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Research article

Compact operators on the new Motzkin sequence spaces

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Abstract: This study aims to construct the BK-spaces $\ell_p(\mathcal{M})$ and $\ell_\infty(\mathcal{M})$ as the domains of the conservative Motzkin matrix $\mathcal{M}$ obtained by using Motzkin numbers. It investigates topological properties, obtains Schauder basis, and then gives inclusion relations. Additionally, it expresses α-, β-, and γ-duals of these spaces and submits the necessary and sufficient conditions of the matrix classes between the described spaces and the classical spaces. In the last part, the characterization of certain compact operators is given with the aid of the Hausdorff measure of non-compactness.

Keywords: Motzkin numbers; sequence spaces; matrix mappings; compact operators; Hausdorff measure of non-compactness
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1. Introduction

All real sequence's space is expressed by ω. Any $\Lambda \subset \omega$ is called a sequence space. Frequently encountered sequence spaces can be denoted as ℓ_∞ (space of bounded sequences), c (space of convergent sequences), c_0 (space of null sequences), and ℓ_p $(1 \leq p < \infty)$ (space of absolutely p-summable sequences). These are Banach spaces with $\|u\|_{\ell_\infty} = \|u\|_c = \|u\|_{c_0} = \sup_{r\in\mathbb{N}} |u_r|$ and $\|u\|_{\ell_p} = (\sum_r |u_r|^p)^{\frac{1}{p}}$ for $u = (u_r) \in \omega$, $\sum_r |u_r| = \sum_{r=0}^{+\infty} |u_r|$ and $\mathbb{N} = \{0, 1, 2, 3, ...\}$. Furthermore, bs and cs denote the bounded and convergent series' spaces, respectively.

A Banach space on which all coordinate functionals κ_r defined as $\kappa_r(u) = u_r$ are continuous is named a BK-space. Consider the sequence $e^{(r)}$ whose r^{th} term is 1 and other terms are zero. If each $u = (u_r) \in \Lambda \subset \omega$ can be expressed uniquely as $u = \sum_r u_r e^{(r)}$, then the space Λ satisfies the AK-property.

Consider the multiplier set $(\Lambda \circledast \Upsilon)$ defined by

$$(\Lambda \circledast \Upsilon) = \Big\{\tau = (\tau_r) \in \omega : \tau u = (\tau_r u_r) \in \Upsilon \quad \text{for all} \quad u \in \Lambda\Big\},$$

for $\Lambda, \Upsilon \subset \omega$. Thus, the α-, β- and γ-duals of Λ are expressed by

$$\Lambda^{\alpha} = (\Lambda \divideontimes \ell_1), \quad \Lambda^{\beta} = (\Lambda \divideontimes cs) \quad \text{and} \quad \Lambda^{\gamma} = (\Lambda \divideontimes bs).$$

B_r refers to the rth row of an infinite matrix $B = (b_{rs})$ with real terms. Additionally, if the series is convergent for all $r \in \mathbb{N}$, $(Bu)_r = \sum_s b_{rs}u_s$ is a B-transform of $u \in \omega$. The infinite matrix B is a matrix mapping from Λ to Υ, if $Bu \in \Upsilon$ for all $u \in \Lambda$. The class of all matrix mappings described from Λ to Υ is given by $(\Lambda : \Upsilon)$. Additionally, $B \in (\Lambda : \Upsilon)$ iff $B_r \in \Lambda^{\beta}$ and $Bu \in \Upsilon$ for all $u \in \Lambda$. The set

$$\Upsilon_B = \{u \in \omega : Bu \in \Upsilon\}, \tag{1.1}$$

is a matrix domain of B in Υ.

Spaces Λ and Υ, between which a norm-preserving bijection can be defined, are linearly norm isomorphic spaces, and this situation is denoted by $\Lambda \cong \Upsilon$.

One of the impressive number sequences is the integer sequences consisting of Motzkin numbers which are named after Theodore Motzkin [21]. In mathematics, the r^{th} Motzkin number refers to the number of distinct chords that can be drawn between r points on a circle without intersecting. It should be noted that it is not necessary for the chord to touch all points on the circle. The Motzkin numbers M_r ($r \in \mathbb{N}$) have various applications in geometry, combinatorics, and number theory, and they are represented by the following sequence:

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, \ldots.$$

The Motzkin numbers satisfy the recurrence relations

$$M_r = M_{r-1} + \sum_{s=0}^{r-2} M_s M_{r-s-2} = \frac{2r+1}{r+1} M_{r-1} + \frac{3r-3}{r+2} M_{r-2}.$$

Another relation provided by the Motzkin numbers is given below:

$$M_{r+2} - M_{r+1} = \sum_{s=0}^{r} M_s M_{r-s}, \text{ for } r \geq 0. \tag{1.2}$$

Furthermore, there are two other relations between Motzkin and Catalan numbers C_r, presented by

$$M_r = \sum_{s=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2s} C_s \quad \text{and} \quad C_{r+1} = \sum_{s=0}^{r} \binom{r}{s} M_s,$$

where $\lfloor . \rfloor$ is the floor function.

The generating function $m(u) = \sum_{r=0}^{+\infty} M_r u^r$ of the Motzkin numbers holds

$$u^2 + [m(u)]^2 + (u-1)m(u) + 1 = 0$$

and is described by

$$m(u) = \frac{1 - u - \sqrt{1 - 2u - 3u^2}}{2u^2}.$$

The expression on Motzkin numbers with the help of integral function is as follows:

$$M_r = \frac{2}{\pi}\int_0^{\pi} \sin^2 u\,(2cosu+1)^r\,du.$$

They have the asymptotic behavior

$$M_r \sim \frac{1}{2\sqrt{\pi}}\left(\frac{3}{r}\right)^{\frac{3}{2}} 3^r, \quad r\to\infty.$$

Moreover, it is satisfied from Aigner [1] that

$$\lim_{r\to+\infty}\frac{M_{r+1}}{M_r} = 3.$$

The idea of obtaining sequence spaces with infinity matrices created with the help of integer sequences, such as Schröder [6,7], Mersenne [8], Catalan [12,13,16], Fibonacci [14,15], Lucas [17,18], Bell [19], Padovan [28], and Leonardo [29], has been used by various authors for this purpose. In this context, sources such as summability theory, sequence spaces, matrix domain studies and the necessary basic concepts can be specified [2, 4, 5, 9, 11, 24].

The Motzkin matrix $\mathcal{M} = (\mathfrak{m}_{rs})_{r,s\in\mathbb{N}}$ constructed with the help of Motzkin numbers and relation (1.2) by Erdem et al. [10] can be written follows:

$$\mathfrak{m}_{rs} := \begin{cases} \dfrac{M_s M_{r-s}}{M_{r+2}-M_{r+1}} & , \quad \text{if } 0\le s\le r, \\ 0 & , \quad \text{if } s>r. \end{cases} \tag{1.3}$$

