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Research article

Hopf bifurcation in a predator-prey model under fuzzy parameters involving prey refuge and fear effects

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Abstract: In ecology, the most significant aspect is that the interactions between predators and prey are extremely complicated. Numerous experiments have shown that both direct predation and the fear induced in prey by the presence of predators lead to a reduction in prey density in predator-prey interactions. In addition, a suitable shelter can effectively stop predators from attacking as well as support the persistence of prey population. There has been less exploration of the effects of not only fear but also refuge factors on the dynamics of predator prey interactions. In this paper, we unveil several conclusions about a predator-prey system with fuzzy parameters, considering the cost of fear in two prey species and the effect of shelter on two prey species and one predator. As the first step of the investigation, the boundedness and non-negativity of the solutions to the system are put forward. Using the Jocabian matrix and Lyapunov function methods, we further analyze the existence and stability of the available equilibria and also the existence of Hopf bifurcation, considering the fear parameter as the bifurcation parameter that has been observed by applying the normal form theory. Finally, numerical simulations help us better understand the dynamics of the model, in which some interesting chaotic phenomena are also exhibited.

Keywords: fear effect; refuge; Hopf bifurcation; fuzzy parameter; chaotic phenomena
Mathematics Subject Classification: 34C23, 34D20

1. Introduction

In population ecology, understanding how predators and primary producers influence nutrient flow relative to each other is important. Ecosystem interactions and predator-prey relationships are governed by predation and the delivery of resource processes. The identification of ecological factors that can alter or control dynamic behavior requires theoretical and experimental research. One way to study these questions is by means of experimental control, and another useful way is via mathematical modeling as well as computer simulations. Over decades of theoretical ecology and biomathematics development, mathematical modeling has become an indispensable tool for scientists in related fields to study ecosystems. Since Lotka [1] and Volterra [2], as cornerstones of theoretical ecology, published the first study of predator-prey dynamics, any species in nature can be a predator or prey, and due to its prevalence, it has become one of the most popular topics for researchers to study [3–5]. Besides, because biological resources are renewable and have the most unique development mechanisms, the over-utilization of biological resources and the destruction of the environment by humans will directly affect the balance of the ecosystem. Maintaining ecological balance and meeting humans material needs have attracted the most attention from researchers focused on the scientific management of renewable resource development [6–8].

Shelter serves as a defense strategy. It refers broadly to a series of behaviors by prey to avoid predators in order to increase their survival rate. The concept of sanctuary was first developed by Maynard-Smith [9] and Gause et al. [10], and its popularity has been very high, garnering widespread attention from many scholars [11–15]. Sih et al. [16] investigated the effects of prey refuge in a three-species model and concluded that the system's stability is strongly related to the refuge. Also, similar findings can be displayed in [17–22]. The two modes of refuge analyzed by Gonzalez-Olivares et al. [17] have diverse stability domains in terms of the parameter space. Qi et al. [21] ensure the stability of the system by varying the strength of the refuge.

Through reviewing a large amount of literature, we begin to consider [23, 24] as a basis for the two prey and one predator species that will be modeled in this article. We assume that at a certain time t, the populations of the two prey and one predator are $x_1(t)$, $x_2(t)$, and $y(t)$, respectively. Based on the above, we construct the following model:

$$\begin{cases} \dfrac{dx_1}{dt} = r_1 x_1 \left(1 - \dfrac{x_1}{K_1}\right) - a_1 x_1 x_2 - c_1(1 - m_1)x_1 y - q_1 E_1 x_1, \\ \dfrac{dx_2}{dt} = r_2 x_2 \left(1 - \dfrac{x_2}{K_2}\right) - a_2 x_1 x_2 - c_2(1 - m_2)x_2 y - q_2 E_2 x_2, \\ \dfrac{dy}{dt} = e_1(1 - m_1)x_1 y + e_2(1 - m_2)x_2 y - dy - q_3 E_3 y. \end{cases} \tag{1.1}$$

The significance of the full parameters is annotated in Table 1.

Table 1. Biological meaning of parameters.

