bigg)\Bigg]+\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\int_a^{\tau}\Psi_\psi^{\alpha_1-1}(\tau,s)|z_{u^*,v^*}(s)|\psi'(s)ds\\
\le\;& \Bigg\{\Bigg[\frac{\Psi_\psi^{\gamma_1-1}(b,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\Bigg]\Bigg[|\Theta_2|\Bigg(\sum_{j=1}^{n}\frac{|\mu_j|}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)}
\end{aligned}
$$

$$
\times \int_a^{\eta_j} \Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j, s)\mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(s)-\psi(a))^{\alpha_1+\alpha_2})\Phi(s)\psi'(s)ds
$$
$$
+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\int_a^b \Psi_\psi^{\alpha_2-1}(b, s)\mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(s)-\psi(a))^{\alpha_1+\alpha_2})\Phi(s)\psi'(s)ds\Bigg)
$$
$$
+\sum_{i=1}^{m}\frac{|\Delta_2||\theta_i|}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)}\int_a^{\xi_i} \Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i, s)\mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(s)-\psi(a))^{\alpha_1+\alpha_2})\Phi(s)\psi'(s)ds
$$
$$
+\frac{|\Delta_2|}{\rho^{\alpha_1}\Gamma(\alpha_1)}\int_a^b \Psi_\psi^{\alpha_1-1}(b, s)\mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(s)-\psi(a))^{\alpha_1+\alpha_2})\Phi(s)\psi'(s)ds\Bigg]
$$
$$
+\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\int_a^b \Psi_\psi^{\alpha_1-1}(b, s)\mathbb{E}_{\alpha_1+\alpha_2}(\bar{\lambda}^*\rho^{-(\alpha_1+\alpha_2)}(\psi(s)-\psi(a))^{\alpha_1+\alpha_2})\Phi(s)\psi'(s)ds\Bigg\}\epsilon.
$$

Assumption ($\mathcal{A}_2$) implies that

$$
\begin{aligned}
&\left|u^*(\tau)-\mathcal{M}_{u^*,v^*}(\tau)-\frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau, s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds\right| \\
\leq\ & \Bigg\{\Bigg[\frac{\Psi_\psi^{\gamma_1-1}(b,a)}{\rho^{\gamma_1-1}\Gamma(\gamma_1)|\Theta_1\Theta_2-\Delta_1\Delta_2|}\Bigg]\Bigg[|\Theta_2|\Bigg(\sum_{j=1}^{n}\frac{|\mu_j|\Psi_\psi^{\varrho_j+\alpha_1-1}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1}\Gamma(\varrho_j+\alpha_1)}+\frac{1}{\rho^{\alpha_2}\Gamma(\alpha_2)}\Bigg) \\
&+|\Delta_2|\Bigg(\sum_{i=1}^{m}\frac{|\theta_i|\Psi_\psi^{\delta_i+\alpha_2-1}(\xi_i,a)}{\rho^{\delta_i+\alpha_2}\Gamma(\delta_i+\alpha_2)}+\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\Bigg)\Bigg]+\frac{1}{\rho^{\alpha_1}\Gamma(\alpha_1)}\Bigg\}\mathfrak{E}_\Phi\epsilon\Phi(\tau) \\
=\ & \sigma_1(\Theta_2,\Delta_2)\mathfrak{E}_\Phi\epsilon\Phi(\tau).
\end{aligned}
$$

Applying the same process, we get the following result

$$
\begin{aligned}
&\left|v^*(\tau)-\mathcal{N}_{u^*,v^*}(\tau)-\frac{1}{\rho^{\alpha_2}}\int_a^\tau \Psi_\psi^{\alpha_2-1}(\tau, s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})\mathbb{G}_{u^*,v^*}(s)\psi'(s)ds\right| \\
\leq\ & \sigma_2(\Delta_1,\Theta_1)\mathfrak{E}_\Phi\epsilon\Phi(\tau).
\end{aligned}
$$

The proof is done. □

Theorem 3.12. *Under the assumption in Theorem 3.2, the proposed coupled system* (1.2) *is ML-UHR stable and therefore generalized ML-UHR stable on* $\mathcal{J}$.

Proof. Suppose that $\epsilon > 0$ and $(u^*, v^*) \in \mathbb{U}\times\mathbb{U}$ is a function corresponding to (3.24) and (3.25). Assume that $(u, v) \in \mathbb{U}\times\mathbb{U}$ is the unique solution of the proposed coupled system (1.2). Then, by Lemma 2.18, we obtain that

$$
u(\tau) = \mathcal{M}_{u,v}(\tau) + \frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau, s)\,\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u,v}(s)\psi'(s)ds, \tag{3.45}
$$
$$
v(\tau) = \mathcal{N}_{u,v}(\tau) + \frac{1}{\rho^{\alpha_2}}\int_a^\tau \Psi_\psi^{\alpha_2-1}(\tau, s)\,\mathbb{E}_{\alpha_2,\alpha_2}(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(s))^{\alpha_2})\mathbb{G}_{u,v}(s)\psi'(s)ds. \tag{3.46}
$$

