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Global error bounds for the extended vertical LCP of $\sum -SDD$ matrices

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Abstract: An error bound for the solution of the $\sum -SDD$ matrix extended vertical linear complementarity problem is given when the $\sum -SDD$ matrix satisfies the row W-property. It is shown by the illustrative example that the new bound is better than those in [1] in some cases.

Keywords: extended vertical linear complementarity problem; error bound; $\sum -SDD$ matrices
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1. Introduction and Preliminaries

The extended vertical linear complementarity problem is to find a vector $x \in R^n$ such that

$$r(x) := \min(A_0x + q_0, A_1x + q_1, \cdots, A_kx + q_k) = 0,$$

or prove that there is no such vector, where the min operater works componentwise for both vectors and matrices. Here, it is abbreviated by EVLCP(A,q), where

$$A = (A_0, A_1, \cdots, A_k), A_l \in R^{n\times n}, l = 0, 1, \cdots, k,$$

is a block matrix and

$$q = (q_0, q_1, \cdots, q_k), q_l \in R^n, l = 0, 1, \cdots, k,$$

is a block vector.

When $k = 1, A_0 = I, q_0 = 0$,the EVLCP(A, q) reduces to linear Complementarity problems (denoted by LCP (A_1, q_1)), and when $A_0 = I, q_0 = 0$, the EVLCP(A, q) comes back to vertical linear complementarity problems (can be found in [2,3]).The results proposed by Gohda and Sznajder (can be found in [4–6]) for the extended vertical linear complementarity problem.

Some experts and scholars have extended the theories of existence, uniqueness, and error bound of the linear complementarity problem to the extended vertical linear complementarity problem. For example, for any vector q , in the LCP (A, q) has unique solutions if and only if matrix A is the

P-matrix (can be found in [7–8]). Gowda and Sznajder extended this to the extended vertical linear complementarity to be replaced by problem [2] in 1994, which became the theoretical basis for the later study of the extended vertical linear complementarity problem.

In [9–12], the algorithms for solving the extended vertical linear complementarity problem are proposed, but the solutions obtained by using these algorithms often have errors; thus, the error estimation of the extended vertical linear complementarity problem is worth studying.

Next, let us review some relevant symbols, concepts, theorems, and lemmas.

When the $B = \left(b_{ij}\right) \in R^{n\times n}$ matrix satisfies $b_{ij} \leq 0$, for any $i \neq j$, B is called Z-matrices; if the principal and sub equations of B are all positive, then B is a P-matrix; if B is a Z-matrix and $B^{-1} > 0$, then B is an M-matrix; if $\tilde{B} = (\tilde{b}_{ij})$ and if the comparison matrix of B is an M-matrix, when $i = j, \tilde{b}_{ij} = \left|b_{ij}\right|$, when $i \neq j, \tilde{b}_{ij} = -\left|b_{ij}\right|$, then B is an H-matrix (can be found in [12]).

Definition 1.1. [13] B matrix is $B = \left(b_{ij}\right) \in R^{n\times n}$, if for any $i, j \in N, |b_{ii}| > r_i(B)$, is called a strictly diagonally dominant (SDD) matrix.

Definition 1.2. [1] If for any $i \in N$,

$$\left(A'_j\right)_{i.} = \left(A_{j_i}\right)_{i.} \in \left\{(A_0)_{i.}, (A_1)_{i.}, \cdots, (A_l)_{i.}\right\} = \left\{(A'_0)_{i.}, (A'_1)_{i.}\right), \cdots, (A'_l)_{i.}\right\},$$

where $\left(A'_j\right)_{i.}$ represents the i-th row of matrix $\left(A'_j\right)$, then the block matrix $A' = \left(A'_0, A'_1, \cdots, A'_k\right)$ is called row rearrangement. Similarly, block vectors q and q' also satisfy the above relationship, respectively and here use $\Re(M)$ and $\Re(q)$ to represent the set of rearranged rows of A and q.

Theorem 1.1. [13] For any block vector $q = (q_0, q_1, \cdots, q_k)$, the EVLCP(A, q) has a unique solution if and only if the block matrix $A = (A_0, A_1, \cdots, A_k)$ has the row W-property, i.e.,

$$\min\left(A_0x, A_1x, \cdots, A_kx\right) \leq 0 \leq \max\left(A_0x_, A_1x, \cdots, A_ix\right) \Rightarrow x = 0,$$

where, the max and min operators work componentwise for both matrices and vectors, 0 represents a zero vector.

In 2009, Zhang et al. (can be found in [1]) applied the property of row rearrangement in block matrices to provid necessary and sufficient conditions for block matrices to have the row properties. The specific expression is as follows:

Lemma 1.1. [1] The block matrix $A = (A_0, A_1, \cdots, A_k)$ has the row W-property if and only if $(I - D)A'_j + DA'_l$ is nonsingular for any two blocks A'_j, A'_l of $A' \in \Re(A)$ and for any $D = \operatorname{diag}(d_i), d_i \in [0, 1], i \in N$.

Furthermore, a global error estimation formula for the EVLCP(A, q) is also provided, as follows:

Let x^* be the solution of the EVLCP(A, q). If the block matrix $A = (A_0, A_1, \cdots, A_k)$ has the row W-property, then for any $x \in R^n$,

$$\|x - x^*\| \leq \alpha(A) \cdot \|r(x)\|, \tag{1.1}$$

where

$$\alpha(A) := \max_{M'\in\Re(M)} \max_{j<l\in\{0,1,\ldots,k\}} \max_{d\in[0,1]^n} \left\|\left[(I - D)A'_j + DA'_l\right]^{-1}\right\|.$$

Apparently, due to the difficulty to compute $\left\|\left[(I-D)A_j^{'}+DA_l^{'}\right]^{-1}\right\|$ exactly, it is upper bound (1.1) for $\alpha(A)$ and cannot be computed easily. Thus, some computable upper bounds for $\alpha(A)$ are given under various matrix norms by considering the structured property for matrices A_j, $j=0,1,\ldots,k$.

