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Research article

Analysis of rumor spreading with different usage ranges in a multilingual environment

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Abstract: This paper investigates rumor propagation in a multilingual environment, taking into account language usage variations. Firstly, a 2I2S2R model is proposed within a heterogeneous network framework that incorporates both immunologic and cross-transmitted mechanisms. Secondly, the paper calculates the basic reproduction number R_0 by the next-generation matrix method. Thirdly, the local asymptotic stability and the global asymptotic stability are further explored, which indicate that whether the rumor continuously spreads or becomes extinct is determined by the threshold. Finally, the numerical simulation and sensitivity analysis are given to illustrate the effectiveness of theoretical results and the influence of model parameters on rumor spreading.

Keywords: rumor spreading; heterogeneous network; multi-lingual environment; stability analysis
Mathematics Subject Classification: 34D05, 37N20

1. Introduction

Recently, the dissemination of rumors has invariably harmed society. Rumors often conceal the truth and cause people to be misled and deceived, which may affect people's judgment and decision-making ability. What is more serious is that rumors often spread unreliable information, creating panic and chaos, which poses a great threat to social stability and security. With the development of social networks, rumors spread more rapidly and widely, so the issue of rumor propagation needs to be given more attention. Hence, understanding the patterns and traits of rumor dissemination empowers governments and individuals to effectively implement appropriate measures, steer public opinion, and safeguard social stability.

Research on rumor propagation has been underway since the last century. In 1965, Daley and Kendall [1] established a connection between rumors and epidemics, introducing the classical DK rumor-spreading model. Subsequently, numerous scholars have identified increasing distinctions between rumor propagation and epidemic transmission, leading to the development of various rumor propagation models, such as, the SIR model [2], the SIHR model [3], the stochastic model [4], the diffusion model [5], and so on [6–9]. Concretely, in reference [2], an enhanced rumor propagation model based on the SIR model was utilized to investigate the rumor issue in complex social networks. Reference [3] introduced a SIHR model, which considered the interplay between forgetting and memory mechanisms. Additionally, an extended rumor spreading model incorporating knowledge education was presented in [6]. Notably, the above research is conducted on the basis of homogeneous networks.

In a homogeneous network, all nodes belong to the same type and share similar features or attributes. However, the applicability of the above studies is limited, as it is challenging to find a social network where all users have the same degree of reality. Especially with the increasing development of social networks, users with different characteristics or attributes are more easily accessible [10–13]. Hence, more research on the spread of rumors on heterogeneous networks has begun to emerge [14–16]. Some heterogeneous network models were introduced to analyze the dynamic behaviors of the rumor propagation [17–20]. For example, an IFCD model was addressed, which took full consideration of the heterogeneity of network users and stochastic disturbances in the network environment [11]. In references [16, 19], some delay rumor propagation models were proposed to study its stability and bifurcation occurs in heterogeneous networks. Anti-rumor mechanism was introduced to control rumor spreading in heterogeneous networks [17]. The dynamical behaviors and control of rumor propagation model incorporating delay were investigated under a heterogeneous social networks [20].

Notably, most research on rumor dissemination focuses on a single-language environment. However, in recent years, particularly in multi-ethnic regions of China, the escalation of social issues due to rumors has underscored the growing significance of studying rumor propagation in multilingual settings. This is related to social stability and national unity. At present, some scholars have redirected their attention towards examining rumor dissemination in multilingual environments, such as, some extended multi-lingual SIR rumor spreading models that were proposed to delve into its dynamical behaviors and control strategies [21,22]. However, the above research landscape on multilingual rumor propagation within homogeneous networks cannot achieve a comprehensive and practical study of the multilingual rumor spreading mechanism.

Subsequently, more dynamic models and control strategies were addressed to analyze the problem of multilingual rumor propagation in the heterogeneous network [6, 23–25]. For instance, a rumor propagation model with two language spreaders was proposed to analyze its stability, which considered the network topology [23]. However, it is worth noting that the above studies, whether based on homogeneous or heterogeneous networks [21–23], mostly treat multiple languages as equally prevalent, assuming consistent usage across all languages. While this approach may be suitable for some situations, it does not reflect the reality of many multilingual areas. For example, in China, Mandarin is widely spoken, but proficiency in other languages is limited, which results in networks with varying language hierarchies. Motivated by these analyses, the language usage variations are taken into account in this paper, and a novel rumor propagation model tailored to such asymmetric language networks is introduced to address the complexities of real-world multilingual settings.

Based on the mechanism described above, this paper focuses on studying the global stability of a multilingual SIR rumor propagation model with unidirectional propagation patterns between different groups. The main contributions of this paper include the following aspects:

1) Compared to existing results [21–23], this paper explores the impact of language usage scope, considers the unidirectional transmission relationship between two language groups, which can fill a gap in the dissemination of multilingual rumors to a certain extent.

2) Based on a one-way propagation relationship in a multilingual environment, a 2I2S2R model is proposed to analyze the dynamics of multilingual rumor propagation by applying the Lyapunov function and the

3) In the numerical simulations, the sensitivity analysis of the basic reproduction number is addressed to further illustrate the influence of model parameters on the process of rumor propagation and provide control strategies for rumor suppression.

Inspired by these analyses, the paper is organized as follows: The network model is introduced in Section 2. The existence and stability of equilibrium solutions are shown in Section 3. Some numerical examples are further addressed to demonstrate the validity of Section 4. The conclusions are given in Section 5.

2. Network model

In this section, we construct a SIR multilingual rumor propagation model. Six states are proposed to indicate the different statuses of the rumor-spreading process. Ignorants-1 ($I^1(t)$) and Ignorants-2 ($I^2(t)$) represent the people who do not know the rumor. Language-1 is the one with a smaller usage range, while Language-2 has a larger one. In our hypothetical environment, people who can speak Language-1 will definitely speak Language-2. Spreaders-1 ($S^1(t)$) represent the people who know Rumor-1, and Spreaders-2 ($S^2(t)$) represent those who know Rumor-2. Removers-1 ($R^1(t)$) and Removers-2 ($R^2(t)$) mean the people who know Rumor-1 and Rumor-2, respectively, but have not spread them. Rumor-1 and Rumor-2 are the same rumor, which are popular in the environments of Language-1 and Language-2, respectively. $I^1(t)$, $S^1(t)$, and $R^1(t)$ form Group-1, and Group-2 consists of $I^2(t)$, $S^2(t)$, and $R^2(t)$. People in Group-1 can speak both Language-1 and Language-2, so people from Group-1 will have a certain probability of transferring to Group-2. The people who are originally in Group-2 can only speak Language-2, for which people in Group-2 will not transfer to Group-1. Next, the state transition is depicted in Figure 1 with the following rules, and the meaning of main parameters is shown in Table 1.

1) A user changes state I^1 to state S^v by believing the rumor with different possible routes of transmission ($v = 1, 2$). An ignorant I^1 becomes $S^v (v = 1, 2)$ with rumor conversion rate $\alpha_v (v = 1, 2)$, and I^1 is connected to one or more $S^v (v = 1, 2)$ with probability $\Phi_v(t) (v = 1, 2)$ at time t. Hence, the infected probability of I^1 becoming S_1 (or S_2) is $\alpha_1 I^1_{k_i}(t)\Phi_1(t)$ (or $\alpha_2 I^1_{k_i}(t)\Phi_2(t)$), respectively.

2) An ignorant I^2 becomes S^2 with rumor conversion rate α_2, and is connected to one or more S^2 with probability $\Phi_2(t)$. Additionally, S^1 can also become S^2 with probability $\alpha_2 S^1_{k_i}(t)\Phi_2(t)$.

3) In this process of rumor propagation, the population enters each group in a ratio of $b_v (v = 1, 2)$. The proportion of users leaving the system in each group is d. I^1 and I^2 can both be directly converted into R^1 and R^2 with probabilities of μ_1 and μ_2, respectively.

Table 1. Main parameters in the 2I2S2R model.