Clearly, $u(a) = u^*(a)$, $u(b) = u^*(b)$, ${}_\rho\mathcal{I}_{a^+}^{\varrho_j;\psi}u(\eta_j) = {}_\rho\mathcal{I}_{a^+}^{\varrho_j;\psi}u^*(\eta_j)$ $v(a) = v^*(a)$, $v(b) = v^*(b)$, and ${}_\rho\mathcal{I}_{a^+}^{\delta_i;\psi}v(\xi_i) = {}_\rho\mathcal{I}_{a^+}^{\delta_i;\psi}v^*(\xi_i)$, $i = 1, 2\ldots, m$, and $j = 1, 2, \ldots, n$, which yields that $\mathcal{M}_{u,v}(\tau) = \mathcal{M}_{u^*,v^*}(\tau)$

and $\mathcal{N}_{u,v}(\tau) = \mathcal{N}_{u^*,v^*}(\tau)$. We obtain that $|\mathcal{M}_{u,v}(\tau) - \mathcal{M}_{u^*,v^*}(\tau)| = 0$. By applying Lemma 3.11 with $|x-y| \leq |x| + |y|$, for any $x, y \in \mathbb{R}$, $\tau \in \mathcal{J}$, implies that

$$\begin{aligned}
& |u^*(\tau) - u(\tau)| \\
\leq\ & \left|u^*(\tau) - \mathcal{M}_{u,v}(\tau) - \frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u,v}(s)\psi'(s)ds\right| \\
\leq\ & \left|u^*(\tau) - \mathcal{M}_{u^*,v^*}(\tau) - \frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\mathbb{F}_{u^*,v^*}(s)\psi'(s)ds\right| \\
& +\frac{1}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1})\left|\mathbb{F}_{u^*,v^*}(s) - \mathbb{F}_{u,v}(s)\right|\psi'(s)ds \\
& +|\mathcal{M}_{u,v}(\tau) - \mathcal{M}_{u^*,v^*}(\tau)| \\
\leq\ & \sigma_1(\Theta_2,\Delta_2)\mathfrak{E}_\Phi\epsilon\Phi(\tau) + \frac{\mathcal{L}}{\rho^{\alpha_1}}\int_a^\tau \Psi_\psi^{\alpha_1-1}(\tau,s)\mathbb{E}_{\alpha_1,\alpha_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(s))^{\alpha_1}) \\
& \times[|u^*(s)-u(s)| + |v^*(s)-v(s)|]\psi'(s)ds.
\end{aligned}$$

Then,

$$\|u^* - u\| \leq \frac{\rho^{\alpha_1}\Gamma(\alpha_1+1)}{\rho^{\alpha_1}\Gamma(\alpha_1+1) - \mathcal{L}\Psi_\psi^{\alpha_1}(b,a)}\left(\sigma_1(\Theta_2,\Delta_2)\mathfrak{E}_\Phi\epsilon\Phi(\tau) + \mathcal{L}_\rho\mathcal{I}_{a^+}^{\alpha_1;\psi}[\|v^*-v\|]\right). \tag{3.47}$$

Using the same procedure, we have

$$\|v^* - v\| \leq \frac{\rho^{\alpha_2}\Gamma(\alpha_2+1)}{\rho^{\alpha_2}\Gamma(\alpha_2+1) - \mathcal{K}\Psi_\psi^{\alpha_2}(b,a)}\left(\sigma_2(\Delta_1,\Theta_1)\mathfrak{E}_\Phi\epsilon\Phi(\tau) + \mathcal{K}_\rho\mathcal{I}_{a^+}^{\alpha_2;\psi}[\|u^*-u\|]\right). \tag{3.48}$$

Utilizing Theorem 2.15 and Corollary 2.17 to (3.47) and (3.48), implies that

$$\begin{aligned}
& \|u^* - u\| \\
\leq\ & \left[\frac{\mathfrak{E}_\Phi\epsilon\Phi(\tau)}{\rho^{\alpha_1}\Gamma(\alpha_1+1) - \mathcal{L}\Psi_\psi^{\alpha_1}(b,a)}\left(\sigma_1(\Theta_2,\Delta_2)\rho^{\alpha_1}\Gamma(\alpha_1+1) + \frac{\mathcal{L}\sigma_2(\Delta_1,\Theta_1)\rho^{\alpha_2}\Gamma(\alpha_2+1)\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1) - \mathcal{K}\Psi_\psi^{\alpha_2}(b,a)}\right)\right] \\
& \times\mathbb{E}_{\alpha_1+\alpha_2}\left(\frac{\mathcal{L}\mathcal{K}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{[\rho^{\alpha_1}\Gamma(\alpha_1+1) - \mathcal{L}\Psi_\psi^{\alpha_1}(b,a)][\rho^{\alpha_2}\Gamma(\alpha_2+1) - \mathcal{K}\Psi_\psi^{\alpha_2}(b,a)]}(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}\right),
\end{aligned}$$