For example, Zhang et al. provided an upper bound on $\alpha_\infty(A)$ when all A_j are SDD matrices in [1], and also provided an upper bound on another $\alpha_\infty(A)$ for a special matrix A under certain conditions.

Theorem 1.2. [1] Suppose that matrices $A_0, A_1, \cdots, A_k$ has the positive diagonals, with the spectral radius

$$\rho\left(\max\left(\Lambda_0^{-1}|Q_0|, \Lambda_1^{-1}|Q_1|, \cdots, \Lambda_\mathrm{k}^{-1}|Q_\mathrm{k}|\right)\right)<1,$$

then $A=(A_0, A_1, \cdots, A_k)$ has the row W-property and

$$\alpha_\infty(A)\le\left\|\left[I-\max_{i=0,1,\cdots,k}\left(\Lambda_i^{-1}|Q_i|\right)\right]^{-1}\max_{i=0,1,\cdots,k}\left(\Lambda_i^{-1}\right)\right\|_\infty,$$

where Λ_i is the diagonal part of A_i, $Q_i=\Lambda_i-A_i$, for $i=0,1,\ldots,k$.

In addition, Zhang et al. presented a computable upper bound for $\alpha(A)$ under the infinity norm in another rspecial class of block matrices.

Theorem 1.3. [1] Suppose that $A_0, A_1, \cdots, A_k$ are SDD matrices and $A=(A_0, A_1, \cdots, A_k)$ has the row W-property , and for each $i\in N, \left(A_j\right)_{ii}(A_l)_{ii}>0$, with any $j<l\in\{0,1,\ldots,k\}$, then

$$\alpha_\infty(A)\le\frac{1}{\min_{i\in N}\{(min(A_0e, A_1e,\ldots,A_ke))_i\}},$$

where $\tilde{A}_i$ is the comparison matrix of A_i, i.e., $\left(\tilde{A}_i\right)_{\tau\tau}=|(A_i)_{\tau\tau}|, \left(\tilde{A}_i\right)_{\tau j}=-\left|(A_i)_{\tau j}\right|$ for $\tau\neq j, (A_i)_{\tau j}$ is the element in the τ-th row and j-th column of A_i, and $(\tilde{A}_i)_{\tau j}$ is the element in the τ-th row and j-th column of $(\tilde{A})_i$.

However, the error estimation formula provided is applicable only to a certain type of special matrix, and it is not easy to verify. Therefore, it is necessary to explore the error estimation formula for solutions of EVLCP(A, q) for other special matrix classes.

In 2021, Wang et al. provided error estimation formulas for the EVLCP(A, q) of B_π^R-matrices and B-matrices in reference [14]. In 2023, Zhao et al. provided error estimation formulas for the EVLCP(A, q) of $S-SDDS-B$ matrix and $S-SDDS$ matrix in reference [15].

In 2013, García Esnaola and Peña first proposed a special class of H-matrices: $\sum$-SDD matrices (see [16] for details) and provided error estimates for their linear complementarity problems with parameters. Its definition is as follows:

Definition 1.3. [16] Let matrix $B=\left(b_{ij}\right)\in C^{\mathrm{n}\times\mathrm{n}}$ if there is a non empty subset S such that the following two conditions hold:

(I) $|b_{ii}|>r_i^S(B), i\in S$;

(II) $\left(|b_{ii}|-r_i^s(B)\right)\left(\left|b_{jj}\right|-r_j^s(B)\right)>r_i^{\bar{s}}(B)r_j^s(B), i\in S, j\in\bar{S}$.

then B is called the $\sum-SDD$ matrix.

Theorem 1.4. [16] If B is an $\sum-SDD$ matrix and S is a nonempty subset of N, then

$$\left\|B^{-1}\right\|_\infty\le\max_{i\in S, j\in\bar{S}}\max\left\{\rho_{ij}^S(B), \rho_{ji}^{\bar{S}}(B)\right\},$$

where

$$\rho_{ij}^{S}(B)=\frac{|b_{ii}|-r_i^S(B)+r_j^S(B)}{\left(|b_{ii}|-r_i^S(B)\right)\left(\left|b_{jj}\right|-r_j^{\bar S}(B)\right)-r_i^{\bar S}(B)r_j^S(B)},$$

$$\rho_{ji}^{\bar S}(B)=\frac{\left|b_{ij}\right|-r_j^{\bar S}(B)+r_i^{\bar S}(B)}{\left(|b_{ii}|-r_i^S(B)\right)\left(\left|b_{jj}\right|-r_j^{\bar S}(B)\right)-r_i^{\bar S}(B)r_j^S(B)}.$$

2. A global error bound for the extended vertical LCP of $\sum -SDD$ Matrix

Proposition 2.1. Let $B=\left(b_{ij}\right)\in C^{n\times n}$ and $M=\left(m_{ij}\right)\in C^{n\times n}$ be $\sum -SDD$ matrices with positive main diagonal elements, and all have the same set $S\subset N$, If $\forall i\in S, \forall j\in\bar S$, satisfies $b_{ij}m_{ij}>0$ (or $b_{ji}m_{ji}>0$) and

$$\left(|b_{ii}|-r_i^S(B)\right)\left(\left|m_{jj}\right|-r_j^{\bar S}(M)\right)>r_i^{\bar S}(B)r_j^S(M),$$

$$\left(|m_{ii}|-r_i^S(M)\right)\left(\left|b_{jj}\right|-r_j^{\bar S}(B)\right)>r_i^{\bar S}(M)r_j^S(B),$$

then $(I-D)B+DM$ is a $\sum -SDD$ matrices, where $D=\operatorname{diag}(d_i), d_i\in[0,1], i\in N$.