Parameters	Meaning
b_v	Coming rate of the group v
d	Leaving rate of the different compartment
α_v	Rumor conversion rate (i.e., cross transmission showed in Figure 1)
β_v	Probability of $S^v_{k_i}$ cured and transformed into $R^v_{k_i}$
μ_v	Immunity rate of Ignorants against rumors
$\langle k \rangle$	The average degree
$\theta(k_i)$	Infectivity of a user with degree k_i
$\Phi_v(t)$	The probabilities that an ignorant whether in Group-1 or Group-2 gets in touch with spreaders

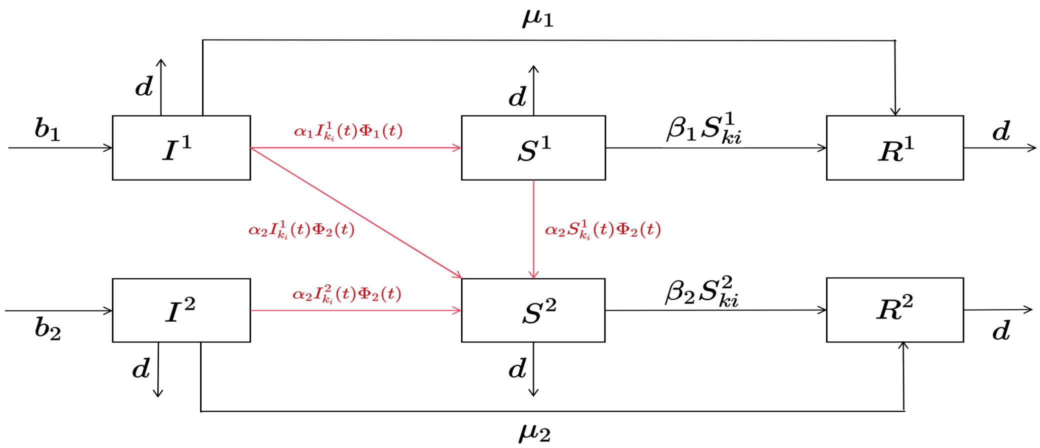

Figure 1. The state transition of 2I2S2R model.

From the above analysis, the 2I2S2R rumor propagation is described as follows:

$$\begin{cases} \dfrac{\mathrm{d}I^1_{k_i}(t)}{\mathrm{d}t} = b_1 - \alpha_1 I^1_{k_i}(t)\Phi_1(t) - \alpha_2 I^1_{k_i}(t)\Phi_2(t) - (\mu_1 + d)I^1_{k_i}(t), \\ \dfrac{\mathrm{d}I^2_{k_i}(t)}{\mathrm{d}t} = b_2 - \alpha_2 I^2_{k_i}(t)\Phi_2(t) - (\mu_2 + d)I^2_{k_i}(t), \\ \dfrac{\mathrm{d}S^1_{k_i}(t)}{\mathrm{d}t} = \alpha_1 I^1_{k_i}(t)\Phi_1(t) - \alpha_2 S^1_{k_i}(t)\Phi_2(t) - (\beta_1 + d)S^1_{k_i}(t), \\ \dfrac{\mathrm{d}S^2_{k_i}(t)}{\mathrm{d}t} = \alpha_2 [I^1_{k_i}(t) + I^2_{k_i}(t) + S^1_{k_i}(t)]\Phi_2(t) - (\beta_2 + d)S^2_{k_i}(t), \\ \dfrac{\mathrm{d}R^1_{k_i}(t)}{\mathrm{d}t} = \mu_1 I^1_{k_i}(t) + \beta_1 S^1_{k_i}(t) - dR^1_{k_i}(t), \\ \dfrac{\mathrm{d}R^2_{k_i}(t)}{\mathrm{d}t} = \mu_2 I^2_{k_i}(t) + \beta_2 S^2_{k_i}(t) - dR^2_{k_i}(t), \end{cases} \tag{2.1}$$

where $\Phi_v(t)$ represents the probabilities that an ignorant person whether in Group-1 or Group-2 gets in touch with spreaders, and $\Phi_v(t)$ is defined as

$$\Phi_v(t) = \frac{\sum_{i=1}^{n} \theta(k_i) Z(k_i) S_{k_i}^{v}(t)}{\langle k \rangle}, \ v = 1, 2. \tag{2.2}$$

It represents the contact rate between group S^v and another group. $Z(k_i)$ is defined as the ratio of people with degree k_i to all users in the network; therefore, $\sum_{i=1}^{n} Z(k_i) = 1$ and $\langle k \rangle = \sum_{i=1}^{n} k_i Z(k_i)$ measures the average degree. The active nodes in the network satisfy $I_{k_i}^1(t) + I_{k_i}^2(t) + S_{k_i}^1(t) + S_{k_i}^2(t) + R_{k_i}^1(t) + R_{k_i}^2(t) = 1$ and $d = b_1 + b_2$.

Remark 1. In contrast to references [2, 3, 21], both the multilingual environment and homogeneous networks are considered in constructing the rumor propagation model (2.1), which is more in line with the actual social networks. Besides, it is note worthy that $\theta(k_i)$ has several cases, such as, $\theta(k_i)$ is a constant [26], $\theta(k_i) = k_i$ [27], or $\theta(k_i)$ being a nonlinear function [28]. Here, we select $\theta(k_i) = k_i^p/(1+k_i^q)$ since a larger degree yields larger infectivity in practice.

3. Existence and analysis of equilibrium solution

In this section, we will calculate equilibrium solutions, including the zero-equilibrium solutions and positive-equilibrium solutions, and analyze their existence and properties. To begin with, the next-generation matrix method [29] is used to calculate the basic reproduction number of the model (2.1).

Let

$$\psi = \left(S_{k_i}^1(t), S_{k_i}^2(t), I_{k_i}^1(t), I_{k_i}^2(t), R_{k_i}^1(t), R_{k_i}^2(t)\right).$$

Then model (2.1) can be written as

$$\frac{d\psi}{dt} = \mathcal{F}(\psi) - \mathcal{V}(\psi),$$

where

$$\mathcal{F}(\psi) = \begin{pmatrix} \alpha_1 I_{k_i}^1(t)\Phi_1(t) \\ \alpha_2(I_{k_i}^1(t) + I_{k_i}^2(t) + S_{k_i}^1(t))\Phi_2(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathcal{V}(\psi) = \begin{pmatrix} \alpha_2 S_{k_i}^1(t)\Phi_2(t) + (\beta_1 + d)S_{k_i}^1(t) \\ (\beta_2 + d)S_{k_i}^2(t) \\ -b_1 + \alpha_1 I_{k_i}^1(t)\Phi_1(t) + \alpha_2 I_{k_i}^1(t)\Phi_2(t) + (\mu_1 + d)I_{k_i}^1(t) \\ -b_2 + \alpha_2 I_{k_i}^2(t)\Phi_2(t) + (\mu_2 + d)I_{k_i}^2(t) \\ -\mu_1 I_{k_i}^1(t) - \beta_1 S_{k_i}^1(t) + dR_{k_i}^1(t) \\ -\mu_2 I_{k_i}^2(t) - \beta_2 S_{k_i}^2(t) + dR_{k_i}^2(t) \end{pmatrix}.$$

Obviously, the zero-equilibrium solution is

$$E_0 = \left\{ \left(\frac{b_1}{d+\mu_1}, \frac{b_2}{d+\mu_2}, 0, 0, \frac{\mu_1 b_1}{d(\mu_1+d)}, \frac{\mu_2 b_2}{d(\mu_2+d)}\right), \cdots, \left(\frac{b_1}{d+\mu_1}, \frac{b_2}{d+\mu_2}, 0, 0, \frac{\mu_1 b_1}{d(\mu_1+d)}, \frac{\mu_2 b_2}{d(\mu_2+d)}\right) \right\},$$

and the Jacobian matrices of $\mathcal{F}(\psi)$ and $\mathcal{V}(\psi)$ at E_0 are

$$\mathcal{DF}(E_0) = \begin{pmatrix} F & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{DV}(E_0) = \begin{pmatrix} V & 0 & 0 \\ J_1 & J_2 & 0 \\ J_3 & J_4 & J_5 \end{pmatrix},$$

where $F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$, $V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$, $V_i = \mathrm{diag}(\beta_i + d)_{n\times n}, i = 1, 2$ and

$$F_1 = \begin{pmatrix} \frac{\alpha_1 I^1_{k_1}(t)\theta(k_1)Z(k_1)}{\langle k\rangle} & \frac{\alpha_1 I^1_{k_1}(t)\theta(k_2)Z(k_2)}{\langle k\rangle} \cdots & \frac{\alpha_1 I^1_{k_1}(t)\theta(k_n)Z(k_n)}{\langle k\rangle} \\ \frac{\alpha_1 I^1_{k_2}(t)\theta(k_1)Z(k_1)}{\langle k\rangle} & \frac{\alpha_1 I^1_{k_2}(t)\theta(k_2)Z(k_2)}{\langle k\rangle} \cdots & \frac{\alpha_1 I^1_{k_2}(t)\theta(k_n)Z(k_n)}{\langle k\rangle} \\ \vdots & \vdots \quad \ddots & \vdots \\ \frac{\alpha_1 I^1_{k_n}(t)\theta(k_1)Z(k_1)}{\langle k\rangle} & \frac{\alpha_1 I^1_{k_n}(t)\theta(k_2)Z(k_2)}{\langle k\rangle} \cdots & \frac{\alpha_1 I^1_{k_n}(t)\theta(k_n)Z(k_n)}{\langle k\rangle} \end{pmatrix}.$$