and

$$\begin{aligned}
& \|v^* - v\| \\
\leq\ & \left[\frac{\mathfrak{E}_\Phi\epsilon\Phi(\tau)}{\rho^{\alpha_2}\Gamma(\alpha_2+1) - \mathcal{K}\Psi_\psi^{\alpha_2}(b,a)}\left(\sigma_2(\Delta_1,\Theta_1)\rho^{\alpha_2}\Gamma(\alpha_2+1) + \frac{\mathcal{K}\sigma_1(\Theta_2,\Delta_2)\rho^{\alpha_1}\Gamma(\alpha_1+1)\Psi_\psi^{\alpha_2}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1) - \mathcal{L}\Psi_\psi^{\alpha_1}(b,a)}\right)\right] \\
& \times\mathbb{E}_{\alpha_1+\alpha_2}\left(\frac{\mathcal{L}\mathcal{K}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{[\rho^{\alpha_1}\Gamma(\alpha_1+1) - \mathcal{L}\Psi_\psi^{\alpha_1}(b,a)][\rho^{\alpha_2}\Gamma(\alpha_2+1) - \mathcal{K}\Psi_\psi^{\alpha_2}(b,a)]}(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}\right).
\end{aligned}$$

Then, we obtain

$$\|(u,v) - (u^*,v^*)\| \leq (\mathfrak{E}_{f_\Phi} + \mathfrak{E}_{g_\Phi})\epsilon\Phi(\tau)\mathbb{E}_{\alpha_1+\alpha_2}(\kappa_{f_\Phi,g_\Phi}(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}),$$

where

$$
\begin{aligned}
\mathfrak{E}_{f_\Phi} &:= \frac{\mathfrak{E}_\Phi}{\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_\psi^{\alpha_1}(b,a)}\Big(\sigma_1(\Theta_2,\Delta_2)\rho^{\alpha_1}\Gamma(\alpha_1+1) \\
&\qquad +\frac{\mathcal{L}\sigma_2(\Delta_1,\Theta_1)\rho^{\alpha_2}\Gamma(\alpha_2+1)\Psi_\psi^{\alpha_1}(b,a)}{\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_\psi^{\alpha_2}(b,a)}\Big), \\
\mathfrak{E}_{g_\Phi} &:= \frac{\mathfrak{E}_\Phi}{\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_\psi^{\alpha_2}(b,a)}\Big(\sigma_2(\Delta_1,\Theta_1)\rho^{\alpha_2}\Gamma(\alpha_2+1) \\
&\qquad +\frac{\mathcal{K}\sigma_1(\Theta_2,\Delta_2)\rho^{\alpha_1}\Gamma(\alpha_1+1)\Psi_\psi^{\alpha_2}(b,a)}{\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_\psi^{\alpha_1}(b,a)}\Big), \\
\kappa_{f_\Phi,g_\Phi} &:= \frac{\mathcal{L}\mathcal{K}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{[\rho^{\alpha_1}\Gamma(\alpha_1+1)-\mathcal{L}\Psi_\psi^{\alpha_1}(b,a)][\rho^{\alpha_2}\Gamma(\alpha_2+1)-\mathcal{K}\Psi_\psi^{\alpha_2}(b,a)]}.
\end{aligned}
$$

Therefore, the proposed coupled system (1.2) is ML-UHR stable. By setting $\epsilon = 1$, it follows that

$$\|(u,v)-(u^*,v^*)\| \le (\mathfrak{E}_{f_\Phi}+\mathfrak{E}_{g_\Phi})\Phi(\tau)\mathbb{E}_{\alpha_1+\alpha_2}(\kappa_{f_\Phi,g_\Phi}(\psi(\tau)-\psi(a))^{\alpha_1+\alpha_2}).$$

Hence, the proposed coupled system (1.2) is generalized ML-UHR stable. The proof is done. □

4. An numerical example

This section shows a numerical example for the accuracy and applicability of the main results. All computations were performed using MATLAB software.

