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Research article

Rényi entropy of past lifetime from lower k-record values

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Abstract: This paper explored the concept of past Rényi entropy within the context of k-record values. We began by introducing a representation of the past Rényi entropy for the n-th lower k-record values, sampled from any continuous distribution function F, concerning the past Rényi entropy of the n-th lower k-record values sampled from a uniform distribution. Then, we delved into the examination of the monotonicity properties of the past Rényi entropy of k-record values. Specifically, we focused on the aging properties of the component lifetimes and investigated how they impacted the monotonicity of the past Rényi entropy. Additionally, we derived an expression for the n-th lower k-records in terms of the past Rényi entropy, specifically when the first lower k-record was less than a specified threshold level, and then investigated several properties of the given formula.

Keywords: k-record values; Rényi entropy; past Rényi entropy; stochastic orders
Mathematics Subject Classification: 94A08, 62P99

1. Introduction

Record values play a crucial role in various fields of application and have been extensively researched. Several books, such as Ahsanullah [1], Arnold et al. [2], and Nevzorov [3], focus on the theory and applications of record values. In athletics, temperature, wind velocity, etc., researchers often rely on available record data to address statistical inference problems related to the parent distribution, but making inferences from records is challenging due to the records occurring rarely in real life situations. The expected waiting time for each subsequent record after the first observation is infinite. To address this issue, k-records, introduced by Dziubdziela and Kopociński [4], can be utilized as they occur more frequently than traditional records. Consider the first 10 observations from David and Nagaraja [5]: 0.464, 0.060, 1.486, 1.022, 1.394, 0.906, 1.179, -1.501, -0.690, and 1.372. The records observed from the data are 0.464 and 1.486. However, one can construct upper k-records from the data, which can be seen in Table 1.

Table 1. Sequences of k-records for $k = 2, 3, 4$.

2-Records	0.060	0.464	1.022	1.394	
3-Records	0.060	0.464	1.022	1.179	1.372
4-Records	0.060	0.464	0.906	1.022	1.179

Consider a sequence of independent and identically distributed (iid) random variables (RVs) $\{X_i, i \geq 1\}$, with a cumulative distribution function (cdf) $F(x)$ and a probability density function (pdf) $f(x)$. It is assumed throughout the paper that the RVs X_i are nonnegative . For a fixed positive integer $k \geq 1$, the nth lower k-record time, denoted as $Z_{n(k)}$ for the sequence $\{X_i, i \geq 1\}$, is defined as follows:

$$Z_{1(k)} = 1,\ Z_{n+1(k)} = \min\{j : j > Z_{n(k)}, X_{k:Z_{n(k)}+k-1} > X_{k:j+k-1}\},\ n = 1, 2, \ldots.$$

Here, the j-th order statistic in a sample of size m is denoted as $X_{j:m}$. Using this notation, the k-th lower record value in the sequence $\{X_i\}, i \geq 1$, is defined as $\mathfrak{L}_{n(k)} = X_{k:Z_{n(k)}+k-1}$. The pdf and cdf of $\mathfrak{L}_{n(k)}$ are commonly denoted as $f_{n(k)}(x)$ and $F_{n(k)}(x)$, respectively, which are shown as follows:

$$f_{n(k)}(x) = \frac{k^n}{\Gamma(n)}[F(x)]^{k-1}[-\log F(x)]^{n-1}f(x),\ x > 0, \tag{1.1}$$

$$F_{n(k)}(x) = [F(x)]^k \sum_{i=0}^{n-1} \frac{[-k\log F(x)]^i}{i!} = \frac{\Gamma(n, -k\log F(x))}{\Gamma(n)},\quad x \geq 0, \tag{1.2}$$

where

$$\Gamma(a, x) = \int_x^\infty u^{a-1}e^{-u}du,\ a, x > 0, \tag{1.3}$$

denotes the upper incomplete gamma function and the complete gamma function is denoted as $\Gamma(\cdot)$ (refer to [6–8] for more information). We denote $\mathcal{V} \sim \Gamma_t(\alpha,\beta)$ to represent a random variable V following a truncated Gamma distribution. The pdf of this distribution is given by:

$$f_{\mathcal{V}}(v) = \frac{\beta^\alpha}{\Gamma(\alpha, t)}v^{\alpha-1}e^{-\beta v},\ v > t, \tag{1.4}$$

where $\alpha > 0$ and $\beta > 0$.

Since Shannon [9] introduced a measure of uncertainty for discrete distributions based on the Boltzmann entropy, there has been significant interest in quantifying uncertainty associated with probability distributions. Numerous studies have explored the Shannon measure of uncertainty and its applications. Consider a nonnegative RV X characterized by its pdf denoted as $f(x)$. Rényi [10] introduced an entropy measure of order γ for the RV X, defined as follows:

$$\mathcal{H}_\gamma(X) = c(\gamma)\log\int_0^\infty f^\gamma(x)dx,\ (\gamma > 0, \gamma \neq 1). \tag{1.5}$$

Here, the expression $\log(\cdot)$ represents the natural logarithm, and $c(\gamma) = \frac{1}{1-\gamma}$. The Shannon differential entropy can be obtained as $\mathcal{H}(X) = \lim_{\gamma\to 1}\mathcal{H}_\gamma(X) = -\mathbb{E}[\log f(X)]$. It is noteworthy that the Rényi entropy serves as a measure of the uniformity of a density function. Higher values of the Rényi entropy indicate increased uncertainty in the density function f and a reduced ability to predict future outcomes of X (see, e.g., Ebrahimi et al. [11]). The Rényi entropy is a one-parameter generalization of Shannon entropy that offers greater flexibility and has diverse applications across various fields. Its versatile nature is evident in areas such as in communication and coding theory Csiszár [12], data mining, detection, segmentation, classification Neemuchwala et al. [13], hypothesis testing Molina and Morales [14], characterization of signals and sequences Vinga and Almeida [15], signal processing Basseville [16], and image matching and registration [13] and the references therein. The presented applications underscore the broad applicability of Rényi entropy and highlight its crucial role in addressing various challenges in the literature.

When assessing the lifetime of a new system, the Rényi entropy $\mathcal{H}_\gamma(X)$ can be a very useful measure for quantifying uncertainty. However, in some cases, it becomes necessary to assess the uncertainty associated with the residual lifetime of the system, denoted as $[X-t|X>t]$. This refers to the situation where the system's lifetime extends beyond a given time t, and it raises questions about the remaining uncertainty. In this case, the concept of residual Rényi entropy has been introduced in the literature as follows (for further details, refer to Gupta and Nanda [17]):

$$\mathcal{H}_\gamma(X;t) = c(\gamma)\log\int_t^\infty \left(\frac{f(x)}{S(t)}\right)^\gamma dx, \tag{1.6}$$

where $S(t) = P(X > t)$ stands for the survival function of X. The notion of $\mathcal{H}\gamma(X;t)$ has garnered significant interest among researchers across diverse scientific and engineering fields. It serves as a generalization of the classical residual Shannon differential entropy, encompassing a broad spectrum of properties and applications. Scholars such as [17–19], and others have extensively investigated the properties of $\mathcal{H}_\gamma(X;t)$. In a recent study, [20] explored the residual lifetime of a coherent system using the concept of Rényi entropy.