It is possible to state the Motzkin matrix more clearly as follows:

$$\mathcal{M} := \begin{bmatrix} \frac{M_0M_0}{M_2-M_1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{M_0M_1}{M_3-M_2} & \frac{M_1M_0}{M_3-M_2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{M_0M_2}{M_4-M_3} & \frac{M_1M_1}{M_4-M_3} & \frac{M_2M_0}{M_4-M_3} & 0 & 0 & 0 & \cdots \\ \frac{M_0M_3}{M_5-M_4} & \frac{M_1M_2}{M_5-M_4} & \frac{M_2M_1}{M_5-M_4} & \frac{M_3M_0}{M_5-M_4} & 0 & 0 & \cdots \\ \frac{M_0M_4}{M_6-M_5} & \frac{M_1M_3}{M_6-M_5} & \frac{M_2M_2}{M_6-M_5} & \frac{M_3M_1}{M_6-M_5} & \frac{M_4M_0}{M_6-M_5} & 0 & \cdots \\ \frac{M_0M_5}{M_7-M_6} & \frac{M_1M_4}{M_7-M_6} & \frac{M_2M_3}{M_7-M_6} & \frac{M_3M_2}{M_7-M_6} & \frac{M_4M_1}{M_7-M_6} & \frac{M_5M_0}{M_7-M_6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Furthermore, it is also known from [10] that the Motzkin matrix $\mathcal{M}$ is conservative, that is, $\mathcal{M}\in(c:c)$ and the inverse $\mathcal{M}^{-1} = (\mathfrak{m}_{rs}^{-1})$ of the Motzkin matrix $\mathcal{M}$ is

$$\mathfrak{m}_{rs}^{-1} := \begin{cases} (-1)^{r-s}\dfrac{M_{s+2}-M_{s+1}}{M_r}\pi_{r-s} & , \quad \text{if } 0\le s\le r, \\ 0 & , \quad \text{if } s>r, \end{cases} \tag{1.4}$$

where $\pi_0 = 0$ and

$$\pi_r = \begin{vmatrix} M_1 & M_0 & 0 & 0 & \cdots & 0 \\ M_2 & M_1 & M_0 & 0 & \cdots & 0 \\ M_3 & M_2 & M_1 & M_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ M_r & M_{r-1} & M_{r-2} & M_{r-3} & \cdots & M_1 \end{vmatrix},$$

for all $r \in \mathbb{N}\backslash\{0\}$. From its definition, it is clear that $\mathcal{M}$ is a triangle. Furthermore, $\mathcal{M}$-transform of a sequence $u = (u_s)$ is stated as

$$v_r := (\mathcal{M}u)_r = \frac{1}{M_{r+2} - M_{r+1}} \sum_{s=0}^{r} M_s M_{r-s} u_s, \quad (r \in \mathbb{N}). \tag{1.5}$$

The idea of constructing new normed sequence spaces as domains of special infinite matrices, as an application of summability theory to sequence spaces, has appeared as a favorite research area in recent years. Creating these infinite matrices with the help of special number sequences and thus obtaining new normed sequence spaces and also examining some of properties (e.g., completeness, inclusion relations, Schauder basis, duals, matrix transformations, compact operators, and core theorems) is newer topic of study since Kara and Başarır [14].

The primary research question of this study is whether it is possible to obtain new normed sequence spaces as the domain of the conservative Motzkin matrix obtained with the help of Motzkin numbers and to examine some of the properties just mentioned above on these spaces.

In this study, two new sequence spaces are obtained as domains of the Motzkin matrix on ℓ_p ($1 \le p < +\infty$) and ℓ_∞. Subsequently, some algebraic and topological properties, inclusion relations, basis, duals, and matrix transformations of these spaces are presented. In the last section, compactness criteria of some matrix operators defined on these spaces are investigated.

2. Motzkin sequence spaces $\ell_p(\mathcal{M})$ and $\ell_\infty(\mathcal{M})$

In this part, we introduce the BK-sequence spaces $\ell_p(\mathcal{M})$ and $\ell_\infty(\mathcal{M})$ with the help of the Motzkin matrix which are linearly isomorphic to ℓ_p and ℓ_∞, respectively. Finally, the Schauder basis of $\ell_p(\mathcal{M})$ and the inclusion relations of the spaces are presented.

Now, we may present the Motzkin sequence spaces $\ell_p(\mathcal{M})$ and $\ell_\infty(\mathcal{M})$ as follows:

$$\ell_p(\mathcal{M}) = \left\{ u = (u_s) \in \omega : \sum_r \left| \frac{1}{M_{r+2} - M_{r+1}} \sum_{s=0}^{r} M_s M_{r-s} u_s \right|^p < \infty \right\},$$

and

$$\ell_\infty(\mathcal{M}) = \left\{ u = (u_s) \in \omega : \sup_{r \in \mathbb{N}} \left| \frac{1}{M_{r+2} - M_{r+1}} \sum_{s=0}^{r} M_s M_{r-s} u_s \right| < \infty \right\},$$

for $1 \le p < +\infty$. Then, $\ell_p(\mathcal{M})$ and $\ell_\infty(\mathcal{M})$ can be rewritten as $\ell_p(\mathcal{M}) = (\ell_p)_{\mathcal{M}}$ and $\ell_\infty(\mathcal{M}) = (\ell_\infty)_{\mathcal{M}}$ with the notation (1.1). The matrix domain $\Upsilon_{\mathcal{M}}$ is a Motzkin sequence space for each normed sequence space Υ.

It should be noted that BK-spaces have a significant role in summability theory. For instance, the matrix operators between BK-spaces are continuous. Additionally, there is a useful method for the characterizations of compact linear operators between the spaces as an application of the Hausdorf measure of non-compactness.

Now we can show that the newly defined spaces are BK-spaces:

Theorem 2.1. *$\ell_p(\mathcal{M})$ and $\ell_\infty(\mathcal{M})$ are BK-spaces with*

$$\|u\|_{\ell_p(\mathcal{M})} = \left(\sum_r \left|\frac{1}{M_{r+2}-M_{r+1}}\sum_{s=0}^{r} M_s M_{r-s} u_s\right|^p\right)^{\frac{1}{p}},$$

and

$$\|u\|_{\ell_\infty(\mathcal{M})} = \sup_{r\in\mathbb{N}} \left|\frac{1}{M_{r+2}-M_{r+1}}\sum_{s=0}^{r} M_s M_{r-s} u_s\right|,$$

respectively.

Proof. In the proof of this theorem, it will be used that ℓ_p and ℓ_∞ are BK-spaces. It is known from Wilansky [27] that Υ_B is BK-space with $\|u\|_{\Upsilon_B} = \|Bu\|_\Upsilon$ if B is triangle and Υ is a BK-space. Consequently, $\ell_p(\mathcal{M})$ and $\ell_\infty(\mathcal{M})$ are BK-spaces with the norms $\|.\|_{\ell_p(\mathcal{M})}$ and $\|.\|_{\ell_\infty(\mathcal{M})}$, respectively. □

Theorem 2.2. *$\ell_p(\mathcal{M}) \cong \ell_p$ and $\ell_\infty(\mathcal{M}) \cong \ell_\infty$.*

Proof. In order to show that two spaces are linearly norm isomorphic, there must be a linear norm preserving bijection between them. The mapping $\mathcal{K} : \ell_p(\mathcal{M}) \to \ell_p$, $\mathcal{K}(u) = \mathcal{M}u$ is linear and since $\mathcal{K}(u) = 0 \Rightarrow u = 0$, $\mathcal{K}$ is injective. Consider the sequences $v = (v_s) \in \ell_p$ and $u = (u_s) \in \omega$ with

$$u_s = \sum_{i=0}^{s} (-1)^{s-i} \frac{M_{i+2}-M_{i+1}}{M_s} \pi_{s-i} v_i. \quad (s \in \mathbb{N}). \tag{2.1}$$

For the equation

$$\begin{aligned}
(\mathcal{M}u)_r &= \frac{1}{M_{r+2}-M_{r+1}} \sum_{s=0}^{r} M_s M_{r-s} u_s \\
&= \frac{1}{M_{r+2}-M_{r+1}} \sum_{s=0}^{r} M_s M_{r-s} \sum_{i=0}^{s} (-1)^{s-i} \frac{M_{i+2}-M_{i+1}}{M_s} \pi_{s-i} v_i \\
&= \frac{1}{M_{r+2}-M_{r+1}} \sum_{i=0}^{r} \left(\sum_{s=0}^{r-i} (-1)^s M_{r-s-i} \pi_s\right) (M_{i+2}-M_{i+1})\, v_i \\
&= \frac{1}{M_{r+2}-M_{r+1}} \left((-1)^0 M_0 \pi_0\right) (M_{r+2}-M_{r+1})\, v_r = v_r,
\end{aligned}$$

for $\sum_{s=0}^{r-i} (-1)^s M_{r-s-i} \pi_s = 0$, if $r \neq i$, then $\mathcal{K}$ is surjective. Additionally, since the relation $\|u\|_{\ell_p(\mathcal{M})} = \|\mathcal{M}u\|_{\ell_p}$ holds, then $\mathcal{K}$ keeps the norm.