Example 4.1. *Consider the following nonlinear coupled Cauchy-type system under nonlocal integral conditions involving the ψ-HPFDO of the form*

$$
\begin{cases}
{}^{H}_{\frac{\sqrt{\pi}}{2}}\mathfrak{D}_{0^+}^{\frac{\pi}{2},\frac{\sqrt{2}}{2};\psi}u(\tau) = \dfrac{-\sqrt{3}}{2}u(\tau)+f(\tau,u(\tau),v(\tau)), & \tau\in(0,2], \\
{}^{H}_{\frac{\sqrt{\pi}}{2}}\mathfrak{D}_{0^+}^{\frac{\pi}{3},\frac{\sqrt{2}}{2};\psi}v(\tau) = -2^e v(\tau)+g(\tau,u(\tau),v(\tau)), & \tau\in(0,2], \\
u(0)=0, \qquad u(2)=\displaystyle\sum_{i=1}^{3}\left(\frac{4i}{3^i}\right){}_{\frac{\sqrt{\pi}}{2}}\mathcal{I}_{0^+}^{\frac{i}{5};\psi}v\left(\frac{i}{5-i}\right), \\
v(0)=0, \qquad v(2)=\displaystyle\sum_{j=1}^{2}\left(\frac{2^j}{10-2j}\right){}_{\frac{\sqrt{\pi}}{2}}\mathcal{I}_{0^+}^{\frac{2+j}{5};\psi}u\left(\frac{2j}{3}\right).
\end{cases}
\tag{4.1}
$$

From the proposed coupled system (4.1), we obtain the following constants $\alpha_1 = \pi/2$, $\alpha_2 = \pi/3$, $\rho = \sqrt{\pi}/2, \beta = \sqrt{2}/2, a = 0, b = 2, \lambda_1 = -\sqrt{3}/2, \lambda_2 = -2^e, \theta_i = 4i/3^i, \delta_i = i/5, \xi_i = i/(5-i), i = 1,2,3,$ $\mu_j = 2^j/(10-2j), \varrho_j = (2+j)/5, \eta_j = 2j/3, j = 1,2$, and $\psi(t) = 4-2\exp(-t)$. By directly computing all constants, we obtain $\Delta_1 \approx 0.671504$, $\Delta_2 \approx 0.075203$, $\Theta_1 \approx 0.758008$, $\Theta_2 \approx 0.392766$, $\omega_{1,1} \approx 1.636261$, $\omega_{1,2} \approx 1.579949$, $\omega_{2,1} \approx 6.097342$, $\omega_{2,2} \approx 5.747051$, $\omega_3(\Theta_1,\Delta_1) \approx 2.421206$, $\omega_3(\Delta_2,\Theta_2) \approx 0.813771$, $\omega_4(\Delta_1,\Theta_1) \approx 3.045738$, $\omega_4(\Theta_2,\Delta_2) \approx 0.817465$, $\Omega_1 \approx 20.512894$, and $\Omega_2 \approx 24.068323$. We will separate it into two cases.

Case I. The given nonlinear functions as suitable for the proposed coupled system (4.1) are as follows

$$f(\tau, u(\tau), v(\tau)) = 3\cos(2\pi t) + \frac{3e^{-2t}}{\ln^4(2t+9)}\left(\frac{|u|}{|u|+9}\right) + \frac{4\sin(\pi t)}{(3t+2)^5}\left(\frac{|v|}{|v|+8}\right), \tag{4.2}$$

$$g(\tau, u(\tau), v(\tau)) = \frac{\log_3(4t+3)}{(2t+1)^3} + \frac{3-2t^2}{3\cos(t)+24}\left(\frac{|u|}{|u|+7}\right) + \frac{2^{1-2t}}{2t+17}\left(\frac{|v|}{|v|+6}\right). \tag{4.3}$$

By applying (4.2) and (4.3), for each u_i, $v_i \in \mathbb{R}$, $(i = 1, 2)$, and $\tau \in [0, 2]$, we have the following results:

$$|f(\tau, u_1, v_1) - f(\tau, u_2, v_2)| \le \frac{1}{64}(|u_1 - u_2| + |v_1 - v_2|),$$
$$|g(\tau, u_1, v_1) - g(\tau, u_2, v_2)| \le \frac{1}{49}(|u_1 - u_2| + |v_1 - v_2|).$$

From assumption $(\mathcal{A}_1)$ in Theorem 3.2, we have $\mathcal{L} = 1/64$ and $\mathcal{K} = 1/49$. Then, $\Omega_1\mathcal{L} + \Omega_2\mathcal{K} \approx 0.811704 < 1$, since all assumptions in Theorem 3.2 are true. Therefore, the proposed coupled system (4.1) has a unique solution on $[0, 2]$. Moreover, we can calculate constants $\Lambda_1(\Theta_2, \Delta_2) \approx 7.184020 > 0$ and $\Lambda_2(\Delta_1, \Theta_1) \approx 21.674867 > 0$. By using all previous constants, we get $\mathfrak{C}_f \approx 7.960149 > 0$, $\mathfrak{C}_g \approx 22.642672 > 0$, and $\kappa_{f,g} \approx 0.000464 > 0$. Then, by conclusion of Theorem 3.10, the proposed coupled system (4.1) is ML-UH stable on $[0, 2]$. By taking $\chi_f(\epsilon) = \mathfrak{C}_f\epsilon$ and $\chi_g(\epsilon) = \mathfrak{C}_g\epsilon$ with $\chi_f(0) = \chi_g(0) = 0$, the proposed coupled system (4.1) is generalized ML-UH stable on $[0, 2]$. Additionally, if $\Phi(\tau) = \Psi_\psi^{\pi/4-1}(\tau, a)$ in (3.40), then