Proof. Since both B and M are $\sum -SDD$ matrice, so for any $i\in S, j\in\bar S$, the following hold:

$$|b_{ii}|>r_i^S(B), \left(|b_{ii}|-r_i^S(B)\right)\left(\left|b_{jj}\right|-r_j^S(B)\right)>r_i^{\bar S}(B)r_j^S(B),$$

$$|m_{ii}|>r_i^S(M), \left(|m_{ii}|-r_i^S(M)\right)\left(\left|m_{jj}\right|-r_j^S(M)\right)>r_i^{\bar S}(M)r_j^S(M).$$

Note that $d_i\in[0,1]$, hence $1-d_i\geq 0$ and $d_i\geq 0$, they are not equal to 0 at the same time. Let $(I-D)B+D=C=(c_{ij})$. Thus for any $i\in S, j\in\bar S$, we have

$$\begin{aligned}
&|c_{ii}|-r_i^S(C)\\
&=(1-d_i)|b_{ii}|+d_i|m_{ii}|-\textstyle\sum_{j\in S/\{i\}}(1-d_i)\left|b_{ij}\right|-\sum_{j\in S/\{i\}}d_i|m_{ii}|\\
&=(1-d_i)\left(|b_{ii}|-r_i^S(B)\right)+d_i\left(|m_{ii}|-r_i^S(M)\right)>0,
\end{aligned}$$

i.e.,

$$|c_{ii}|>r_i^S(C),$$

and for any $i\in S, j\in\bar S$, we have

$$\begin{aligned}
&\left(|c_{ii}|-r_i^S(C)\right)\left(\left|c_{jj}\right|-r_j^{\bar S}(C)\right)\\
&=\left[(1-d_i)\left(|b_{ii}|-r_i^S(B)\right)+d_i\left(|m_{ii}|-r_i^S(M)\right)\right]\times\left[\left(1-d_j\right)\left(\left|b_{jj}\right|-r_j^{\bar S}(B)\right)+d_j\left(\left|m_{jj}\right|-r_j^{\bar S}(M)\right)\right]\\
&=(1-d_i)\left(1-d_j\right)\left(|b_{ii}|-r_i^S(B)\right)\left(\left|b_{jj}\right|-r_j^{\bar S}(B)\right)+(1-d_i)d_j\left(|b_{ii}|-r_i^S(B)\right)\left(|m_{jj}|-r_j^{\bar S}(M)\right)\\
&+d_i\left(1-d_j\right)\left(|m_{ii}|-r_i^S(M)\right)\left(\left|b_{jj}\right|-r_j^{\bar S}(B)\right)+d_id_j\left(|m_{ii}|-r_i^S(M)\right)\left(\left|m_{jj}\right|-r_j^{\bar S}(M)\right)\\
&>(1-d_i)\left(1-d_j\right)r_i^{\bar S}(B)r_j^S(B)+(1-d_i)d_jr_i^{\bar S}(B)r_j^S(M)+d_i\left(1-d_j\right)r_i^{\bar S}(M)r_j^S(B)+d_id_jr_i^{\bar S}(M)r_j^S(M)\\
&=\left[(1-d_i)r_i^{\bar S}(B)+d_ir_i^{\bar S}(M)\right]\left[\left(1-d_j\right)r_j^S(B)+d_jr_j^S(M)\right]\\
&=r_i^{\bar S}(C)r_j^S(C),
\end{aligned}$$

form Definition 1.3 the conclusion follows.

According to Proposition 2.1, Lemma 1.1 and the fact that a $\sum -SDD$ matrix is nonsingular(can be found in [17] for), block matrix composed of $\sum -SDD$ matrix has the row W-property.

Proposition 2.2. $A_0, A_1, \cdots, A_k$ are $\sum -SSD$ matrices, and each $A_{l_i}\,(l_i = 0, 1, \cdots, k)$ satisfies the condition of Proposition 2.1, then each $A_l^{'}$ in $A^{'} \in \mathfrak{R}(A)$ is a $\sum -SDD$ matrix, $A = (A_0, A_1, \cdots, A_k)$ has the row W-property.

Proof. From Definition 1.2, for the i-th row $\left(A_l^{'}\right)_{i.}$ $(i \in N)$ of $A_l^{'}$, there exist $l_i \in \{0, 1, \cdots, k\}$, such that $\left(A_l'\right)_{i.} = \left(A_{ji}\right)_{i.}$. Since A_{l_i} is a $\sum -SDD$ matrix, for any $i \in S, j \in \bar{S}$, we have

$$\begin{aligned}
&\left|(A_{l_i})_{ii}\right| > r_i^s\,(A_{l_i})\,, \\
&\left(\left|(A_{l_i})_{ii}\right| - r_i^s\,(A_{l_i})\right)\left(\left|(A_{l_i})_{jj}\right| - r_j^{\bar{s}}\,(A_{l_i})\right) > r_i^{\bar{S}}\,(A_{l_i})\, r_j^S\,(A_{l_i})\,,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left|\left(A_l^{'}\right)_{ii}\right| > r_i^s\left(A_l^{'}\right), \\
&\left(\left|\left(A_l^{'}\right)_{ii}\right| - r_i^s\left(A_l^{'}\right)\right)\left(\left|\left(A_l^{'}\right)_{jj}\right| - r_j^{\bar{s}}\left(A_l^{'}\right)\right) > r_i^{\bar{S}}\left(A_l^{'}\right) r_j^S\left(A_l^{'}\right),
\end{aligned}$$

from Definition 1.3, $A_l^{'}$ is a $\sum -SDD$ matrix.