Parameters	Biological meaning
r_1, r_2	Growth rates of prey x_1 and prey x_2
K_1, K_2	Carrying capacity of prey x_1 and prey x_2
a_1, a_2	Interspecific competition between prey x_1 and prey x_2
c_1, c_2	Predation coefficients for prey x_1 and prey x_2
m_1, m_2	Refuge rates of prey x_1 and prey x_2
e_1, e_2	Conversion factors for prey x_1 and prey x_2
q_1, q_2, q_3	Captureability factors for prey x_1, prey x_2 and predator y
E_1, E_2, E_3	Harvesting efforts for prey x_1, prey x_2 and predator y
d	Predator y mortality rate

Most species in nature, including humans, are influenced by fear. Fear may cause an abnormal state and behavior to arise. As usual, prey have an innate fear of predators. The ecology of fear is related to combining the optimal behavior of prey and predators with their population densities [25, 26]. In view of reality, it is a fact that prey fear predators, which is seen as a psychological effect that can have a lasting impact on prey populations. This psychological influence is often easy to overlook, but it is necessary to consider it in the context of practical ecology [27]. Wang et al. [28] first considered the effect of the fear factor on the model and first proposed the fear of prey $F(k, y)$. Afterwards, some researchers have investigated the effects of the fear effect and predator interferences in some three-dimensional systems as well as explored the generation of Hopf bifurcation conditions in the presence of a fear parameter as a bifurcation parameter [29–32]. Zanette et al. [33] observed that prey will reduce reproducing because of fear of being killed by predators, thus decreasing the risk of being killed after giving birth, which also leads directly to a decline in prey birth rates. According to the above discussion, our paper considers the different fears k_i caused by predators for the two prey species.

In reality, when prey feel the crisis of being hunted, they will reproduce less and increase their survival rate. These conditions about the fear factor $F(k_i, y)$ $(i = 1, 2)$ are listed as follows:

1) $F(0, y) = 1$: prey production does not decrease when the prey does not fear the predator;

2) $F(k_i, 0) = 1$: even though the prey will develop a fear of predators and there will be no predators, prey production will still not decline;

3) $\lim\limits_{k_i\to\infty} F(k_i, y) = 0$: when the prey's fear of the predator is very high, this will result in the prey production tending to zero;

4) $\lim\limits_{y\to\infty} F(k_i, y) = 0$: prey have a fear of predators, and when predator numbers are too large, this can also lead to prey production tending to zero;

5) $\frac{\partial F(k_i,y)}{\partial k_i} < 0$: the greater the prey's fear of predators, the less productive it will be;

6) $\frac{\partial F(k_i,y)}{\partial y} < 0$: predators are inversely proportional to their prey.

For ease of analysis, we draw on Wang et al. [28] to consider the fear effect:

$$F(k_i, y) = \frac{1}{1 + k_i y}(i = 1, 2), \tag{1.2}$$

obviously, $F(k_i, y)$ $(i = 1, 2)$ in (1.2) satisfies conditions **1**)–**6**). Based on the above conditions, this study will consider the effect of fear on system (1.1) to obtain system (1.3).

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{r_1 x_1}{1 + k_1 y}\left(1 - \dfrac{x_1}{K_1}\right) - a_1 x_1 x_2 - c_1(1 - m_1)x_1 y - q_1 E_1 x_1, \\ \dfrac{dx_2}{dt} = \dfrac{r_2 x_2}{1 + k_2 y}\left(1 - \dfrac{x_2}{K_2}\right) - a_2 x_1 x_2 - c_2(1 - m_2)x_2 y - q_2 E_2 x_2, \\ \dfrac{dy}{dt} = e_1(1 - m_1)x_1 y + e_2(1 - m_2)x_2 y - dy - q_3 E_3 y. \end{cases} \tag{1.3}$$

Notably, most biological parameters in much of the literature are fixed constants. However, in reality, the survival of species is full of unknowns, and all data are not always constant, which can lead to deviations from the ideal model with fixed parameters. In order to make the model more relevant and the results more accurate, we cannot just consider fixed parameters. Therefore, to make the study more convincing, it is necessary to target imprecise parameters. Professor Zadeh [34], who first proposed the fuzzy set theory, also argued that the application of fuzzy differential equations is a more accurate method for modeling biological dynamics in the absence of accurate data conditions [35]. Moreover, the first introduction of the idea of fuzzy derivatives came from Chang and Zadeh [36]. Further, Kaleva [37] studied the generalized fuzzy derivatives based on Hukuhara differentiability, the Zadeh extension principle, and the strong generalized differentiability concept. Bede et al. [38] employed the notion of strongly generalized differentiability to investigate fuzzy differential equations. Khastan and Nieto [39] solved the margin problem for fuzzy differential equations in their article. Motivated by the method of Pal [13] and Wang [23], we assume that the imprecise parameters $\widetilde{r_1}$, $\widetilde{r_2}$, $\widetilde{a_1}$, $\widetilde{a_2}$, $\widetilde{c_1}$, $\widetilde{c_2}$, $\widetilde{e_1}$, $\widetilde{e_1}$ and $\widetilde{d}$ represent all triangular fuzzy numbers (the relevant theories of fuzzy sets are detailed in Appendix A), then the system (1.3) can be written as