$$\int_0^2 \Psi_\psi^{\frac{\pi}{2}-1}(2, s)\mathbb{E}_{\frac{\pi}{2}}((\sqrt{\pi})^{-\frac{\pi}{2}}(\psi(2) - \psi(s))^{\frac{\pi}{2}})\Psi_\psi^{\frac{\pi}{4}-1}(s, 0)\psi'(s)ds$$
$$\le \Psi_\psi^{\frac{\pi}{2}}(2, 0)\frac{\Gamma\left(\frac{\pi}{2}\right)\Gamma\left(\frac{\pi}{4}\right)}{\Gamma\left(\frac{\pi}{2} + \frac{\pi}{4}\right)}\Psi_\psi^{\frac{\pi}{4}-1}(2, 0).$$

This yields that $\mathfrak{C}_\Phi = \Psi_\psi^{\frac{\pi}{2}}(2, 0)\frac{\Gamma(\frac{\pi}{2})\Gamma(\frac{\pi}{4})}{\Gamma(\frac{\pi}{2}+\frac{\pi}{4})} \approx 4.374427$. From all previous constants, we can compute the values $\sigma_1(\Theta_2, \Delta_2) \approx 9.219534 > 0$, $\sigma_2(\Delta_1, \Theta_1) \approx 29.663994 > 0$, $\mathfrak{C}_{f_\Phi} \approx 44.906414 > 0$, $\mathfrak{C}_{g_\Phi} \approx 135.465425 > 0$, and $\kappa_{f_\Phi, g_\Phi} \approx 0.000464 > 0$. Hence, by the conclusion of Theorem 3.12, the proposed coupled system (4.1) is ML-UHR stable on $[0, 2]$. Finally, if $\epsilon = 1$, the proposed coupled system (4.1) is generalized ML-UHR stable on $[0, 2]$.

Case II. The given linear functions as suitable for the proposed coupled system (4.1) are as follows

$$\mathbb{F}_{u,v}(\tau) = \Psi_\psi^{\frac{12}{5}}(\tau, a), \qquad \text{and} \qquad \mathbb{G}_{u,v}(\tau) = \Psi_\psi^{\frac{17}{10}}(\tau, a).$$

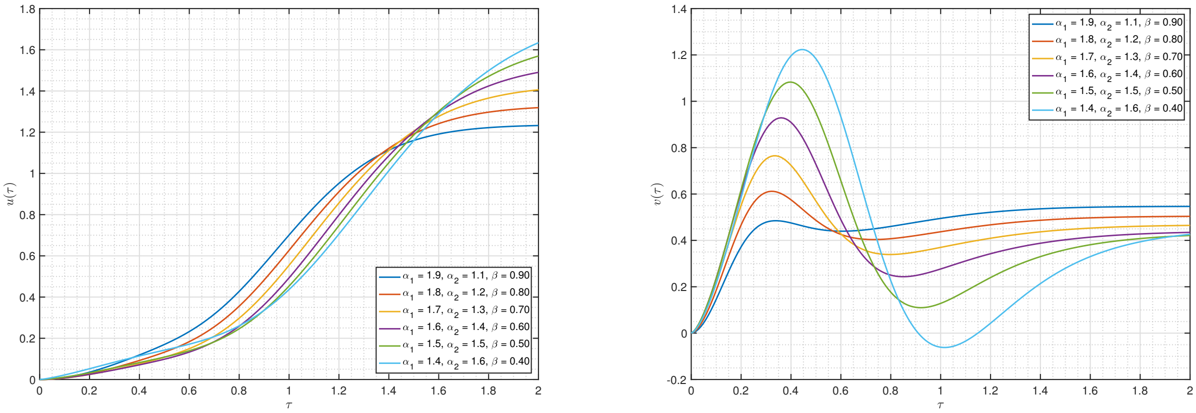
From Lemma 2.18, the implicit solution of the proposed coupled system (4.1) is given by

$$u(\tau) = \left[\frac{e^{\frac{\rho-1}{\rho}\psi(a)}\Psi_\psi^{\gamma_1-1}(\tau, a)}{\rho^{\gamma_1-1}(\Theta_1\Theta_2 - \Delta_1\Delta_2)}\,\mathbb{E}_{\alpha_1,\gamma_1}(\lambda_1\rho^{-\alpha_1}(\psi(\tau) - \psi(a))^{\alpha_1})\right]$$
$$\times\left[\Theta_2\left(\frac{\Psi_\psi^{\alpha_2+\frac{17}{10}}(b, a)}{\rho^{\alpha_2}}\Gamma\left(\frac{27}{10}\right)\,\mathbb{E}_{\alpha_2,\alpha_2+\frac{27}{10}}(\lambda_2\rho^{-\alpha_2}(\psi(b) - \psi(a))^{\alpha_2})\right.\right.$$

