AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 24385–24393.
DOI: 10.3934/math.20241187
Received: 01 July 2024
Revised: 29 July 2024
Accepted: 31 July 2024
Published: 19 August 2024

Research article

On H'-splittings of a handlebody

Yan Xu, Bing Fang and Fengchun Lei*

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China

* **Correspondence:** Email: fclei@dlut.edu.cn.

Abstract: Let M be a compact connected orientable 3-manifold and F be a compact connected orientable surface properly embedded in M. If F cuts M into two handlebodies X and Y (i.e., $M = X \cup_F Y$), then we say that F is an H'-splitting surface for M and call $X \cup_F Y$ an H'-splitting for M. When the H'-splitting surface F is incompressible in a handlebody H, a characteristic of an H'-splitting $H_1 \cup_F H_2$ to denote H is already known. In the present paper, we generalize the above result as follows: Let H be a handlebody of genus $g \geq 1$, $X \cup_F Y$ an H'-splitting for H. Then, either $X \cup_F Y$ is stabilized, or there exists a reducing system $\mathcal{J}_1 \cup \mathcal{K}_1$ of F, such that $\mathcal{J}_1$ is quasi-primitive in Y and $\mathcal{K}_1$ is quasi-primitive in X. Combining the result with the known result, we obtain a characteristic of an H'-splitting $H_1 \cup_F H_2$ to denote a handlebody.

Keywords: H'-splitting; Heegaard splitting; incompressible surface; primitivity; quasi-primitivity
Mathematics Subject Classification: 57N10

1. Introduction

It is a well known fact that each compact connected orientable 3-manifold M admits a Heegaard splitting $V \cup_F W$, where F is an orientable closed surface embedded in the interior of M which cuts M into two compression bodies V and W with $\partial_+ V = F = \partial_+ W$. In 1970, Downing [1] proved that each compact connected 3-manifold M with nonempty boundary has a decomposition as $H_1 \cup_F H_2$, where H_1 and H_2 are two handlebodies with the same genus, and $F = H_1 \cap H_2$ is a connected surface properly embedded in M. In 1973, Roeling [2] discussed such handlebody-splittings for 3-manifolds with connected boundaries. Later, Suzuki [3] slightly modified the results of Downing [1] and Roeling [2], and formulated a Haken type theorem for these handlebody-splittings in the way of Casson-Gordon [4].

Let M be a compact connected orientable 3-manifold and F be a compact connected orientable surface properly embedded in M. If F cuts M into two handlebodies X and Y (not necessarily with the same genus), that is, $M = X \cup_F Y$, then we say that F is an H'-splitting surface for M and call $X \cup_F Y$ an H'-splitting of M. It is clear that if M is closed, then the H'-splitting $X \cup_F Y$ is just a Heegaard splitting for M. If M is with non-empty boundary, then the H'-splitting $X \cup_F Y$ is distinct from a Heegaard splitting of M.

It has been shown in [5] that each compact connected orientable 3-manifold admits an H'-splitting (i.e., H'-splittings, similar to Heegaard splittings, which are common structures of 3-manifolds). Thus, it follows that it is a new way to construct all compact connected orientable 3-manifolds.

The Casson-Gordon Theorem [4] on weakly reducible Heegaard splittings has been generalized to the H'-splitting case in [5]. On the other hand, there exist examples (refer to [5]) to show that Haken's lemma does not hold in the H'-splitting case in general. This implies that the properties of H'-splitting structures for the 3-manifolds with boundaries are quite different from the Heegaard splitting structures and Downing's handlebody-splitting structures as above.

A characteristic of an H'-splitting $H_1 \cup_F H_2$ to denote a handlebody has been described in [6], where F is incompressible in both H_1 and H_2 (see Theorem 2.7 in Section 2 or [6] for the detail). In the present paper, we generalize the above result, regardless that F is compressible or incompressible in H, as follows: Let H be a handlebody of genus $g \geq 1$, $X \cup_F Y$ an H'-splitting for H. Then, either $X \cup_F Y$ is stabilized or there exists a reducing system $\mathcal{J}_1 \cup \mathcal{K}_1$ of F, such that $\mathcal{J}_1$ is quasi-primitive in Y and $\mathcal{K}_1$ is quasi-primitive in X. (refer to Sections 2 and 3 for the definitions). Combining the result with Theorem 2.7, we obtain a characteristic of an H'-splitting $X \cup_F Y$ to denote a handlebody.