Next, according to the elementary transformation of a matrix, one has

$$F_1 \to \frac{1}{\langle k\rangle}\begin{pmatrix} \alpha_1 \sum_{i=1}^n \theta(k_i)Z(k_i)I^1_{k_i}(t) & \alpha_1 I^1_{k_1}(t)\theta(k_2)Z(k_2) & \cdots \alpha_1 I^1_{k_1}(t)\theta(k_n)Z(k_n) \\ 0 & 0 \quad \cdots & 0 \\ \vdots & \vdots \quad \vdots & \vdots \\ 0 & 0 \quad \cdots & 0 \end{pmatrix}.$$

Similarly, it follows

$$F_2 \to \frac{1}{\langle k\rangle}\begin{pmatrix} \alpha_2 \sum_{i=1}^n \theta(k_i)Z(k_i)(I^1_{k_i}(t) + I^2_{k_i}(t) + S^1_{k_i}(t)) & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence, the basic reproduction number R_0 of the model is computed as

$$R_0 = \rho(\mathcal{F}\mathcal{V}^{-1}) = \max\{R_{01}, R_{02}\},$$

where

$$R_{01} = \frac{\alpha_1 b_1 \sum_{i=1}^n \theta(k_i)Z(k_i)}{\langle k\rangle(\beta_1 + d)(\mu_1 + d)}, \quad R_{02} = \frac{\alpha_2(\frac{b_1}{d+\mu_1} + \frac{b_2}{d+\mu_2})\sum_{i=1}^n \theta(k_i)Z(k_i)}{\langle k\rangle(\beta_2 + d)}.$$

Theorem 3.1. *For the basic reproduction number of the model (2.1), the equilibrium solution is unique in the following three different cases:*

(i) *If* $R_0 < 1$, *model (2.1) only has a zero-equilibrium solution.*

(ii) *If* $R_{01} < 1$ *and* $R_{02} > 1$, *model (2.1) has a unique positive-equilibrium solution.*

(iii) *If* $R_{01} > 1$ *and* $R_{02} > 1$, *model (2.1) has a unique positive-equilibrium solution.*

Proof. (i) Obviously, when $R_0 < 1$, model (2.1) only has a zero-equilibrium solution, E_0.

(ii) When $R_{01} < 1, R_{02} > 1$, define the solution as E_1^*, i.e.,

$$E_1^* = (I_{1k_i}^{1*}, I_{1k_i}^{2*}, S_{1k_i}^{1*}, S_{1k_i}^{2*}, R_{1k_i}^{1*}, R_{1k_i}^{2*}),$$

where

$$S_{1k_i}^{1*} = 0, \quad I_{1k_i}^{1*} = \frac{b_1}{\alpha_2\Phi_2^* + \mu_1 + d}, \quad I_{1k_i}^{2*} = \frac{b_2}{\alpha_2\Phi_2^* + \mu_2 + d},$$

$$S_{1k_i}^{2*} = \frac{\alpha_2(I_{1k_i}^{1*} + I_{1k_i}^{2*})\Phi_2^*}{\beta_2 + d}, \quad R_{1k_i}^{1*} = \frac{\mu_1 I_{1k_i}^{1*}}{d}, \quad R_{1k_i}^{2*} = \frac{\mu_2 I_{1k_i}^{2*} + \beta_2 S_{1k_i}^{2*}}{d}.$$

It is obviously that $\Phi_1^* = 0$, and

$$\Phi_2^* = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\frac{\alpha_2(\frac{b_1}{\alpha_2\Phi_2^*+\mu_1+d} + \frac{b_2}{\alpha_2\Phi_2^*+\mu_2+d})}{\beta_2 + d}\Phi_2^*. \tag{3.1}$$

Further, we set

$$G_1(\Phi_2^*) = 1 - \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\frac{\alpha_2(\frac{b_1}{\alpha_2\Phi_2^*+\mu_1+d} + \frac{b_2}{\alpha_2\Phi_2^*+\mu_2+d})}{\beta_2 + d},$$

and it yields that to any $\Phi_2^* > 0$, $G_1'(\Phi_2^*) > 0$ and $G_1'(+\infty) = 1$. Then, we have

$$\lim_{\Phi_2^*\to 0^+} G_1(\Phi_2^*) = 1 - \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\frac{\alpha_2(\frac{b_1}{\mu_1+d} + \frac{b_2}{\mu_2+d})}{\beta_2 + d} = 1 - R_{02} \le 0.$$

It proves that model (2.1) has a unique equilibrium solution when $R_{01} < 1, R_{02} > 1$.

(iii) When $R_{01} > 1, R_{02} > 1$, define the solution as E_2^*,

$$E_2^* = (I_{2k_i}^{1*}, I_{2k_i}^{2*}, S_{2k_i}^{1*}, S_{2k_i}^{2*}, R_{2k_i}^{1*}, R_{2k_i}^{2*}),$$

where

$$I_{2k_i}^{1*} = \frac{b_1}{\alpha_1\Phi_1^* + \alpha_2\Phi_2^* + \mu_1 + d}, \quad I_{2k_i}^{2*} = \frac{b_2}{\alpha_2\Phi_2^* + \mu_2 + d},$$

$$S_{2k_i}^{1*} = \frac{\alpha_1 I_{2k_i}^{1*}\Phi_1^*}{\alpha_2\Phi_2^* + \beta_1 + d}, \quad S_{2k_i}^{2*} = \frac{\alpha_2(I_{2k_i}^{1*} + I_{2k_i}^{2*} + S_{2k_i}^{1*})\Phi_2^*}{\beta_2 + d},$$

$$R_{2k_i}^{1*} = \frac{\mu_1 I_{2k_i}^{1*} + \beta_1^{1*} S_{2k_i}^{1*}}{d}, \quad R_{2k_i}^{2*} = \frac{\mu_2 I_{2k_i}^{2*} + \beta_2 S_{2k_i}^{2*}}{d}.$$

Next, expanding $S_{2k_i}^{1*}$ and $S_{2k_i}^{2*}$, we obtain

$$S_{2k_i}^{1*} = \frac{\alpha_1 b_1 \Phi_1^*}{(\alpha_2\Phi_2^* + \beta_1 + d)(\alpha_1\Phi_1^* + \alpha_2\Phi_2^* + \mu_1 + d)}$$

and

$$S_{2k_i}^{2*} = \frac{\alpha_2\Phi_2^*}{\beta_2 + d}[\frac{b_1}{\alpha_1\Phi_1^* + \alpha_2\Phi_2^* + \mu_1 + d} + \frac{b_2}{\alpha_2\Phi_2^* + \mu_2 + d} + \frac{b_1\alpha_1\Phi_1^*}{(\alpha_1\Phi_1^* + \alpha_2\Phi_2^* + \mu_1 + d)(\alpha_2\Phi_2^* + \mu_2 + d)}].$$

Similarly to case (ii), one has

$$\begin{aligned}
\Phi_1^* &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)S_{2k_i}^{1*}\\
&= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\frac{\alpha_1 b_1\Phi_1^*}{(\alpha_2\Phi_2^*+\beta_1+d)(\alpha_1\Phi_1^*+\alpha_2\Phi_2^*+\mu_1+d)},\\
\Phi_2^* &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\frac{\alpha_2\Phi_2^*}{\beta_2+d}[\frac{b_1}{\alpha_1\Phi_1^*+\alpha_2\Phi_2^*+\mu_1+d}+\frac{b_2}{\alpha_2\Phi_2^*+\mu_2+d}\\
&\quad+\frac{b_1\alpha_1\Phi_1^*}{(\alpha_1\Phi_1^*+\alpha_2\Phi_2^*+\mu_1+d)(\alpha_2\Phi_2^*+\mu_2+d)}].\\
G_2(\Phi_1^*,\Phi_2^*) &= 1-\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\frac{\alpha_1 b_1}{(\alpha_2\Phi_2^*+\beta_1+d)(\alpha_1\Phi_1^*+\alpha_2\Phi_2^*+\mu_1+d)},
\end{aligned}$$

and

$$\begin{aligned}
G_3(\Phi_1^*,\Phi_2^*) &= 1-\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\frac{\alpha_2}{\beta_2+d}[\frac{b_1}{\alpha_1\Phi_1^*+\alpha_2\Phi_2^*+\mu_1+d}\\
&\quad+\frac{b_2}{\alpha_2\Phi_2^*+\mu_2+d}+\frac{b_1\alpha_1\Phi_1^*}{(\alpha_1\Phi_1^*+\alpha_2\Phi_2^*+\mu_1+d)(\alpha_2\Phi_2^*+\mu_2+d)}].
\end{aligned}$$