Let $A_j^{'}, A_l^{'}$ be any two blocks in $A^{'} \in \mathfrak{R}(A)$, then $A_j^{'}, A_l^{'}$ are all $\sum -SDD$ matrices. According to Proposition 2.1, $(I-D)\,A_j^{'} + DA_l^{'}$ is a $\sum -SDD$ matrix for any $D = \operatorname{diag}(d_i)\,, d_i \in [0,1], i \in N$, and thus $(I-D)\,A_j^{'} + DA_l^{'}$ is nonsingular. From Lemma 1.1, block matrix $A = (A_0, A_1, \cdots, A_k)$ has the row W-property.

Next, we provide an upper bounds for $\alpha_\infty(A)$ with each $A_l, l = 0, 1, 2, \cdots, k$ being a $\sum -SDD$ matrix.

Theorem 2.1. Let $A = (A_0, A_1, \cdots, A_k)$, if each $A_{l_i}\,(l_i = 0, 1, \ldots, k)$ is a $\sum -SDD$ matrix and satisfies the condition of Proposition 2.1, then

$$\begin{aligned}
&\max\nolimits_{M' \in \mathfrak{R}(M)} \max\nolimits_{j<l\in\{0,1,\ldots,k\}} \max\nolimits_{d\in[0,1]^n} \left\|\left[(I-D)A_j^{'} + DA_l^{'}\right]^{-1}\right\| \\
&\leq \max \max\nolimits_{i\in S, \tau\in\bar{S}} \left\{ \frac{4\max_{p=0,1,\ldots,k}\left\{\left(\alpha_{max}^p\right)_{\rho_{max}^S}\right\}\cdot\max_{p=0,1,\ldots,k}\left\{\left(\beta_{min}^p\right)_{A_p}^{\rho_{min}^S}\right\}}{\min_{p=0,1,\ldots,k}\left\{\left(\beta_{min}^p\right)_{A_p}^{\rho_{min}^S}\right\}^2}, \frac{4\max_{p=0,1,\ldots,k}\left\{\left(\alpha_{max}^p\right)_{\rho_{max}^{\bar{S}}}\right\}\cdot\max_{p=0,1,\ldots,k}\left\{\left(\beta_{min}^p\right)_{A_p}^{\rho_{min}^S}\right\}}{\min_{p=0,1,\ldots,k}\left\{\left(\beta_{min}^p\right)_{A_p}^{\rho_{min}^S}\right\}^2} \right\},
\end{aligned}$$

where

$$\begin{aligned}
&(\alpha_{\max}^p)_{\rho_{\min}^S} = \max\nolimits_{i\in S,\tau\in\bar{S}} \left\{\left(\alpha_{i\tau}^p\right)_{\rho_{i\tau}^S}\right\}, \\
&\left(\alpha_{i\tau}^p\right)_{\rho_{\tau i}^S} = \left|\left(A_p\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A_p\right) + r_i^{\bar{S}}\left(A_p\right), \\
&\left(\beta_{min}^p\right)_{A_p}^{\rho_{min}^S} = \min\nolimits_{i\in S,\tau\in\bar{S}} \left(\beta_{i\tau}^p\right)_{A_p}^{\rho_{i\tau}^S}.
\end{aligned}$$

Proof. For any two blocks $A_j^{'}, A_l^{'}$ in $A^{'} \in \mathfrak{R}(A)$, and any $D = \operatorname{diag}(d_i)\,, d_i \in [0,1], i \in N$. According to Propositions 2.1 and 2.2, it can be inferred that $A_j^{'}, A_l^{'}$ and A_D all are $\sum -SDD$ matrices with positive main diagonal elements. According to Theorem 1.4,

$$\left\|A_D^{-1}\right\|_\infty = \left\|\left((I-D)A_j^{'} + DA_l^{'}\right)^{-1}\right\| \leq \max_{i\in S} \max_{\tau\in\bar{S}} \left\{\rho_{i\tau}^S(A_D), \rho_{\tau i}^{\bar{S}}(A_D)\right\},$$

where any $i \in S, \tau \in \bar{S}$,

$$\rho_{i\tau}^S(A_D) = \frac{|a_{ii}| - r_i^S(A_D) + r_\tau^S(A_D)}{\left(|a_{ii}| - r_i^S(A_D)\right)\left(|a_{\tau\tau}| - r_\tau^{\bar{S}}(A_D)\right) - r_i^{\bar{S}}(A_D) r_\tau^S(A_D)},$$

$$\rho_{\tau i}^{\bar{S}}(A_D) = \frac{|a_{\tau\tau}| - r_\tau^{\bar{S}}(A_D) + r_\tau^{\bar{S}}(A_D)}{\left(|a_{ii}| - r_i^S(A_D)\right)\left(|a_{\tau\tau}| - r_\tau^{\bar{S}}(A_D)\right) - r_i^{\bar{S}}(A_D) r_\tau^S(A_D)}.$$

