Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 23971–23978.
DOI:10.3934/math.20241165
Received: 16 May 2024
Revised: 26 July 2024
Accepted: 30 July 2024
Published: 13 August 2024

Research article

On some m-th root metrics

Xiaoling Zhang[1,*], **Cuiling Ma**[1] **and Lili Zhao**[2]

[1] College of Mathematics and Systems Science, Xinjiang University, Urumqi 830017, China
[2] School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, China

* **Correspondence:** Email: zhangxiaoling@xju.edu.cn.

Abstract: The Ricci curvature in Finsler geometry naturally generalizes the Ricci curvature in Riemannian geometry. In this paper, we study the m-th root metric with weakly isotropic scalar curvature and obtain that its scalar curvature must vanish. Further, we prove that a locally conformally flat cubic Finsler metric with weakly isotropic scalar curvature must be locally Minkowskian.

Keywords: scalar curvature; weakly isotropic scalar curvature; m-th root Finsler metrics; locally conformally flat; cubic metrics
Mathematics Subject Classification: 53C30, 53C60

1. Introduction

The Ricci curvature in Finsler geometry naturally generalizes the Ricci curvature in Riemannian geometry. However, in Finsler geometry, there are several versions of the definition of scalar curvature because the Ricci curvature tensor is defined in different forms. Here we adopt the definition of scalar curvature, which was introduced by Akbar–Zadeh [1, see (2.1)]. Tayebi [11] characterized general fourth-root metrics with isotropic scalar curvature. Moreover, he studied Bryant metrics with isotropic scalar curvature. Later, a locally conformally flat (α,β)-metric with weakly isotropic scalar curvature was studied by Chen–Xia [4]. They proved that its scalar curvature must vanish. Recently, Cheng–Gong [5] proved that if a Randers metric is of weakly isotropic scalar curvature, then it must be of isotropic S-curvature. Furthermore, they concluded that when a locally conformally flat Randers metric is of weakly isotropic scalar curvature, it is Minkowskian or Riemannian. Very recently, Ma–Zhang–Zhang [8] showed that the Kropina metric with isotropic scalar curvature is equivalent to an Einstein Kropina metric according to the navigation data.

Shimada [9] first developed the theory of m-th root metrics as an interesting example of Finsler metrics, immediately following Matsumoto and Numata's theory of cubic metrics [7]. It is applied to biology as an ecological metric by Antonelli [2]. Later, many scholars studied these metrics ([3, 6, 10–

12], etc). In [13], cubic Finsler manifolds in dimensions two or three were studied by Wegener. He only abstracted his PhD thesis and barely did all the calculations in that paper. Kim and Park [6] studied the m-th root Finsler metrics which admit $(\alpha,\beta)-$types. In [12], Tayebi–Razgordani–Najafi showed that if the locally conformally flat cubic metric is of relatively isotropic mean Landsberg curvature on a manifold M of dimension $n(\geq 3)$, then it is a Riemannian metric or a locally Minkowski metric. Tripathia–Khanb–Chaubey [10] considered a cubic (α,β)-metric which is a special class of p-power Finsler metric, and obtained the conditions under which the Finsler space with such special metric will be projectively flat. Further, they obtained in which case this Finsler space will be a Berwald space or Douglas space.

In this paper, we mainly focus on m-th root metrics with weakly isotropic scalar curvature and obtain the following results:

Theorem 1.1. *Let the $m(\geq 3)$-th root metric F be of weakly isotropic scalar curvature. Then its scalar curvature must vanish.*

Let $A := a_{i_1 i_2 \cdots i_m}(x) y^{i_1} y^{i_2} \cdots y^{i_m}$. If $A = F^m$ is irreducible, then the further result is obtained as follows:

Theorem 1.2. *Let $F = \sqrt[m]{A}$ be the $m(\geq 3)$-th root metric. Assume that A is irreducible. Then the following are equivalent: (i) F is of weakly isotropic scalar curvature; (ii) its scalar curvature vanishes; (iii) it is Ricci-flat.*

Based on Theorem 1.1, we obtain the result for locally conformally flat cubic Finsler metrics as following:

Theorem 1.3. *Let F be a locally conformally flat cubic Finsler metric on a manifold M of dimension $n(\geq 3)$. If F is of weakly isotropic scalar curvature, then F must be locally Minkowskian.*

2. Preliminaries

In this section, we mainly introduce several geometric quantities in Finsler geometry and several results that will be used later.