It is easy to obtain that for all Φ_1^*,

$$\frac{\partial G_2\left(\Phi_1^*,\Phi_2^*\right)}{\partial\Phi_1^*}>0,\ \lim_{\Phi_1^*\to+\infty}G_2(\Phi_1^*,\Phi_2^*)=1,\ \frac{\partial G_3\left(\Phi_1^*,\Phi_2^*\right)}{\partial\Phi_2^*}>0,\ \lim_{\Phi_2^*\to+\infty}G_3(\Phi_1^*,\Phi_2^*)=1.$$

Moreover, it shows that

$$\lim_{\Phi_1^*\to 0+}G_2(\Phi_1^*,0)=1-\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\frac{\alpha_1 b_1}{(\beta_1+d)(\mu_1+d)}=1-R_{01}<0,$$

$$\lim_{\Phi_2^*\to 0+}G_3(0,\Phi_2^*)=1-\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\frac{\alpha_2}{\beta_2+d}[\frac{b_1}{\mu_1+d}+\frac{b_2}{\mu_2+d}]=1-R_{02}<0.$$

Hence, model (2.1) has a unique positive-equilibrium solution if $R_{01}>1$ and $R_{02}>1$. □

Theorem 3.2. *The zero-equilibrium solution E_0 of the model (2.1) is locally asymptotically stable if $R_0<1$.*

Proof. According to stability theory [30], we first derive the Jacobin matrix $J(E_0)$

$$J(E_0)=\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n}\\ A_{21} & A_{22} & \cdots & A_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ A_{n1} & A_{n2} & \cdots & A_{nn}\end{pmatrix}, \tag{3.2}$$

where

$$f_1 = \frac{\alpha_1 b_1 \theta(k_i) Z(k_i)}{\langle k\rangle (d+\mu_1)} - \beta_1 - d, \quad f_2 = \frac{\alpha_2 \theta(k_i) Z(k_i)}{\langle k\rangle}(\frac{b_1}{d+\mu_1} + \frac{b_2}{d+\mu_2}) - \beta_2 - d,$$

and

$$A_{ii} = \begin{pmatrix} f_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_2 & 0 & 0 & 0 & 0 \\ -\frac{\alpha_1 b_1 \theta(k_i) Z(k_i)}{\langle k\rangle (d+\mu_1)} & -\frac{\alpha_2 b_1 \theta(k_i) Z(k_i)}{\langle k\rangle (d+\mu_1)} & -\mu_1 - d & 0 & 0 & 0 \\ 0 & -\frac{\alpha_2 b_2 \theta(k_i) Z(k_i)}{\langle k\rangle (d+\mu_2)} & 0 & -\mu_2 - d & 0 & 0 \\ \beta_1 & 0 & \mu_1 & 0 & -d & 0 \\ 0 & \beta_2 & 0 & \mu_2 & 0 & -d \end{pmatrix},$$

$$A_{ij} = \begin{pmatrix} \frac{\alpha_1 b_1 \theta(k_i) Z(k_i)}{\langle k\rangle (d+\mu_1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha_2 \theta(k_i) Z(k_i)}{\langle k\rangle}(\frac{b_1}{d+\mu_1} + \frac{b_2}{d+\mu_2}) & 0 & 0 & 0 & 0 \\ -\frac{\alpha_1 b_1 \theta(k_i) Z(k_i)}{\langle k\rangle (d+\mu_1)} & -\frac{\alpha_2 b_1 \theta(k_i) Z(k_i)}{\langle k\rangle (d+\mu_1)} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha_2 b_2 \theta(k_i) Z(k_i)}{\langle k\rangle (d+\mu_2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further, we have

$$\begin{aligned} & |\lambda - J(E_0)| \\ = \; & (\lambda + \beta_1 + d)^{n-1} (\lambda + \beta_2 + d)^{n-1} (\lambda + \mu_1 + d)^{2n} (\lambda + \mu_2 + d)^{2n} \\ & \times [\lambda + (\beta_1 + d)(1 - \frac{1}{\langle k\rangle} \sum_{i=1}^{n} \theta(k_i) Z(k_i) \frac{\alpha_1 b_1}{(\beta_1 + d)(\mu_1 + d)})] \\ & \times (\lambda + (\beta_2 + d)[1 - \frac{1}{\langle k\rangle} \sum_{i=1}^{n} \theta(k_i) Z(k_i) \frac{\alpha_2}{\beta_2 + d}(\frac{b_1}{\mu_1 + d} + \frac{b_2}{\mu_2 + d})]). \end{aligned}$$

Then, the eigenvalues of (3.2) are $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = -(\beta_1 + d)$, $\lambda_n = \lambda_{n+1} = \cdots = \lambda_{2n-2=} - (\beta_2 + d)$, $\lambda_{2n-1} = \lambda_{2n} = \cdots = \lambda_{4n-2} = -(\mu_1 + d)$, $\lambda_{4n-1} = \lambda_{4n} = \cdots = \lambda_{6n-2} = -(\mu_2 + d)$, $\lambda_{6n-1} = -(\beta_1 + d))(1 - R_{01})$, $\lambda_{6n} = -(\beta_2 + d))(1 - R_{02})$. Since $R_0 < 1$, it is obvious that for any i from 1 to $6n$, $\lambda_i < 0$. Therefore, the zero-equilibrium solution E_0 is locally asymptotically stable in model (2.1). □

Theorem 3.3. *The zero-equilibrium solution E_0^* of model (2.1) is globally asymptotically stable if $R_0 < 1$.*

Proof. According to model (2.1) and [31], one has

$$\begin{aligned} \frac{\mathrm{d}I_{k_i}^1(t)}{\mathrm{d}t} &= b_1 - \alpha_1 I_{k_i}^1(t)\Phi_1(t) - \alpha_2 I_{k_i}^1(t)\Phi_2(t) - (\mu_1 + d) I_{k_i}^1(t) \\ &\leq b_1 - (\mu_1 + d) I_{k_i}^1(t), \end{aligned} \tag{3.3}$$

which means that $\sup I_{k_i}^1(t) \leq \frac{b_1}{\mu_1 + d} = \tilde{I}^1$.

Next, amuse $\epsilon_1 > 0$ that is sufficiently small. So for $t \to +\infty$, we have

$$\sup I_{k_i}^1(t) \leq \tilde{I}^1 + \epsilon_1,$$

and

$$\frac{\mathrm{d}S^{1}_{k_i}(t)}{\mathrm{d}t} \leq \alpha_1 \Phi_1(t)(\tilde{I}^1 + \epsilon_1) - (d+\beta_1)S^{1}_{k_i}(t).$$

From the principle of comparison, we set a new function $Q^1_{k_i}(t)$, $Q^1_{k_i}(0) = S^{1}_{k_i}(0) = 0$ and

$$\frac{\mathrm{d}Q^1_{k_i}(t)}{\mathrm{d}t} = \alpha_1 \tilde{\Phi}_1(t)(\tilde{I}^1 + \epsilon_1) - (d+\beta_1)Q^1_{k_i}(t),$$

where

$$\tilde{\Phi}_1(t) = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)\,Z(k_i)\,Q^1_{k_i}(t).$$

Construct a Lyapunov function

$$V_1(t) = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)\,Z(k_i)\,Q^1_{k_i}(t). \tag{3.4}$$

Then, it derives that

$$\begin{aligned}\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)\,Z(k_i)\,[\alpha_1\tilde{\Phi_1}(t)(\tilde{I}^1 + \epsilon_1) - (d+\beta_1)Q^1_{k_i}(t)] \\ &= \tilde{\Phi}_1(t)(\beta_1 + d)[R_{01} + \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{\theta(k_i)\,Z(k_i)\alpha_1\epsilon_1}{\beta_1 + d} - 1].\end{aligned}$$

