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*Research article*

# A reliable analytic technique and physical interpretation for the two-dimensional nonlinear Schrödinger equations

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**Abstract:** Nonlinear Schrödinger equations are a key paradigm in nonlinear research, attracting both mathematical and physical attention. This work was primarily concerned with the usage of a reliable analytic technique in order to solve two models of (2+1)-dimensional nonlinear Schrödinger equations. By applying a comprehensible wave transformation, every nonlinear model was simplified to an ordinary differential equation. A number of critical solutions were observed that correlated to various parameters. The provided approach has various advantages, including reducing difficult computations and succinctly presenting key results. Some 2D and 3D graphical representations regarding presented solitons were considered for the appropriate values of the parameters. We also showed the effect of the physical parameters on the dynamical behavior of the presented solutions. Finally, the proposed approach may be expanded to tackle increasingly complicated problems in applied science.

**Keywords:** two-dimensional nonlinear Schrödinger equations; closed-form solutions; solitons
**Mathematics Subject Classification:** 35C08, 35Q40, 35Q55

## 1. Introduction

Nonlinear partial differential equations (NPDEs) are commonly used to describe natural phenomena, which are strongly related to fundamental principles in applied sciences [1–3]. One of the areas that scientists find most intriguing in the current period is nonlinear phenomena [4, 5]. Environmental factors influencing the movement or flow of fluids or gases, for example, necessitate the use of PDEs to include and deliberate all cogent constraints in order to ensure an accurate description of this process, while keeping in mind the inherent nonlinearity of these processes when describing them with NPDEs [6–8]. Thus, in the quest for long-term scientific, technical, and industrial advancements, the multidisciplinary study of NPDEs cannot be abandoned. The solutions to NPDEs give information on the variables that influenced the behavior of these physical phenomena [9, 10]. Moreover, nonlinear wave equations enable a variety of innovative solutions that are significantly different from linear wave issues [11–14]. However, analytical and numerical approximation for the NPDEs is still in early stages of research. Developing effective numerical algorithms is crucial for solving these problems [15–18].

Mathematicians and other researchers have given solitonic investigations a great deal of attention since they find them to be very applicable to their work [19–22]. Solitons are nondispersive long-wave particles that move in packets at constant velocity. Soliton theory is closely related to modern physics and is utilized to explain several physical challenges at the forefront of this dynamic discipline. Recent years have seen significant progress in analytic solutions to highly nonlinear differential equations, particularly the NPDEs. This is because trying to determine the dynamic behaviors of models described by these NPDEs can become a mirage if their analyticity is not sufficiently entrenched [23–27].

The nonlinear Schrödinger (NLS) equations are used to represent numerous nonlinear processes in nonlinear optics and other scientific fields [28–30]. These equations explain wave propagation in mediums with dispersive and nonlinear features. The NLS equation affects the phase and dispersion of self-modulations in optical Kerr applications [31, 32]. Using statistical mechanics domains, the NLS complex field dynamics have been examined in [33]. The propagation of dark and bright solitons in optical fibres depends on the balance of self-phase modulation and group velocity dispersive effects [34–36]. Localized nonlinear potentials have been used to study the splitting of breather solitons of the NLS equation [37]. The nonlocal NLS describes the propagation of paraxial beams in nonlocal medium [38–40]. The propagation of the superimposed field of Laguerre-Gaussian and Hermite-Gaussian solitons in a nonlocal NLS was studied [41].

In this work, we first consider the (2+1)-dimensional nonlinear Schrödinger (2D-NLS) equation [42]:

$$iu_t + u_{xx} + u_{yy} + \delta \mid u \mid^2 u = 0, \tag{1.1}$$

where $u(x, y, t)$ denotes the complex wave, $x, y$ are the position variables and $t$ is the time variable, and $\delta$ denotes the nonlinear coefficient. This model appears in several areas of physics, including electromagnetic wave propagation [43], quantum mechanics [44], and design of certain optoelectronic devices [45]. Najafi and Arbabi introduced exact traveling wave solutions for model (1.1), using the sine-cosine technique.