Let M be an $n(\geq 3)$-dimensional smooth manifold. The points in the tangent bundle TM are denoted by (x,y), where $x \in M$ and $y \in T_xM$. Let (x^i, y^i) be the local coordinates of TM with $y = y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \longrightarrow [0,+\infty)$ such that

(1) F is smooth in $TM \backslash \{0\}$;

(2) $F(x,\lambda y) = \lambda F(x,y)$ for any $\lambda > 0$;

(3) The fundamental quadratic form $g = g_{ij}(x,y) dx^i \otimes dx^j$, where

$$g_{ij}(x,y) = [\frac{1}{2} F^2(x,y)]_{y^i y^j}$$

is positively definite. We use the notations: $F_{y^i} := \frac{\partial F}{\partial y^i}$, $F_{x^i} := \frac{\partial F}{\partial x^i}$, $F^2_{y^i y^j} := \frac{\partial^2 F^2}{\partial y^i \partial y^j}$.

Let F be a Finsler metric on an n-dimensional manifold M, and let G^i be the geodesic coefficients of F, which are defined by

$$G^i := \frac{1}{4} g^{ij} (F^2_{x^k y^j} y^k - F^2_{x^j}),$$

where $(g^{ij}) = (g_{ij})^{-1}$. For any $x \in M$ and $y \in T_xM \backslash \{0\}$, the Riemann curvature $R_y := R^i{}_k(x,y) \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i{}_k := 2G^i_{x^k} - G^i_{x^jy^k}y^j + 2G^jG^i_{y^jy^k} - G^i_{y^j}G^j_{y^k}.$$

The Ricci curvature **Ric** is the trace of the Riemann curvature defined by

$$\mathbf{Ric} := R^k{}_k.$$

The Ricci tensor is

$$\mathbf{Ric}_{ij} := \frac{1}{2}\mathbf{Ric}_{y^iy^j}.$$

By the homogeneity of **Ric**, we have $\mathbf{Ric} = \mathbf{Ric}_{ij}y^iy^j$. The scalar curvature $\mathbf{r}$ of F is defined as

$$\mathbf{r} := g^{ij}\mathbf{Ric}_{ij}. \tag{2.1}$$

A Finsler metric is said to be of weakly isotropic scalar curvature if there exists a 1-form $\theta = \theta_i(x)y^i$ and a scalar function $\chi = \chi(x)$ such that

$$\mathbf{r} = n(n-1)(\frac{\theta}{F} + \chi). \tag{2.2}$$

An (α,β)-metric is a Finsler metric of the form

$$F = \alpha\phi(s),$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form, $s := \frac{\beta}{\alpha}$ and $b :=\parallel \beta \parallel_\alpha < b_0$. It has been proved that $F = \alpha\phi(s)$ is a positive definite Finsler metric if and only if $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying the following condition:

$$\phi(s) - s\phi'(s) + (B - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0, \tag{2.3}$$

where $B := b^2$.