We have

$$\begin{aligned}
&|a_{ii}| - r_i^S(A_D) + r_\tau^S(A_D) \\
&= \left|(1-d_i)\left(A'_j\right)_{ii} + d_i\left(A'_l\right)_{ii}\right| - (1-d_i)\, r_i^S\left(A'_j\right) - d_i r_i^S\left(A'_l\right) + (1-d_\tau)\, r_\tau^S\left(A'_j\right) + d_i r_i^S\left(A'_l\right) \\
&< \left(\left|\left(A'_j\right)_{ii}\right| - r_i^S\left(A'_j\right) + r_\tau^S\left(A'_j\right)\right) + \left(\left|\left(A'_l\right)_{ii}\right| - r_i^S\left(A'_l\right) + r_\tau^S\left(A'_l\right)\right) \\
&= \left(\alpha_{i\tau}^j\right)'_{\rho_{i\tau}^S} + \left(\alpha_{i\tau}^l\right)'_{\rho_{i\tau}^S} \\
&\leq 2\max\nolimits_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{max}^S},
\end{aligned}$$

where $t = i, l$, and

$$\left(\alpha_{\max}^t\right)'_{\rho_{max}^S} = \max\left\{\left(\alpha_{i\tau}^t\right)'_{\rho_{i\tau}^S}\right\}, \left(\alpha_{i\tau}^t\right)'_{\rho_{i\tau}^S} = \left|\left(A'_t\right)_{ii}\right| - r_i^S\left(A'_t\right) + r_\tau^S\left(A'_t\right).$$

Similarly, we have

$$\begin{aligned}
&|a_{\tau\tau}| - r_\tau^{\bar{S}}(A_D) + r_i^{\bar{S}}(A_D) \\
&= \left|(1-d_\tau)\left(A'_j\right)_{\tau\tau} + d_\tau\left(A'_l\right)_{ii}\right| - (1-d_\tau)\, r_\tau^{\bar{S}}\left(A'_j\right) - d_\tau r_\tau^{\bar{S}}\left(A'_l\right) + (1-d_i)\, r_i^{\bar{S}}\left(A'_j\right) + d_i r_i^{\bar{S}}\left(A'_l\right) \\
&< \left(\left|\left(A'_j\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A'_j\right) + r_\tau^S\left(A'_j\right)\right) + \left(\left|\left(A'_l\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A'_l\right) + r_i^{\bar{S}}\left(A'_l\right)\right) \\
&= \left(\alpha_{\tau i}^j\right)'_{\rho_{i\tau}^{\bar{S}}} + \left(\alpha_{i\tau}^l\right)'_{\rho_{\tau i}^{\bar{S}}} \\
&\leq 2\max\nolimits_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{max}^{\bar{S}}},
\end{aligned}$$

where $t = j, l$, and

$$\left(\alpha_{\max}^t\right)'_{\rho_{max}^{\bar{S}}} = \max\left\{\left(\alpha_{\tau i}^t\right)'_{\rho_{i\tau}^{\bar{S}}}\right\}, \left(\alpha_{\tau i}^t\right)'_{\rho_{\tau i}^{\bar{S}}} = \left|\left(A'_t\right)_{\tau\tau}\right| - r_i^{\bar{S}}\left(A'_t\right) + r_i^{\bar{S}}\left(A'_t\right).$$

Therefore, it can be concluded that

$$\begin{aligned}
&\left(|a_{ii}| - r_i^S(A_D)\right)\left(|a_{\tau\tau}| - r_\tau^{\bar{S}}(A_D)\right) - r_i^{\bar{S}}(A_D)\, r_\tau^S(A_D) \\
&= \left(\left|(1-d_i)\left(A'_j\right)_{ii} + d_i\left(A'_l\right)_{ii}\right| - (1-d_i)\, r_i^S\left(A'_j\right) - d_i r_i^S\left(A'_l\right)\right) \\
&\times\left(\left|(1-d_\tau)\left(A'_j\right)_{\tau\tau} + d_\tau\left(A'_l\right)_{\tau\tau}\right| - (1-d_\tau)\, r_\tau{}^{\bar{S}}\left(A'_j\right) - d_{\tau_\tau} r_\tau^{\bar{S}}\left(A'_l\right)\right) \\
&-\left((1-d_i)\, r_i^{\bar{S}}\left(A'_j\right) + d_i r_i^{\bar{S}}\left(A'_l\right)\right)\times\left((1-d_\tau)\, r_\tau^S\left(A'_j\right) + d_\tau r_\tau^S\left(A'_l\right)\right) \\
&= (1-d_i)(1-d_\tau)\left(\left|\left(A'_j\right)_{ii}\right| - r_i^S\left(A'_j\right)\right)\left(\left|\left(A'_j\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A'_j\right)\right) \\
&+ (1-d_i)\, d_\tau\left(\left|\left(A'_j\right)_{ii}\right| - r_i^S\left(A'_j\right)\right)\left(\left|\left(A'_l\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A'_l\right)\right) \\
&+ d_i(1-d_\tau)\left(\left|\left(A'_l\right)_{ii}\right| - r_i^S\left(A'_j\right)\right)\left(\left|\left(A'_l\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A'_j\right)\right) \\
&+ d_i d_\tau\left(\left|\left(A'_l\right)_{ii}\right| - r_i^S\left(A'_l\right)\right)\left(\left|\left(A'_l\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A'_l\right)\right) \\
&- (1-d_i)(1-d_\tau)\, r_i^{\bar{S}}\left(A'_j\right) r_\tau^S\left(A'_j\right) - (1-d_i)\, d_\tau r_i^{\bar{S}}\left(A'_j\right) r_\tau^S\left(A'_l\right) \\
&- d_i(1-d_\tau)\, r_i^{\bar{S}}\left(A'_l\right) r_\tau^S\left(A'_j\right) - d_i d_\tau r_i^{\bar{S}}\left(A'_l\right) r_\tau^S\left(A'_l\right)
\end{aligned}$$