Past and future uncertainties are inherent in real-world systems. The notion of past entropy captures the uncertainty associated with previous events, while residual entropy quantifies uncertainty regarding future events. The concept of past time or inactivity time is very important because it can be employed in describing stochastic processes with real-world applications. This concept refers to the time elapsed after an event (such as product failure, task completion, or automobile accident) until the observation or reporting occurs, by knowing that the event happens at or before the observation time t. In forensic and actuarial science, the time elapsed since failure is important for predicting the actual time of failure. So, the study of past entropy and its statistical applications has garnered attention in the literature. The works of Di Crescenzo and Longobardi [21], Nair and Sunoj [22], and Gupta et al. [23] have made substantial contributions to the understanding of the properties and applications of past entropy in the domain of order statistics. Consider an RV X which represents the lifetime of a system. Recall that the pdf of $X_t = [X|X < t]$ is expressed as $f_t(x) = \frac{f(x)}{F(t)}$, where $0 < x < t$. This pdf provides the conditional probability distribution of X given that it is less than t, with $F(t)$ representing the cdf of X. In this context, we recall the concept of past Rényi entropy (PRE) at time t for X, defined for all $\gamma > 0$ as follows:

$$\overline{\mathcal{H}}_\gamma(X_t) = c(\gamma)\log\int_0^t f_t^\gamma(x)dx = c(\gamma)\log\int_0^t \left(\frac{f(x)}{F(t)}\right)^\gamma dx \tag{1.7}$$

$$= c(\gamma)\log\left[\int_0^1 f_t^{\gamma-1}(F_t^{-1}(u))du\right],\ t>0, \tag{1.8}$$

where $F_t^{-1}(u) = \inf\{x; F_t(x) \geq u\}$ stands for the quantile function of $F_t(x) = F(x)/F(t),\ 0 \leq x \leq t$ and $U \sim U(0, 1)$ for all $\gamma > 0$. It is important to note that the PRE $\overline{\mathcal{H}}\gamma(X_t)$ lies within the range of $[-\infty, \infty]$. When an item is observed to fail at time t, the PRE $\overline{\mathcal{H}}\gamma(X_t)$ quantifies the level of uncertainty regarding its past lifetime.

The study of record values, pioneered by Chandler [24], has gained significant attention across various practical domains. Notably, Glick [25] provides an example involving the breaking strength of wooden beams. For a comprehensive understanding of the theory and application of record values, refer to Ahsanullah [1], Arnold et al. [2], and the referenced sources therein. Statistical inference based on record data poses significant challenges due to the rarity of record occurrences and the infinite expected waiting time for subsequent records after the first one. For instance, in actuarial science, when analyzing insurance claims in nonlife insurance (as discussed in Kamps [26]). To address these challenges, the model of k-record statistics, introduced by Dziubdziela and Kopociński [4], offers a suitable alternative.

Numerous researchers have examined various information properties of record values. In this case, Zarezadeh and Asadi [27] investigated properties of Rényi entropy for order statistics and record values. Habibi et al. [28] examined Kullback-Leibler information of such records, while Abbasnejad and Arghami [29] focused on Rényi information. Baratpur et al. [30] studied information properties of records using Shannon entropy and mutual information, providing entropy bounds. Jose and Sathar [31] discussed Rényi entropy and important properties of k-records from continuous distributions. They also proposed a simple estimator and demonstrated applications using real-life data. Asha and Chacko [32] explored properties of PRE for k-record values from absolutely continuous distributions. Shrahili and Kayid [33] investigated the residual Tsallis entropy of lower record values from iid RVs. Building upon this prior work, our paper further investigates and presents detailed results on past Rényi entropy of k-records from continuous distributions.

This paper's findings are structured as follows: In Section 2, we first present a representation of the PRE of the n-th lower k-record values sampled from any continuous distribution function F, in terms of the PRE of the n-th lower k-record values sampled from a uniform distribution. We investigate various results including the monotonicity and aging properties of the proposed measure. In Section 3, we delve into the examination and derivation of different properties of past Rényi entropy for the n-th lower k-records, where the first k-record is less than a specified threshold level. Finally, in Section 4, we provide the conclusion of the paper.

Throughout this paper, we consider nonnegative RVs denoted by X and Y. These variables have absolutely cdfs denoted by $F(x)$ and $G(x)$ and pdfs denoted by $f(x)$ and $g(x)$, respectively. The terms "increasing" and "decreasing" are used in a non-strict sense.

2. Results on past Rényi entropy of k-record values

In this section, we concentrate on exploring the past Rényi entropy of the RV $\mathfrak{L}_{n(k)}$. This measure quantifies the uncertainty inherent in the density of $[t - \mathfrak{L}_{n(k)}|\mathfrak{L}_{n(k)} \leq t]$ and provides insights into the predictability of past lifetimes for the n-th lower k-records. It allows us to assess the predictability of the system's inactivity time. To enhance computational efficiency, we introduce a lemma that establishes a formula connecting the PRE of order statistics in the uniform case with the imperfect beta function. This connection is of practical significance as it simplifies the calculation of the PRE. We omit the proof of this lemma, as it involves straightforward computations based on the definition of the PRE.

Lemma 2.1. *Let $\{U_i, i = 1, 2, \ldots\}$ be a sequence of iid RVs adopted from a uniformly distributed population. In addition, let $\mathfrak{L}^{\star}_{n(k)}$ denote the n-th lower k-records of $\{U_i, i = 1, 2, \ldots\}$. Then*

$$\overline{\mathcal{H}}_\gamma(\mathfrak{L}^{\star}_{n(k)}; t) = c(\gamma) \log \frac{k^{n\gamma}\Gamma(\gamma(n-1)+1, -(\gamma(k-1)+1)\log t)}{(\gamma(k-1)+1)^{\gamma(n-1)+1}\Gamma^\gamma(n, -k\log t)}, \; 0 < t < 1.$$

Proof. Substituting the pdf and cdf of uniform distributions from Eq (1.1) into Eq (1.7) yields the following expression:

$$\begin{aligned}
\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)}; t) &= c(\gamma) \log \int_0^t \left(\frac{k^n u^{k-1}[-\log u]^{n-1}}{\Gamma(n, -k\log t)}\right)^\gamma du \\
&= c(\gamma) \log \frac{k^{n\gamma}}{\Gamma^\gamma(n, -k\log t)} \int_0^t u^{\gamma(k-1)}[-\log u]^{\gamma(n-1)} du.
\end{aligned}$$

By substituting $z = -\log u$, we arrive at the following expression:

$$\begin{aligned}
\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)}; t) &= c(\gamma) \log \frac{k^{n\gamma}}{\Gamma^\gamma(n, -k\log t)} \int_{-\log t}^{\infty} z^{\gamma(n-1)} e^{-(\gamma(k-1)+1)z} dz \\
&= c(\gamma) \log \frac{k^{n\gamma}}{(\gamma(k-1)+1)^{\gamma(n-1)+1}\Gamma^\gamma(n, -k\log t)} \int_{-(\gamma(k-1)+1)\log t}^{\infty} x^{\gamma(n-1)} e^{-x} dx \\
&= c(\gamma) \log \frac{k^{n\gamma}\Gamma(\gamma(n-1)+1, -(\gamma(k-1)+1)\log t)}{(\gamma(k-1)+1)^{\gamma(n-1)+1}\Gamma^\gamma(n, -k\log t)}, \; 0 < t < 1.
\end{aligned}$$

The second equality follows from the change of variable $x = (\gamma(k-1)+1)z$ while the last equality is derived using Eq (1.3) and completes the proof. □

The utilization of the widely recognized imperfect gamma function, facilitated by the lemma presented, allows researchers to effortlessly calculate the PRE of record values from a uniform distribution. It improves the usability and applicability of PRE in different scenarios. The forthcoming theorem will establish the connection of the PRE of n-th lower k-record values $\mathfrak{L}_{n(k)}$ and the PRE of n-th lower k-record values derived from a uniform distribution.