Let $F = \sqrt[3]{a_{ijk}(x)y^iy^jy^k}$ be a cubic metric on a manifold M of dimension $n \geq 3$. By choosing a suitable non-degenerate quadratic form $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and one-form $\beta = b_i(x)y^i$, it can be written in the form

$$F = \sqrt[3]{p\beta\alpha^2 + q\beta^3},$$

where p and q are real constants such that $p + qB \neq 0$ (see [6]). The above equation can be rewritten as

$$F = \alpha(ps + qs^3)^{\frac{1}{3}},$$

which means that F is also an (α,β)-metric with $\phi(s) = (ps + qs^3)^{\frac{1}{3}}$. Then, by (2.3), we obtain

$$-p^2B + p(4p + 3qB)s^2 > 0. \tag{2.4}$$

Two Finsler metrics F and $\widetilde{F}$ on a manifold M are said to be conformally related if there is a scalar function $\kappa = \kappa(x)$ on M such that $F = e^{\kappa(x)}\widetilde{F}$. Particularly, an (α,β)-metric $F = \alpha\phi(\frac{\beta}{\alpha})$ is said to be conformally related to a Finsler metric $\widetilde{F}$ if $F = e^{\kappa(x)}\widetilde{F}$ with $\widetilde{F} = \widetilde{\alpha}\phi(\widetilde{s}) = \widetilde{\alpha}\phi(\frac{\widetilde{\beta}}{\widetilde{\alpha}})$. In the following, we always use symbols with a tilde to denote the corresponding quantities of the metric $\widetilde{F}$. Note that $\alpha = e^{\kappa(x)}\widetilde{\alpha}, \beta = e^{\kappa(x)}\widetilde{\beta}$, thus $\widetilde{s} = s$.

A Finsler metric that is conformally related to a locally Minkowski metric is said to be locally conformally flat. Thus, a locally conformally flat (α,β)-metric F has the form $F = e^{\kappa(x)}\widetilde{F}$, where $\widetilde{F} = \widetilde{\alpha}\phi(\frac{\widetilde{\beta}}{\widetilde{\alpha}})$ is a locally Minkowski metric.

Denoting

$$\begin{aligned} &r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),\\ &r^i{}_j := a^{il} r_{lj}, \quad s^i{}_j := a^{il} s_{lj},\\ &r_j := b^i r_{ij}, \quad r := b^i r_i, \quad s_j := b^i s_{ij},\\ &r_{00} := r_{ij} y^i y^j, \quad s^i{}_0 := s^i{}_j y^j, \quad s_0 := s_i y^i, \end{aligned}$$

where $b^i := a^{ij} b_j$, $b_{i|j}$ denotes the covariant differentiation with respect to α.

Let G^i and G^i_α denote the geodesic coefficients of F and α, respectively. The geodesic coefficients G^i of $F = \alpha\phi(\frac{\beta}{\alpha})$ are related to G^i_α by

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + (-2Q\alpha s_0 + r_{00})(\Psi b^i + \Theta \alpha^{-1} y^i),$$

where

$$\begin{aligned} &Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[(\phi - s\phi) + (B - s^2)\phi'']},\\ &\Psi := \frac{\phi''}{2[(\phi - s\phi') + (B - s^2)\phi'']}. \end{aligned}$$

Assume that $F = \alpha\phi(\frac{\beta}{\alpha})$ is conformally related to a Finsler metric $\widetilde{F} = \widetilde{\alpha}\phi(\frac{\widetilde{\beta}}{\widetilde{\alpha}})$ on M, i.e., $F = e^{\kappa(x)}\widetilde{F}$. Then

$$a_{ij} = e^{2\kappa(x)}\widetilde{a}_{ij}, b_i = e^{\kappa(x)}\widetilde{b}_i, \widetilde{b} :=\| \widetilde{\beta} \|_{\widetilde{\alpha}} = \sqrt{\widetilde{a}_{ij}\widetilde{b}^i\widetilde{b}^j} = b.$$