The proof can be done similarly for the other spaces.

□

As a result of the isomorphism between the mentioned spaces, connections can be established between some properties. However, since the new concepts presented here may enable different perspectives and open a new door to generalizations, it is useful to express some basic theorems for newly defined spaces.

For a normed sequence space $(\Lambda, \|.\|)$ and $(\eta_r) \in \Lambda$, (η_r) is a Schauder basis for Λ if for any $u \in \Lambda$, there is a unique sequence (σ_r) of scalars such that

$$\left\| u - \sum_{s=0}^{r} \sigma_s \eta_s \right\| \longrightarrow 0,$$

as $r \to +\infty$. This can be stated as $u = \sum_s \sigma_s \eta_s$.

The inverse image of the basis $(e^{(s)})_{s\in\mathbb{N}}$ of ℓ_p becomes the basis of $\ell_p(\mathcal{M})$ $(1 \le p < +\infty)$ since $\mathcal{K} : \ell_p(\mathcal{M}) \to \ell_p$ is an isomorphism in Theorem 2.2. Therefore, it will be given the following result without proof.

Theorem 2.3. *The set* $\eta^{(s)} = \left(\eta_r^{(s)}\right) \in \ell_p(\mathcal{M})$ *expressed by*

$$\eta_r^{(s)} := \begin{cases} (-1)^{r-s} \dfrac{M_{s+2} - M_{s+1}}{M_r} \pi_{r-s} &, \quad \text{if } 0 \le s \le r, \\ \\ 0 &, \quad \text{if } s > r, \end{cases}$$

is a Schauder basis for $\ell_p(\mathcal{M})$ *and the unique representation of any* $u \in \ell_p(\mathcal{M})$ *is stated as* $u = \sum_s \sigma_s \eta^{(s)}$ *for* $1 \le p < +\infty$ *and* $\sigma_s = (\mathcal{M}u)_s$.

It is worth noting that since every normed linear space with a Schauder basis is separable, the sequence space $\ell_p(\mathcal{M})$ is seperable.

Theorem 2.4. *The inclusion* $\ell_p(\mathcal{M}) \subset \ell_{\tilde{p}}(\mathcal{M})$ *strictly holds for* $1 \le p < \tilde{p} < +\infty$.

Proof. For the first part, it is sufficient to show that every element taken from $\ell_p(\mathcal{M})$ is in $\ell_{\tilde{p}}(\mathcal{M})$. Let us take $u = (u_s) \in \ell_p(\mathcal{M})$ such that $\mathcal{M}u \in \ell_p$. Furthermore, it is known that $\ell_p \subset \ell_{\tilde{p}}$ for $1 \le p < \tilde{p} < +\infty$. Then, $\mathcal{M}u \in \ell_{\tilde{p}}$ and $u = (u_s) \in \ell_{\tilde{p}}(\mathcal{M})$.

If $\tilde{v} = \mathcal{M}\tilde{u} \in \ell_{\tilde{p}} \setminus \ell_p$, then the rest of the proof is completed. □

Theorem 2.5. $\ell_\infty \subset \ell_\infty(\mathcal{M})$.

Proof. For the proof, it is necessary to show that every element taken from ℓ_∞ is in $\ell_\infty(\mathcal{M})$. For $u = (u_s) \in \ell_\infty$, it is clear that

$$\begin{aligned} \|u\|_{\ell_\infty(\mathcal{M})} &= \sup_{r\in\mathbb{N}} \left| \frac{1}{M_{r+2} - M_{r+1}} \sum_{s=0}^{r} M_s M_{r-s} u_s \right| \\ &\le \|u\|_\infty \sup_{r\in\mathbb{N}} \left| \frac{1}{M_{r+2} - M_{r+1}} \sum_{s=0}^{r} M_s M_{r-s} \right| \\ &= \|u\|_\infty < +\infty. \end{aligned}$$

Thus, $u \in \ell_\infty(\mathcal{M})$. □

3. Dual spaces

In third part of the article, duals of new spaces will be found.

The idea of dual space plays an important role in the representation of linear functionals and the characterization of matrix transformations between sequence spaces. Consider that $\xi \in \{\alpha,\beta,\gamma\}$. In that case,

$$\ell_1^\xi = \ell_\infty, \quad \ell_\infty^\xi = c_0^\xi = c^\xi = \ell_1 \text{ and } \ell_p^\xi = \ell_q, \text{ where } 1 < p,q < +\infty \text{ with } q = \frac{p}{p-1}.$$

Additionally, for any $\Lambda \in \omega$, $\Lambda^\alpha \subset \Lambda^\beta \subset \Lambda^\gamma$. If $(b_{rs})_{s\in\mathbb{N}} \in \Lambda^\beta$ for all $r \in \mathbb{N}$, then, the B-transform $(Bu)_r = \sum_s b_{rs}u_s$ of any sequence $u = (u_s) \in \Lambda$ is convergent for an infinite matrix B.

Now, consider the conditions (3.1)–(3.12) as

$$\sup_{s\in\mathbb{N}} \sum_r |b_{rs}| < \infty, \tag{3.1}$$

$$\sup_{s\in\mathbb{N}} \sum_r |b_{rs}|^p < \infty, \tag{3.2}$$

$$\sup_{r,s\in\mathbb{N}} |b_{rs}| < \infty, \tag{3.3}$$

$$\lim_{r\to+\infty} b_{rs} \text{ exists for all } s \in \mathbb{N}, \tag{3.4}$$

$$\lim_{r\to+\infty} b_{rs} = 0, \tag{3.5}$$

$$\sup_{E\in\mathcal{F}} \sum_s \left|\sum_{r\in E} b_{rs}\right|^q < \infty, \tag{3.6}$$

$$\sup_{r\in\mathbb{N}} \sum_s |b_{rs}|^q < \infty, \tag{3.7}$$

$$\sup_{E\in\mathcal{F}} \sum_r \left|\sum_{s\in E} b_{rs}\right| < \infty, \tag{3.8}$$

$$\sup_{E\in\mathcal{F}} \sum_r \left|\sum_{s\in E} b_{rs}\right|^p < \infty, \tag{3.9}$$

$$\sup_{r\in\mathbb{N}} \sum_s |b_{rs}| < \infty, \tag{3.10}$$

$$\lim_{r\to+\infty} \sum_s |b_{rs}| = \sum_s \left|\lim_{r\to+\infty} b_{rs}\right|, \tag{3.11}$$

$$\lim_{r\to+\infty} \sum_s |b_{rs}| = 0, \tag{3.12}$$

where, the $\mathcal{F} \subset \mathbb{N}$ is finite and $1 < p < +\infty$. It can be now presented the table created using [26] and characterizing some matrix classes:

Let us consider the sets $\varpi_1 - \varpi_7$ which will be used to compute the duals:

$$\varpi_1 = \left\{\tau = (\tau_s) \in \omega : \sup_{E\in\mathcal{F}} \sum_s \left|\sum_{r\in E} (-1)^{r-s} \frac{M_{s+2} - M_{s+1}}{M_r} \pi_{r-s}\tau_r\right|^q < \infty\right\},$$

$$\varpi_2 = \left\{\tau=(\tau_s)\in\omega : \sup_{s\in\mathbb{N}}\sum_r\left|(-1)^{r-s}\frac{M_{s+2}-M_{s+1}}{M_r}\pi_{r-s}\tau_r\right|<\infty\right\},$$

$$\varpi_3 = \left\{\tau=(\tau_s)\in\omega : \sup_{E\in\mathcal{F}}\sum_r\left|\sum_{s\in E}(-1)^{r-s}\frac{M_{s+2}-M_{s+1}}{M_r}\pi_{r-s}\tau_r\right|<\infty\right\},$$

$$\varpi_4 = \left\{\tau=(\tau_s)\in\omega : \lim_{r\to+\infty}\sum_{i=s}^{r}(-1)^{i-s}\frac{M_{s+2}-M_{s+1}}{M_i}\pi_{i-s}\tau_i \text{ exists for each } s\in\mathbb{N}\right\},$$

$$\varpi_5 = \left\{\tau=(\tau_s)\in\omega : \sup_{r\in\mathbb{N}}\sum_s\left|\sum_{i=s}^{r}(-1)^{i-s}\frac{M_{s+2}-M_{s+1}}{M_i}\pi_{i-s}\tau_i\right|^q<\infty\right\},$$

$$\varpi_6 = \left\{\tau=(\tau_s)\in\omega : \sup_{r,s\in\mathbb{N}}\left|\sum_{i=s}^{r}(-1)^{i-s}\frac{M_{s+2}-M_{s+1}}{M_i}\pi_{i-s}\tau_i\right|<\infty\right\},$$

$$\varpi_7 = \left\{\tau=(\tau_s)\in\omega : \lim_{r\to+\infty}\sum_s\left|\sum_{i=s}^{r}(-1)^{i-s}\frac{M_{s+2}-M_{s+1}}{M_i}\pi_{i-s}\tau_i\right| = \sum_s\left|\sum_{i=s}^{\infty}(-1)^{i-s}\frac{M_{s+2}-M_{s+1}}{M_i}\pi_{i-s}\tau_i\right|\right\}.$$

Theorem 3.1. *The following statements hold:*

(i) $\left(\ell_p(\mathcal{M})\right)^\alpha = \varpi_1,\ (1<p<+\infty)$,

(ii) $(\ell_1(\mathcal{M}))^\alpha = \varpi_2$,

(iii) $(\ell_\infty(\mathcal{M}))^\alpha = \varpi_3$.

Proof. **(i)** From (1.5),

$$\begin{aligned}\tau_r u_r &= \tau_r\left(\sum_{s=0}^{r}(-1)^{r-s}\frac{M_{s+2}-M_{s+1}}{M_r}\pi_{r-s}\nu_s\right)\\ &= \left(\sum_{s=0}^{r}(-1)^{r-s}\frac{M_{s+2}-M_{s+1}}{M_r}\pi_{r-s}\tau_r\right)\nu_s = (G\nu)_r\,,\end{aligned} \tag{3.13}$$

for all $r\in\mathbb{N}$ and $u\in\ell_p(\mathcal{M})$, where the infinite matrix $G=(g_{rs})$ can described as

$$g_{rs} := \begin{cases}(-1)^{r-s}\dfrac{M_{s+2}-M_{s+1}}{M_r}\pi_{r-s}\tau_s, & \text{if } 0\le s\le r,\\ 0, & \text{if } s>r.\end{cases}$$

Then by (3.13), $\tau u=(\tau_r u_r)\in\ell_1$ while $u\in\ell_p(\mathcal{M})$ iff $G\nu\in\ell_1$ while $\nu\in\ell_p$. Thus, $\tau\in\left(\ell_p(\mathcal{M})\right)^\alpha$ iff $G\in(\ell_p:\ell_1)$. From Table 1, $\left(\ell_p(\mathcal{M})\right)^\alpha=\varpi_1$ for $1<p<+\infty$.

The proofs of **(ii)** and **(iii)** can be seen similarly to the first part through the aid of the conditions of the classes $(\ell_1:\ell_1)$ and $(\ell_\infty:\ell_1)$, respectively, from Table 1. Therefore, they are ommitted. □

Table 1. Characterizations of $(\Lambda : \Upsilon)$, where $\Lambda, \Upsilon \in \{\ell_1, \ell_p, \ell_\infty, c, c_0\}$ and $1 < p < +\infty$.

$(\Lambda \downarrow: \Upsilon \rightarrow)$	ℓ_1	ℓ_p	ℓ_∞	c	c_0
ℓ_1	(3.1)	(3.2)	(3.3)	(3.3),(3.4)	(3.3),(3.5)
ℓ_p	(3.6)	•	(3.7)	(3.4),(3.7)	(3.5),(3.7)
ℓ_∞	(3.8)	(3.9)	(3.10)	(3.4),(3.11)	(3.12)
c	(3.8)	(3.9)	(3.10)	•	•
c_0	(3.8)	(3.9)	(3.10)	•	•

Note: The symbol "•" represents the conditions of the classes that are unknown or not interest to this study.

Theorem 3.2. *The following statements hold:*

(i) $\left(\ell_p(\mathcal{M})\right)^\beta = \varpi_4 \cap \varpi_5$, $(1 < p < +\infty)$,

(ii) $(\ell_1(\mathcal{M}))^\beta = \varpi_4 \cap \varpi_6$,

(iii) $(\ell_\infty(\mathcal{M}))^\beta = \varpi_4 \cap \varpi_7$.

Proof. **(i)** Consider that $\tau = (\tau_s) \in \omega$ and $u \in \ell_p(\mathcal{M})$ with $\nu \in \ell_p$ as (1.5). By considering the Eq (2.1), we can see that

$$\begin{aligned}\psi_r = \sum_{s=0}^{r} \tau_s u_s &= \sum_{s=0}^{r} \tau_s \left(\sum_{i=0}^{s} (-1)^{s-i} \frac{M_{i+2} - M_{i+1}}{M_s} \pi_{s-i} \right) \nu_i \\ &= \sum_{s=0}^{r} \left(\sum_{i=s}^{r} (-1)^{i-s} \frac{M_{s+2} - M_{s+1}}{M_i} \pi_{i-s} \tau_i \right) \nu_s \\ &= (O\nu)_r,\end{aligned} \tag{3.14}$$

for $O = (o_{rs})$ is expressed by

$$o_{rs} := \begin{cases} \displaystyle\sum_{i=s}^{r} (-1)^{i-s} \frac{M_{s+2} - M_{s+1}}{M_i} \pi_{i-s} \tau_i & , \quad 0 \leq s \leq r, \\ 0 & , \quad s > r, \end{cases} \tag{3.15}$$

for every $r, s \in \mathbb{N}$. Then, from (3.14), $\tau u \in cs$ while $u = (u_s) \in \ell_p(\mathcal{M})$ iff $\psi = (\psi_r) \in c$ while $\nu \in \ell_p$. In this case, $\tau \in \left(\ell_p(\mathcal{M})\right)^\beta$ iff $O \in \left(\ell_p : c\right)$. Consequently, from the conditions of $\left(\ell_p : c\right)$ in Table 1, the proof is complete. □

The proofs of the **(ii)** and **(iii)** can easily be seen similarly to the first part through the aid of the conditions of the classes $(\ell_1 : c)$ and $(\ell_\infty : c)$, respectively, from the Table 1. Therefore, they are ommitted too.

Theorem 3.3. *The following statements hold:*

(i) $\left(\ell_p(\mathcal{M})\right)^\gamma = \varpi_5, (1 < p < +\infty)$,

(ii) $(\ell_1(\mathcal{M}))^\gamma = \varpi_6$,

(iii) $(\ell_\infty(\mathcal{M})^\gamma = \varpi_5$ *with* $q = 1$.

Proof. This can be done similarly with Theorem 3.2 by considering with together the classes $(\ell_p : \ell_\infty)$, $(\ell_1 : \ell_\infty)$, and $(\ell_\infty : \ell_\infty)$ from Table 1 with $O = (o_{rs})$ expressed by (3.15). □

4. Matrix mappings

This section offers to submit the matrix classes related Motzkin sequence spaces described in this study. The theorem it will be written now forms the fundamental of this section.