Theorem 2.1. *Consider a sequence of iid RVs $\{X_i\}, i \geq 1$, having the common cdf F and pdf f. Assume that $\mathfrak{L}_{n(k)}$ is the n-th lower record value of the sequence $\{X_i\}$. The past Rényi entropy of $\mathfrak{L}_{n(k)}$ is obtained as:*

$$\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)}; t) = \overline{\mathcal{H}}_\gamma(\mathfrak{L}^{\star}_{n(k)}; F(t)) + c(\gamma) \log \mathbb{E}[f^{\gamma-1}(F^{-1}(e^{-V_{n(k)}}))], \; t \in (0, +\infty), \tag{2.1}$$

so that $V_{n(k)} \sim \Gamma_{-\log F(t)}(\gamma(n-1)+1, \gamma(k-1)+1)$.

Proof. By employing a change of variable $u = F(x)$, and using Eqs (1.1), (1.2), and (1.7), we obtain the following expression:

$$\begin{aligned}
\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)}; t) &= c(\gamma) \log \int_0^t \left(\frac{f_{\mathfrak{L}_{n(k)}}(x)}{F_{\mathfrak{L}_{n(k)}}(t)}\right)^\gamma dx \\
&= c(\gamma) \log \int_0^t \left(\frac{k^n F^{k-1}(x)[-\log F(x)]^{n-1} f(x)}{\Gamma(n, -k\log F(t))}\right)^\gamma dx
\end{aligned}$$

$$= c(\gamma)\log\frac{k^{n\gamma}}{\Gamma^{\gamma}(n,-k\log F(t))}\int_0^t F^{\gamma(k-1)}(x)[-\log F(x)]^{\gamma(n-1)}f^{\gamma}(x)dx$$
$$= c(\gamma)\log\frac{k^{n\gamma}}{\Gamma^{\gamma}(n,-k\log F(t))}\int_0^{F(t)} u^{\gamma(k-1)}(-\log u)^{\gamma(n-1)}f^{\gamma-1}(F^{-1}(u))du.$$

Applying the transformation $z = -\log u$, we obtain the following expression:

$$\begin{aligned}\overline{\mathcal{H}}_{\gamma}(\mathfrak{L}_{n(k)};t) &= c(\gamma)\log\frac{k^{n\gamma}}{\Gamma^{\gamma}(n,-k\log F(t))}\int_{-\log F(t)}^{\infty} z^{\gamma(n-1)}e^{-(\gamma(k-1)+1)z}f^{\gamma-1}(F^{-1}(e^{-z}))dz \\ &= c(\gamma)\log\frac{k^{n\gamma}\Gamma(\gamma(n-1)+1,-(\gamma(k-1)+1)\log F(t))}{(\gamma(k-1)+1)^{\gamma(n-1)+1}\Gamma^{\gamma}(n,-k\log F(t))} \\ &+ c(\gamma)\log\int_{-\log F(t)}^{\infty}\frac{(\gamma(k-1)+1)^{\gamma(n-1)+1}z^{\gamma(n-1)}e^{-(\gamma(k-1)+1)z}f^{\gamma-1}(F^{-1}(e^{-z}))}{\Gamma(\gamma(n-1)+1,-(\gamma(k-1)+1)\log F(t))}dz \\ &= \overline{\mathcal{H}}_{\gamma}(\mathfrak{L}^{\star}_{n(k)};F(t)) + c(\gamma)\log\mathbb{E}[f^{\gamma-1}(F^{-1}(e^{-V_{n(k)}}))],\ t>0. \end{aligned} \tag{2.2}$$

The proof is completed upon recalling Lemma 2.1 and, hence, the theorem. □

Let us consider an example to illustrate that not all distributions are monotone in terms of $\overline{\mathcal{H}}_{\gamma}(\mathfrak{L}_{n(k)};t)$.

Example 2.1. Consider a nonnegative random variable X, characterized by the following cdf:

$$F(x) = \begin{cases} \exp\{-\frac{1}{2}-\frac{1}{x}\}, & \text{if } 0 \le x < 1 \\ \exp\{-2+\frac{x^2}{2}\}, & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \ge 2. \end{cases} \tag{2.3}$$

It is challenging to derive an explicit expression for $\overline{\mathcal{H}}\gamma(\mathfrak{L}_{n(k)};t)$, and, as a result, we are compelled to lean on numerical computation procedures.

We have plotted the relationship between $\overline{\mathcal{H}}\gamma(\mathfrak{L}_{n(k)};t)$ and t for the case where $n = 5$, which is shown in Figure 1. We considered various values of k ranging from 1 to 5 and chose $\gamma = 0.5$ and $\gamma = 5$ as examples. The graph clearly demonstrates that $\overline{\mathcal{H}}\gamma(\mathfrak{L}_{n(k)};t)$ is not a monotonic function for all values of γ, as depicted in Figure 1.

The forthcoming theorem presents a significant result regarding the monotonicity of the PRE for k-record values, assuming that the underlying random variable X exhibits the property of decreasing reversed hazard rate (DRHR). In particular, we recall that a random variable X is said to possess DRHR if the hazard ratio $\tau(x) = f(x)/F(x)$ decreases monotonically for all $x > 0$. This analysis provides new senses into the demeanor of PRE in the context of DRHR, contributing to a deeper understanding of this important class of stochastic notion. For references on the DRHR properties of record values, k-record values, and generalized order statistics, the readers are refereed to, e.g., Ahmadi and Balakrishnan [34], Wang and Zhao [35], and Zamani and Madadi [36] and the references therein.