Since $R_{01} < 1$ and the definition of ϵ_1, we select a small enough ϵ_1, so we can obtain that $\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} \leq 0$. Similarly, one has

$$\frac{\mathrm{d}I^2_{k_i}(t)}{\mathrm{d}t} \leq b_2 - (\mu_2 + d)I^2_{k_i}(t), \tag{3.5}$$

and $\sup I^2_{k_i}(t) \leq \frac{b_2}{\mu_2+d} = \tilde{I}^2$. Besides, construct a small enough $\epsilon_2 > 0$, $I^2_{k_i}(t) \leq \tilde{I}^2 + \epsilon_2$, $0 < S^{1}_{k_i}(t) < \epsilon_2$. Hence, it follows that

$$\frac{\mathrm{d}S^{2}_{k_i}(t)}{\mathrm{d}t} \leq \alpha_2\Phi_2(t)(\tilde{I}^1 + \epsilon_1 + \tilde{I}^2 + \epsilon_2) - (d+\beta_2)S^{2}_{k_i}(t). \tag{3.6}$$

Then we set a new function, $Q^2_{k_i}(t)$, which satisfies

$$Q^2_{k_i}(0) = S^{2}_{k_i}(0) = 0,\ \frac{\mathrm{d}Q^2_{k_i}(t)}{\mathrm{d}t} = \alpha_2\tilde{\Phi}_2(t)(\tilde{I}^1 + \epsilon_1 + \tilde{I}^2 + \epsilon_2) - (d+\beta_2)Q^2_{k_i}(t),$$

where

$$\tilde{\Phi}_2(t) = \frac{\sum_{i=1}^{n}\theta(k_i)\,Z(k_i)\,Q^2_{k_i}(t)}{\langle k\rangle}.$$

Similarly, construct a Lyapunov function

$$V_2(t) = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)\,Z(k_i)\,Q^2_{k_i}(t).$$

Then, one has

$$\begin{aligned}
\frac{\mathrm{d}V_2(t)}{\mathrm{d}t} &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)[\alpha_2\tilde{\Phi}_2(t)(\tilde{I}^1+\epsilon_1+\tilde{I}^2+\epsilon_2)-(d+\beta_2)Q^2_{k_i}(t)] \\
&= \tilde{\Phi}_2(t)(\beta_2+d)[\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{\theta(k_i)Z(k_i)\alpha_2(\tilde{I}^1+\epsilon_1+\tilde{I}^2+\epsilon_2)}{\beta_2+d}-1] \\
&= \tilde{\Phi}_2(t)(\beta_2+d)[\frac{\alpha_2(\frac{b_1}{d+\mu_1}+\frac{b_2}{d+\mu_2})\sum_{i=1}^{n}\theta(k_i)Z(k_i)}{\langle k\rangle(\beta_2+d)} \\
&\quad +\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{\theta(k_i)Z(k_i)\alpha_2(\epsilon_1+\epsilon_2)}{\beta_2+d}-1] \\
&= \tilde{\Phi}_2(t)(\beta_2+d)[R_{02}+\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{\theta(k_i)Z(k_i)\alpha_2(\epsilon_1+\epsilon_2)}{\beta_2+d}-1].
\end{aligned}$$

Since $R_{02} < 1$ and the definition of ϵ_1 and ϵ_2, select small enough ϵ_1 and ϵ_2. So we can obtain that $\frac{\mathrm{d}V_2(t)}{\mathrm{d}t} \leq 0$. Similarly, we construct a small enough $\epsilon_3 > 0$, for $t \to +\infty$, $0 < S^1_{k_i}(t) < \epsilon_3$, $0 < S^2_{k_i}(t) < \epsilon_3$. Hence,

$$\frac{\mathrm{d}I^1_{k_i}(t)}{\mathrm{d}t} \geq b_1 - [(\alpha_1+\alpha_2)\frac{\sum_{i=1}^{n}\theta(k_i)Z(k_i)\epsilon_3}{\langle k\rangle}+\mu_1+d]I^1_{k_i}(t).$$

Then,

$$\inf I^1_{k_i}(t) \leq \frac{b_1}{(\alpha_1+\alpha_2)\frac{\sum_{i=1}^{n}\theta(k_i)Z(k_i)\epsilon_3}{\langle k\rangle}+\mu_1+d} = \tilde{I}^1.$$

Set $\epsilon_3 \to 0$, so it follows $\inf I^1_{k_i}(t) \leq \frac{b_1}{\mu_1+d} = \sup T^1_{k_i}(t)$ for $t \to +\infty$. Hence, E^*_0 is globally asymptotically stable if $R_0 < 1$. □

Lemma 3.4. *If $R_{01} < 1$ and $R_{02} > 1$, the positive-equilibrium solution of model (2.1) satisfies $S^1_{k_i}(t) = S^{1*}_{1k_i} = \Phi_1(t) = \Phi^*_1 = 0$ when $t \to +\infty$,*

$$S^{2*}_{1k_i} = \frac{\alpha_2(I^{1*}_{1k_i}+I^{2*}_{1k_i})\Phi^*_2}{\beta_2+d} \to \beta_2+d = \frac{\alpha_2(I^{1*}_{1k_i}+I^{2*}_{1k_i})\Phi^*_2}{S^{2*}_{1k_i}} = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\alpha_2(I^{1*}_{1k_i}+I^{2*}_{1k_i}). \tag{3.7}$$

Proof. From the definition of $\Phi_1(t)$, there is

$$\begin{aligned}
\dot{\Phi}_1(t) &\leq \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)[\alpha_1 I^1_{k_i}(t)\Phi_1(t)-(\beta_1+d)S^1_{k_i}(t)] \\
&\leq \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)[\frac{\alpha_1 b_1\Phi_1(t)}{\mu_1+d}-(\beta_1+d)S^1_{k_i}(t)] \\
&= \Phi_1(t)(\beta_1+d)(R_{01}-1) < 0.
\end{aligned}$$

which means that (3.7) holds based on reference [32]. □

Theorem 3.5. *The positive-equilibrium solution E^*_1 of model (2.1) is globally asymptotically stable if $R_{01} < 1$ and $R_{02} > 1$.*

Proof. Construct the Lyapunov $V_3(t)$ as

$$\begin{aligned} V_3(t) &= \frac{1}{2\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{1k_i}^{1*}}\theta(k_i)Z(k_i)(I_{k_i}^{1}(t)-I_{1k_i}^{1*})^2+\left(\Phi_2(t)-\Phi_2^*-\Phi_2^*\ln\left(\frac{\Phi_2(t)}{\Phi_2^*}\right)\right)\\ &\quad+\frac{1}{2\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{1k_i}^{2*}}\theta(k_i)Z(k_i)(I_{k_i}^{2}(t)-I_{1k_i}^{2*})^2. \end{aligned}$$

According to Lemma 3.4, it follows that

$$\begin{aligned} \frac{\mathrm{d}V_3(t)}{\mathrm{d}t} &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{1k_i}^{1*}}\theta(k_i)\,Z(k_i)\left(I_{k_i}^{1}(t)-I_{1k_i}^{1*}\right)\dot I_{k_i}^{1}(t)+\frac{\Phi_2(t)-\Phi_2^*}{\Phi_2(t)}\dot\Phi_2(t)\\ &\quad+\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{1k_i}^{2*}}\theta(k_i)\,Z(k_i)\left(I_{k_i}^{2}(t)-I_{1k_i}^{2*}\right)\dot I_{k_i}^{2}(t)\\ &= f_1+f_2+f_3, \end{aligned}$$