Further, we have

$$\begin{aligned} &b_{i|j} = e^{\kappa(x)}(\widetilde{b}_{i\|j} - \widetilde{b}_j\kappa_i + \widetilde{b}_l\kappa^l\widetilde{a}_{ij}),\\ &{}^{\alpha}\Gamma^l_{ij} = {}^{\widetilde{\alpha}}\widetilde{\Gamma}^l_{ij} + \kappa_j\delta^l_i + \kappa_i\delta^l_j - \kappa^l\widetilde{a}_{ij},\\ &r_{ij} = e^{\kappa(x)}\widetilde{r}_{ij} + \frac{1}{2}e^{\kappa(x)}(-\widetilde{b}_j\kappa_i - \widetilde{b}_i\kappa_j + 2\widetilde{b}_l\kappa^l\widetilde{a}_{ij}),\\ &s_{ij} = e^{\kappa(x)}\widetilde{s}_{ij} + \frac{1}{2}e^{\kappa(x)}(\widetilde{b}_i\kappa_j - \widetilde{b}_j\kappa_i),\\ &r_i = \widetilde{r}_i + \frac{1}{2}(\widetilde{b}_l\kappa^l\widetilde{b}_i - b^2\kappa_i), \ r = e^{-\kappa(x)}\widetilde{r},\\ &s_i = \widetilde{s}_i + \frac{1}{2}(b^2\kappa_i - \widetilde{b}_l\kappa^l\widetilde{b}_i),\\ &r^i{}_i = e^{-\kappa(x)}\widetilde{r}^i{}_i + (n-1)e^{-\kappa(x)}\widetilde{b}_i\kappa^i,\\ &s^j{}_i = e^{-\kappa(x)}\widetilde{s}^j{}_i + \frac{1}{2}e^{-\kappa(x)}(\widetilde{b}^j\kappa_i - \widetilde{b}_i\kappa^j). \end{aligned}$$

Here $\widetilde{b}_{i\|j}$ denotes the covariant derivatives of $\widetilde{b}_i$ with respect to $\widetilde{\alpha}$, ${}^{\alpha}\Gamma^m{}_{ij}$ and ${}^{\widetilde{\alpha}}\widetilde{\Gamma}^m{}_{ij}$ denote Levi–Civita connections with respect to α and $\widetilde{\alpha}$, respectively. In the following, we adopt the notations $\kappa_i := \frac{\partial\kappa(x)}{\partial x^i}$, $\kappa_{ij} := \frac{\partial^2\kappa(x)}{\partial x^i\partial x^j}$, $\kappa^i := \widetilde{a}^{ij}\kappa_j$, $\widetilde{b}^i := \widetilde{a}^{ij}\widetilde{b}_j$, $f := \widetilde{b}_i\kappa^i$, $f_1 := \kappa_{ij}\widetilde{b}^i y^j$, $f_2 := \kappa_{ij}\widetilde{b}^i\widetilde{b}^j$, $\kappa_0 := \kappa_i y^i$, $\kappa_{00} := \kappa_{ij} y^i y^j$ and $\| \nabla\kappa \|^2_{\widetilde{\alpha}} := \widetilde{a}^{ij}\kappa_i\kappa_j$.

Lemma 2.1. (**[4]**) *Let $F = e^{\kappa(x)}\widetilde{F}$, where $\widetilde{F} = \widetilde{\alpha}\phi(\frac{\widetilde{\beta}}{\widetilde{\alpha}})$ is locally Minkowskian. Then the Ricci curvature of F is determined by*

$$\boldsymbol{Ric} = D_1 \| \nabla\kappa \|^2_{\widetilde{\alpha}}\widetilde{\alpha} + D_2\kappa_0^2 + D_3\kappa_0 f\widetilde{\alpha} + D_4 f^2\widetilde{\alpha}^2 + D_5 f_1\widetilde{\alpha} + D_6\widetilde{\alpha}^2 + D_7\kappa_{00},$$

where $D_k(k = 1, ..., 7)$ is listed in Lemma 3.2 in [4].
Lemma 2.2. (**[4]**) *Let $F = e^{\kappa(x)}\widetilde{F}$, where $\widetilde{F} = \widetilde{\alpha}\phi(\frac{\widetilde{\beta}}{\widetilde{\alpha}})$ is locally Minkowskian. Then the scalar curvature of F is determined by*

$$\boldsymbol{r} = \frac{1}{2}e^{-2\kappa(x)}\rho^{-1}[\Sigma_1 - (\tau + \eta\lambda^2)\Sigma_2 - \frac{\lambda\eta}{\widetilde{\alpha}}\Sigma_3 - \frac{\eta}{\widetilde{\alpha}^2}\Sigma_4],$$