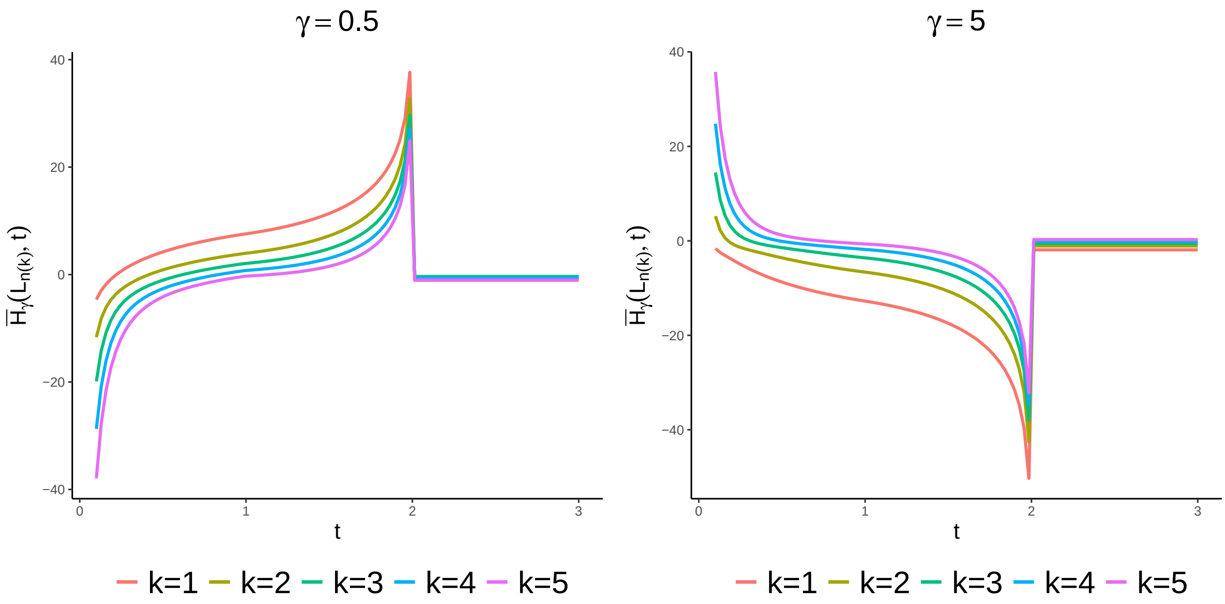

Figure 1. Graphical representation of $\overline{\mathcal{H}}\gamma(\mathfrak{L}_{n(k)}; t)$ as a function of t for different values of γ.

Theorem 2.2. *Let X be DRHR. Thus, for all* $\gamma > 0$*, the function* $\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)}; t)$ *is increasing in t.*

Proof. By utilizing the expressions given in (1.1) and (1.2), we can express the RHR function of $\mathfrak{L}_{n(k)}$ as follows:

$$\tau_{n(k)}(t) = \frac{f_{n(k)}(t)}{F_{n(k)}(t)} = \zeta_{n(k)}(t)\tau(t),\ \ t > 0, \tag{2.4}$$

where

$$\zeta_{n(k)}(t) = \frac{k^n[-\log F(t)]^{n-1}/\Gamma(n)}{\sum_{i=0}^{n-1}\frac{[-k\log F(t)]^i}{i!}}. \tag{2.5}$$

It is evident that the function $\zeta_{n(k)}(t)$ is monotonically decreasing with respect to t. Therefore, under the assumption that X possesses the DRHR property, we can infer that $\mathfrak{L}_{n(k)}$ also exhibits the DRHR property. As a result, we can conclude that the PRE $\overline{\mathcal{H}}\gamma(\mathfrak{L}_{n(k)}; t)$, for all $\gamma > 0$, is increasing in t, in line with the findings presented in Theorem 1 of Kayid and Shrahili [37]. This confirms the favored result and ends the proof. □

The subsequent example indicates the utility of Theorem 2.2.

Example 2.2. Consider a sequence of iid RVs $\{X_i\}, i \geq 1$, where each X_i follows a Fréchet distribution with the cdf given by:

$$F(x) = e^{-x^{-3}},\ \ x > 0. \tag{2.6}$$

By observing that $F^{-1}(u) = (-\log u)^{-1/3}$ for $0 < u < 1$, we can proceed to calculate as

$$\mathbb{E}[f^{\gamma-1}(F^{-1}(e^{-V_{n(k)}}))] = \frac{3^{\gamma-1}(\gamma(k-1)+1)^{\gamma(n-1)+1}\Gamma(\gamma(n+\frac{1}{3})-\frac{1}{3}, \gamma k t^{-3})}{(\gamma k)^{\gamma(n+\frac{1}{3})-\frac{1}{3}}\Gamma(\gamma(n-1)+1, t^{-3}(\gamma(k-1)+1))},$$

and

$$\overline{\mathcal{H}}_\gamma(\mathfrak{L}^{\star}_{n(k)}; F(t)) \ = \ c(\gamma)\log\frac{k^{n\gamma}\Gamma(\gamma(n-1)+1, t^{-3}(\gamma(k-1)+1)))}{(\gamma(k-1)+1)^{\gamma(n-1)+1}\Gamma^{\gamma}(n, kt^{-3})}.$$

Using (2.1), we get

$$\overline{\mathcal{H}}_{\gamma}(\mathfrak{L}_{n(k)};t) = c(\gamma)\log\left[\frac{3^{\gamma-1}k^{n\gamma}\Gamma(\gamma(n+\frac{1}{3})-\frac{1}{3},\gamma kt^{-3})}{(\gamma k)^{\gamma(n+\frac{1}{3})-\frac{1}{3}}\Gamma^{\gamma}(n,kt^{-3})}\right],\ n \geq 1. \tag{2.7}$$

In order to examine the behavior of the PRE $\overline{\mathcal{H}}\gamma(\mathfrak{L}_{n(k)};t)$, we focus on the case where $n = 5$. We plot $\overline{\mathcal{H}}\gamma(\mathfrak{L}_{n(k)};t)$ as a function of t, considering different values of $k = 1, 2, \cdots, 5$ and choosing $\gamma = 0.5$ and $\gamma = 2$. The resulting plots are depicted in Figure 2. The observed trends in the plots align with the findings of Theorem 2.2, which establishes that the PRE decreases as t increases when the random variable X exhibits the DRHR property.

The subsequent theorem presents a significant result regarding the closure property of increasing PRE of distributions when forming lower k-record values.