where

$$\begin{aligned} f_1 &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{1k_i}^{1*}}\theta(k_i)\,Z(k_i)\left(I_{k_i}^{1}(t)-I_{1k_i}^{1*}\right)[\alpha_2 I_{1k_i}^{1*}\Phi_2^*+(\mu_1+d)I_{1k_i}^{1*}\\ &\quad-\alpha_2 I_{k_i}^{1}(t)\Phi_2(t)-(\mu_1+d)I_{k_i}^{1}(t)]\\ &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}-\frac{1}{I_{1k_i}^{1*}}\theta(k_i)\,Z(k_i)\left(I_{k_i}^{1}(t)-I_{1k_i}^{1*}\right)[(\mu_1+d)(I_{k_i}^{1}(t)-I_{1k_i}^{1*})\\ &\quad+\alpha_2 I_{k_i^1(t)}\Phi_2(t)-\alpha_2 I_{1k_i}^{2*}\Phi_2(t)+\alpha_2 I_{1k_i}^{2*}\Phi_2(t)-\alpha_2 I_{1k_i}^{2*}\Phi_2^*]\\ &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}[-\frac{1}{I_{1k_i}^{1*}}\theta(k_i)\,Z(k_i)\,(\alpha_2\Phi_2(t)+\mu_1+d)\left(I_{k_i}^{1}(t)-I_{1k_i}^{1*}\right)^2\\ &\quad-\theta(k_i)Z(k_i)\alpha_2(\Phi_2(t)-\Phi_2^*)(I_{k_i}^{1}(t)-I_{1k_i}^{1*})],\\ f_2 &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{1k_i}^{2*}}\theta(k_i)\,Z(k_i)\left(I_{k_i}^{2}(t)-I_{1k_i}^{2*}\right)(b_2-\alpha_2 I_{k_i}^{2}(t)\Phi_2\\ &\quad-(\mu_2+d)I_{k_i^2(t)})\\ &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}[-\frac{1}{I_{1k_i}^{2*}}\theta(k_i)\,Z(k_i)\,(\alpha_2\Phi_2(t)+\mu_2+d)\left(I_{k_i}^{2}(t)-I_{1k_i}^{2*}\right)^2\\ &\quad-\theta(k_i)Z(k_i)\alpha_2(\Phi_2(t)-\Phi_2^*)(I_{k_i}^{2}(t)-I_{1k_i}^{2*})],\\ f_3 &= (\Phi_2(t)-\Phi_2^*)[\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\alpha_2(I_{k_i}^{1}(t)+I_{k_i}^{2}(t))-(\beta_2+d)]\\ &= (\Phi_2(t)-\Phi_2^*)[\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\alpha_2(I_{k_i}^{1}(t)+I_{k_i}^{2}(t))\\ &\quad-\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\alpha_2(I_{1k_i}^{1*}+I_{1k_i}^{2*})]. \end{aligned}$$

Hence, one further has

$$\begin{aligned}\frac{\mathrm{d}V_3(t)}{\mathrm{d}t} &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}[-\frac{1}{I_{1k_i}^{1*}}\theta(k_i)Z(k_i)(\alpha_2\Phi_2(t)+\mu_1+d)\left(I_{k_i}^{1}(t)-I_{1k_i}^{1*}\right)^2\\&\quad-\frac{1}{I_{1k_i}^{2*}}\theta(k_i)Z(k_i)(\alpha_2\Phi_2(t)+\mu_2+d)\left(I_{k_i}^{2}(t)-I_{1k_i}^{2*}\right)^2]\\&\le 0.\end{aligned}$$

Therefore, we can conclude that the positive-equilibrium solution E_1^* of model (2.1) is globally asymptotically stable if $R_{01} < 1$ and $R_{02} > 1$. □

Lemma 3.6. *If $R_{01} > 1$ and $R_{02} > 1$, when $t \to +\infty$, the positive-equilibrium solution of model (2.1) satisfies*

$$\beta_1 + d = \frac{\alpha_1 I_{2k_i}^{1*}\Phi_1^*}{S_{2k_i}^{1*}} - \alpha_2\Phi_2^* = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\alpha_1\theta(k_i)Z(k_i)I_{2k_i}^{1*} - \alpha_2\Phi_2^*,$$

and

$$\beta_2 + d = \frac{\alpha_2(I_{2k_i}^{1*}+I_{2k_i}^{2*}+S_{2k_i}^{1*})\Phi_2^*}{S_{2k_i}^{2*}} = \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\alpha_2\theta(k_i)Z(k_i)(I_{2k_i}^{1*}+I_{2k_i}^{2*}+S_{2k_i}^{1*}).$$

Theorem 3.7. *The positive-equilibrium solution E_1^* of model (2.1) is globally asymptotically stable if $R_{01} > 1$ and $R_{02} > 1$.*

Proof. Construct the Lyapunov $V_4(t)$ as

$$\begin{aligned}V_4(t) &= \frac{1}{2\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{2k_i}^{1*}}\theta(k_i)Z(k_i)(I_{k_i}^{1}(t)-I_{2k_i}^{1*})^2\\&\quad+\frac{1}{2\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{2k_i}^{2*}}\theta(k_i)Z(k_i)(I_{k_i}^{2}(t)-I_{2k_i}^{2*})^2\\&\quad+\left(\Phi_1(t)-\Phi_1^*-\Phi_1^*\ln\left(\frac{\Phi_1(t)}{\Phi_1^*}\right)\right)+\left(\Phi_2(t)-\Phi_2^*-\Phi_2^*\ln\left(\frac{\Phi_2(t)}{\Phi_2^*}\right)\right).\end{aligned}$$

Then, based on Lemma 3.6, it yields that

$$\begin{aligned}\frac{\mathrm{d}V_4(t)}{\mathrm{d}t} &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{2k_i}^{1*}}\theta(k_i)Z(k_i)\left(I_{k_i}^{1}(t)-I_{2k_i}^{1*}\right)\dot{I}_{k_i}^{1}(t)+\frac{\Phi_1(t)-\Phi_1^*}{\Phi_1(t)}\dot{\Phi}_1(t)\\&\quad+\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{2k_i}^{2*}}\theta(k_i)Z(k_i)\left(I_{k_i}^{2}(t)-I_{2k_i}^{2*}\right)\dot{I}_{k_i}^{2}(t)+\frac{\Phi_2(t)-\Phi_2^*}{\Phi_2(t)}\dot{\Phi}_2(t)\\&= g_1+g_2+g_3+g_4,\end{aligned}$$

where

$$\begin{aligned}g_1 &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{2k_i}^{1*}}\theta(k_i)Z(k_i)\left(I_{k_i}^{1}(t)-I_{2k_i}^{1*}\right)[\alpha_2 I_{2k_i}^{1*}\Phi_2^*+\alpha_1 I_{2k_i}^{1*}\Phi_1^*\\&\quad+(\mu_1+d)I_{2k_i}^{1*}-\alpha_2 I_{k_i}^{1}(t)\Phi_2(t)-\alpha_1 I_{k_i}^{1}(t)\Phi_1(t)-(\mu_1+d)I_{k_i}^{1}(t)]\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{2k_i}^{1*}}\theta(k_i)Z(k_i)(\alpha_1\Phi_1(t)+\alpha_2\Phi_2(t)+\mu_1+d)\left(I_{k_i}^{1}(t)-I_{2k_i}^{1*}\right)^2\\
&\quad -\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)[\alpha_2(\Phi_2(t)-\Phi_2^*)+\alpha_1(\Phi_1(t)-\Phi_1^*)](I_{k_i}^{1}(t)-I_{2k_i}^{1*}),\\
g_2 &= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}-\frac{1}{I_{2k_i}^{2*}}\theta(k_i)Z(k_i)(\alpha_2\Phi_2(t)+\mu_2+d)\left(I_{k_i}^{2}(t)-I_{2k_i}^{2*}\right)^2\\
&\quad -\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\alpha_2(\Phi_2(t)-\Phi_2^*)(I_{k_i}^{2}(t)-I_{2k_i}^{2*}),\\
g_3 &= (\Phi_1(t)-\Phi_1^*)\frac{1}{\Phi_1(t)}\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)[\alpha_1 I_{k_i}^{1}(t)\Phi_1(t)-\alpha_2 S_{k_i}^{1}(t)\Phi_2(t)\\
&\quad -(\beta_1+d)S_{k_i}^{1}(t)]\\
&= (\Phi_1(t)-\Phi_1^*)[\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\alpha_1 I_{k_i}^{1}(t)-\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\alpha_1 I_{2k_i}^{1*}\\
&\quad +\alpha_2\Phi_2^*-\alpha_2\Phi_2(t)]\\
&= \frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)\alpha_1(I_{k_i}^{1}(t)-I_{1k_i}^{1*})(\Phi_1(t)-\Phi_1^*)\\
&\quad -\alpha_2(\Phi_2(t)-\Phi_2^*)(\Phi_1(t)-\Phi_1^*),\\
g_4 &= (\Phi_2(t)-\Phi_2^*)\frac{1}{\Phi_2(t)}\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\theta(k_i)Z(k_i)[\alpha_2(I_{k_i}^{1}(t)+I_{k_i}^{2}(t)+S_{k_i}^{1}(t))\Phi_2(t)\\
&\quad -(\beta_2+d)S_{k_i}^{2}(t)]\\
&= (\Phi_2(t)-\Phi_2^*)\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\alpha_2\theta(k_i)Z(k_i)[(I_{k_i}^{1}(t)-I_{2k_i}^{1*})+(I_{k_i}^{2}(t)-I_{2k_i}^{2*})]\\
&\quad +(\Phi_2(t)-\Phi_2^*)\alpha_2(\Phi_1(t)-\Phi_1^*).
\end{aligned}
$$