where

$$\tau := \frac{\delta}{1 + \delta B}, \quad \eta := \frac{\mu}{1 + Y^2\mu}, \quad \lambda := \frac{\varepsilon - \delta s}{1 + \delta B},$$
$$\delta := \frac{\rho_0 - \varepsilon^2\rho_2}{\rho}, \quad \varepsilon := \frac{\rho_1}{\rho_2}, \quad \mu := \frac{\rho_2}{\rho},$$
$$Y := \sqrt{A_{ij}Y^iY^j}, \quad A_{ij} := a_{ij} + \delta b_i b_j,$$
$$\rho := \phi(\phi - s\phi'), \quad \rho_0 := \phi\phi'' + \phi'\phi',$$
$$\rho_1 := -s(\phi\phi'' + \phi'\phi') + \phi\phi', \quad \rho_2 := s[s(\phi\phi'' + \phi'\phi') - \phi\phi'],$$

and $\Sigma_i(i = 1, ..., 4)$ are listed in the proof of Lemma 3.3 in [4].
Lemma 2.3. (**[14]**) *Let m-th root metric $F = \sqrt[m]{a_{i_1 i_2 \cdots i_m}(x)y^{i_1}y^{i_2}\cdots y^{i_m}}$ be a Finsler metric on a manifold of dimension n. Then the Ricci curvature of F is a rational function in y.*

3. Proof of main theorems

In this section, we will prove the main theorems. Firstly, we give the proof of Theorem 1.1.
The proof of Theorem 1.1. For an m-th root metric $F = \sqrt[m]{a_{i_1 i_2 \cdots i_m}(x)y^{i_1}y^{i_2}\cdots y^{i_m}}$ on a manifold M, the inverse of the fundamental tensor of F is given by (see [14])

$$g^{ij} = \frac{1}{(m-1)F^2}(AA^{ij} + (m-2)y^iy^j), \tag{3.1}$$

where $A_{ij} = \frac{1}{m(m-1)}\frac{\partial^2 A}{\partial y^i \partial y^j}$ and $(A^{ij}) = (A_{ij})^{-1}$. Thus, $F^2 g^{ij}$ are rational functions in y.

By Lemma 2.3, the Ricci curvature **Ric** of m-th root metric is a rational function in y. Thus, $\mathbf{Ric}_{ij} := \mathbf{Ric}_{y^iy^j}$ are rational functions. According to (2.1), we have

$$F^2\mathbf{r} = F^2 g^{ij}\mathbf{Ric}_{ij}. \tag{3.2}$$

This means that $F^2\mathbf{r}$ is a rational function in y.

On the other hand, if F is of weakly isotropic scalar curvature, according to (2.2), we obtain

$$F^2\mathbf{r} = n(n-1)(\theta F + \chi F^2),$$

where θ is a 1-form and χ is a scalar function. The right side of the above equation is an irrational function in y. Comparing it with (3.2), we have $\mathbf{r} = 0$. ■

In the following, the proof of Theorem 1.2 is given.

The proof of Theorem 1.2. By Theorem 1.1, we conclude that F is of weakly isotropic scalar curvature if and only if its scalar curvature vanishes. So we just need to prove that (ii) is equivalent to (iii). Assume that the scalar curvature vanishes. Hence, by (3.1), $0 = \mathbf{r} = g^{ij}\mathbf{Ric}_{ij} = \frac{1}{m-1}F^{-2}(AA^{ij} + (m-2)y^iy^j)\mathbf{Ric}_{ij}$ holds. It means that

$$0 = (AA^{ij} + (m-2)y^iy^j)\mathbf{Ric}_{ij} = AA^{ij}\mathbf{Ric}_{ij} + (m-2)\mathbf{Ric}.$$

Since A is irreducible, **Ric** must be divided by A. Thus, $\mathbf{Ric} = 0$.