Theorem 4.1. *Let us consider the* $\Lambda, \Upsilon \in \omega$, *infinite matrices* $H^{(r)} = \left(h_{is}^{(r)}\right)$ *and* $H^* = (h^*_{rs})$ *described as*

$$h_{is}^{(r)} := \begin{cases} \displaystyle\sum_{j=s}^{i} (-1)^{j-s} \frac{M_{s+2} - M_{s+1}}{M_j} \pi_{j-s} b_{rj} & , \quad 0 \leq s \leq i, \\ \\ 0 & , \quad s > i, \end{cases} \tag{4.1}$$

and

$$h^*_{rs} = \sum_{j=s}^{\infty} (-1)^{j-s} \frac{M_{s+2} - M_{s+1}}{M_j} \pi_{j-s} b_{rj}, \tag{4.2}$$

for all $r, s \in \mathbb{N}$. *In that case,* $B = (b_{rs}) \in (\Lambda(\mathcal{M}) : \Upsilon)$ *if and only if* $H^{(r)} \in (\Lambda : c)$ *and* $H^* \in (\Lambda : \Upsilon)$.

Proof. Let us consider $B = (b_{rs}) \in (\Lambda(\mathcal{M}) : \Upsilon)$ and $u \in \Lambda(\mathcal{M})$. In that case,

$$\begin{aligned} \sum_{s=0}^{i} b_{rs} u_s &= \sum_{s=0}^{i} b_{rs} \left(\sum_{j=0}^{s} (-1)^{s-j} \frac{M_{j+2} - M_{j+1}}{M_s} \pi_{s-j} \nu_j \right) \\ &= \sum_{s=0}^{i} \left(\sum_{j=s}^{i} (-1)^{j-s} \frac{M_{s+2} - M_{s+1}}{M_j} \pi_{j-s} b_{rj} \right) \nu_s = \sum_{s=0}^{i} h_{is}^{(r)} \nu_s, \end{aligned} \tag{4.3}$$

for all $i, r \in \mathbb{N}$. Since Bu exists, then $H^{(r)} \in (\Lambda : c)$. By passing limit for $i \to +\infty$ in the relation (4.3), $Bu = H^*\nu$. Since $Bu \in \Upsilon$, $H^*\nu \in \Upsilon$ and so $H^* \in (\Lambda : \Psi)$.

Conversely, let us suppose that $H^{(r)} \in (\Lambda : c)$ and $H^* \in (\Lambda : \Upsilon)$. Then, $h^*_{rs} \in \Lambda^\beta$ which gives us $(b_{rs})_{s\in\mathbb{N}} \in (\Lambda(\mathcal{M}))^\beta$ for all $r \in \mathbb{N}$. Hence, Bu exists for all $u \in \Lambda(\mathcal{M})$. Therefore, from relation (4.3) for $i \to +\infty$, $Bu = H^*\nu$. Thus $B \in (\Lambda(\mathcal{M}) : \Upsilon)$, which is desired result. □

Corollary 4.2. *Consider the infinite matrices* $H^{(r)} = (h_{is}^{(r)})$ *and* $H^* = (h^*_{rs})$ *described with the relations (4.1) and (4.2), respectively. Then, the conditions of the classes* $(\Gamma(\mathcal{M}) : \Psi)$ *can be deduced from Table 2, where* $\Lambda \in \{\ell_1, \ell_p, \ell_\infty\}$, $\Upsilon \in \{\ell_1, \ell_p, \ell_\infty, c, c_0\}$ *and* $1 < p < +\infty$.

Theorem 4.3. *Consider the infinite matrices* $\tilde{B} = (\tilde{b}_{rs})$ *and* $B = (b_{rs})$ *described with the relation*

$$\tilde{b}_{rs} = \sum_{j=0}^{r} \frac{M_j M_{r-j}}{M_{r+2} - M_{r+1}} b_{js}. \tag{4.4}$$

In that case, $B \in (\Lambda : \Upsilon(\mathcal{M}))$ *iff* $\tilde{B} \in (\Lambda : \Upsilon)$ *for* $\Lambda \in \left\{\ell_1, \ell_p, \ell_\infty, c, c_0\right\}$ *and* $\Upsilon \in \left\{\ell_1, \ell_p, \ell_\infty\right\}$.

Table 2. Characterizations of the classes $(\Lambda(\mathcal{M}) : \Upsilon)$, where $\Lambda \in \{\ell_1, \ell_p, \ell_\infty\}$, $\Upsilon \in \{\ell_1, \ell_p, \ell_\infty, c, c_0\}$ and $1 < p < +\infty$.

$(\Lambda(\mathcal{M}) \downarrow: \Upsilon \rightarrow)$	ℓ_1	ℓ_p	ℓ_∞	c	c_0
$\ell_1(\mathcal{M})$	$(3.3)^r, (3.4)^r$ $(3.1)^*$	$(3.3)^r, (3.4)^r$ $(3.2)^*$	$(3.3)^r, (3.4)^r$ $(3.3)^*$	$(3.3)^r, (3.4)^r$ $(3.3)^*, (3.4)^*$	$(3.3)^r, (3.4)^r$ $(3.3)^*, (3.5)^*$
$\ell_p(\mathcal{M})$	$(3.4)^r, (3.7)^r$ $(3.6)^*$	•	$(3.4)^r, (3.7)^r$ $(3.7)^*$	$(3.4)^r, (3.7)^r$ $(3.4)^*, (3.7)^*$	$(3.4)^r, (3.7)^r$ $(3.5)^*, (3.7)^*$
$\ell_\infty(\mathcal{M})$	$(3.4)^r, (3.11)^r$ $(3.8)^*$	$(3.4)^r, (3.11)^r$ $(3.9)^*$	$(3.4)^r, (3.11)^r$ $(3.10)^*$	$(3.4)^r, (3.11)^r$ $(3.4)^*, (3.11)^*$	$(3.4)^r, (3.11)^r$ $(3.12)^*$

Note: Conditions $(\lambda)^r$ and $(\lambda)^*$ represent the condition (λ) hold with the matrices $H^{(r)}$ and H^*, respectively, for $3.1 \leq \lambda \leq 3.12$.

Proof. Consider that the infinite matrices $\tilde{B}$ and B described with the relation (4.4), $\Lambda \in \left\{\ell_1, \ell_p, \ell_\infty, c, c_0\right\}$, and $\Upsilon \in \left\{\ell_1, \ell_p, \ell_\infty\right\}$. For any $u = (u_s) \in \Lambda$,

$$\sum_{s=0}^{\infty} \tilde{b}_{rs} u_s = \sum_{j=0}^{r} \frac{M_j M_{r-j}}{M_{r+2} - M_{r+1}} \sum_{s=0}^{\infty} b_{js} u_s.$$

This means that $\tilde{B}_r(u) = \mathcal{M}_r(Bu)$ for all $r \in \mathbb{N}$, which implies that $Bu \in \Upsilon(\mathcal{M})$ iff $\tilde{B}u \in \Upsilon$ for every $u \in \Lambda$. Thus, $B \in (\Lambda : \Upsilon(\mathcal{M}))$ if and only if $\tilde{B} \in (\Lambda : \Upsilon)$. □

Corollary 4.4. *Consider the infinite matrices $\tilde{B} = (\tilde{b}_{rs})$ and $B = (b_{rs})$ described with the relation (4.4). In that case, the necessary and sufficient conditions for the classes $(\Lambda : \Upsilon(\mathcal{M}))$ can be found in Table 3, where $\Lambda \in \left\{\ell_1, \ell_p, \ell_\infty, c, c_0\right\}$ and $\Upsilon \in \left\{\ell_1, \ell_p, \ell_\infty\right\}$.*

Table 3. Characterizations of $(\Lambda : \Upsilon(\mathcal{M}))$, where $\Lambda \in \left\{\ell_1, \ell_p, \ell_\infty, c, c_0\right\}$ and $\Upsilon \in \left\{\ell_1, \ell_p, \ell_\infty\right\}$.