Further, one has

$$
\begin{aligned}
&\frac{\mathrm{d}V_4(t)}{\mathrm{d}t}\\
&= -\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{2k_i}^{1*}}\theta(k_i)Z(k_i)(\alpha_1\Phi_1(t)+\alpha_2\Phi_2(t)+\mu_1+d)\left(I_{k_i}^{1}(t)-I_{2k_i}^{1*}\right)^2\\
&\quad -\frac{1}{\langle k\rangle}\sum_{i=1}^{n}\frac{1}{I_{2k_i}^{2*}}\theta(k_i)Z(k_i)(\alpha_2\Phi_2(t)+\mu_2+d)\left(I_{k_i}^{2}(t)-I_{2k_i}^{2*}\right)^2\\
&\le 0.
\end{aligned}
$$

Therefore, we can conclude that the positive-equilibrium solution E_2^* of model (2.1) is globally asymptotically stable if $R_{01} > 1$ and $R_{02} > 1$. □

Remark 2. In fact, the basic reproduction number R_0 of model (2.1) is determined by the values of R_{01} and R_{02}, which means that there exist more situations of rumor existence (see Theorem 3.1). Notedly, rumor spreaders $S^1(t)$ and $S^2(t)$ will be extinct with $R_0 < 1$ (see Theorem 3.2), but the spreading of rumors is more complex when $R_0 > 1$. Hence, to further explore its dynamics, Lyapunov function indirect and direct methods are used to analyze local asymptotically stability (see Theorem 3.2) and global asymptotically stability (see Theorems 3.3, 3.5, and 3.7), respectively.

4. Numerical examples

In this section, we use numerical simulation to analyze the dynamic characteristics of the proposed rumor propagation model.

Combined with practical problems and existing results [8, 33, 34], the initial state of a rumor-spreading network typically comprises a predominant number of ignorant individuals, a small group of spreaders, and an even smaller contingent of removers in general. Furthermore, depending on the specific assumptions made regarding the model's context, it is often observed that the population of spreaders (Group-2) is more substantial than initially anticipated. Drawing on the insights from reference [23], suppose that the network obeys power law distribution, and choose $k_i = i$, $i = 1, 2, \cdots, 200$, and $Z(k_i) = \frac{k_i^{-2}}{1.6399}$, so it satisfies that $\sum_{i=1}^{n} Z(k_i) = 1$. Hence, it is easy to obtain that the average degree $\langle k \rangle = 3.5844$. As is mentioned above (see Remark 1), $\theta(k_i) = k_i^p/(1 + k_i^q)$, and select $p = 0.5, q = 0.5$. Moreover, to further demonstrate the effect of the parameters (see Figures 2–5), choose the following series of initial values for the model (2.1): $I_{k_i}^1(0) = 0.3 + \frac{k_i}{3200}$, $I_{k_i}^2(0) = 0.4 + \frac{k_i}{3200}$, $S_{k_i}^1(0) = 0.1 - \frac{k_i}{3200}$, $S_{k_i}^2(0) = 0.15 - \frac{k_i}{6400}$, $R_{k_i}^1(0) = 0.03 - \frac{k_i}{6400}$, $R_{k_i}^2(0) = 0.02 - \frac{k_i}{6400}$, $k_i = i, i = 1, 2, \cdots, 200$.

4.1. *Stability of zero-equilibrium solution*

Combined with Theorem 3.3 and the actual problem, choose $\alpha_1 = 0.6, \alpha_2 = 0.5, b_1 = 0.005, b_2 = 0.005, d = 0.01, \mu_1 = 0.005, \mu_2 = 0.005, \beta_1 = 0.05$ and $\beta_2 = 0.05$. By simple calculation, it can deduce that $R_0 \approx 0.873 < 1$. From Theorem 3.3, we can know that the zero-equilibrium solution E_0^* of model (2.1) is globally asymptotically stable if $R_0 < 1$. It is shown in Figure 2, which takes $k = 50$ as an example.

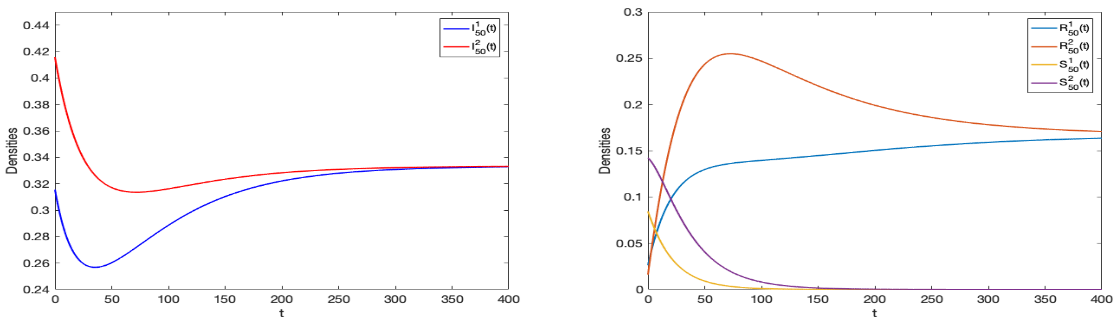

Figure 2. The stability of zero-equilibrium solution E_0 with $R_0 < 1$ and $k = 50$.

Moreover, Figure 3 shows the dynamic state at all degrees from 1 to 200. From Figure 3, we can easily detect that $I_{k_i}^1(t)$ tends to $\frac{b_1}{\mu_1+d} = \frac{1}{3}$ and $I_{k_i}^2(t)$ tends to $\frac{b_2}{\mu_2+d} = \frac{1}{3}$, which is consistent with the conditions of Theorem 3.3. It also shows that the zero-equilibrium solution E_0 is globally asymptotically stable for any $k \in [1, 200]$.

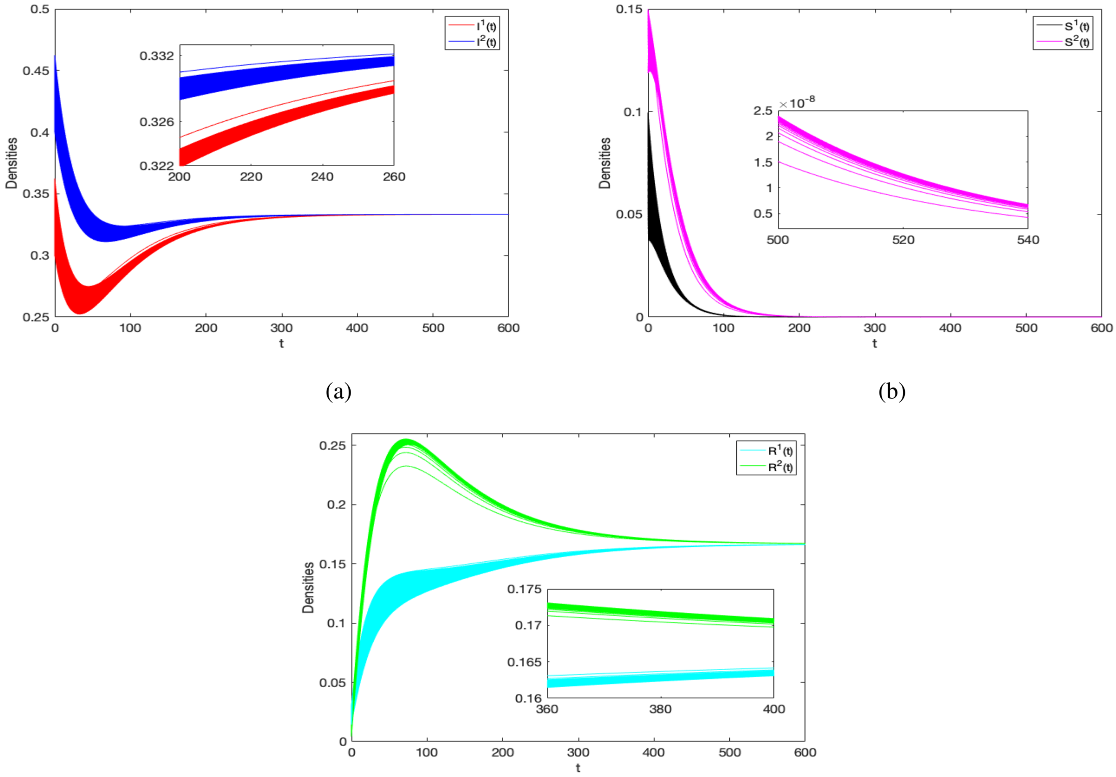

Figure 3. The stability of zero-equilibrium solution E_0 with $R_0 < 1$ and $k \in [1, 200]$.