Second, we consider the (2+1)-dimensional hyperbolic nonlinear Schrödinger (HNLS) model [46–48]:

$$iq_y + \frac{1}{2}(q_{xx} - q_{tt}) + \mid q \mid^2 q = 0, \tag{1.2}$$

where $q(x, y, t)$ represents the complex wave, $x, y$ are the position variables, and $t$ is the time variable. The dynamics of optical soliton propagation in mono-mode optical fibres are described by this model. The importance of studying the HNLS equation has led numerous researchers to use it as a standard model in their research. Ai-Lin and Ji employed the Lie group symmetry technique to find Lie point symmetries and exact traveling solutions for the HNLS model [49]. Aliyu et al. applied the solitary wave ansatz to investigate optical solitary waves for model (1.2) [46]. Durur et al. proposed periodic and singular wave solutions to the HNLS equation utilizing the projected approach [47].

The remainder of this work is constructed as follows. Section 2 provides the closed form solutions of the form $L_1\, Q'' + L_2\, Q^3 + L_3\, Q = 0$. In physics and applied mathematics, this form allows important and meaningful impacts. Section 3 provides the solutions for the 2D-NLS equation. Section 4 provides the solutions for the HNLS equation. The explanation of the provided solutions is shown in Section 5. Some 2D and 3D graphs for the solutions generated for suitable free parameter values are also included. Finally, in Section 6, a conclusion was made and assembled based on the unparalleled study results, along with suggestions on potential areas of the presented results that may be further investigated in the future.

## 2. Closed form of solutions

We present the following equation's closed form solutions:

$$L_1\, Q'' + L_2\, Q^3 + L_3\, Q = 0. \tag{2.1}$$

Using the solver reported in [23], the solutions for the Eq (2.1) are given as
(i) When $L_3 = 0$:

$$Q_{1,2}(x,y,t) = \left( \mp \sqrt{\frac{-L_2}{2L_1}}\,(\zeta + \varrho) \right)^{-1}. \tag{2.2}$$

(ii) When $\frac{L_3}{L_1} < 0$:

$$Q_{3,4}(x,y,t) = \pm \sqrt{\frac{L_3}{L_2}}\, \tan\left( \sqrt{\frac{-L_3}{2L_1}}\,(\zeta + \varrho) \right), \tag{2.3}$$

and

$$Q_{5,6}(x,y,t) = \pm \sqrt{\frac{L_3}{L_2}}\, \cot\left( \sqrt{\frac{-L_3}{2L_1}}\,(\zeta + \varrho) \right). \tag{2.4}$$

(iii) When $\frac{L_3}{L_1} > 0$:

$$Q_{7,8}(x,y,t) = \pm \sqrt{\frac{-L_3}{L_2}}\, \tanh\left( \sqrt{\frac{L_3}{2L_1}}\,(\zeta + \varrho) \right), \tag{2.5}$$

and

$$Q_{9,10}(x,y,t) = \pm \sqrt{\frac{-L_3}{L_2}}\, \coth\left( \sqrt{\frac{L_3}{2L_1}}\,(\zeta + \varrho) \right). \tag{2.6}$$

Here, $\varrho$ is an arbitrary constant.

## 3. Solutions of the 2D-NLS equation

Using the transformation

$$u(x,y,t) = U(\xi) e^{i(bx+ay+ct)}, \quad \xi = x + y + wt, \tag{3.1}$$

$w$ and $c$ denote the speed and frequency of the solitary wave. We have

$$\begin{aligned} iu_t &= (-cU + iwU')e^{i(bx+ay+ct)}, \\ u_{xx} &= (U'' + 2ibU' - b^2U)e^{i(bx+ay+ct)}, \\ u_{yy} &= (U'' + 2iaU' - a^2U)e^{i(bx+ay+ct)}. \end{aligned} \tag{3.2}$$