The other parts of the paper is organized as follows. In Section 2, some necessary preliminaries are given. In Section 3, the statements of the main results and their proofs are given. It is worth noting that a refined version (see Lemma 2.9) of the Haken's lemma in disk case plays an essential role in the the proof of Theorem 3.3.

2. Preliminaries

In this section, we will review some notions and fundamental facts about 3-manifolds that will be used in Section 3. All the 3-manifolds considered in the paper are assumed to be compact and orientable. The concepts and terminologies which are not defined in the paper are all standard (refer to, for example, [7–9]).

Let M be a 3-manifold. A 2-sphere S embedded in M is essential in M if S does not bound a 3-ball in M; otherwise, S is inessential in M. M is reducible if M contains an essential 2-sphere; otherwise, M is irreducible.

Let M be a compact 3-manifold, and F a 2-sided surface properly embedded in M or $F \subset \partial M$. If there exists a disk $D \subset M$ such that $D \cap F = \partial D$ and ∂D is essential in F, then we say that F is compressible in M. Such a disk D is called a compressing disk of F. F is incompressible if F is not compressible in M and no component of F is an inessential 2-sphere, parallel to a disk in ∂M, or a disk in ∂M. If ∂M is compressible in M, then M is said to be ∂-reducible.

A handlebody H is a 3-manifold such that there exists a collection $\mathcal{D} = \{D_1, \cdots, D_n\}$ of pairwise disjoint disks properly embedded in H such that the manifold obtained by cutting H open along $\mathcal{D}$ is a 3-ball. $\mathcal{D}$ is called a complete system of disks for H, and n is called the genus of H.

Let S be an orientable closed surface, and $\mathcal{J} = \{J_1, \cdots, J_k\}$ a collection of pairwise disjoint simple closed curves (s.c.c.) on S. Let C be the 3-manifold obtained by adding 2-handles to $S \times I$ along $\mathcal{J} \times 0$, then capping of any resulting 2-spheres with 3-balls. C is called a compression body. In C, set $\partial_+ C = S \times 1$ and $\partial_- C = \partial C - \partial_+ C$. $\mathcal{J}$ is naturally extended to a collection $\mathcal{D} = \{D_1, \cdots, D_k\}$ of pairwise disjoint disks properly embedded in C. $\mathcal{D}$ is called a defining system of disks for C. It is clear that if $\partial_- C = \emptyset$, C is a handlebody; and if $\partial_- C \neq \emptyset$, the manifold obtained by cutting C open along $\mathcal{D}$ is homeomorphic to $\partial_- C \times I$. $S \times I$ is called a trivial compression body.

Let $\mathcal{D} = \{D_1, ..., D_k\}$ be a defining disk system for a compression body C, and Δ a disk in C such that for some i, $1 \leq i \leq k$, $\Delta \cap D_j = \emptyset$ for $j \neq i$, and $\Delta \cap D_i = \alpha$ is an arc in $\partial\Delta$ properly embedded in D_i, $\Delta \cap \partial_+ C = \beta$ is an arc in $\partial\Delta$, and $\alpha \cap \beta = \partial\alpha = \partial\beta$, $\alpha \cup \beta = \partial\Delta$. α cuts D_i into two disks D_{i1} and D_{i2}. Set $D_i' = D_{i1} \cup \Delta$ and $D_i'' = D_{i2} \cup \Delta$, and move D_i' by a small isotopy such that $D_i' \cap D_i'' = \emptyset$. Set $\mathcal{D}' = (\mathcal{D} \backslash \{D_i\}) \cup \{D_i', D_i''\}$. It is clear that $\mathcal{D}'$ is also a defining disk system for C. We say that $\mathcal{D}'$ is a slide of $\mathcal{D}$ along Δ.