$$\begin{cases} \dfrac{\widetilde{dx_1}}{dt} = \dfrac{\widetilde{r_1}x_1}{1+k_1y}\left(1-\dfrac{x_1}{K_1}\right) - \widetilde{a_1}x_1x_2 - \widetilde{c_1}(1-m_1)x_1y - q_1E_1x_1, \\ \dfrac{\widetilde{dx_2}}{dt} = \dfrac{\widetilde{r_2}x_2}{1+k_2y}\left(1-\dfrac{x_2}{K_2}\right) - \widetilde{a_2}x_1x_2 - \widetilde{c_2}(1-m_2)x_2y - q_2E_2x_2, \\ \dfrac{\widetilde{dy}}{dt} = \widetilde{e_1}(1-m_1)x_1y + \widetilde{e_2}(1-m_2)x_2y - \widetilde{d}y - q_3E_3y, \end{cases} \tag{1.4}$$

we cut these imprecise parameters $\widetilde{r_1}$, $\widetilde{r_2}$, $\widetilde{a_1}$, $\widetilde{a_2}$, $\widetilde{c_1}$, $\widetilde{c_2}$, $\widetilde{e_1}$, $\widetilde{e_1}$, and $\widetilde{d}$ by using α-level. System (1.4) can be expressed as follows:

$$\begin{cases} \left(\dfrac{dx_1}{dt}\right)_L^\alpha = \dfrac{r_{1L}^\alpha x_1}{1+k_1y} - \dfrac{r_{1R}^\alpha}{1+k_1y}\dfrac{x_1^2}{K_1} - a_{1R}^\alpha x_1x_2 - c_{1R}^\alpha(1-m_1)x_1y - q_1E_1x_1, \\ \left(\dfrac{dx_1}{dt}\right)_R^\alpha = \dfrac{r_{1R}^\alpha x_1}{1+k_1y} - \dfrac{r_{1L}^\alpha}{1+k_1y}\dfrac{x_1^2}{K_1} - a_{1L}^\alpha x_1x_2 - c_{1L}^\alpha(1-m_1)x_1y - q_1E_1x_1, \\ \left(\dfrac{dx_2}{dt}\right)_L^\alpha = \dfrac{r_{2L}^\alpha x_2}{1+k_2y} - \dfrac{r_{2R}^\alpha}{1+k_2y}\dfrac{x_2^2}{K_2} - a_{2R}^\alpha x_1x_2 - c_{2R}^\alpha(1-m_2)x_2y - q_2E_2x_2, \\ \left(\dfrac{dx_2}{dt}\right)_R^\alpha = \dfrac{r_{2R}^\alpha x_2}{1+k_2y} - \dfrac{r_{2L}^\alpha}{1+k_2y}\dfrac{x_2^2}{K_2} - a_{2L}^\alpha x_1x_2 - c_{2L}^\alpha(1-m_2)x_2y - q_2E_2x_2, \\ \left(\dfrac{dy}{dt}\right)_L^\alpha = e_{1L}^\alpha(1-m_1)x_1y + e_{2L}^\alpha(1-m_2)x_2y - d_R^\alpha y - q_3E_3y, \\ \left(\dfrac{dy}{dt}\right)_R^\alpha = e_{1R}^\alpha(1-m_1)x_1y + e_{2R}^\alpha(1-m_2)x_2y - d_L^\alpha y - q_3E_3y. \end{cases} \tag{1.5}$$