Substituting Eq (3.2) into Eq (1.1) and distinguishing the real part yields

$$2U'' + \delta U^3 - (a^2 + b^2 + c)\, U = 0\,, \tag{3.3}$$

whereas imaginary part yields $w = -2(a + b)$. The solutions of Eq (3.3) are

$$U_{1,2}(x, y, t) = \left(\mp \sqrt{\frac{-\delta}{4}}\,(\xi + \varrho)\right)^{-1} \quad \delta < 0. \tag{3.4}$$

Thus, the solutions for Eq (1.1) are

$$u_{1,2}(x, y, t) = \left(\mp \sqrt{\frac{-\delta}{4}}\,(x + y + wt + \varrho)\right)^{-1} e^{i(bx+ay+ct)}, \delta < 0. \tag{3.5}$$

$$U_{3,4}(\xi) = \pm \sqrt{\frac{-4\Gamma}{\delta}} \tan\left(\sqrt{\Gamma}\,(\xi + \varrho)\right), \Gamma > 0, \delta < 0. \tag{3.6}$$

and

$$U_{5,6}(\xi) = \pm \sqrt{\frac{-4\Gamma}{\delta}} \cot\left(\sqrt{\Gamma}\,(\xi + \varrho)\right), \Gamma > 0, \delta < 0. \tag{3.7}$$

Thus, the solutions for Eq (1.1) are

$$u_{3,4}(\xi) = \pm \sqrt{\frac{-4\Gamma}{\delta}}\, e^{i(bx+ay+ct)} \tan\left(\sqrt{\Gamma}\,(x + y + wt + \varrho)\right), \Gamma > 0, \delta < 0. \tag{3.8}$$

and

$$u_{5,6}(\xi) = \pm \sqrt{\frac{-4\Gamma}{\delta}}\, e^{i(bx+ay+ct)} \cot\left(\sqrt{\Gamma}\,(x + y + wt + \varrho)\right), \Gamma > 0, \delta < 0. \tag{3.9}$$

$$U_{7,8}(\xi) = \pm \sqrt{\frac{4\Gamma}{\delta}} \tanh\left(\sqrt{-\Gamma}\,(\xi + \varrho)\right), \Gamma < 0, \delta < 0. \tag{3.10}$$

and

$$U_{9,10}(\xi) = \pm \sqrt{\frac{4\Gamma}{\delta}} \coth\left(\sqrt{-\Gamma}\,(\xi + \varrho)\right), \Gamma < 0, \delta < 0. \tag{3.11}$$

Thus, the solutions for Eq (1.1) are

$$u_{3,4}(\xi) = \pm \sqrt{\frac{4\Gamma}{\delta}}\, e^{i(bx+ay+ct)} \tanh\left(\sqrt{-\Gamma}\,(x + y + wt + \varrho)\right), \Gamma < 0, \delta < 0. \tag{3.12}$$

and

$$u_{5,6}(\xi) = \pm \sqrt{\frac{4\Gamma}{\delta}}\, e^{i(bx+ay+ct)} \coth\left(\sqrt{-\Gamma}\,(x + y + wt + \varrho)\right), \Gamma < 0, \delta < 0. \tag{3.13}$$

Here, $\Gamma = \frac{a^2+b^2+c}{4}$.

## 4. Solutions of the HNLS equation

Using the transformation

$$q(x,y,t) = Q(\zeta)e^{i(x+\alpha_2 y+\beta_2 t)}, \quad \zeta = x+\alpha_1 y+\beta_1 t, \tag{4.1}$$

$\beta_1$ and $\beta_2$ represent the speed and frequency of the solitary wave. We have

$$\begin{aligned} iq_y &= (i\alpha_1 Q' - \alpha_2 Q)e^{i(x+\alpha_2 y+\beta_2 t)}, \\ q_{xx} &= (Q'' + 2iQ' - Q)e^{i(x+\alpha_2 y+\beta_2 t)}, \\ q_{tt} &= (\beta_1^2 Q'' + 2i\beta_1\beta_2 Q' - \beta_2^2 Q)e^{i(x+\alpha_2 y+\beta_2 t)}. \end{aligned} \tag{4.2}$$