Let M_i be a compact connected 3-manifold, $F_i \subset \partial M_i$ a connected surface such that no component of ∂F_i bounds a disk in ∂M_i, $i = 1, 2$, and $h : F_1 \to F_2$ a homeomorphism. Set $M = M_1 \cup_h M_2$, $F_1 = F = F_2$ in M, and call M an amalgamation of M_1 and M_2 along F. M is also denoted as $M_1 \cup_F M_2$, and F is called a splitting surface of M.

Clearly, if F is a disk, M is a boundary connected sum of M_1 and M_2, and is also denoted by $M_1 \#_\partial M_2$; if both M_1 and M_2 are compression bodies and $\partial_+ M_1 = F = \partial_+ M_2$, then $M_1 \cup_F M_2$ is a Heegaard splitting for M, and F is a Heegaard surface of M; if both M_1 and M_2 are handlebodies, then $M_1 \cup_F M_2$ is called an H'-splitting for M, and F is an H'-splitting surface of M (here, F is not necessarily closed).

It is well known that any compact connected orientable 3-manifold admits a Heegaard splitting [8]. It has been shown in [5] that any compact connected orientable 3-manifold admits an H'-splitting.

For $M = M_1 \cup_F M_2$, suppose that F is connected and compressible in both M_1 and M_2. Let D_i be a compressing disk of F in M_i, $i = 1, 2$. We say that F is stabilized if $|\partial D_1 \cap \partial D_2| = 1$; reducible if $\partial D_1 = \partial D_2$; and weakly reducible if $\partial D_1 \cap \partial D_2 = \emptyset$. Otherwise, F is unstabilized, irreducible, or strongly irreducible, respectively.

Let $V \cup_F W$ be a Heegaard splitting of genus g for M, and $T \cup_{\mathbb{T}} T'$ the Heegaard splitting of genus 1 for S^3. The connected sum $(V \cup_F W)\#(T \cup_{\mathbb{T}} T')$ is a Heegaard splitting of genus $g + 1$ for M, and is called an elementary stabilization of $V \cup_F W$. A Heegaard splitting $V' \cup_{F'} W'$ is called a stabilization of $V \cup_F W$ if it is obtained by a finite number of elementary stabilization from $V \cup_F W$.

In the following, we collect some known facts which will be used in Section 3.

Lemma 2.1. [10] Let H be a handlebody of genus $n \geq 2$ and F be an incompressible surface in H. Then, the manifold obtained by cutting H open along F is a union of handlebodies.

Lemma 2.2. [10] Let A be a spanning annulus in the compression body C. Then, there exists a defining system $\mathcal{D}$ of disks for C such that A is disjoint from any disk in $\mathcal{D}$.

The following is the uniqueness theorem of Heegaard splittings for S^3, due to Waldhausen [11].

Theorem 2.3. Any positive genus Heegaard splitting of S^3 is stabilized.

A handlebody H has a natural Heegaard splitting: A surface F in int(H) which is parallel to ∂H splits H into a handlebody ($\cong H$) and a trivial compression body. Call it the trivial splitting of H. A Heegaard splitting of H is called standard if it is a stabilization of the trivial splitting.

Two consequences of Theorem 2.3 are as follows.

Theorem 2.4. [12] Any Heegaard splitting of a handlebody H of positive genus is standard.

Theorem 2.5. [10] Let M be an irreducible 3-manifold and $X \cup_F Y$ be a Heegaard splitting for M. Suppose that $X \cup_F Y$ is reducible. Then, $X \cup_F Y$ is stabilized.

Let H be a handlebody and $\mathcal{J}$ be a collection of pairwise disjoint s.c.c. in ∂H. Denote by $H(\mathcal{J})$ the 3-manifold obtained by attaching 2-handles to H along the curves in $\mathcal{J}$.

Definition 2.6. Let $\mathcal{J} = \{J_1, \cdots, J_m\}$ be a collection of simple closed curves in the boundary of a handlebody H of genus n.

(1) If $\{[J_1], \cdots, [J_m]\} \subset \pi_1(H)$ (after some conjugation) can be extended to a basis of $\pi_1(H)$, then we say that $\mathcal{J}$ is *primitive* in H.

(2) If the curves in $\mathcal{J}$ are pairwise disjoint, and $H(\mathcal{J})$ is a handlebody (of genus $n - m$), then we say that $\mathcal{J}$ is *quasi-primitive* in H.