Introducing weighted sum, we change (1.5) to (1.6)

$$\begin{cases} \dfrac{dx_1}{dt} = w_1\left(\dfrac{dx_1}{dt}\right)_L^\alpha + w_2\left(\dfrac{dx_1}{dt}\right)_R^\alpha, \\ \dfrac{dx_2}{dt} = w_1\left(\dfrac{dx_2}{dt}\right)_L^\alpha + w_2\left(\dfrac{dx_2}{dt}\right)_R^\alpha, \\ \dfrac{dy}{dt} = w_1\left(\dfrac{dy}{dt}\right)_L^\alpha + w_2\left(\dfrac{dy}{dt}\right)_R^\alpha, \end{cases} \tag{1.6}$$

where w_1 and w_2 are satisfied with $w_1 + w_2 = 1$, and $w_1, w_2 \geq 0$. Simplifying the system (1.6), we obtain

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{A_1}{1+k_1y}x_1 - \dfrac{A_2}{1+k_1y}\dfrac{x_1^2}{K_1} - A_3x_1x_2 - A_4(1-m_1)x_1y - q_1E_1x_1, \\ \dfrac{dx_2}{dt} = \dfrac{B_1}{1+k_2y}x_2 - \dfrac{B_2}{1+k_2y}\dfrac{x_2^2}{K_2} - B_3x_1x_2 - B_4(1-m_2)x_2y - q_2E_2x_2, \\ \dfrac{dy}{dt} = C_1(1-m_1)x_1y + C_2(1-m_2)x_2y - C_3y - q_3E_3y, \end{cases} \tag{1.7}$$

where

$$\begin{aligned} &A_1 = w_1r_{1L}^{\alpha} + w_2r_{1R}^{\alpha},\ \ A_2 = w_1r_{1R}^{\alpha} + w_2r_{1L}^{\alpha},\ \ A_3 = w_1a_{1R}^{\alpha} + w_2a_{1L}^{\alpha}, \\ &A_4 = w_1c_{1R}^{\alpha} + w_2c_{1L}^{\alpha},\ \ B_1 = w_1r_{2L}^{\alpha} + w_2r_{2R}^{\alpha},\ \ B_2 = w_1r_{2R}^{\alpha} + w_2r_{2L}^{\alpha}, \\ &B_3 = w_1a_{2R}^{\alpha} + w_2a_{2L}^{\alpha},\ \ B_4 = w_1c_{2R}^{\alpha} + w_2c_{2L}^{\alpha},\ \ C_1 = w_1e_{1L}^{\alpha} + w_2e_{1R}^{\alpha}, \\ &C_2 = w_1e_{2L}^{\alpha} + w_2e_{2R}^{\alpha},\ \ C_3 = w_1d_R^{\alpha} + w_2d_L^{\alpha}. \end{aligned}$$

The rest of the paper is shown below: In Section 2, we first prove the nonnegativity and boundedness of the system (1.7). Sections 3 and 4 discuss all possible equilibria and give conditions for the local asymptotic stability and global asymptotic stability of the equilibria. Immediately after that, in Section 5, we analyze the Hopf bifurcation by using the normal form theory. In Section 6, we numerically simulate the theoretical results of Sections 4 and 5. Finally, the article ends with detailed conclusions.

2. Nonnegativity and boundedness

In this section, we give the following theorem to ensure the boundedness and nonnegativity of the solutions of the system (1.7).

Theorem 2.1. *Provided that the initial values $x_1(0) > 0$, $x_2(0) > 0$, and $y(0) > 0$, all solutions of system (1.7) are nonnegative.*

Proof. It is not difficult to find that the right half of the system (1.7) fulfills the local Lipschitzian condition. Integrating both sides of the system (1.7) at the same time yields

$$\begin{aligned} x_1(t) &= x_1(0)\left[\exp\int_0^t \left(\frac{A_1}{1+k_1y} - \frac{A_2}{1+k_1y}\frac{x_1}{K_1} - A_3x_2 - A_4(1-m_1)y - q_1E_1\right)ds\right] > 0, \\ x_2(t) &= x_2(0)\left[\exp\int_0^t \left(\frac{B_1}{1+k_2y} - \frac{B_2}{1+k_2y}\frac{x_2}{K_2} - B_3x_1 - B_4(1-m_2)y - q_2E_2\right)ds\right] > 0, \\ y(t) &= y(0)\left[\exp\int_0^t (C_1(1-m_1)x_1 - C_2(1-m_2)x_2 - C_3 - q_3E_3)ds\right] > 0. \end{aligned} \tag{2.1}$$