$$
\begin{aligned}
& -\sum_{j=1}^{n}\frac{\mu_j\Psi_{\psi}^{\varrho_j+\alpha_1+\frac{12}{5}}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1}}\Gamma\left(\frac{17}{5}\right)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1+\frac{17}{5}}\big(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(a))^{\alpha_1}\big)\Bigg)\\
& -\Delta_2\Bigg(\sum_{i=1}^{m}\frac{\theta_i\Psi_{\psi}^{\delta_i+\alpha_2+\frac{17}{10}}(\xi_i,a)}{\rho^{\delta_i+\alpha_2}}\Gamma\left(\frac{27}{10}\right)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2+\frac{27}{10}}\big(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(a))^{\alpha_2}\big)\\
& -\frac{\Psi_{\psi}^{\alpha_1+\frac{12}{5}}(b,a)}{\rho^{\alpha_1}}\Gamma\left(\frac{17}{5}\right)\mathbb{E}_{\alpha_1,\alpha_1+\frac{17}{5}}\big(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(a))^{\alpha_1}\big)\Bigg)\Bigg]\\
& +\frac{e^{\frac{\rho-1}{\rho}\psi(a)}\Psi_{\psi}^{\alpha_1+\frac{12}{5}}(\tau,a)}{\rho^{\alpha_1}}\Gamma\left(\frac{17}{5}\right)\mathbb{E}_{\alpha_1,\alpha_1+\frac{17}{5}}\big(\lambda_1\rho^{-\alpha_1}(\psi(\tau)-\psi(a))^{\alpha_1}\big),
\end{aligned} \tag{4.4}
$$

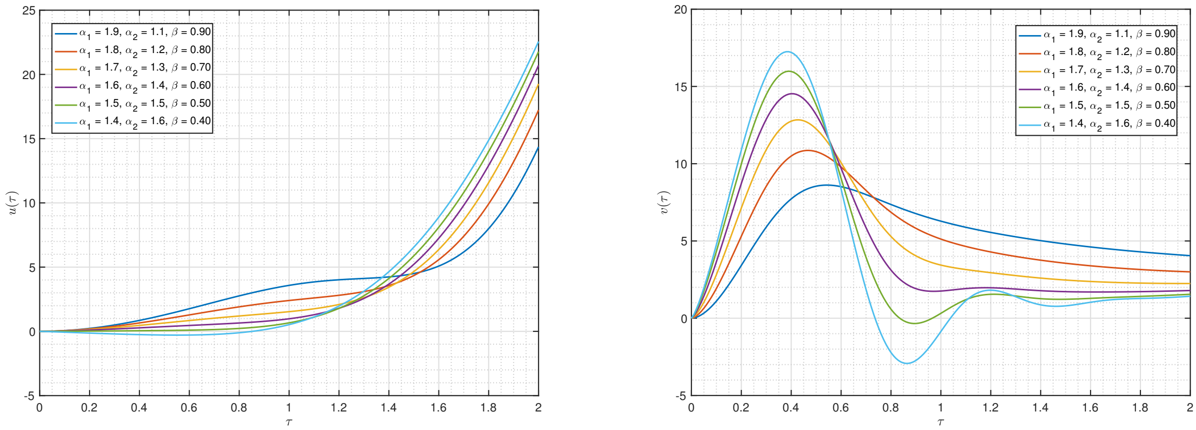
$$
\begin{aligned}
v(\tau) = & \left[\frac{e^{\frac{\rho-1}{\rho}\psi(a)}\Psi_{\psi}^{\gamma_2-1}(\tau,a)}{\rho^{\gamma_2-1}(\Theta_1\Theta_2-\Delta_1\Delta_2)}\mathbb{E}_{\alpha_2,\gamma_2}\big(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(a))^{\alpha_2}\big)\right]\\
& \times\Bigg[\Delta_1\Bigg(\frac{\Psi_{\psi}^{\alpha_2+\frac{17}{10}}(b,a)}{\rho^{\alpha_2}}\Gamma\left(\frac{27}{10}\right)\mathbb{E}_{\alpha_2,\alpha_2+\frac{27}{10}}\big(\lambda_2\rho^{-\alpha_2}(\psi(b)-\psi(a))^{\alpha_2}\big)\\
& -\sum_{j=1}^{n}\frac{\mu_j\Psi_{\psi}^{\varrho_j+\alpha_1+\frac{12}{5}}(\eta_j,a)}{\rho^{\varrho_j+\alpha_1}}\Gamma\left(\frac{17}{5}\right)\mathbb{E}_{\alpha_1,\varrho_j+\alpha_1+\frac{17}{5}}\big(\lambda_1\rho^{-\alpha_1}(\psi(\eta_j)-\psi(a))^{\alpha_1}\big)\Bigg)\\
& -\Theta_1\Bigg(\sum_{i=1}^{m}\frac{\theta_i\Psi_{\psi}^{\delta_i+\alpha_2+\frac{17}{10}}(\xi_i,a)}{\rho^{\delta_i+\alpha_2}}\Gamma\left(\frac{27}{10}\right)\mathbb{E}_{\alpha_2,\delta_i+\alpha_2+\frac{27}{10}}\big(\lambda_2\rho^{-\alpha_2}(\psi(\xi_i)-\psi(a))^{\alpha_2}\big)\\
& -\frac{\Psi_{\psi}^{\alpha_1+\frac{12}{5}}(b,a)}{\rho^{\alpha_1}}\Gamma\left(\frac{17}{5}\right)\mathbb{E}_{\alpha_1,\alpha_1+\frac{17}{5}}\big(\lambda_1\rho^{-\alpha_1}(\psi(b)-\psi(a))^{\alpha_1}\big)\Bigg)\Bigg]\\
& +\frac{e^{\frac{\rho-1}{\rho}\psi(a)}\Psi_{\psi}^{\alpha_2+\frac{17}{10}}(\tau,a)}{\rho^{\alpha_2}}\Gamma\left(\frac{27}{10}\right)\mathbb{E}_{\alpha_2,\alpha_2+\frac{27}{10}}\big(\lambda_2\rho^{-\alpha_2}(\psi(\tau)-\psi(a))^{\alpha_2}\big).
\end{aligned} \tag{4.5}
$$