$(\Lambda \downarrow: \Upsilon(\mathcal{M})) \rightarrow)$	$\ell_1(\mathcal{M})$	$\ell_p(\mathcal{M})$	$\ell_\infty(\mathcal{M})$
ℓ_1	(3.1)	(3.2)	(3.3)
ℓ_p	(3.6)	•	(3.7)
ℓ_∞	(3.8)	(3.9)	(3.10)
c	(3.8)	(3.9)	(3.10)
c_0	(3.8)	(3.9)	(3.10)

Note: Conditions hold with the matrix $\tilde{B} = (\tilde{b}_{rs})$.

5. Compactness by Hausdorff measure of non-compactness

Measures of non-compactness play an important role in functional analysis. They are important tools in metric fixed point theory, the theory of operator equations in Banach spaces, and the characterizations of classes of compact operators. They are also applied in the studies of various kinds of differential and integral equations. For instance, the characterization of compact operators between BK-spaces benefits from Hausdorff measure of non-compactness.

Consider the normed space Λ and the unit sphere $\mathcal{D}_\Lambda$ in Λ. The notation $\|u\|_\Lambda^\diamond$ is expressed by

$$\|u\|_\Lambda^\diamond = \sup_{x \in \mathcal{D}_\Lambda} \left| \sum_s u_s x_s \right|,$$

for a BK-space $\Lambda \supset \Omega$ and $u = (u_s) \in \omega$ for all finite sequences' space, Ω, provided that the series is finite and $u \in \Lambda^{\beta}$.

Lemma 5.1. *[20] The following are satisfied:*

(i) $\ell_\infty^{\beta} = \ell_1$ *and* $\|u\|_{\ell_\infty}^{\diamond} = \|u\|_{\ell_1}$, $\forall u \in \ell_1$.

(ii) $\ell_1^{\beta} = \ell_\infty$ *and* $\|u\|_{\ell_1}^{\diamond} = \|u\|_{\ell_\infty}$, $\forall u \in \ell_\infty$.

(iii) $\ell_p^{\beta} = \ell_q$ *and* $\|u\|_{\ell_p}^{\diamond} = \|u\|_{\ell_q}$, $\forall u \in \ell_q$.

The set of all bounded linear mappings from Λ to Υ is denoted by $\mathfrak{C}(\Lambda : \Upsilon)$.

Lemma 5.2. *[20] There exists $\mathcal{K}_B \in \mathfrak{C}(\Lambda : \Upsilon)$ as $\mathcal{K}_B(u) = Bu$ for BK-spaces Λ and Υ, as well as for all $u \in \Lambda$ and $B \in (\Lambda : \Upsilon)$.*

Lemma 5.3. *[20] If $B \in (\Lambda : \Upsilon)$, then $\|\mathcal{K}_B\| = \|B\|_{(\Lambda:\Upsilon)} = \sup_{r\in\mathbb{N}} \|B_r\|_{\Lambda}^{\diamond} < \infty$ for $\Upsilon \in \{c_0, c, \ell_\infty\}$ and the BK-space $\Lambda \supset \Omega$.*

The Hausdorff measure of non-compactness of a bounded set $\mathcal{P}$ in the metric space Λ is stated with

$$\chi(\mathcal{P}) = \inf\left\{\epsilon > 0 : \mathcal{P} \subset \cup_{j=1}^{r} Q(u_j, n_j), u_j \in \Lambda, n_j < \epsilon, r \in \mathbb{N}\backslash 0\right\},$$

where $Q(u_j, n_j)$ is the open ball centred at u_j and radius n_j for each $j = 1, 2, ..., r$. In-depth information on the subject can be obtained from [20] and its references.

Theorem 5.4. *[25] Consider that $\mathcal{P} \subset \ell_p$ is bounded and mapping $\Psi_n : \ell_p \longrightarrow \ell_p$ stated by $\Psi_n(u) = (u_0, u_1, u_2, ..., u_n, 0, 0, ...)$ for all $u = (u_s) \in \ell_p$, $1 \leq p < +\infty$ and $n \in \mathbb{N}$. Then,*

$$\chi(\mathcal{P}) = \lim_{n\to\infty}\left(\sup_{u\in\mathcal{P}} \|(\mathcal{I} - \Psi_n)(u)\|_{\ell_p}\right),$$

for the identity operator $\mathcal{I}$ on ℓ_p.

A linear mapping $\mathcal{K}$ is compact if $(\mathcal{K}(u))$ has a convergent subsequence in Υ for all $u = (u_s) \in \ell_\infty \cap \Lambda$ for the Banach spaces Λ and Υ.

The Hausdorff measure of non-compactness $\|\mathcal{K}\|_\chi$ of $\mathcal{K}$ is expressed with $\|\mathcal{K}\|_\chi = \chi(\mathcal{K}(\mathcal{D}_\Lambda))$. In that case, $\mathcal{K}$ is compact iff $\|\mathcal{K}\|_\chi = 0$.

The studies [3,22,23] can be given as examples of studies on sequence spaces considered in terms of compactness and Hausdorff measure of non-compactness relationship.

The following results will given for the sequences $x = (x_s)$ and $y = (y_s)$ which are elements of ω and are attached to each other by the relation

$$y_s = \sum_{j=s}^{\infty}(-1)^{j-s}\frac{M_{s+2} - M_{s+1}}{M_j}\pi_{j-s}x_j, \tag{5.1}$$

for all $s \in \mathbb{N}$.

Lemma 5.5. *Let us consider the sequence* $x = (x_s) \in (\ell_p(\mathcal{M}))^\beta$ *for* $1 \leq p \leq \infty$. *In that case,* $y = (y_s) \in \ell_q$ *and*

$$\sum_s x_s u_s = \sum_s y_s v_s, \tag{5.2}$$

for all $u = (u_s) \in \ell_p(\mathcal{M})$.

Lemma 5.6. *Let us consider the sequence* $y = (y_s)$ *described with relation (5.1). In that case, the following statements hold:*

(i) $\|x\|^\diamond_{\ell_\infty(\mathcal{M})} = \sum_s |y_s| < \infty$ *for all* $x = (x_s) \in (\ell_\infty(\mathcal{M}))^\beta$.

(ii) $\|x\|^\diamond_{\ell_1(\mathcal{M})} = \sup_s |y_s| < \infty$ *for all* $x = (x_s) \in (\ell_1(\mathcal{M}))^\beta$.

(iii) $\|x\|^\diamond_{\ell_p(\mathcal{M})} = \left(\sum_s |y_s|^q\right)^{\frac{1}{q}} < \infty$ *for all* $x = (x_s) \in \left(\ell_p(\mathcal{M})\right)^\beta$ *and* $1 < p < +\infty$.

Proof. Only a proof of the first part will be given because the other parts are similar.
(i) From Lemma 5.5, $y = (y_s) \in \ell_1$ and (5.2) holds for $x = (x_s) \in (\ell_\infty(\mathcal{M}))^\beta$ and for all $u = (u_s) \in \ell_\infty(\mathcal{M})$. Since $\|u\|_{\ell_\infty(\mathcal{M})} = \|v\|_{\ell_\infty}$ with (1.5), then $u \in \mathcal{D}_{\ell_\infty(\mathcal{M})}$ if and only if $v \in \mathcal{D}_{\ell_\infty}$. Thus, we can write the equality $\|x\|^\diamond_{\ell_\infty(\mathcal{M})} = \sup_{u\in\mathcal{D}_{\ell_\infty(\mathcal{M})}} \left|\sum_s x_s u_s\right| = \sup_{v\in\mathcal{D}_{\ell_\infty}} \left|\sum_s y_s v_s\right| = \|y\|^\diamond_{\ell_\infty}$. By the aid of the Lemma 5.1, it follows that $\|x\|^\diamond_{\ell_\infty(\mathcal{M})} = \|y\|^\diamond_{\ell_\infty} = \|y\|_{\ell_1} = \sum_s |y_s| < \infty$. □

Lemma 5.7. *[22] Considering the BK-space* $\Lambda \supset \Omega$*;*

(i) *If* $B \in (\Lambda : \ell_\infty)$*, then* $0 \leq \|\mathcal{K}_B\|_\chi \leq \limsup_r \|B_r\|^\diamond_\Lambda$ *and* $\mathcal{K}_B$ *is compact if* $\lim_r \|B_r\|^\diamond_\Lambda = 0$.