If the solution curve starts at any internal point of $\mathbb{R}_+^3 = \{(x_1(t), x_2(t), y(t)) \in \mathbb{R}^3 : x_1(t) \geq 0,\ x_2(t) \geq 0,\ y(t) \geq 0\}$, then $x_1(t)$, $x_2(t)$, and $y(t)$ will always be nonnegative. □

Theorem 2.2. *Assume that the initial values $x_1(0)$, $x_2(0)$, and $y(0)$ are all greater than zero. The feasible region Ω is a positive invariant set of the system (1.7) defined by*

$$\Omega = \left\{(x_1(t), x_2(t), y(t)) \in R_+^3 : \frac{C_1}{A_4}x_1(t) + \frac{C_2}{B_4}x_2(t) + y(t) \leq \frac{\phi}{\mu}\right\},$$

$$\left(i.e\ \ \Omega = \left\{(x_1(t), x_2(t), y(t)) \in R_+^3 : \frac{w_1 e_{1L}^{\alpha} + w_2 e_{1R}^{\alpha}}{w_1 c_{1R}^{\alpha} + w_2 c_{1L}^{\alpha}} x_1(t) + \frac{w_1 e_{2L}^{\alpha} + w_2 e_{2R}^{\alpha}}{w_1 c_{2R}^{\alpha} + w_2 c_{2L}^{\alpha}} x_2(t) + y(t) \le \frac{\phi}{\mu}\right\}\right),$$

where $\mu = min\{q_1E_1, q_2E_2, C_3 + q_3E_3\}$.

Proof. Define a function

$$W(t) = \frac{C_1}{A_4}x_1(t) + \frac{C_2}{B_4}x_2(t) + y(t). \tag{2.2}$$

After taking the derivative on both sides of (2.2), we obtain

$$\frac{dW}{dt} = \frac{C_1}{A_4}\frac{dx_1}{dt} + \frac{C_2}{B_4}\frac{dx_2}{dt} + \frac{dy}{dt}. \tag{2.3}$$

Furthermore, we can obtain

$$\begin{aligned}
\frac{dW}{dt} + \mu W = & \frac{C_1}{A_4(1+k_1y)}\left(A_1x_1 - \frac{A_2x_1^2}{K_1}\right) - \frac{C_1A_3}{A_4}x_1x_2 - C_1(1-m_1)x_1y \\
& + \frac{C_2}{B_4(1+k_2y)}\left(B_1x_2 - \frac{B_2x_2^2}{K_2}\right) - \frac{C_2B_3}{B_4}x_1x_2 - C_2(1-m_2)x_2y \\
& + C_1(1-m_1)x_1y + C_2(1-m_2)x_2y - C_3y - q_3E_3y + \frac{C_1\mu}{A_4}x_1 \\
& - \frac{C_1}{A_4}q_1E_1x_1 - \frac{C_2}{B_4}q_2E_2x_2 + \frac{C_2\mu}{B_4}x_2 + \mu y, \\
= & \frac{C_1}{A_4(1+k_1y)}\left(A_1x_1 - \frac{A_2x_1^2}{K_1}\right) + \frac{C_2}{B_4(1+k_2y)}\left(B_1x_2 - \frac{B_2x_2^2}{K_2}\right) \\
& + \frac{C_1}{A_4}x_1(\mu - q_1E_1) + \frac{C_2}{B_4}x_2(\mu - q_2E_2) + y(\mu - C_3 - q_3E_3) \\
& - \left(\frac{C_1A_3}{A_4} + \frac{C_2B_3}{B_4}\right)x_1x_2,
\end{aligned} \tag{2.4}$$

where $\mu = \min\{q_1E_1, q_2E_2, C_3 + q_3E_3\}$. Let $\phi_1 = \frac{A_1^2K_1}{4A_2}, \phi_2 = \frac{B_1^2K_2}{4B_2}, \phi \equiv \frac{C_1}{A_4}\phi_1 + \frac{C_2}{B_4}\phi_2$, we have

$$\frac{dW}{dt} + \mu W \le \frac{C_1}{A_4}\phi_1 + \frac{C_2}{B_4}\phi_2 = \phi. \tag{2.5}$$

Therefore, it can be deduced that

$$W \le \frac{\phi}{\mu} + Ne^{-\mu t}, \tag{2.6}$$

where N is a positive constant. Then we can further obtain

$$\limsup_{t\to\infty} W \le \frac{\phi}{\mu}, \tag{2.7}$$

which indicates that the feasible domain Ω is a positive invariant set. □

3. Existence of biological equilibria

In this section, we discuss the existence of all equilibria in the system (1.7). All equilibria for system (1.7) are provided by

(1) Trivial equilibrium $P_1 = (0, 0, 0)$.