Substituting Eq (4.2) into Eq (1.2) and distinguishing the real part yields

$$(1-\beta_1^2)Q'' + 2\,Q^3 + (\beta_2^2 - 2\alpha_2 - 1)\,Q = 0\,, \tag{4.3}$$

whereas the imaginary part yields $\alpha_1 = \beta_1\beta_2 - 1$. The solutions of Eq (4.3) are

$$Q_{1,2}(\zeta) = \left(\mp\sqrt{\frac{1}{\beta_1^2-1}}\,(\zeta+\varrho)\right)^{-1}, \beta_1 > 1 \text{ or } \beta_1 < -1. \tag{4.4}$$

Thus, the solutions for Eq (1.2) are

$$q_{1,2}(x,y,t) = \left(\mp\sqrt{\frac{1}{\beta_1^2-1}}(x+\alpha_1 y+\beta_1 t+\varrho)\right)^{-1} e^{i(x+\alpha_2 y+\beta_2 t)}, \beta_1 > 1 \text{ or } \beta_1 < -1, \tag{4.5}$$

$$Q_{3,4}(\zeta) = \pm\sqrt{\frac{\Lambda}{2}}\tan\left(\sqrt{\frac{\Lambda}{2(\beta_1^2-1)}}\,(\zeta+\varrho)\right), \Lambda > 0, \beta_1 > 1 \text{ or } \beta_1 < -1, \tag{4.6}$$

and

$$Q_{5,6}(\zeta) = \pm\sqrt{\frac{\Lambda}{2}}\cot\left(\sqrt{\frac{\Lambda}{2(\beta_1^2-1)}}\,(\zeta+\varrho)\right), \Lambda > 0, \beta_1 > 1 \text{ or } \beta_1 < -1. \tag{4.7}$$

Thus, the solutions for Eq (1.2) are

$$q_{3,4}(x,y,t) = \pm\sqrt{\frac{\Lambda}{2}}\,e^{i(x+\alpha_2 y+\beta_2 t)}\tan\left(\sqrt{\frac{\Lambda}{2(\beta_1^2-1)}}(x+\alpha_1 y+\beta_1 t+\varrho)\right), \tag{4.8}$$

and

$$q_{5,6}(x,y,t) = \pm\sqrt{\frac{\Lambda}{2}}\,e^{i(x+\alpha_2 y+\beta_2 t)}\cot\left(\sqrt{\frac{\Lambda}{2(\beta_1^2-1)}}(x+\alpha_1 y+\beta_1 t+\varrho)\right), \tag{4.9}$$

where $\Lambda > 0, \beta_1 > 1$ or $\beta_1 < -1$.

$$Q_{7,8}(\zeta) = \pm\sqrt{\frac{-\Lambda}{2}}\tanh\left(\sqrt{\frac{-\Lambda}{2(\beta_1^2-1)}}\,(\zeta+\varrho)\right), \tag{4.10}$$

and

$$Q_{9,10}(\zeta) = \pm\sqrt{\frac{-\Lambda}{2}}\coth\left(\sqrt{\frac{-\Lambda}{2(\beta_1^2-1)}}\,(\zeta+\varrho)\right). \tag{4.11}$$

Thus, the solutions for Eq (1.2) are

$$q_{7,8}(x,y,t) = \pm\sqrt{\frac{-\Lambda}{2}}\,e^{i(x+\alpha_2 y+\beta_2 t)}\tanh\left(\sqrt{\frac{-\Lambda}{2(\beta_1^2-1)}}(x+\alpha_1 y+\beta_1 t+\varrho)\right), \tag{4.12}$$

and

$$q_{9,10}(x,y,t) = \pm\sqrt{\frac{-\Lambda}{2}}\,e^{i(x+\alpha_2 y+\beta_2 t)}\coth\left(\sqrt{\frac{-\Lambda}{2(\beta_1^2-1)}}(x+\alpha_1 y+\beta_1 t+\varrho)\right), \tag{4.13}$$

where $\Lambda < 0, \beta_1 > 1$ or $\beta_1 < -1$. Here, $\Lambda = \beta_2^2 - 2\alpha_2 - 1$.