(ii) *If* $B \in (\Lambda : c_0)$*, then* $\|\mathcal{K}_B\|_\chi = \limsup_r \|B_r\|^\diamond_\Lambda$ *and* $\mathcal{K}_B$ *is compact if and only if* $\lim_r \|B_r\|^\diamond_\Lambda = 0$.

(iii) *If* $B \in (\Lambda : \ell_1)$*, then*

$$\lim_j \left(\sup_{E\in\mathcal{F}_j} \left\|\sum_{r\in E} B_r\right\|^\diamond_\Lambda\right) \leq \|\mathcal{K}_B\|_\chi \leq 4.\lim_j \left(\sup_{E\in\mathcal{F}_j} \left\|\sum_{r\in E} B_r\right\|^\diamond_\Lambda\right),$$

and $\mathcal{K}_B$ *is compact iff* $\lim_j \left(\sup_{E\in\mathcal{F}_j} \|\sum_{r\in E} B_r\|^\diamond_\Lambda\right) = 0$, *where* $\mathcal{F}$ *represents the family of all finite subsets of* $\mathbb{N}$ *and* $\mathcal{F}_j$ *is the subcollection of* $\mathcal{F}$ *consisting of subsets of* $\mathbb{N}$ *with elements that are greater than j.*

The matrices $H^* = (h^*_{rs})$ and $B = (b_{rs})$ connected with (4.2) will be considered in the continuation of the study under the assumption that the series is convergent.

Lemma 5.8. *Let* $\Upsilon \subset \omega$ *and* $B = (b_{rs})$ *be an infinite matrix. If* $B \in (\ell_p(\mathcal{M}) : \Upsilon)$*, then* $H^* \in (\ell_p : \Upsilon)$ *and* $Bu = H^* v$ *hold for all* $u \in \ell_p(\mathcal{M})$ *and* $1 \leq p \leq +\infty$.

Proof. It is obvious from Lemma 5.5. So, we omit it. □

Theorem 5.9. *For* $1 < p < +\infty$*:*

(i) *If $B \in (\ell_p(\mathcal{M}) : \ell_\infty)$, then*

$$0 \le \|\mathcal{K}_B\|_\chi \le \limsup_r \left(\sum_s |h^*_{rs}|^q \right)^{\frac{1}{q}},$$

and $\mathcal{K}_B$ is compact if

$$\lim_r \left(\sum_s |h^*_{rs}|^q \right)^{\frac{1}{q}} = 0.$$

(ii) *If $B \in (\ell_p(\mathcal{M}) : c_0)$, then*

$$\|\mathcal{K}_B\|_\chi = \limsup_r \left(\sum_s |h^*_{rs}|^q \right)^{\frac{1}{q}},$$

and $\mathcal{K}_B$ is compact iff

$$\lim_r \left(\sum_s |h^*_{rs}|^q \right)^{\frac{1}{q}} = 0.$$

(iii) *If $B \in (\ell_p(\mathcal{M}) : \ell_1)$, then*

$$\lim_j \|B\|^{(j)}_{(\ell_p(\mathcal{M}):\ell_1)} \le \|\mathcal{K}_B\|_\chi \le 4. \lim_j \|B\|^{(j)}_{(\ell_p(\mathcal{M}):\ell_1)},$$

and $\mathcal{K}_B$ is compact iff

$$\lim_j \|B\|^{(j)}_{(\ell_p(\mathcal{M}):\ell_1)} = 0,$$

where $\|B\|^{(j)}_{(\ell_p(\mathcal{M}):\ell_1)} = \sup_{E \in \mathcal{F}_j} \left(\sum_s \left| \sum_{r \in E} h^*_{rs} \right|^q \right)^{\frac{1}{q}}$ for all $j \in \mathbb{N}$.

Proof. **(i)** Let $B \in (\ell_p(\mathcal{M}) : \ell_\infty)$ and $u = (u_s) \in \ell_p(\mathcal{M})$. Since the series $\sum_s b_{rs} u_s$ converges for each $r \in \mathbb{N}$, $B_r \in (\ell_p(\mathcal{M}))^\beta$. From Lemma 5.6 (iii), we reach that $\|B_r\|^\diamond_{\ell_p(\mathcal{M})} = (\sum_s |h^*_{rs}|^q)^{\frac{1}{q}}$. In that case, from Lemma 5.7 (i), we see that

$$0 \le \|\mathcal{K}_B\|_\chi \le \limsup_r \left(\sum_s |h^*_{rs}|^q \right)^{\frac{1}{q}},$$

and $\mathcal{K}_B$ is compact if

$$\lim_r \left(\sum_s |h^*_{rs}|^q \right)^{\frac{1}{q}} = 0.$$

(ii) Suppose that $B \in (\ell_p(\mathcal{M}) : c_0)$. Since $\|B_r\|^\diamond_{\ell_p(\mathcal{M})} = (\sum_s |h^*_{rs}|^q)^{\frac{1}{q}}$ for each $r \in \mathbb{N}$ and from Lemma 5.7 (ii), we see that

$$\|\mathcal{K}_B\|_\chi = \limsup_r \left(\sum_s |h^*_{rs}|^q \right)^{\frac{1}{q}},$$

and $\mathcal{K}_B$ is compact iff

$$\lim_r \left(\sum_s |h^*_{rs}|^q \right)^{\frac{1}{q}} = 0.$$

(iii) Let $B \in (\ell_p(\mathcal{M}) : \ell_1)$. From Lemma 5.6, $\|\sum_{r\in E} B_r\|^{\diamond}_{\ell_p(\mathcal{M})} = \|\sum_{r\in E} H^*_r\|^{\diamond}_{\ell_q}$. Thus, from Lemma 5.7 (iii), we see that

$$\lim_j \left(\sup_{E\in\mathcal{F}_j} \sum_s \left| \sum_{r\in E} h^*_{rs} \right|^q \right)^{\frac{1}{q}} \leq \|\mathcal{K}_B\|_\chi \leq 4. \lim_j \left(\sup_{E\in\mathcal{F}_j} \sum_s \left| \sum_{r\in E} h^*_{rs} \right|^q \right)^{\frac{1}{q}},$$

and $\mathcal{K}_B$ is compact iff

$$\lim_j \left(\sup_{E\in\mathcal{F}_j} \sum_s \left| \sum_{r\in E} h^*_{rs} \right|^q \right)^{\frac{1}{q}} = 0.$$

□

Theorem 5.10. *The following statements hold:*

(i) *If* $B \in (\ell_\infty(\mathcal{M}) : \ell_\infty)$, *then*

$$0 \leq \|\mathcal{K}_B\|_\chi \leq \limsup_r \sum_s |h^*_{rs}|,$$

and $\mathcal{K}_B$ *is compact if*

$$\lim_r \sum_s |h^*_{rs}| = 0.$$

(ii) *If* $B \in (\ell_\infty(\mathcal{M}) : c_0)$, *then*

$$\|\mathcal{K}_B\|_\chi = \limsup_r \sum_s |h^*_{rs}|,$$

and $\mathcal{K}_B$ *is compact iff*

$$\lim_r \sum_s |h^*_{rs}| = 0.$$

(iii) *If* $B \in (\ell_\infty(\mathcal{M}) : \ell_1)$, *then*

$$\lim_j \|B\|^{(j)}_{(\ell_\infty(\mathcal{M}):\ell_1)} \leq \|\mathcal{K}_B\|_\chi \leq 4. \lim_j \|B\|^{(j)}_{(\ell_\infty(\mathcal{M}):\ell_1)},$$

and $\mathcal{K}_B$ *is compact iff*

$$\lim_j \|B\|^{(j)}_{(\ell_\infty(\mathcal{M}):\ell_1)} = 0,$$

where $\|B\|^{(j)}_{(\ell_\infty(\mathcal{M}):\ell_1)} = \sup_{E\in\mathcal{F}_j} \left(\sum_s \left| \sum_{r\in E} h^*_{rs} \right| \right)$.