(2) Axial equilibrium $P_2 = (x_1^{\varsigma}, 0, 0)$ exists if $A_1 > q_1E_1$, where $x_1^{\varsigma} = \frac{K_1(A_1 - q_1E_1)}{A_2}$.

(3) Axial equilibrium $P_3 = (0, x_2^{\Upsilon}, 0)$ exists if $B_1 > q_2E_2$, where $x_2^{\Upsilon} = \frac{K_2(B_1 - q_2E_2)}{B_2}$.

(4) Axial equilibrium $P_4 = (x_1^{\Psi}, 0, y^{\Psi})$ exists if $A_1 \geq \frac{A_2 x_1^{\Psi}}{K_1} + q_1E_1$ and $\sqrt{\Delta_1} > A_4(1 - m_1) + k_1q_1E_1$, where

$$\begin{aligned} x_1^{\Psi} &= \frac{C_3 + q_3E_3}{C_1(1 - m_1)}, \quad y^{\Psi} = \frac{\sqrt{\Delta_1} - (A_4(1 - m_1) + k_1q_1E_1)}{2k_1A_4(1 - m_1)}, \\ \Delta_1 &= 4\left(k_1A_4(1 - m_1)\right)\left(A_1 - \frac{A_2x_1^{\Psi}}{K_1} - q_1E_1\right) + (A_4m_1 - A_4 - k_1q_1E_1)^2. \end{aligned} \tag{3.1}$$

(5) Axial equilibrium $P_5 = (0, x_2^{\nu}, y^{\nu})$ exists if $B_1 \geq \frac{B_2 x_2^{\nu}}{K_2} + q_2E_2$ and $\sqrt{\Delta_2} > B_4(1 - m_2) + k_2q_2E_2$, where

$$\begin{aligned} x_2^{\nu} &= \frac{C_3 + q_3E_3}{C_2(1 - m_2)}, \quad y^{\nu} = \frac{\sqrt{\Delta_2} - (B_4(1 - m_2) + k_2q_2E_2)}{2k_2B_4(1 - m_2)}, \\ \Delta_2 &= 4\left(k_2B_4(1 - m_2)\right)\left(B_1 - \frac{B_2x_2^{\nu}}{K_2} - q_2E_2\right) + (B_4m_2 - B_4 - k_2q_2E_2)^2. \end{aligned} \tag{3.2}$$

(6) Axial equilibrium $P_6 = (x_1^{\iota}, x_2^{\iota}, 0)$ exists if $B_1 > B_3x_1^{\iota} + q_2E_2$, $K_1K_2A_3B_3 > A_2B_2$ and $A_3B_1K_2 + B_2q_1E_1 > A_3K_2q_2E_2 + A_1B_2$, where

$$x_1^{\iota} = \frac{A_3K_1K_2(B_1 - q_2E_2) + K_1B_2(q_1E_1 - A_1)}{K_1K_2A_3B_3 - A_2B_2}, x_2^{\iota} = \frac{K_2(B_1 - B_3x_1^{\iota} - q_2E_2)}{B_2}. \tag{3.3}$$

(7) Internal equilibrium $P_7 = (x_1^*, x_2^*, y^*)$ exists, and its value will be given in the proof of Theorem 3.1.

Theorem 3.1. *When $g_3 > 0$ and $g_4g_5 < 0$ are met, there is an internal equilibrium P_7.*

Proof. We derive that from the second equation of the system (1.7)

$$g_1y^2 + g_2y + g_3 = 0, \tag{3.4}$$

where

$$g_1 = -k_2B_4(1 - m_2), \;\; g_2 = -k_2B_3x_1 - B_4(1 - m_2) - q_2E_2k_2, \;\; g_3 = \left(B_1 - \frac{x_2}{K_2}B_2 - B_3x_1 - q_2E_2\right).$$