In the Definition 2.6, if the curves in $\mathcal{J}$ are pairwise disjoint, it is clear that $\mathcal{J}$ is primitive implies that $\mathcal{J}$ is quasi-primitive. But the converse is generally not true except for $m = 1$. It is a theorem in [13] that if all subsets $\mathcal{J}'$ of $\mathcal{J}$ are quasi-primitive in H, then $\mathcal{J}$ is primitive in H.

The following theorem gives a characteristic of an H'-splitting $X \cup_F Y$ for a handlebody H, where F is incompressible in H.

Theorem 2.7 ([6])**.** Let $X \cup_F Y$ be an H'-splitting for a 3-manifold M, where $g(X), g(Y) \geq 2$. Suppose that F is incompressible in both X and Y. Then, M is a handlebody if and only if there exists a basis curve set $\mathcal{J} = \{J_1, \cdots, J_m\}$ of $\pi_1(F)$ with a partition $(\mathcal{J}_1, \mathcal{J}_2)$ of $\mathcal{J}$ such that $\mathcal{J}_1$ is primitive in X and $\mathcal{J}_2$ is primitive in Y.

A properly embedded annulus A in a compression body C is called a spanning annulus if A is incompressible in C and the two components of ∂A are lying in $\partial_+ C$ and $\partial_- C$ respectively.

The following is a well-known fact, refer to [10] for a proof.

Theorem 2.8. Let V be a compression body with $\partial_- V \neq \emptyset$ and F be an incompressible, ∂-incompressible surface properly embedded in M. Then, each component of F is either a spanning annulus, an essential disk, or parallel to a component of $\partial_- V$ in V.

Let P_b be a connected planar surface with b boundary components, $b > 0$. Let Γ be a collection of pairwise disjoint and non-parallel simple arcs properly embedded in P_b, such that the surface obtained by cutting P_b along Γ is a union of m disks, it is clear that $m \leq b - 1$.

Haken's lemma states that any Heegaard splitting of a reducible 3-manifold is reducible. It was generalized to the ∂-reducible 3-manifolds as follows: For any Heegaard splitting $V \cup_S W$ of a ∂-reducible 3-manifold M, there exists a compression disk D for ∂M in M, such that $D \cap S$ is a single circle. The following is a stronger version of this result, and an outline proof is included for convenience.

Lemma 2.9. Let M be a connected 3-manifold with a Heegaard splitting $V \cup_S W$ and F be a sub-surface of $\partial_- V$. Suppose that F is incompressible in V and compressible in M. Then, there exists a compresing disk D for F in M, such that $D \cap S$ is a single circle.

Proof. By assumption, $F \subset \partial_- V$ is incompressible in V and compressible in M, so V is non-trivial. Any compressing disk E of F in M intersects S non-empty. Let Γ be a spine of W such that Γ is in general position with $E \cap W$. Thus $E \cap W$ consists of pairwise disjoint disks in W. Choose such a compressing disk D of F in M, such that $|D \cap S|$ is minimal among all such disks in M. Set $P = D \cap V$. P is a connected planar surface. We may further assume that P is incompressible in V. If P is an annulus, then the lemma holds.

Assume $m = |S \cap P| \geq 2$. By Theorem 2.8, P is ∂-compressible in V. Let Δ be a ∂-compresing disk for P. Push D along Δ by isotopy to get D_1, and denote $D_1 \cap V$ by P_1. Then, P_1 is the surface obtained by doing the boundary compressing P in V along Δ. If P_1 is still ∂-compressible in V, do the similar operation as above. After a finite number such operations, we can isotope D to D' in M such that $D' \cap V$ consists of a spanning annulus and a collection of essential disks in V. It is easy to see $m' = |P' \cap S| \leq m$. Thus, $Q' = D' \cap W$ is a connected planar surface properly embedded in W with $\partial Q' \subset S$.

We may further assume that Q' is incompressible in M. Similarly, we can isotope D' to D'' in M such that $Q'' = D'' \cap W$ consists of a collection of essential disks in W with $m'' = |Q'' \cap S| < |Q' \cap S| \leq m$, a contradiction to the minimality of $|D \cap S|$. □

3. Main results

A similar result to Theorem 2.5 for a reducible H'-splitting of an irreducible 3-manifold holds as follows.