## 5. Physical interpretation

The nonlinear Schrödinger (NLS) equation is a sophisticated model that has captured the interest of mathematicians and physicists due to its potential applications in applied science and new physics. This model can accurately represent many complicated natural processes. The solutions that supply the equation owing to the structure of the Schrödinger equation are complex wave solutions. It was stated that research on higher dimensional models is more accurate in explaining physical phenomena that are not linear, like Bose-Einstein condensations, ultrafast nonlinear optics, optoelectronic devices, coastal water motions, etc.

This study uses the closed form solutions for a commonly used NPDE to solve the two models of 2D-NLS equations. More specifically, engineers, mathematicians, and physicists can utilize this form as a box solver. The main advantages of this solver over the others are that it avoids laborious and complicated calculations and provides a greater range of applications for solving other natural science equations without boundary or initial conditions. This approach displays hyperbolic, trigonometric, and rational solution forms. These solutions give wave pictures in applied sciences that describe complex processes. For example, these solutions were developed to improve optical wave performance in fibres. Figures 1 and 2 represent the 2D and 3D periodic wave for solution (3.12). Figure 3 illustrates that reducing $\delta$ reduced the amplitude of the periodic wave of solution (3.12). Furthermore, there is no shift or reverse in the periodic amplitude in Figure 3. Figures 4 and 5 represent the 2D and 3D periodic wave for solution (4.12). The 3D graphical representation 6 of $\mid q_7(x,y,t) \mid$ depicts the 2D localized soliton wave solution of (4.12). Finally, Figure 7 clarifies the 2D localized soliton wave solution of (4.12) for different values of $t$.

Ultimately, the results demonstrate the effectiveness of the proposed method and its capacity to provide a large number of wave solutions for NPDEs, which will be helpful in the study of physics' solitary theory. This solver's capacity to handle a broad variety of nonlinear fractional differential equation models is one of its main features. We also plan for considering some important physical properties and physical structures, such as positivity preservation, maximum principle, long time behavior, singular solutions, etc [50–53].