It follows from the Descartes law of signs that Eq (3.4) has one and only one solution y^* greater than zero if and only if $g_3 > 0$, i.e., $B_1 > \frac{x_2}{K_2}B_2 + B_3x_1 + q_2E_2$. Substituting y^* into the algebra expression on the right side of the first equation of the system (1.7) equals zero; furthermore, we obtain

$$x_2 = \frac{A_1}{A_3 + A_3k_1y^*} - \frac{A_2x_1}{K_1A_3 + K_1A_3k_1y^*} - \frac{A_4(1 - m_1)y^*}{A_3} - \frac{q_1E_1}{A_3}. \tag{3.5}$$

Introduce (3.5) into the right side of the third equation of the system (1.7), which meets zero, it simplifies to obtain

$$g_4x_1 - g_5 = 0, \tag{3.6}$$

where

$$g_4 = \left[C_1(1-m_1) - \frac{A_2C_2(1-m_2)}{K_1A_3 + K_1A_3k_1y^*}\right],$$
$$g_5 = \left[\frac{A_1C_2(1-m_2)}{A_3 + A_3k_1y^*} - \frac{A_4C_2(1-m_1)(1-m_2)y^*}{A_3} - \frac{C_2q_3E_3(1-m_2)}{A_3} - C_3 - q_3E_3\right].$$

Reusing the Descartes law of signs, we can assert that there exists at least one positive solution x_1^* of Eq (3.6) if and only if $g_4g_5 < 0$. And then we can deduce that

$$x_2^* = \frac{A_1}{A_3 + A_3k_1y^*} - \frac{A_2x_1^*}{K_1A_3 + K_1A_3k_1y^*} - \frac{A_4(1-m_1)y^*}{A_3} - \frac{q_1E_1}{A_3}.$$

then the interior equilibrium $P_7(x_1^*, x_2^*, y^*)$ exists. □

4. Stability analysis

In this section, the Jocabian matrix will be used to prove the local stability of all equilibria. Moreover, we prove the global stability of the internal equilibrium P_7 by constructing a Lyapunov function.

4.1. Local stability

The Jocabian matrix for system (1.7) is given below:

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}, \tag{4.1}$$

where

$$\begin{aligned}
M_{11} &= \frac{A_1}{1+k_1y} - \frac{2A_2}{1+k_1y}\frac{x_1}{K_1} - A_3x_2 - A_4(1-m_1)y - q_1E_1, \quad M_{12} = -A_3x_1, \\
M_{13} &= -\frac{k_1A_1x_1}{(1+k_1y)^2} + \frac{k_1A_2}{(1+k_1y)^2}\frac{x_1^2}{K_1} - A_4(1-m_1)x_1, \quad M_{21} = -B_3x_2, \\
M_{22} &= \frac{B_1}{1+k_2y} - \frac{2B_2}{1+k_2y}\frac{x_2}{K_2} - B_3x_1 - B_4(1-m_2)y - q_2E_2, \\
M_{23} &= -\frac{k_2B_1x_2}{(1+k_2y)^2} + \frac{k_2B_2}{(1+k_2y)^2}\frac{x_2^2}{K_2} - B_4(1-m_2)x_2, \\
M_{31} &= C_1(1-m_1)y, \quad M_{32} = C_2(1-m_2)y, \quad M_{33} = C_1(1-m_1)x_1 + C_2(1-m_2)x_2 - C_3 - q_3E_3.
\end{aligned} \tag{4.2}$$

Through simple calculation, we directly draw the conclusion that trivial and axial equilibria are locally asymptotically stable:

(**1**) $P_1(0,0,0)$ is locally asymptotically stable if

$$\frac{A_1}{q_1}-E_1<0 \quad \text{and} \quad \frac{B_1}{q_2}-E_2<0. \tag{4.3}$$

(**2**) $P_2(x_1^{\varsigma},0,0)$ is locally asymptotically stable if

$$\frac{B_1}{q_2}-E_2<\frac{B_3x_1^{\varsigma}}{q_2}<\frac{B_3(C_3+q_3E_3)}{C_1(1-m_1)q_2}. \tag{4.4}$$

(**3**) $P_3(0,x_2^{\Upsilon},0)$ is locally asymptotically stable if

$$\frac{A_1}{q_1}-E_1<\frac{A_3x_2^{\Upsilon}}{q_1}<\frac{A_3(C_3+q_3E_3)}{C_2(1-m_2)q_1}. \tag{4.5}$$