Theorem 3.1. Let M be an irreducible 3-manifold, and $X \cup_F Y$ an H'-splitting for M. Suppose that $X \cup_F Y$ is reducible. Then $X \cup_F Y$ is stabilized.

Proof. By assumption, $X \cup_F Y$ is reducible. There exists an essential circle α in F, and α bounds a disk D in X and a disk E in Y, respectively. We divide it into three cases to discuss.

Case 1. α is non-separating in F. Then, there exists a s.c.c. β in F such that β meets α in one point. Let N be a regular neighborhood of $D \cup E \cup \beta$ in M. Then, N is a once-punctured $S^2 \times S^1$, contradicting to that M is irreducible. See Figure 1 (a) below. Thus, Case 1 cannot happen.

Case 2. α is separating in F, and cuts F into two surfaces F_1 and F_2 with $(\partial F_i) \cap \partial M \neq \emptyset$, $i = 1, 2$. In the case, $D \cup E$ is a separating 2-sphere in M, which cuts M into M_1 and M_2, and neither M_1 nor M_2 is a 3-ball, again contradicting to that M is irreducible. See Figure 1 (b) below. Thus, Case 2 cannot happen.

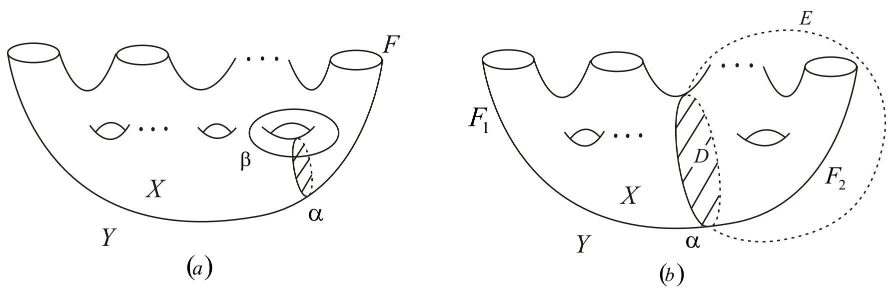

Figure 1. Case (1) and Case (2).

Case 3. α is separating in F, and cuts out of a once-punctured surface F' of positive genus from F with $\partial F' = \alpha$. In the case, D cuts out of a handlebody H_1 from X and E cuts out of a handlebody H_2 from Y, $H_1 \cap F = F' = H_2 \cap F$. Set $M' = H_1 \cup_{F'} H_2$, then $\partial M' = D \cup E$. By the irreducibility of M, M' is a

3-ball. By capping of M' with a 3-ball B^3, we get $M'' = M' \cup_\partial B^3 \cong S^3$, and F' naturally extended to a Heegaard surface of positive genus for M''. By Theorem 2.3, such a Heegaard splitting is stabilized. It follows that $X \cup_F Y$ is stabilized. □

The following is a direct consequence of Theorem 3.1:

Corollary 3.2. Let H be a handlebody of genus $g \geq 1$. If $X \cup_F Y$ is a reducible H'-splitting for H, then $X \cup_F Y$ is stabilized.

Let H be a handlebody of genus g and J be an s.c.c. on ∂H. If there exists a disk D properly embedded in H with $|J \cap \partial D| = 1$, we call J a longitude of H. It is clear that $X \#_\partial Y$ is a handlebody if and only if both X and Y are handlebodies. For an H'-splitting $X \cup_A Y$ for a 3-manifold M, where A is an annulus, it is known (refer to [14] for a proof) that M is a handlebody if and only if the core curve of A is a longitude of either X or Y. In particular, if H_J is the manifold obtained by adding a 2-handle to the handlebody along an s.c.c. J on ∂H, then H_J is a handlebody if and only if J is a longitude of H (or equivalently, J is primitive in H).