$$
\begin{aligned}
&> (1-d_i)(1-d_\tau)\left(\left|\left(A_j'\right)_{ii}\right| - r_i^S\left(A_j'\right)\right)\left(\left|\left(A_j'\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A_j'\right)\right)\\
&+ (1-d_i)\, d_\tau\left(\left|\left(A_j'\right)_{ii}\right| - r_i^S\left(A_j'\right)\right)\left(\left|\left(A_l'\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A_l'\right)\right)\\
&+ d_i d_\tau\left(\left|\left(A_l'\right)_{ii}\right| - r_i^S\left(A_l'\right)\right)\left(\left|\left(A_l'\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A_l'\right)\right)\\
&- (1-d_i)(1-d_\tau)\, r_i^{\bar{S}}\left(A_j'\right) r_\tau^S\left(A_j'\right) - (1-d_i)\, d_\tau r_i^{\bar{S}}\left(A_j'\right) r_\tau^S\left(A_l'\right)\\
&- d_i(1-d_\tau)\, r_i^{\bar{S}}\left(A_l'\right) r_\tau^S\left(A_j'\right) - d_i d_\tau r_i^{\bar{S}}\left(A_l'\right) r_\tau^S\left(A_l'\right)\\
&= (1-d_i)(1-d_\tau)\left[\left(\left|\left(A_j'\right)_{ii}\right| - r_i^S\left(A_j'\right)\right)\left(\left|\left(A_j'\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A_j'\right)\right) - r_i^{\bar{S}}\left(A_j^i\right) r_\tau^S\left(A_j'\right)\right]\\
&+ d_i d_\tau\left[\left(\left|\left(A_l'\right)_{ii}\right| - r_i^S\left(A_l'\right)\right)\left(\left|\left(A_l'\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A_l'\right)\right) - r_i^{\bar{S}}\left(A_l'\right) r_\tau^S\left(A_l'\right)\right]\\
&= (1-d_i)(1-d_\tau)\left(\beta_{i\tau}^j\right)_{A_j'}^{\rho_{i\tau}^S} + d_i d_\tau\left(\beta_{i\tau}^l\right)_{A_l'}^{\rho_{i\tau}^S}\\
&\geq (1-d_i)(1-d_\tau)\left(\beta_{\min}^j\right)_{A_j'}^{\rho_{min}^S} + d_i d_\tau\left(\beta_{\min}^l\right)_{A_l'}^{\rho_{min}^S}\\
&> 0,
\end{aligned}
$$

where $t = j, l$, and

$$
(\beta_{\max}^t)_{A_t'}^{\rho_{min}^S} = \min_{i\in S,\tau\in S}\left\{(\beta_{i\tau}^t)_{A_t'}^{\rho_{i\tau}^S}\right\},
$$

$$
(\beta_{\tau i}^t)_{M_t'}^{\rho_{i\tau}^S} = \left(\left|\left(A_t'\right)_{ii}\right| - r_i^S\left(A_t'\right)\right)\left(\left|\left(A_t'\right)_{\tau\tau}\right| - r_\tau^{\bar{S}}\left(A_t'\right)\right) - r_i^{\bar{S}}\left(A_t'\right) r_\tau^S\left(A_t'\right).
$$

Therefore,

$$
\begin{aligned}
\rho_{i\tau}^S(A_D) &= \frac{|a_{ii}| - r_i^S(A_D) + r_\tau^S(A_D)}{\left(|a_{ii}| - r_i^S(A_D)\right)\left(|a_{\tau\tau}| - r_\tau^{\bar{S}}(A_D)\right) - r_i^{\bar{S}}(A_D) r_\tau^S(A_D)}\\
&< \frac{2\max_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{\max}^S}}{(1-d_i)(1-d_\tau)\left(\beta_{\min}^j\right)_{M_j'}^{\rho_{\min}^S} + d_i d_\tau\left(\beta_{\min}^l\right)_{M_l'}^{\rho_{min}^S}}\\
&\leq \frac{2\max_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{\max}^S}}{\dfrac{\left[min\left\{\left(\beta_{min}^j\right)_{A_j'}^{\rho_{min}^S}, \left(\beta_{min}^l\right)_{A_l'}^{\rho_{min}^S}\right\}\right]^2}{2max\left\{\left(\beta_{min}^j\right)_{A_j'}^{\rho_{min}^S}, \left(\beta_{min}^l\right)_{A_l'}^{\rho_{min}^S}\right\}}}\\
&= \frac{4\max_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{\max}^S} \cdot max\left\{\left(\beta_{min}^j\right)_{A_j'}^{\rho_{min}^S}, \left(\beta_{min}^l\right)_{A_l'}^{\rho_{min}^S}\right\}}{\left[min\left\{\left(\beta_{min}^j\right)_{A_j'}^{\rho_{min}^S}, \left(\beta_{min}^l\right)_{A_l'}^{\rho_{min}^S}\right\}\right]^2}.
\end{aligned}
$$