4.2. *Stability of positive-equilibrium solution*

To further explore the effect of language usage variations on the multi-lingual rumor spreading, the following different parameters are selected with actual and empirical results [23, 34], and two cases are explored based on Theorems 3.5 and 3.7 as follows:

Case 1. Stability of positive-equilibrium solution for $R_{01} < 1$ and $R_{02} > 1$.

Choose $\alpha_1 = 0.4, \alpha_2 = 0.72, b_1 = 0.01, b_2 = 0.01, d = 0.02, \mu_1 = 0.01, \mu_2 = 0.01, \beta_1 = 0.03, \beta_2 = 0.02$, and it yields $R_{01} \approx 0.3250 < 1$, $R_{02} \approx 1.4624 > 1$. Figure 4 depicts that positive equilibrium E_1^* is globally asymptotically stable for $k \in [1, 200]$. Notedly, $S^1_{k_i}(t)$ tends to 0 rapidly, and $S^2_{k_i}(t)$ remains prevalent thereafter.

Case 2. Stability of positive-equilibrium solution for $R_{01} > 1$ and $R_{02} > 1$.

Choose $\alpha_1 = 0.55, \alpha_2 = 0.25, b_1 = 0.003, b_2 = 0.003, d = 0.006, \mu_1 = 0.002, \mu_2 = 0.002, \beta_1 = 0.0028, \beta_2 = 0.005$. Then, it derives that $R_{01} \approx 2.856 > 1$ and $R_{02} \approx 2.007 > 1$. Figure 5 shows that positive-equilibrium solution E_2^* is globally asymptotically stable for any $k \in [1, 200]$, and two cases of rumors are rare but still prevalent as time goes by.

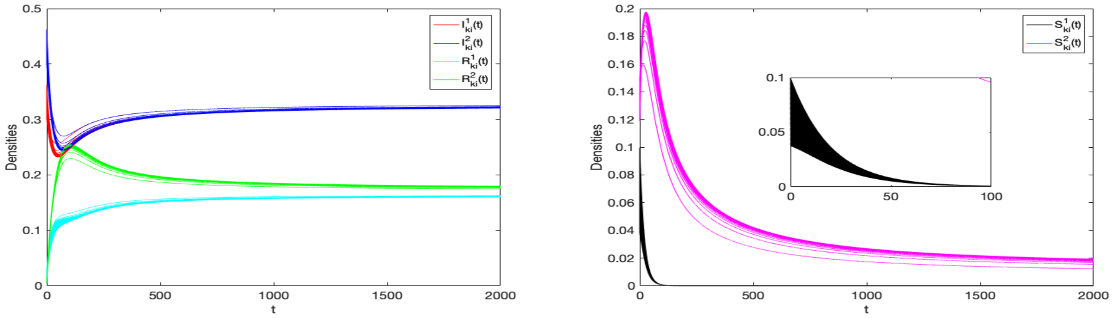

Figure 4. The stability of positive-equilibrium solution E_1^* with $R_{01} < 1$ and $R_{02} > 1$.

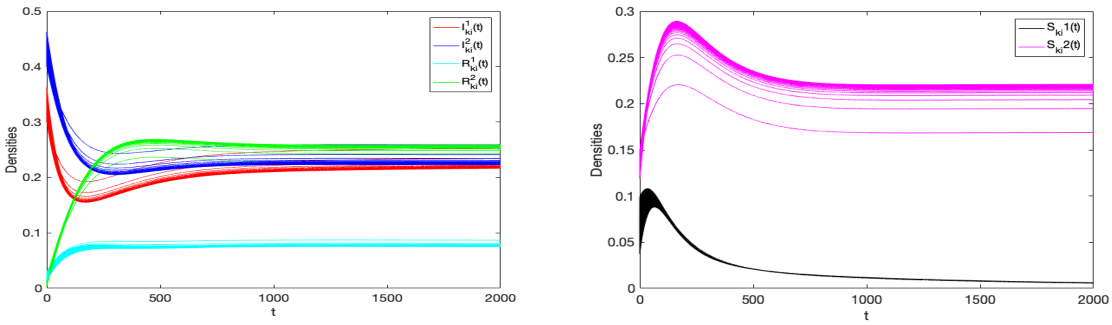

Figure 5. The stability of positive-equilibrium solution E_2^* with $R_{01} > 1$ and $R_{02} > 1$.

Remark 3. Note that the stability of the zero-equilibrium solution and positive-equilibrium solutions of the model (2.1) is discussed, respectively. When $R_0 < 1$, the system has a zero-equilibrium solution, and the rumor gradually disappears (see Figure 3). However, when the system has a positive-equilibrium solution, its stability is more complex. When $R_{01} < 1$ and $R_{02} > 1$, the rumor propagated in language 1 disappears, while the rumor propagated in Language-2 still prevalent (see Figure 4). Conversely, when $R_{01} > 1$ and $R_{02} > 1$, rumors propagated in both languages continue to exist over time (see Figure 5). Therefore, the study on the propagation of rumors in a multilingual environment holds significant research value.

4.3. Sensitivity analysis

To further analyze the different factors contribution to the rumor spreading, a sensitivity analysis is further shown here [35]. The normalized forward sensitivity index of a variable u depends differentiably on a parameter p. It is defined as:

$$\gamma_p^u := \frac{\partial u}{\partial p} \times \frac{u}{p}.$$

Next, in terms of the uncertainty of the basic reproductive number, we choose the parameter values from Section 4.1. Since the value R_0 is related to R_{01}, R_{02}, we need to perform their sensitivity analyses, respectively. Here, we only show its results for R_{02} due to their similarity (see Table 2).

Table 2. Sensitivity index of R_{01} to parameter values of model (2.1).

Parameter	Sensitivity index
$\langle k \rangle$	−1.000
α_1	0
α_2	+1.000
b_i	+0.500
β_1	0
β_2	−0.833
μ_i	−0.167
d	−1.333

From Table 2, it is evident that a 1% reduction in the leaving rate $\langle k \rangle$ leads to a 1% increase in R_{02}, while a 1% reduction in β_2 results in a 0.833% increase in R_{02}. Conversely, a 1% reduction in the incoming rate $b_i (i = 1, 2)$ decreases R_{02} by 0.5%. Additionally, increasing $\mu_i (i = 1, 2)$ impacts rumor spreading, further reducing R_{02}.

Remark 4. In summary, various effective strategies can be employed to decrease R_{02}, such as enhancing the leaving rate d, regulating the transmission rate among individuals, etc. These measures can be implemented through network consensus monitoring, educational campaigns, and other interventions. However, some specific control strategies are not proposed in the model, and an optimal solution to suppress rumours is also not provided here. Therefore, we will carry out an in-depth study on this aspect in the future.

5. Conclusions

The dynamical behaviors of the multilingual rumor propagation 2I2S2R model have been analyzed under heterogeneous networks. The basic reproduction number by the next-generation matrix method has been calculated, and the stability has been explored in different cases. Moreover, numerical simulations have been provided to further show the dynamic characteristics of the model. However, the research background presented in this paper is somewhat idealized and does not adequately address several critical real-world factors, including time delays, stochastic phenomena, and the influence of government surveillance. To enhance the robustness of our findings, in future research, we aim to develop a more comprehensive multilingual rumor propagation model that reflects the complexities of real-world scenarios. This will enable us to engage in a more in-depth discussion regarding the control mechanisms associated with the suppression and elimination of rumors.

Author contributions

Methodology, L.H. and J.W.; formal analysis, L.H. and J.W; resources, J.W.; writing original draft preparation, L.H. and J.W.; writing-review and editing, J.L. and T.M.; supervision, T.M.; software, T.M. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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