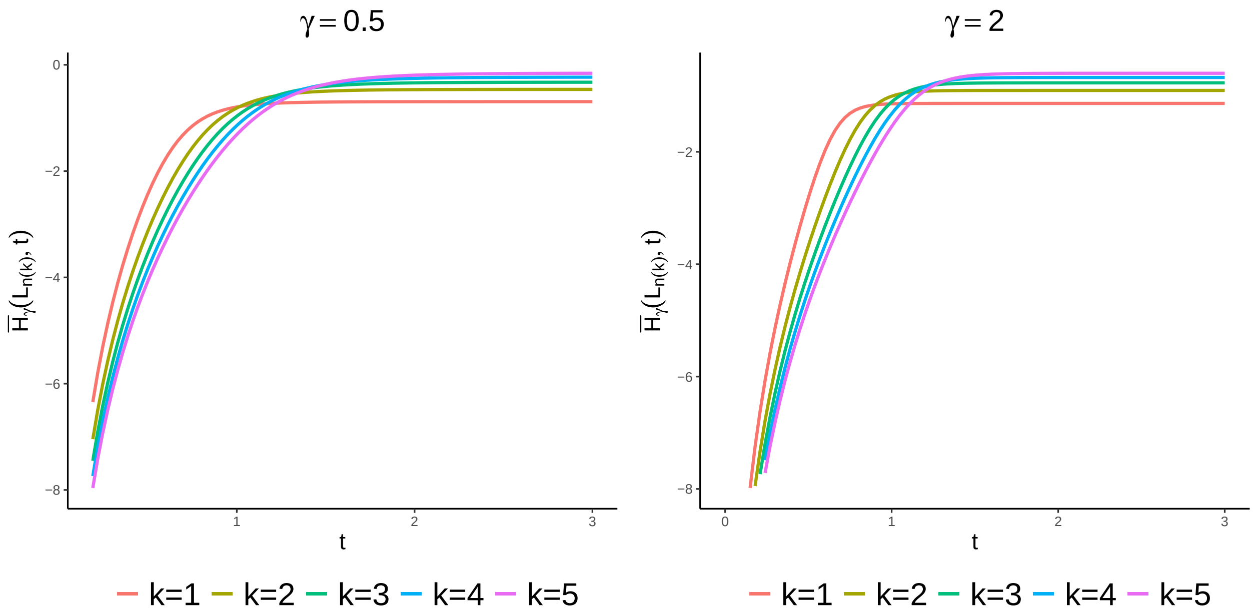

Figure 2. The graph of $\overline{\mathcal{H}}_{\gamma}(\mathfrak{L}_{n(k)};t)$ for $\gamma = 0.5$ (left panel) and $\gamma = 2$ (right panel) as a function of t.

Theorem 2.3. *Assuming that $\overline{\mathcal{H}}_{\gamma}(X;t)$ is increasing in t, then $\overline{\mathcal{H}}_{\gamma}(\mathfrak{L}_{n(k)};t)$ is also increasing in t for all $\gamma > 0$.*

Proof. By referring to Eq (2.4), we observe that the RHR function of $\mathfrak{L}_{n(k)}$ can be written as $\tau_{n(k)}(t) = \zeta_{n(k)}(t)\tau(t)$, where $\zeta_{n(k)}(t)$ is defined in (2) and $t > 0$. It is clear that the values of $\zeta_{n(k)}(t)$ decrease as t increases and takes values within the range $(0, 1)$. Additionally, we readily notice that

$$\lim_{t\to\infty}\frac{F_{n(k)}(t)}{F(t)} = 0.$$

Consequently, the assumptions of Theorem 3.1 of Mahmoudi and Asadi [38] are satisfied, establishing that $\overline{\mathcal{H}}_{\gamma}(\mathfrak{L}_{n(k)};t)$ increases for all $\gamma > 0$ concerning t. □

The subsequent example indicates the utility of Theorem 2.3.

Example 2.3. Let us consider a sequence of iid RVs $\{X_i\}, i \geq 1$ that follow a beta distribution with cdf $F(x) = x^2,\ 0 < x < 1$. It is straightforward to observe that

$$\overline{\mathcal{H}}_\gamma(X;t) = \frac{\gamma}{1-\gamma}\log\frac{2}{\gamma+1} + \log t, \quad 0 < t < 1,$$

which increases with increasing t. Additionally, employing (2.1), we can derive the following expression

$$\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)};t) = c(\gamma)\log\frac{(2k)^{n\gamma}\Gamma(\gamma(n-1)+1, -(\gamma(2k-1)+1)\log t))}{(\gamma(2k-1)+1)^{\gamma(n-1)+1}\Gamma^\gamma(n, -2k\log t)},\ 0 < t < 1.$$

To investigate the behavior of the PRE $\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)};t)$, we consider the case where $n = 5$. We plot the graph of $\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)};t)$ for different values of $\gamma = 0.5$ and $\gamma = 2$, while varying t, and considering different values of $k = 1, 2, \cdots, 5$. The resulting plots are presented in Figure 3. It shows that the PRE decreases as t increases.

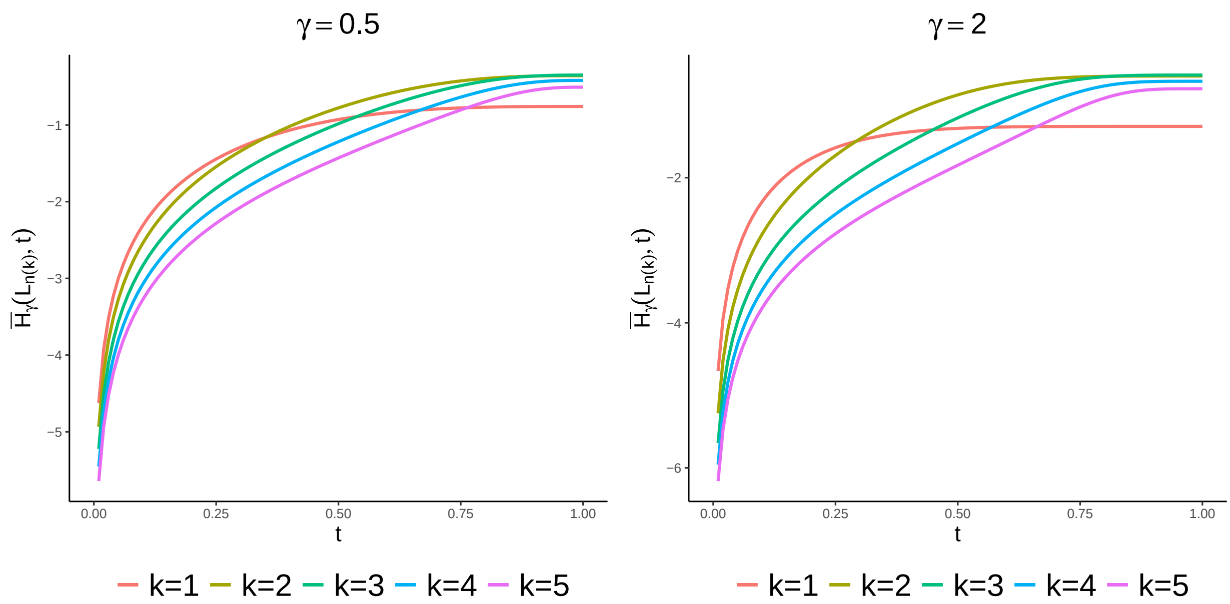

Figure 3. The graph of $\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)};t)$ for $\gamma = 0.5$ (left panel) and $\gamma = 2$ (right panel) as a function of t.

Several researchers, including Kochar [39], Raqab and Amin [40, 41], Khaledi [42], and Khaledi and Shojaei [43], have investigated stochastic comparisons of upper record values. The preservation properties of the n-th lower k-record values under the DRHR property were investigated by Kundu et al. [44]. They have shown that if the n-th upper k-record values exhibit the DRHR property, then the $(n-1)$-th upper k-record values also possess the DRHR property. Moreover, they demonstrated that if the n-th lower k-record has DRHR property, then the n-th lower l-record has also DRHR property for $l > k$. Now, we obtain some results for the n-th lower k-record values and then provide some results on the monotone property of PRE. The following theorem shows that DRHR property passes from $\mathfrak{L}_{n-1(k)}$ to $\mathfrak{L}_{n(k)}$.