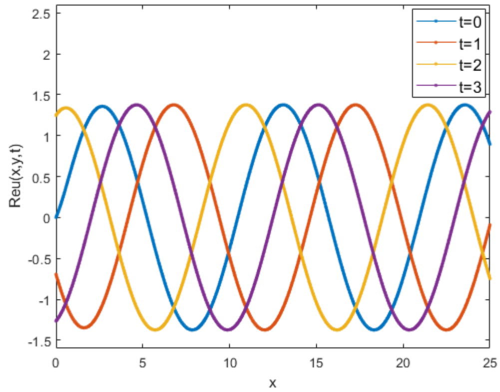


**Figure 1.** 2D periodic wave solution of (3.12) for different values of $t$.

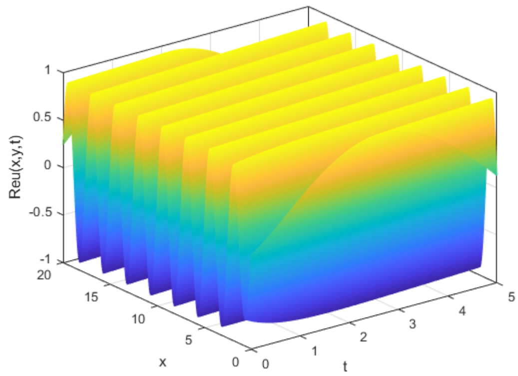


**Figure 2.** 3D periodic wave solution of (3.12).

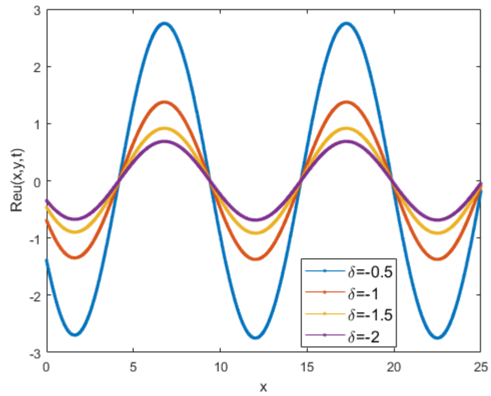


**Figure 3.** 2D periodic wave solution of (3.12) for different values of $\delta$.

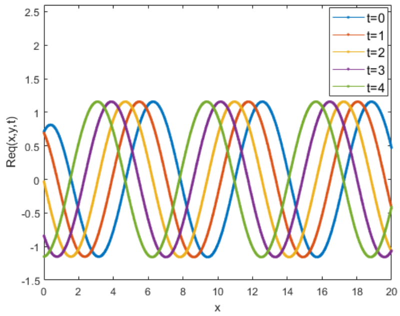


**Figure 4.** 2D periodic wave solution of (4.12) for different values of $t$.

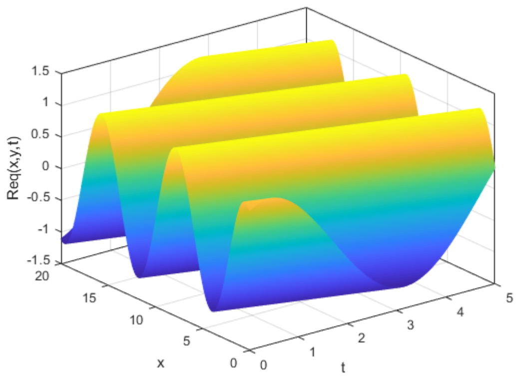


**Figure 5.** 3D periodic wave solution of (4.12).

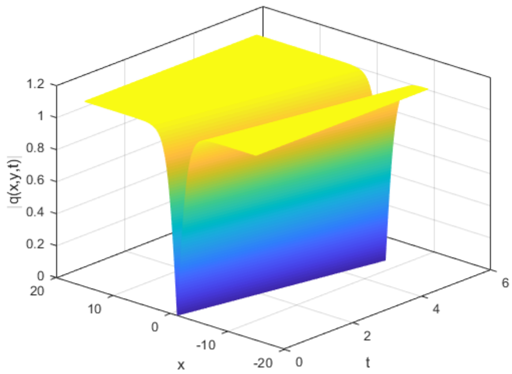


**Figure 6.** 3D localized soliton wave solution of (4.12).

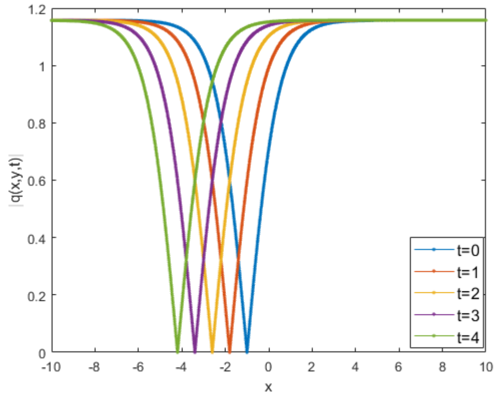


**Figure 7.** 2D localized soliton wave solution of (4.12) for different values of $t$.

## 6. Conclusions

This article successfully retrieved the necessary wave solutions to the governing models using the reliable analytical method, namely, we considered the 2D-NLS equation and the HNLS equation. There are three types of solutions: hyperbolic, trigonometric, and rational. Certain constraints are also provided for validating the results. Graphical illustrations of some solutions are provided to clarify these behaviors. We also clarify the physical parameter's influence on the dynamic of the provided solutions. We anticipate that the results presented will be valuable in understanding some physical phenomena for various nonlinear mathematical physics models. Many scientists will use the closed form as a box solver to solve a number of additional complicated models that emerge in the applied sciences.

## Author contributions

Mahmoud A. E. Abdelrahman: Conceptualization, Data curation, Formal analysis, Writing-original draft; H. S. Alayachi: Conceptualization, Data curation, Formal analysis, Writing-original draft.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare that they have no competing interests.

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