Simulations of solutions (4.4) and (4.5) for the proposed coupled system (4.1) via a variety of functions $\psi(t)$, α_1, α_2, and β are shown in Figures 1–4. In Figure 1, we obtain that if we set $\psi(\tau) = 4^{\beta} - 2e^{-\frac{\alpha_1}{\alpha_2}\tau^2}$ and the values of α_1 increases from 1.4 to 1.9, α_2 decreases from 1.6 to 1.1, and β increases from 0.40 to 0.90, the corresponding graphs of $u(\tau)$ increases (Figure 1a), while $v(\tau)$ oscillates and tends to be stable over the long term (Figure 1b). In Figure 2, we obtain that if we set $\psi(\tau) = \alpha_2\sqrt{\tau^{\alpha_1+\alpha_2+\beta}}$ and the values of α_1 increases from 1.4 to 1.9, α_2 decreases from 1.6 to 1.1, and β increases from 0.40 to 0.90, the corresponding graphs of $u(\tau)$ increases (Figure 2a) while $v(\tau)$ oscillates and tends to be stable over the long term (Figure 2b). In Figure 3, we obtain that if we set $\psi(\tau) = \log_2(\alpha_1\tau^{\alpha_2} + \beta)$ and the values of α_1 increases from 1.4 to 1.9, α_2 decreases from 1.6 to 1.1, and β increases from 0.40 to 0.90, the corresponding graphs of $u(\tau)$ increases (Figure 3a), while $v(\tau)$ oscillates and tends to be stable over the long term (Figure 3b). In Figure 4, we obtain that if we set $\psi(\tau) = \left|2\sin^2\left(\frac{\sqrt{(\pi\tau)}}{2(\alpha_1+\alpha_2+\beta)}\right)\right|$ and the values of α_1 increases from 1.4 to 1.9, α_2 decreases from 1.6 to 1.1, and β increases from 0.40 to 0.90, the corresponding graphs of $u(\tau)$ (Figure 4a) and $v(\tau)$ (Figure 4a) decrease and increase, and tend to be stable over the long term (Figure 4b).



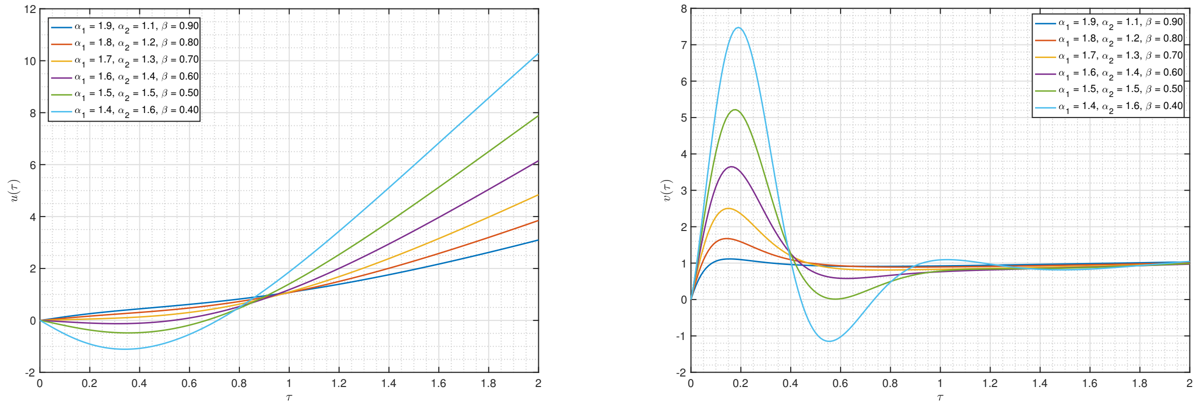
(a) **(b)**

Figure 1. Graphical simulations $u(\tau)$ and $v(\tau)$ of Example 4.1 via $\psi(\tau) = 4^{\beta} - 2e^{-\frac{\alpha_1}{\alpha_2}\tau^2}$ when α_1, α_2, and β are varied.