Theorem 2.4. *If $\mathfrak{L}_{n-1(k)}$ is DRHR, then $\mathfrak{L}_{n(k)}$ is also DRHR.*

Proof. Consider the function $\phi(t) = -\log F(t) \geq 0$, for all $t > 0$. Then, from Eq (2.4), one can write $\tau_{n(k)}(t) = \theta(t)\tau_{n-1(k)}(t)$ such that

$$\begin{aligned}\theta^{-1}(t) &= \frac{n-1}{k}\frac{\sum_{i=0}^{n-1}\frac{[k\phi(t)]^i}{i!}}{\phi(t)\sum_{i=0}^{n-2}\frac{[k\phi(t)]^i}{i!}} \\ &= \frac{1}{k}\left[\frac{n-1}{\phi(t)} + \frac{\frac{[k\phi(t)]^{n-2}}{(n-2)!}}{\sum_{i=0}^{n-2}\frac{[k\phi(t)]^i}{i!}}\right].\end{aligned}$$

Since $\phi(t)$ is a decreasing function of t, one can easily see that $\theta^{-1}(t)$ is an increasing function of t and hence $\theta(t)$ is a decreasing function of t. Thus, Lemma 2.1 of Kundu et al. [44] completes the proof. □

Let us consider a sequence of iid random variables $X_i, i \geq 1$ with cdf F and pdf f. We denote the n-th lower k-record value and l-record value as $\mathfrak{L}_{n(k)}$ and $\mathfrak{L}_{n(l)}$, respectively, and their hazard rates as $\tau_{n(k)}(t)$ and $\tau_{n(l)}(t)$. In the following theorem, we present a significant result indicating that if the PRE of n-th lower k-record values is increasing, then the PRE of n-th lower l-record values is also increasing when $k > l$.

Theorem 2.5. *If $\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(l)}; t)$ is increasing in t, then $\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)}; t)$ is also decreasing in t for all $\gamma > 0$ and $k > l$.*

Proof. It is assumed that $\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(l)}; t)$ increases as t increases. Moreover, we can establish the relationship $\tau_{n(k)}(t) = \eta_{k,l,n-1}(\phi(t))\tau_{n(l)}(t)$, where $\eta_{k,l,n-1}(\phi(t))$ is given by

$$\eta_{k,l,n-1}(\phi(t)) = \left(\frac{k}{l}\right)^n \frac{\sum_{i=0}^{n-1}\frac{[k\phi(t))]^i}{i!}}{\sum_{i=0}^{n-1}\frac{[l\phi(t)]^i}{i!}},\ t > 0,$$

and $\phi(t) = -\log F(t)$. Note that $\phi(t)$ is a decreasing function of $t > 0$. Moreover, the function

$$\eta_{k,l,n-1}(x) = \left(\frac{k}{l}\right)^n \frac{\sum_{i=0}^{n-1}\frac{[kx)]^i}{i!}}{\sum_{i=0}^{n-1}\frac{[lx]^i}{i!}},\ x > 0,$$

is an increasing function of $x > 0$ when $k > l$, as shown in Lemma 2.1 of Raqab and Amin [40]. Furthermore, the range of $\eta_{k,l,n-1}(x)$ is a subset of $(0, 1)$. Consequently, $\eta_{k,l,n-1}(\phi(t))$ is a decreasing function of $t > 0$ and its range is a subset of $(0, 1)$, which implies that the conditions of Theorem 3.1 of Mahmoudi and Asadi [38] are satisfied, thereby completing the proof. □

3. Conditional Rényi entropy of k-records

In the subsequent analysis, our focus lies on evaluating the past nth lower k-records denoted as $t - \mathfrak{L}_{n(k)},\ t \geq 0$, subject to the condition $\mathfrak{L}_{1(k)} \leq t$. Here, $\mathfrak{L}_{1(k)}$ corresponds to the first lower k-record, which is equivalent to $X_{k:k}$. Therefore, the condition $\mathfrak{L}_{1(k)} \leq t$ implies that the first lower k-record is less than the specified threshold $t > 0$. Consequently, the cdf of $\mathfrak{L}^t_{n(k)} = [t - \mathfrak{L}_{n(k)} | \mathfrak{L}_{1(k)} \leq t]$ can be expressed as (as presented in Tavangar and Asadi [45])

$$F_{\mathfrak{L}^t_{n(k)}}(x) = P(t - \mathfrak{L}_{n(k)} \leq x | \mathfrak{L}_{1(k)} \leq t),$$

$$= \frac{\Gamma(n, -k\log(F(t-x)/F(t)))}{\Gamma(n)}, \tag{3.1}$$

for all $0 \le x \le t$. It follows that

$$f_{\mathfrak{L}^t_{n(k)}}(x) = \frac{k^n}{\Gamma(n)}\left(\frac{F(t-x)}{F(t)}\right)^{k-1}\left(-\log\frac{F(t-x)}{F(t)}\right)^{n-1}\frac{f(t-x)}{F(t)}, \quad 0 \le x \le t. \tag{3.2}$$

It is important to note that the conditional distribution $[t - \mathfrak{L}_{n(k)}|\mathfrak{L}_{1(k)} \le t]$ represents the unconditional distribution of the $(n+1)$-th lower value of k-records from the distribution $F(t-x)/F(t)$. In the subsequent analysis, the primary objective is to investigate the Rényi entropy associated with the random variable $\mathfrak{L}^t_{n(k)}$, which quantifies the level of uncertainty inherent in the density of $[t-\mathfrak{L}_{n(k)}|\mathfrak{L}_{1(k)} \le t]$. To accomplish this, we introduce the function $F_t(x) = F(x)/F(t)$, where $0 \le x \le t$. The probability integral transformation $V_{n(k)} = F_t(\mathfrak{L}^t_{n(k)})$ plays a crucial role in this approach. The transformation $V_{n(k)} = F_t(\mathfrak{L}^t_{n(k)})$ is of significant importance and possesses the following pdf:

$$g_{n(k)}(u) = \frac{k^n}{\Gamma(n)}u^{k-1}(-\log u)^{n-1}, \quad 0 < u < 1. \tag{3.3}$$

The subsequent theorem provides a derived term for the Rényi entropy of $\mathfrak{L}^t_{n(k)}$ through the use of the aforementioned transformations.

Theorem 3.1. *The Rényi entropy for $\mathfrak{L}^t_{n(k)}$ can be written in the following expression:*

$$\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{n(k)}) = \omega(\gamma)\log\left[\int_0^1 g^\gamma_{n(k)}(u)f_t^{\gamma-1}(F_t^{-1}(u))du\right], \ t > 0, \tag{3.4}$$

for all $\gamma > 0$ where $F_t^{-1}(u) = \inf\{x; F_t(x) \ge u\}$ and $U \sim U(0,1)$ for all $\gamma > 0$.