Conversely, if $\mathbf{Ric} = 0$, then by the definition of $\mathbf{r}$ we have $\mathbf{r} = 0$. ■

Based on Theorem 1.1, we can prove Theorem 1.3 for locally conformally flat cubic metrics.

The proof of Theorem 1.3. Assume that the locally conformally flat cubic metric F is of weakly isotropic scalar curvature. Then, by Lemma 2.2 and Theorem 1.1, we obtain the scalar curvature vanishes, i.e.,

$$\Sigma_1 - (\tau + \eta\lambda^2)\Sigma_2 - \frac{\lambda\eta}{\widetilde{\alpha}}\Sigma_3 - \frac{\eta}{\widetilde{\alpha}^2}\Sigma_4 = 0.$$

Further, by detailed expressions of $\Sigma_i (i = 1, \cdots 4)$, the above equation can be rewritten as

$$\frac{B(4p + 3qB)\kappa_0^2 - 4(4p + 3qB)\widetilde{\beta}\kappa_0 f + 4p\widetilde{\alpha}^2 f^2}{(4p + 3qB)^8\widetilde{\alpha}^2 s^2\gamma^7} + \frac{T}{\gamma^6} = 0, \tag{3.3}$$

where $\gamma := pB\widetilde{\alpha}^2 - (4p + 3qB)\widetilde{\beta}^2$ and T has no γ^{-1}.

Thus, the first term of (3.3) can be divided by γ. It means that there is a function $h(x)$ on M such that

$$B(4p + 3qB)\kappa_0^2 - 4(4p + 3qB)\widetilde{\beta}\kappa_0 f + 4p\widetilde{\alpha}^2 f^2 = h(x)\gamma.$$

The above equation can be rewritten as

$$B(4p + 3qB)\kappa_0^2 - 4(4p + 3qB)\widetilde{\beta}\kappa_0 f + 4p\widetilde{\alpha}^2 f^2 = h(x)[pB\widetilde{\alpha}^2 - (4p + 3qB)\widetilde{\beta}^2]. \tag{3.4}$$

Differentiating (3.4) with y^i yields

$$B(4p + 3qB)\kappa_0\kappa_i - 2(4p + 3qB)(\widetilde{b}_i\kappa_0 + \widetilde{\beta}\kappa_i)f + 4p\widetilde{a}_{il}y^l f^2 = h(x)[pB\widetilde{a}_{il}y^l - (4p + 3qB)\widetilde{\beta}\widetilde{b}_i]. \tag{3.5}$$

Differentiating (3.5) with y^j yields

$$B(4p + 3qB)\kappa_i\kappa_j - 2(4p + 3qB)(\widetilde{b}_i\kappa_j + \widetilde{b}_j\kappa_i)f + 4p\widetilde{a}_{ij}f^2 = h(x)[pB\widetilde{a}_{ij} - (4p + 3qB)\widetilde{b}_i\widetilde{b}_j].$$

Contracting the above with $\widetilde{b}^i\widetilde{b}^j$ yields

$$Bf^2(8p + 9qB) = 3B^2h(x)(p + qB).$$

Thus, we have

$$h(x) = \frac{(8p + 9qB)f^2}{3B(p + qB)}. \tag{3.6}$$

Substituting (3.6) into (3.5) and contracting (3.5) with $\widetilde{b}^i$ yield

$$(4p + 3qB)f(f\widetilde{\beta} - B\kappa_0) = 0. \tag{3.7}$$

Furthermore, by (2.4) and $4p + 3qB \neq 0$, we have $f(f\widetilde{\beta} - B\kappa_0) = 0$.

Case I: $f = 0$. It means $h(x) = 0$ by (3.6). Thus, one has that $\kappa_i = 0$ by (3.4), which means

$$\kappa = constant.$$

Case II: $f \neq 0$. It implies that $f\widetilde{\beta} - B\kappa_0 = 0$. Substituting it into (3.4), we obtain

$$\widetilde{\beta}^2 = -B\widetilde{\alpha}^2,$$

which does not exist.