Theorem 3.3. Let H be a handlebody of genus $g \geq 1$, $X \cup_F Y$ an H'-splitting for H. Suppose that F is weakly reducible in H. Then either $X \cup_F Y$ is stabilized, or there exists a collection $\mathcal{D} = \{D_1, \cdots, D_m\}$ ($\mathcal{E} = \{E_1, \cdots, E_n\}$, resp.) of pairwise disjoint compressing disks of F in X (in Y, resp.), such that $\partial\mathcal{D} \cap \partial\mathcal{E} = \emptyset$, and if we denote by F' the surface obtained by compressing F in H along $\mathcal{D} \cup \mathcal{E}$, then F' is incompressible in H. Moreover, if $\mathcal{D}'$ ($\mathcal{E}'$, resp.) is the subset of $\mathcal{D}$ ($\mathcal{E}$, resp.) which consists of only the non-separating disks in X (in Y, resp.), then $\partial\mathcal{D}'$ is quasi-primitive in Y and $\partial\mathcal{E}'$ is quasi-primitive in X.

Proof. By assumption, F is weakly reducible in H. There exist compresing disks $D_1 \subset X$ and $E_1 \subset Y$ of F with $\partial D_1 \cap \partial E_1 = \emptyset$. If ∂D_1 is parallel to ∂E_1, then $X \cup_F Y$ is reducible (therefore, stabilized, by Corollary 3.2). Extend D_1 and E_1 to a collection $\mathcal{D} = \{D_1, \cdots, D_m\}$ ($\mathcal{E} = \{E_1, \cdots, E_n\}$, resp.) of pairwise disjoint compressing disks of F in X (in Y, resp.) in such a way that $\partial\mathcal{D} \cap \partial\mathcal{E} = \emptyset$, and the following conditions are satisfied:

(1) If we denote by F_X (F_Y, resp.) the surface obtained from F by compressing X (Y, resp.) along $\mathcal{D}$ ($\mathcal{E}$, resp.), then no component of F_X (F_Y, resp.) whose boundary is a subset of the set of the cutting sections of $\mathcal{D}$ ($\mathcal{E}$, resp.) is a planar surface.

(2) If Δ is a compresing disk of F in X (Y, resp.) with $\partial\Delta \cap (\partial\mathcal{D} \cup \partial\mathcal{E}) = \emptyset$, then $\partial\Delta$ cuts out of a planar surface P from F_X (F_Y, resp.) with $\partial P - \{\partial\Delta\} \subset \partial\mathcal{D}$ ($\partial P - \{\partial\Delta\} \subset \partial\mathcal{E}$, resp.).

(3) If F' is the surface obtained from F by compressing F along $\mathcal{D} \cup \mathcal{E}$, $C(F') = -\chi(F')$, the comlexity of F', is minimal over all such $\mathcal{D} \cup \mathcal{E}$.

By (1) and (2), each ∂D_i (∂E_j, resp.) either is non-separating in F, or cuts F into two pieces, each of which intersects ∂H non-empty.

Denote by $\widetilde{F}$ the surface obtained by cutting F open along $\partial\mathcal{D} \cup \partial\mathcal{E}$. If $\widetilde{F}$ has a planar component Q with $\partial Q = L_1 \cup L_2$, where L_1 is a subset of the set of the cutting sections of $\partial\mathcal{D}$ and L_2 is a subset of the set of the cutting sections of $\partial\mathcal{E}$, then let α be an s.c.c. on Q such that α cuts Q into Q_1 and Q_2 with $\partial Q_1 - \alpha = L_1$ and $\partial Q_2 - \alpha = L_2$. Therefore, by above (1), $L_1, L_2 \neq \emptyset$. Thus, α is essential in F and α bounds disks in both X and Y. Hence, $X \cup_F Y$ is reducible (therefore, stabilized, by Corollary 3.2). In the following, we assume that $\widetilde{F}$ has no such planar component.