Proof. By recalling relations (1.7) and (3.2) and by applying $u = F_t(z)$, we can establish the following relationship:

$$\begin{aligned}
\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{n(k)}) &= \omega(\gamma)\log\left[\int_0^t \left(f_{\mathfrak{L}^t_{n(k)}}(x)\right)^\gamma dx\right] \\
&= \omega(\gamma)\log\left[\int_0^t \left(\frac{k^n}{\Gamma(n)}\left(\frac{F(t-x)}{F(t)}\right)^{k-1}\left(-\log\frac{F(t-x)}{F(t)}\right)^{n-1}\frac{f(t-x)}{F(t)}\right)^\gamma dx\right] \\
&= \omega(\gamma)\log\left[\int_0^t \left(\frac{k^n}{\Gamma(n)}[F_t(z)]^{k-1}[-\log F_t(z)]^{n-1}f_t(z)\right)^\gamma dz\right] \text{(by taking } z = t-x) \\
&= \omega(\gamma)\log\left[\int_0^1 \left(\frac{k^n}{\Gamma(n)}u^{k-1}(-\log u)^{n-1}\right)^\gamma \left(f_t(F_t^{-1}(u))\right)^{\gamma-1} dx\right] \\
&= \omega(\gamma)\log\left[\int_0^1 g^\gamma_{n(k)}(u)\left(f_t(F_t^{-1}(u))\right)^{\gamma-1} du\right].
\end{aligned}$$

The final equality is derived by recognizing $g_{n(k)}(u)$ as the pdf of $V_{n(k)}$, as specified in Eq (3.3). By incorporating this result, we conclude the proof successfully. □

The next theorem examines how aging affects the PRE of k-record values.

Theorem 3.2. *Let X have DRHR property. So, $\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{n(k)})$ is increasing in t for all $\gamma > 0$.*

Proof. It is evident that $f_t(F_t^{-1}(x)) = x\tau_t(F_t^{-1}(x))$. So, Eq (3.4) can be rewritten as

$$e^{(1-\gamma)\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{n(k)})} = \int_0^1 g^\gamma_{n(k)}(u)u^{\gamma-1}\left(\tau_t(F_t^{-1}(u))\right)^{\gamma-1} du, \tag{3.5}$$

for all $\gamma > 0$. The relationship $F_t^{-1}(u) = F^{-1}(uF(t))$ holds true for all $0 < u < 1$, and it can be easily verified. Consequently, we obtain the following expression:

$$\tau_t(F_t^{-1}(u)) \;\; = \;\; \tau(F^{-1}(uF(t))),\; 0 < u < 1.$$

If $t_1 \leq t_2$, it follows that $F^{-1}(uF(t_1)) \leq F^{-1}(uF(t_2))$. Consequently, when X exhibits the property of DRHR, for all $\gamma > 1(0 < \gamma \leq 1)$, we can establish the following inequality:

$$\begin{aligned}\int_0^1 g^\gamma_{n(k)}(u)u^{\gamma-1}\left(\tau_{t_1}(F_{t_1}^{-1}(u))\right)^{\gamma-1} du \quad &= \quad \int_0^1 g^\gamma_{n(k)}(u)u^{\gamma-1}\left(\tau(F^{-1}(uF(t_1)))\right)^{\gamma-1} du \\ &\geq (\leq) \quad \int_0^1 g^\gamma_{n(k)}(u)u^{\gamma-1}\left(\tau(F^{-1}(uF(t_2)))\right)^{\gamma-1} du \\ &= \quad \int_0^1 g^\gamma_{n(k)}(u)u^{\gamma-1}\left(\tau_{t_2}(F_{t_2}^{-1}(u))\right)^{\gamma-1} du,\end{aligned}$$

for all $t_1 \leq t_2$. Using (3.5), we get

$$e^{(1-\gamma)\overline{\mathcal{H}}_\gamma(\mathfrak{L}^{t_1}_{n(k)})} \geq (\leq) e^{(1-\gamma)\overline{\mathcal{H}}_\gamma(\mathfrak{L}^{t_2}_{n(k)})},$$

for all $\gamma > 1(0 < \gamma \leq 1)$. This implies that $\overline{\mathcal{H}}_\gamma(\mathfrak{L}^{t_1}_{n(k)}) \leq \overline{\mathcal{H}}_\gamma(\mathfrak{L}^{t_2}_{n(k)})$ for all $\gamma > 0$, and this completes the proof. □

We should remark that the DRHR property of X in Theorem 3.2 leads to the amount of probability $P(X > t - \delta | X \leq t)$ decreasing with t for some positive very small δ. It is therefore to be expected that the extent of the uncertainty and surprise (and thus the value of PRE) increases with t if a value of X is obtained close to t, provided that X is smaller than t.

The outcomes derived from Theorems 3.1 and 3.2 are demonstrated in the following example.

Example 3.1. Let us consider a sequence of iid RVs $\{X_i\}, i \geq 1$, with the following cdf as $F(x) = e^{-1/x},\; x > 0$. Here, we can show that

$$\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{n(k)}) = c(\gamma)\log\int_0^1\left(\frac{1}{t} - \log u\right)^{2(\gamma-1)} u^{\gamma-1}g^\gamma_{n(k)}(u)du,\; t > 0.$$

Figure 4 presents a plot showcasing the PRE $\overline{\mathcal{H}}_\gamma(\mathfrak{L}_{n(k)}; t)$ for a specific scenario where $n = 5$. The plot includes two different values of γ, namely, 0.5 and 2. Various values of k ranging from 1 to 5 are considered, along with different values of t. The results demonstrate that the Rényi entropy of $\mathfrak{L}^t_{n(k)}$ increases as the time t increases. It is important to mention that the distribution under investigation displays the property of decreasing RHRs.

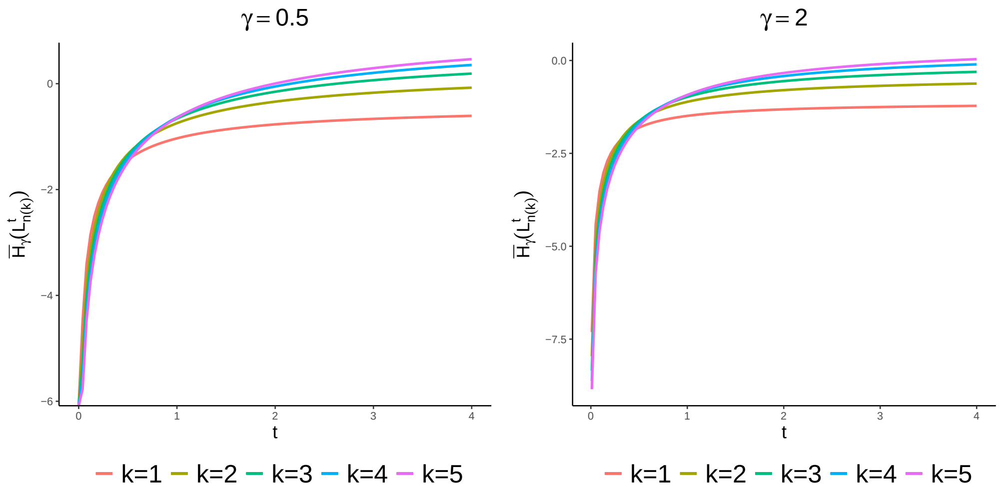

Figure 4. The values of $\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{5(k)})$ for the Frêchet distribution concerning t for $\gamma = 0.5$ and $\gamma = 2$ for $k = 1, 2, 3, 4, 5$.