Proof. This can be proven in the same manner as Theorem 5.9, so we omit the proof here. □

Theorem 5.11. **(i)** *If* $B \in (\ell_1(\mathcal{M}) : \ell_\infty)$, *then*

$$0 \leq \|\mathcal{K}_B\|_\chi \leq \limsup_r \left(\sup_s |h^*_{rs}| \right),$$

and $\mathcal{K}_B$ *is compact if*

$$\lim_r \left(\sup_s |h^*_{rs}| \right) = 0.$$

(ii) *If $B \in (\ell_1(\mathcal{M}) : c_0)$, then*

$$\|\mathcal{K}_B\|_\chi = \limsup_r \left(\sup_s |h^*_{rs}|\right),$$

and $\mathcal{K}_B$ is compact iff

$$\lim_r \left(\sup_s |h^*_{rs}|\right) = 0.$$

Proof. It can be seen this in a similar way to the proof of Theorem 5.9. □

Lemma 5.12. *[22] If Λ has AK property or $\Lambda = \ell_\infty$ and $B \in (\Lambda : c)$, then*

$$\frac{1}{2}\limsup_r \|B_r - b\|^\diamond_\Lambda \le \|\mathcal{K}_B\|_\chi \le \limsup_r \|B_r - b\|^\diamond_\Lambda,$$

and $\mathcal{K}_B$ is compact iff

$$\lim_r \|B_r - b\|^\diamond_\Lambda = 0,$$

where $b = (b_s)$ and $b_s = \lim_r b_{rs}$.

Theorem 5.13. *If $B \in (\ell_p(\mathcal{M}) : c)$ for $1 < p < +\infty$, then*

$$\frac{1}{2}\limsup_r \left(\sum_s \left|h^*_{rs} - h^*_s\right|^q\right)^{\frac{1}{q}} \le \|\mathcal{K}_B\|_\chi \le \limsup_r \left(\sum_s \left|h^*_{rs} - h^*_s\right|^q\right)^{\frac{1}{q}},$$

and $\mathcal{K}_B$ is compact iff

$$\lim_r \left(\sum_s \left|h^*_{rs} - h^*_s\right|^q\right)^{\frac{1}{q}} = 0.$$

Proof. Let $B \in (\ell_p(\mathcal{M}) : c)$. In that case, it is obtained that $H^* \in (\ell_p : c)$ by Lemma 5.8. By the aid of the Lemma 5.12, we reache that

$$\frac{1}{2}\limsup_r \|H^*_r - h^*\|^\diamond_{\ell_p} \le \|\mathcal{K}_B\|_\chi \le \limsup_r \|H^*_r - h^*\|^\diamond_{\ell_p}.$$

This implies that, by Lemma 5.6 (iii),

$$\frac{1}{2}\limsup_r \left(\sum_s \left|h^*_{rs} - h^*_s\right|^q\right)^{\frac{1}{q}} \le \|\mathcal{K}_B\|_\chi \le \limsup_r \left(\sum_s \left|h^*_{rs} - h^*_s\right|^q\right)^{\frac{1}{q}},$$

holds. Hence, we conclude with Lemma 5.12 that $\mathcal{K}_B$ is compact iff

$$\lim_r \left(\sum_s \left|h^*_{rs} - h^*_s\right|^q\right)^{\frac{1}{q}} = 0.$$

□

Theorem 5.14. *If $B \in (\ell_\infty(\mathcal{M}) : c)$, in this case*

$$\frac{1}{2}\limsup_r \left(\sum_s \left| h^*_{rs} - h^*_s \right| \right) \leq \|\mathcal{K}_B\|_\chi \leq \limsup_r \left(\sum_s \left| h^*_{rs} - h^*_s \right| \right),$$

and $\mathcal{K}_B$ is compact iff

$$\lim_r \left(\sum_s \left| h^*_{rs} - h^*_s \right| \right) = 0.$$

Proof. It can be seen this in a similar way to the proof of Theorem 5.13. □

Theorem 5.15. *If $B \in (\ell_1(\mathcal{M}) : c)$, then*

$$\frac{1}{2}\limsup_r \left(\sup_s \left| h^*_{rs} - h^*_s \right| \right) \leq \|\mathcal{K}_B\|_\chi \leq \limsup_r \left(\sup_s \left| h^*_{rs} - h^*_s \right| \right),$$

and $\mathcal{K}_B$ is compact iff

$$\lim_r \left(\sup_s \left| h^*_{rs} - h^*_s \right| \right) = 0.$$

Proof. It can be seen this also in a similar way to the proof of Theorem 5.13. □

6. Conclusions

In this study, as an example of the application of matrix summability methods to Banach spaces, two new sequence spaces are constructed as the domains of the Motzkin matrix operator defined by Erdem et al. [10] in the sequence spaces ℓ_p and ℓ_∞, some algebraic and topological properties of these spaces are revealed, their duals are calculated, some matrix classes concerning the new spaces are characterized and finally, the compactness criteria of some operators on these spaces are expressed with the help of the Hausdorff measure of non-compactness.

In our future work, we plan to act with the idea expressed above and obtain new normed and paranormed sequence spaces in this direction.

Creating infinite matrices with the help of special number sequences and thus obtaining new normed or paranormed sequence spaces and also examining some properties in these spaces (e.g., completeness, inclusion relations, Schauder basis, duals, matrix transformations, compact operators and core theorems) can be suggested as an idea to researchers who want to study in this field.

List of abbreviations and symbols

M_r	: r^{th} Motzkin number
$\mathcal{M}$	: Motzkin matrix
ω	: space of all real sequences
ℓ_∞	: space of bounded sequences
c	: space of convergent sequences
c_0	: space of null sequences
ℓ_p	: space of absolutely p-summable sequences
bs	: space of bounded series
cs	: space of convergent series
$(\Lambda : \Upsilon)$	: class of matrix mappings described from Λ to Υ
Υ_B	: matrix domain of B in Υ
$\Lambda \cong \Upsilon$	: linear isomorphism between the spaces Λ and Υ
$\lfloor . \rfloor$	: floor function
$\mathbb{N}$	: set of positive integers with zero
$\mathcal{F}$	: collection of all finite subsets of $\mathbb{N}$
$\mathcal{F}_j$	: subcollection of $\mathcal{F}$ consisting of subsets of $\mathbb{N}$ with elements that are greater than j
Λ^α	: α-dual of the sequence space Λ
Λ^β	: β-dual of the sequence space Λ
Λ^γ	: γ-dual of the sequence space Λ
$\mathcal{D}_\Lambda$	: unit sphere of the normed space Λ
$\mathfrak{C}(\Lambda : \Upsilon)$	: set of all bounded linear mappings from Λ to Υ
$\chi(\mathcal{P})$	: Hausdorff measure of non-compactness of a bounded set $\mathcal{P}$
$\|\mathcal{K}\|_\chi$	: Hausdorff measure of non-compactness of a linear mapping $\mathcal{K}$
$Q(u_j, n_j)$	: open ball centred at u_j with radius n_j

Use of AI tools declaration

The author declares he is not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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