Let N_1 (N_2, resp.) be a regular neighborhood of $\mathcal{D}$ in X ($\mathcal{E}$ in Y, resp.) such that $(N_1 \cap F) \bigcap (N_2 \cap F) = \emptyset$. Set $X^* = \overline{X \setminus N_1}$, $Y^* = \overline{Y \setminus N_2}$. X^* (Y^*, resp.) is in fact the manifold obtained by compressing X along $\mathcal{D}$ (Y along $\mathcal{E}$, resp.), therefore it is a union of handlebodies. Assume that after compressing X along $\mathcal{D}$ (Y along $\mathcal{E}$, resp.), F is changed into F_1 (F_2, resp.). Since F is separating in H, each component of F_1 (F_2, resp.) is separating in H. If F_1 (F_2, resp.) has a component which is a closed surface of positive genus, it bounds a handlebody in X^* (Y^*, resp.), contradicting to the choice of $\mathcal{D}$ ($\mathcal{E}$, resp.). Thus, each component of F_1 (F_2, resp.) has non-empty boundary.

Set $X' = X^* \cup N_2$, $Y' = Y^* \cup N_1$, and $F' = X' \cap Y'$. Then, $H = X' \cup_{F'} Y'$. By above (1), no component of $\partial X'$ ($\partial Y'$, resp.) is a 2-sphere. Suppose that F' has k components $F'_1, \cdots, F'_k$. It follows that each F'_i is separating in H, $1 \leq i \leq k$. With no loss, assume that F' cuts H into $k+1$ pieces $M_1, \cdots, M_{k+1}$, and

$$H = ((M_1 \cup_{F'_1} M_2) \cup_{F'_2} \cdots) \cup_{F'_k} M_{k+1},$$

where each M_i is the 3-manifold obtained by adding some 2-handles (possibly empty) to a handlebody in X^* or Y^*, $1 \leq i \leq k+1$.

We now show that F' is incompressible in H. Otherwise, some F'_i is compressible in M_i. Without loss of generality, assume that M_i is a 3-manifold obtained by adding a subset N' (2-handles) of N_2 to a handlebody H^* in X^* along the curves on a component S' of F_X. Say $N' = \{\eta(E_{i_1}), ..., \eta(E_{i_s})\}$. Let S be a surface in the interior of H^* which is parallel to ∂H^*. Then, S is a Heegaard surface in M_i. In fact, S splits M_i into a handlebody $V \cong H^*$ and a compression body W, and $\mathcal{E}' = \{E_{i_1}, ..., E_{i_s}\}$ can be extended to a defining disk system for W. Note that $F'_i \subset \partial\_W$. By Lemma 2.9, there exists a compressing disk D of F'_i in M_i, such that $D \cap S$ is an essential circle in S. Set $A = D \cap W$, A is a spanning annulus in W. See Figure 2 below. By Lemma 2.2, there exists a defining disk system $\mathcal{E}^*$ for W, such that A is disjoint from each disk in $\mathcal{E}^*$. Assume that A and $\mathcal{E}'$ are in general position. By an innermost argument, we may further assume that $\Lambda = A \cap \bigcup_{j=1}^{s} E_{i_j}$ has no circle components. Let γ be an arc component of Λ which is outermost in A. Then, γ cuts out of a disk Δ from A with $\text{int}(\Delta) \cap \bigcup_{j=1}^{s} E_{i_j} = \emptyset$. Slide $\mathcal{E}'$ along Δ to a new defining disk system for W with less intersection with A. After finite such slides, we can obtain the defining disk system $\mathcal{E}^*$ for W which is disjoint from A. Since both the component of ∂A lying in S and $\partial \mathcal{E}'$ are lying in the parallel copy S'' of S' in S, it follows that $\partial \mathcal{E}^* \subset S''$. It is clear that there is a subset $\tilde{\mathcal{E}}$ of $\mathcal{E}^*$, such that $\tilde{\mathcal{E}}$ is a defining disk system for W and $\tilde{\mathcal{E}}$ contains s disks. Set $\mathcal{D}_1 = \mathcal{D} \cup \{D\}$, and $\mathcal{E}_1 = (\mathcal{E} \backslash \mathcal{E}') \bigcup (\tilde{\mathcal{E}} \cap Y)$. Note that ∂D is essential in F'. Denote by F'' the surface obtained from F by compressing F in H along $\mathcal{D}_1 \cup \mathcal{E}_1$, it is clear that $C(F'') < C(F')$, a contradiction to the minimality of $C(F')$.