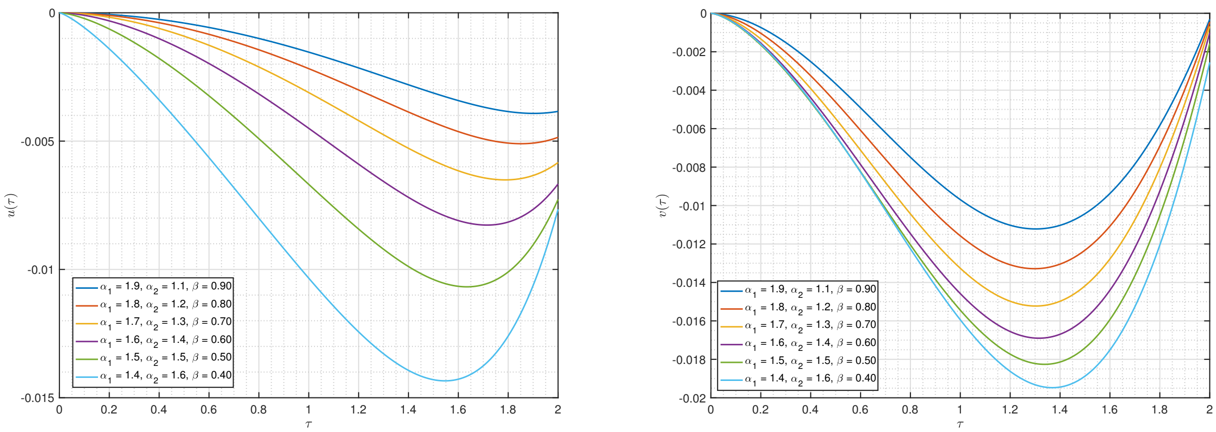
(a) **(b)**

Figure 2. Graphical simulations $u(\tau)$ and $v(\tau)$ of Example 4.1 via $\psi(\tau) = \alpha_2\sqrt{\tau^{\alpha_1+\alpha_2+\beta}}$ when α_1, α_2, and β are varied.



(a) **(b)**

Figure 3. Graphical simulations $u(\tau)$ and $v(\tau)$ of Example 4.1 via $\psi(\tau) = \log_2(\alpha_1\tau^{\alpha_2} + \beta)$ when α_1, α_2, and β are varied.



(a) **(b)**

Figure 4. Graphical simulations $u(\tau)$ and $v(\tau)$ of Example 4.1 via $\psi(\tau) = \left|2\sin^2\left(\frac{\sqrt{(\pi\tau)}}{2(\alpha_1+\alpha_2+\beta)}\right)\right|$ when α_1, α_2, and β are varied.

5. Conclusions

Finding the qualitative properties of differential equations under fractional-order is one of the best challenging tasks in the theory of these equations. Integral inequalities are the major keys that help investigate their properties. In this paper, we developed and established a novel extended coupled Gronwall inequality in the framework of ψ-HPFOs. We studied the exactness and applicability of the main results by analyzing the uniqueness of solutions to the nonlinear coupled ψ-HPF Cauchy-type system by applying fixed point theory of Banach's type with some properties of the ML functions. Additionally to what has been finished in this paper, one can utilize the established extended coupled Gronwall inequality to establish certain additional qualitative properties for these solutions, such as ML-UH stability, ML-UHR stability, and their generalization. Finally, a numerical example is provided to show the effectiveness and practicability of our theoretical criteria.

Furthermore, our main results are not only innovative in the framework of the proposed coupled system at hand, but they also provide some novel specific circumstances by modifying the factors involved. The benefit of this work is that, with a single work, we may decide on numerous scenarios based on the parameter values ρ, k, α, β, and another function $\psi(\tau)$ without having to resort to another investigation. We expect that the concepts offered here will positively inspire further applications in applied sciences and engineering in the stated direction. It is important to note that the ψ-HPFDO is the most generalized form of the HPFDO [11] and the ψ-HFDO [8].

Author contributions

Weerawat Sudsutad: Problem statement, conceptualization, methodology, investigation, writing the original draft, writing, reviewing, and editing; Jutarat Kongson: Methodology, investigation, writing, reviewing, and editing; Chatthai Thaiprayoon: Supervision, methodology, investigation, writing, reviewing, and editing; Nantapat Jarasthitikulchai, Marisa Kaewsuwan: Investigation, writing, reviewing, and editing. All the authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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