Similarly, it can be inferred that

$$
\rho_{i\tau}^{\bar{S}}(A_D) \leq \frac{4\max_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{\max}^{\bar{S}}} \cdot max\left\{\left(\beta_{min}^j\right)_{A_j'}^{\rho_{min}^S}, \left(\beta_{min}^l\right)_{A_l'}^{\rho_{min}^S}\right\}}{\left[min\left\{\left(\beta_{min}^j\right)_{A_j'}^{\rho_{min}^S}, \left(\beta_{min}^l\right)_{A_l'}^{\rho_{min}^S}\right\}\right]^2},
$$

then

$$\max_{M'\in\Re(M)}\max_{j<l\in\{0,1,\ldots,k\}}\max_{d\in[0,1]^n}\left\|\left[(I-D)A_j'+DA_l'\right]^{-1}\right\|$$
$$\leq max\max_{i\in S,\tau\in\bar{S}}\left\{\frac{4\max_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{\max}^S}\cdot max\left\{\left(\beta_{min}^j\right)_{A_j'}^{\rho_{min}^S},\left(\beta_{min}^l\right)_{A_l'}^{\rho_{min}^S}\right\}}{\left[min\left\{\left(\beta_{min}^j\right)_{A_j'}^{\rho_{min}^S},\left(\beta_{min}^l\right)_{A_l'}^{\rho_{min}^S}\right\}\right]^2},\frac{4\max_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{\max}^{\bar{S}}}\cdot max\left\{\left(\beta_{min}^j\right)_{A_j'}^{\rho_{min}^S},\left(\beta_{min}^l\right)_{A_l'}^{\rho_{min}^S}\right\}}{\left[min\left\{\left(\beta_{min}^j\right)_{A_j'}^{\rho_{min}^S},\left(\beta_{min}^l\right)_{A_l'}^{\rho_{min}^S}\right\}\right]^2}\right\}.$$

Furthermore, from Definition 1.2, we can regard A_j' and A_l' as two blocks in a row rearrangement fo $A=(A_0,A_1,\ldots,A_k)$, and thus for $t=j$ or $t=l$ and for any $i\in N$, there exists $t_i\in\{0,1,\cdots,k\}$ such that

$$(\alpha_{i\tau}^t)'_{\rho_{i\tau}^S}=\left(\alpha_{i\tau}^{t_i}\right)_{\rho_{i\tau}^S},(\alpha_{\tau i}^t)'_{\rho_{\tau i}^S}=\left(\alpha_{\tau i}^{t_i}\right)_{\rho_{\tau i}^{\bar{S}}},(\beta_{i\tau}^t)_{A_t'}^{\rho_{i\tau}^S}=\left(\beta_{i\tau}^{t_i}\right)_{A_{t_i}}^{\rho_{i\tau}^S},$$

we have

$$\max_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{\max}^S}=\max_{t=j,l}\left\{\max_{i=S,\tau\in\bar{S}}\left(\alpha_{i\tau}^t\right)'_{\rho_{i\tau}^S}\right\}=\max_{i\in S,\tau\in S}\left\{\max_{t=j,l}\left(\alpha_{i\tau}^{t_i}\right)_{\rho_{i\tau}^S}\right\}$$
$$\leq\max_{i\in S,\tau\in\bar{S}}\left\{\max_{p=0,1,\cdots,k}\left(\alpha_{i\tau}^p\right)_{\rho_{i\tau}^S}\right\}=\max_{p=0,1,\cdots,k}\left\{\max_{i\in S,\tau\in\bar{S}}\left(\alpha_{i\tau}^p\right)_{\rho_{i\tau}^S}\right\}=\max_{p=0,1,\cdots,k}\left\{\left(\alpha_{max}^p\right)_{\rho_{max}^S}\right\},$$

and

$$\min\left\{\left(\beta_{\min}^j\right)_{A_j'}^{\rho_{\min}^S},\left(\beta_{\min}^l\right)_{A_l'}^{\rho_{\min}^S}\right\}=\min_{t=j,l}\left\{\min_{i\in S,\tau\in\bar{S}}\left(\beta_{i\tau}^t\right)_{A_t'}^{\rho_{i\tau}^S}\right\}=\min_{t=j,l}\left\{\min_{i\in S,\tau\in\bar{S}}\left(\beta_{i\tau}^{t_i}\right)_{A_{t_i}}^{\rho_{i\tau}^S}\right\}$$
$$=\min_{i\in S,\tau\in S}\left\{\min_{t=j,l}\left(\beta_{i\tau}^{t_i}\right)_{A_{t_i}}^{\rho_{i\tau}^S}\right\}\geq\min_{i\in S,\tau\in\bar{S}}\left\{\min_{p=0,1,\cdots,k}\left(\beta_{i\tau}^p\right)_{A_p}^{\rho_{i\tau}^S}\right\}=\min_{p=0,1,\cdots,k}\left\{\min_{i\in S,\tau\in\bar{S}}\left(\beta_{i\tau}^p\right)_{A_p}^{p_{i\tau}^S}\right\}$$
$$=\min_{p=0,1,\cdots,k}\left\{\left(\beta_{\min}^p\right)_{A_p}^{\rho_{\min}^S}\right\},$$

and

$$\max\left\{\left(\beta_{\min}^j\right)_{A_j'}^{\rho_{\min}^S},\left(\beta_{\min}^l\right)_{A_l'}^{\rho_{\min}^S}\right\}\leq\max_{p=0,1,\cdots,k}\left\{\left(\beta_{\min}^p\right)_{A_p}^{\rho_{\min}^S}\right\},$$