Above all, we have $\kappa = constant$. Thus we conclude that the conformal transformation must be homothetic. ■

Author contributions

X. Zhang: Conceptualization, Validation, Formal analysis, Resources, Software, Investigation, Methodology, Supervision, Writing-original draft, Writing-review and editing, Project administration, Funding acquisition; C. Ma: Formal analysis, Software, Investigation, Supervision, Writing-review and editing; L. Zhao: Resources, Investigation, Supervision. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Xiaoling Zhang's research is supported by National Natural Science Foundation of China (No. 11961061, 11761069).

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. H. Akbar-Zadeh, Sur les espaces de Finsler *à* courbures sectionnelles constantes, *Bulletins de l'Académie Royale de Belgique*, **74** (1988), 281–322. https://doi.org/10.3406/barb.1988.57782
2. P. L. Antonelli, R. S. Ingarden, M. Matsumoto, *The theory of Sprays and Finsler space with applications in physics and biology*, Dordrecht: Springer, 1993. https://doi.org/10.1007/978-94-015-8194-3
3. G. Aranasiu, M. Neagu, On Cartan spaces with the m-th root metric $K(x,p) = \sqrt[m]{a^{i_1 i_2 \ldots i(m)}(x) p_{i_1} p_{i_2} \cdots p_{i_m}}$, *Hypercomplex Numbers in Geometry and Physics*, **2** (2009), 67–73.

4. B. Chen, K. Xia, On conformally flat polynomial (α,β)-metrics with weakly isotropic scalar curvature, *J. Korean Math. Soc.*, **56** (2019), 329–352. https://doi.org/10.4134/JKMS.j180186
5. X. Cheng, Y. Gong, The Randers metrics of weakly isotropic scalar curvature, *Acta Mathematica Sinica (Chinese Series)*, **64** (2021), 1027–1036. https://doi.org/10.12386/A2021sxxb0085
6. B. Kim, H. Park, The m-th root Finsler metrics admitting $(\alpha,\beta)-$types, *Bull. Korean Math. Soc.*, **41** (2004), 45–52. https://doi.org/10.4134/BKMS.2004.41.1.045
7. M. Matsumoto, S. Numara, On Finsler spaces with a cubic metric, *Tensor (N. S.)*, **33** (1979), 153–162.
8. Y. Ma, X. Zhang, M. Zhang, Kropina metrics with isotropic scalar curvature via navigation data, *Mathematics*, **12** (2024), 505. https://doi.org/10.3390/math12040505
9. H. Shimada, On Finsler spaces with metric $L = \sqrt[m]{a_{i_1 i_2 \ldots i_m} y^{i_1} y^{i_2} \ldots y^{i_m}}$, *Tensor (N. S.)*, **33** (1979), 365–372.
10. B. K. Tripathi, S. Khanb, V. Chaubey, On projectively flat Finsler space with a cubic (α,β)-metric, *Filomat*, **37** (2023), 8975–8982. https://doi.org/10.2298/FIL2326975T
11. A. Tayebi, On generalized 4-th root metrics of isotropic scalar curvature, *Math. Slovaca*, **68** (2018), 907–928. https://doi.org/10.1515/ms-2017-0154
12. A. Tayebi, M. Razgordani, B. Najafi, On conformally flat cubic (α,β)-metrics, *Vietnam J. Math.*, **49** (2021), 987–1000. https://doi.org/10.1007/s10013-020-00389-0
13. V. J. M. Wegener, Untersuchungen der zwei- und dreidimensionalen Finslerschen Raumemit der Grundform $L = \sqrt[3]{a_{ijk} x'^i x'^k x'^l}$, *Proc. K. Akad. Wet. (Amsterdam)*, **38** (1935), 949–955.
14. Y. Yu, Y You, On Einstein m-th root metrics, *Differ. Geom. Appl.*, **28** (2010), 290–294. https://doi.org/10.1016/j.difgeo.2009.10.011