In the subsequent theorem, we derive a lower bound for the Rényi entropy of $\mathfrak{L}^t_{n(k)}$ in the case of $\gamma > 1$, and an upper bound in the case of $0 < \gamma < 1$. These bounds are established with respect to the PRE of the parent distribution, denoted as $\overline{\mathcal{H}}_\gamma(X;t)$.

Theorem 3.3. *When* $\gamma > 1$ $(0 < \gamma < 1)$, *we have*

$$\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{n(k)}) \geq (\leq) \frac{\gamma}{1-\gamma} \log g_{n(k)}(v^\star) + \overline{\mathcal{H}}_\gamma(X;t), \tag{3.6}$$

where $g_{n(k)}(v^\star)$ *and* $v^\star = e^{-\frac{n-1}{k-1}}$.

Proof. By observing the mode of $g_{n(k)}(v)$ as $v^\star = e^{-\frac{n-1}{k-1}}$, we can deduce that $g_{n(k)}(v) \leq g_{n(k)}(v^\star)$ for $0 < v < 1$. Consequently, for $\gamma > 1$ or $0 < \gamma < 1$, we can establish the following inequality:

$$\begin{aligned}
\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{n(k)}) &= c(\gamma) \log \int_0^1 g^\gamma_{n(k)}(v) \left(f_t(F_t^{-1}(u))\right)^{\gamma-1} dv \\
&\geq (\leq) c(\gamma) \log \int_0^1 (g_{n(k)}(v^\star))^\gamma \left(f_t(F_t^{-1}(u))\right)^{\gamma-1} dv \\
&= \frac{\gamma}{1-\gamma} \log g_{n(k)}(v^\star) + \overline{\mathcal{H}}_\gamma(X;t).
\end{aligned}$$

The final equality is obtained by employing (1.8), which leads to the desired result. □

The application of the bounds provided in Theorem 3.3 is investigated in the following example.

Example 3.2. Consider a sequence of iid random variables $\{X_i\}, i \geq 1$, where each X_i follows a standard exponential distribution with the cdf $F(x) = 1 - e^{-x}$ for $x > 0$. From (1.7), we can obtain

$$\overline{\mathcal{H}}_\gamma(X;t) = c(\gamma) \left[\log(1 - e^{-\gamma t}) - \gamma \log(1 - e^{-t}) - \log(\gamma)\right], \ t > 0.$$

The goal of this example is to verify a lower bound for the Rényi entropy of $\mathfrak{L}^t_{8(4)}$ in the case of $\gamma > 1$ and an upper bound in the case of $0 < \gamma < 1$. After performing the necessary calculations, we get

$v^\star = e^{-\frac{n-1}{k-1}} = 0.09697197$. Consequently, we find that $g_{8(4)}(v^\star) = 4.465042$. Therefore, based on the findings of Theorem 3.3, a lower bound emanated for the Rényi entropy of $\mathfrak{L}^t_{8(4)}$ in the case of $\gamma > 1$ as

$$\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{8(4)}) \geq \frac{1.5\gamma}{1-\gamma} + c(\gamma)\left[\log(1-e^{-\gamma t}) - \gamma\log(1-e^{-t}) - \log(\gamma)\right], \tag{3.7}$$

and an upper bound for $0 < \gamma < 1$ as

$$\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{8(4)}) \leq \frac{1.5\gamma}{1-\gamma} + c(\gamma)\left[\log(1-e^{-\gamma t}) - \gamma\log(1-e^{-t}) - \log(\gamma)\right], \tag{3.8}$$

for all $t > 0$. By assuming an exponential distribution, we have calculated the lower bound for $\overline{\mathcal{H}}\gamma(\mathfrak{L}^t_{8(4)})$ using (3.7) in the left panel and the upper bound using (3.8) in the right panel. Moreover, we obtained the exact value of $\overline{\mathcal{H}}\gamma(\mathfrak{L}^t_{8(4)})$ directly from (3.4). The results are presented in Figure 5.

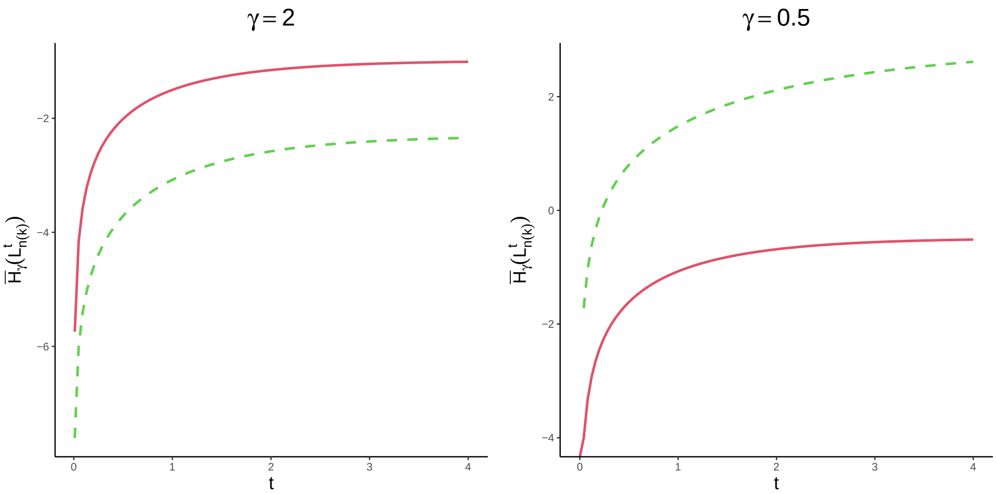

Figure 5. Exact value and the given bounds of $\overline{\mathcal{H}}_\gamma(\mathfrak{L}^t_{n(k)})$ for the exponential distribution with respect to time t.

4. Conclusions

The paper presented some information properties of k-record values to quantify uncertainty using the concept of past Rényi entropy. Specifically, we first provided an expression for the PRE of k-record values and then delved into the monotonicity properties of the PRE of k-record values, considering the aging properties of the component lifetimes. This examination enhanced our understanding of how the PRE behaves as the lifetimes of the components change. Moreover, we studied the preservation properties of PRE of k-record values in terms of the DRHR property. These findings revealed that the DRHR property of the n-th lower k-records is transmitted to the previous ones. Additionally, we extended this result to encompass k-record values, demonstrating that the DRHR property is passed from $\mathfrak{L}_{n-1(k)}$ to $\mathfrak{L}_{n(k)}$. Furthermore, we established that if the PRE of n-th lower k-record values is increasing, then the PRE of n-th lower l-record values is also increasing when $k > l$. In addition, we derived an expression for the past Rényi entropy of the n-th lower k-records given that the first lower k-record is less than a threshold level t. Through the investigation of several properties of this formula, we gained insights into the behavior and characteristics of Rényi entropy in the context of k-record values. These findings offer insights on record value information, contributing to the understanding of PRE in k-record values. The results presented in this paper expand the existing body of knowledge in this field and can be generalized to the other area of information theory, for example, cumulative past entropy, fractional generalized cumulative past entropy, and record values including generalized order statistics.

Author contributions

Mansour Shrahili: Conceptualization, methodology, writing-original draft; Mohamed Kayid: Writing-review and editing, software, validation. All authors contributed equally to the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

There is no conflict of interest declared by the authors.

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