thus

$$\frac{4\max_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{\max}^S}\cdot\max\left\{\left(\beta_{\min}^j\right)_{A_j'}^{\rho_{\min}^S},\left(\beta_{\min}^l\right)_{A_l'}^{\rho_{\min}^S}\right\}}{\left[\min\left\{\left(\beta_{\min}^j\right)_{A_j'}^{\rho_{\min}^S},\left(\beta_{\min}^l\right)_{A_l'}^{\rho_{\min}^S}\right\}\right]^2}\leq\frac{4\max_{p=0,1,\cdots k}\left\{\left(\alpha_{\max}^p\right)_{\rho_{\max}^S}\right\}\cdot\max_{p=0,1,\cdots,k}\left\{\left(\beta_{\min}^p\right)_{A_p}^{\rho_{\min}^S}\right\}}{\min_{p=0,1,\cdots,k}\left\{\left(\beta_{\min}^p\right)_{A_p}^{\rho_{\min}^S}\right\}^2}.$$

Similarly, it can be concluded that

$$\frac{4\max_{t=j,l}\left(\alpha_{\max}^t\right)'_{\rho_{\max}^{\bar{S}}}\cdot\max\left\{\left(\beta_{\min}^j\right)_{A_j'}^{\rho_{\min}^S},\left(\beta_{\min}^l\right)_{A_l'}^{\rho_{\min}^S}\right\}}{\left[\min\left\{\left(\beta_{\min}^j\right)_{A_j'}^{\rho_{\min}^S},\left(\beta_{\min}^l\right)_{A_l'}^{\rho_{\min}^S}\right\}\right]^2}\leq\frac{4\max_{p=0,1,\cdots k}\left\{\left(\alpha_{\max}^p\right)_{\rho_{\max}^{\bar{S}}}\right\}\cdot\max_{p=0,1,\cdots,k}\left\{\left(\beta_{\min}^p\right)_{A_p}^{\rho_{\min}^S}\right\}}{\min_{p=0,1,\cdots,k}\left\{\left(\beta_{\min}^p\right)_{A_p}^{\rho_{\min}^S}\right\}^2},$$

then for any A_j' and A_l' for $A'\in\Re(A)$, satisfy

$$\max_{M'\in\Re(M)}\max_{j<l\in\{0,1,\ldots,k\}}\max_{d\in[0,1]^n}\left\|\left[(I-D)A_j'+DA_l'\right]^{-1}\right\|$$
$$\leq\max\max_{i\in S,\tau\in\bar{S}}\left\{\frac{4\max_{p=0,1,\cdots k}\left\{\left(\alpha_{\max}^p\right)_{\rho_{\max}^S}\right\}\cdot\max_{p=0,1,\cdots,k}\left\{\left(\beta_{\min}^p\right)_{A_p}^{\rho_{\min}^S}\right\}}{\min_{p=0,1,\cdots,k}\left\{\left(\beta_{\min}^p\right)_{A_p}^{\rho_{\min}^S}\right\}^2},\frac{4\max_{p=0,1,\cdots k}\left\{\left(\alpha_{\max}^p\right)_{\rho_{\max}^{\bar{S}}}\right\}\cdot\max_{p=0,1,\cdots,k}\left\{\left(\beta_{\min}^p\right)_{A_p}^{\rho_{\min}^S}\right\}}{\min_{p=0,1,\cdots,k}\left\{\left(\beta_{\min}^p\right)_{A_p}^{\rho_{\min}^S}\right\}^2}\right\},$$

from the arbitrariness of $A_j^{'}$ and $A_l^{'}$, the conclusion follows.

Next, we give a numerical example to illustrate that our results have advantages over some existing results.

Example 1. Let $A = (A_0, A_1, A_2)$, where

$$A_0 = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{pmatrix}, A_1 = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 5 & 1 \\ 2 & 1 & 7 \end{pmatrix}, A_2 = \begin{pmatrix} 3 & 1.5 & 1 \\ 2 & 6 & 3 \\ 1.9 & 1 & 3 \end{pmatrix},$$

are all $\sum -SDD$ matrices and satisfies the condition of Proposition 2.1, thus $A = (A_0, A_1, A_2)A = (A_0, A_1, A_2)$ has the row W-property. Then by Theorem 2.1 we can get

$$\alpha_\infty(A) \leq 2.0378.$$

By Theorem 1.2, Since $\rho\left(\max\left(\Lambda_0^{-1}|Q_0|, \Lambda_1^{-1}|Q_1|, \cdots, \Lambda_k^{-1}|Q_k|\right)\right) = 0.8760 < 1$, we get

$$\alpha_\infty(A) \leq 10.$$

By Theorem 1.3 we get

$$\alpha_\infty(A) \leq 2.5210.$$

According to the calculation example, it can be seen that new bound in Theorem 2.1 is sharper than those in Theorems 1.2 and 1.3 given by Zhang et al. in [1] in some cases.

3. Conclusions

In this paper, I first apply the properties of $\sum -SDD$ matrices to prove that block matrices composed of $\sum -SDD$ matrices have row W-properties under certain conditions and obtains the extension of $\sum -SDD$ matrices under these conditions for the error bound of the solution to the vertical linear complementarity problem. In the process of research, it is found that the error bounds of extended vertical linear complementarity problems for other types of matrices need to be further studied and explored, such as N-type matrices and CKV-type matrices.

Use of AI tools declaration

The author declares he/she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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