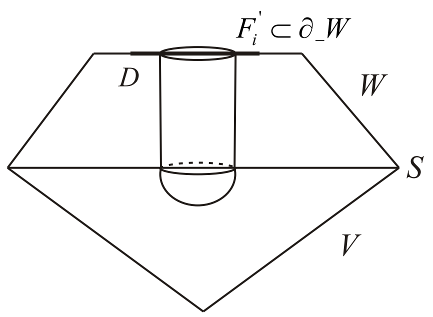

Figure 2. The Heegaard splitting $V \cup_S W$ of M_i.

Thus, F' is incompressible in H. It follows from Lemma 2.1 that each M_i is a handlebody, $1 \leq i \leq k+1$. If $\mathcal{D}'$ ($\mathcal{E}'$, resp.) is the subset of $\mathcal{D}$ ($\mathcal{E}$, resp.) which consists of only the non-separating disks in X (in Y, resp.), it follows that $\partial\mathcal{D}'$ is quasi-primitive in Y and $\partial\mathcal{E}'$ is quasi-primitive in X.

This completes the proof. □

By a similar arguments to the proof of Theorem 3.3, we have the following theorem. The proof is omitted.

Theorem 3.4. Let H be a handlebody of genus $g \geq 1$, $X \cup_F Y$ an H'-splitting for H. Suppose that F is compressible in X (or Y) and incompressible in Y (or X), or F is compressible in both X and Y, and is strongly irreducible in H. Then, there exists a collection $\mathcal{D} = \{D_1, \cdots, D_m\}$ ($\mathcal{E} = \{E_1, \cdots, E_n\}$, resp.) of pairwise disjoint compressing disks of F in X (in Y, resp.), such that if we denote by F' the surface obtained by compressing F in H along $\mathcal{D}$ ($\mathcal{E}$, resp.), then F' is incompressible in H. Moreover, if $\mathcal{D}'$ ($\mathcal{E}'$, resp.) is the subset of $\mathcal{D}$ ($\mathcal{E}$, resp.) which consists of only the non-separating disks in X (in Y, resp.), then $\partial\mathcal{D}'$ is quasi-primitive in Y ($\partial\mathcal{E}'$ is quasi-primitive in X, resp.).

Use the notations as in Theorems 3.3 and 3.4. We call $\partial\mathcal{D}' \cup \partial\mathcal{E}'$ a reducing system of F (in Theorem 3.4, $\partial\mathcal{D}'$ or $\partial\mathcal{E}' = \emptyset$). Combining Theorem 3.3, Theorem 3.4, and Theorem 2.7, we have the following direct corollary.

Theorem 3.5. Let H be a handlebody of genus $g \geq 1$, $X \cup_F Y$ an H'-splitting for H. Then, either $X \cup_F Y$ is stabilized, or there exists a reducing system $\mathcal{J}_1 \cup \mathcal{K}_1$ of F, such that $\mathcal{J}_1$ is quasi-primitive in Y, and $\mathcal{K}_1$ is quasi-primitive in X. Moreover, if the incompressible surface F' obtained by compressing F in H along the disks in H bounded by $\mathcal{J}_1 \cup \mathcal{K}_1$ is connected, then there exists a basis curve set $\mathcal{L}$ of $\pi_1(F')$ with a partition $(\mathcal{J}_2, \mathcal{K}_2)$ of $\mathcal{L}$ such that $\mathcal{J}_2$ is primitive in X and $\mathcal{K}_2$ is primitive in Y.

We remark that a similar conclusion as in Theorem 3.5 holds when F' not connected. We omit the statement.

4. Conclusions

We describe a characteristic of an H'-splitting $X \cup_F Y$ to denote a handlebody H, where F may be compressible in H. This generalizes an earlier result (Theorem 2.7) in which the H'-surface in a handlebody is assumed to be incompressible.

Author contributions

Yan Xu: Writing the original draft, Investigation, Methodology; Bing Fang: Writing-review & editing the draft, Investigation; Fengchun Lei: Conceptualization, Methodology, Supervision, Writing-review & editing the draft. All authors agreed to publish the final version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by a grant (No.12071051) of NSFC.

The authors are grateful to the referees for their insightful comments and suggestions, which helped to improve the papers presentation.

Conflict of interest

The authors declare no conflict of interest.

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