(**4**) $P_4(x_1^{\Psi},0,y^{\Psi})$ is locally asymptotically stable if

$$\frac{B_1}{q_2(1+k_2y^{\Psi})}-E_2<\frac{B_3x_1^{\Psi}+B_4(1-m_2)y^{\Psi}}{q_2}. \tag{4.6}$$

(**5**) $P_5(0,x_2^{\nu},y^{\nu})$ is locally asymptotically stable if

$$\frac{A_1}{q_1(1+k_1y^{\nu})}-E_1<\frac{A_3x_2^{\nu}+A_4(1-m_1)y^{\nu}}{q_1}. \tag{4.7}$$

(**6**) $P_6(x_1^{\iota},x_2^{\iota},0)$ is locally asymptotically stable if

$$C_1(1-m_1)x_1^{\iota}+C_2(1-m_2)x_2^{\iota}<C_3+q_3E_3. \tag{4.8}$$

We draw the conclusion that the internal equilibrium $P_7(x_1^*,x_2^*,y^*)$ is locally asymptotically stable from the proof of Theorem 4.1.

Theorem 4.1. *The internal equilibrium P_7 is locally asymptotically stable if it exists and the following conditions are fulfilled:*

$$\psi_1>0,\psi_1\psi_2>0,\psi_3>0. \tag{4.9}$$

Proof. The Jocabian matrix of system (1.7) at (x_1^*,x_2^*,y^*) is

$$\begin{pmatrix} L_1 & L_2 & L_3 \\ L_4 & L_5 & L_6 \\ L_7 & L_8 & L_9 \end{pmatrix}, \tag{4.10}$$

where

$$\begin{aligned}
L_1 &= -\frac{A_2}{(1+k_1y^*)}\frac{x_1^*}{K_1}<0, \quad L_2=-A_3x_1^*<0,\\
L_3 &= -\frac{k_1A_1x_1^*}{(1+k_1y^*)^2}-A_4(1-m_1)x_1^*+\frac{k_1A_2}{(1+k_1y^*)^2}\frac{(x_1^*)^2}{K_1},\\
L_4 &= -B_3x_2^*<0, \quad L_5=-\frac{B_2}{(1+k_2y^*)}\frac{x_2^*}{K_2}<0,\\
L_6 &= -\frac{k_2B_1x_2^*}{(1+k_2y^*)^2}-B_4(1-m_2)x_2^*+\frac{k_2B_2}{(1+k_2y^*)^2}\frac{(x_2^*)^2}{K_2},\\
L_7 &= C_1(1-m_1)y^*>0, \quad L_8=C_2(1-m_2)y^*>0, \quad L_9=0.
\end{aligned} \tag{4.11}$$

Therefore, the characteristic equation at P_7 can be expressed as

$$\eta^3 + \psi_1\eta^2 + \psi_2\eta + \psi_3 = 0, \tag{4.12}$$

where

$$\begin{aligned}
\psi_1 &= -L_1 - L_5,\\
\psi_2 &= L_1L_5 - L_6L_8 - L_3L_7 - L_2L_4,\\
\psi_3 &= L_8(L_1L_6 - L_3L_4) + L_7(L_3L_5 + L_2L_6).
\end{aligned} \tag{4.13}$$

The Routh-Hurwitz criterion shows that the internal equilibrium P_7 is locally asymptotically stable; the following conditions need to be met: $\psi_1 > 0$, $\psi_1\psi_2 > 0$, and $\psi_3 > 0$. □

4.2. Global stability

This subsection studies the global asymptotic stability of interior equilibrium P_7.

Theorem 4.2. *If condition* $4\Gamma_1\Gamma_2 l_1 l_2 A_2 B_2(1 + k_1y)(1 + k_2y) > (l_1A_3 + l_2B_3)^2$ *(i.e.* $4\Gamma_1\Gamma_2 l_1 l_2(w_1r_{1R}^\alpha + w_2r_{1L}^\alpha)(w_1r_{2R}^\alpha + w_2r_{2L}^\alpha)(1 + k_1y)(1 + k_2y) > (l_1(w_1a_{1R}^\alpha + w_2a_{1L}^\alpha) + l_2(w_1a_{2R}^\alpha + w_2a_{2L}^\alpha))^2$*) holds, then* P_7 *is globally asymptotically stable.*

Proof. We construct a Lyapunov function:

$$V(x_1, x_2, y) = l_1\left[x_1 - x_1^* - x_1^*\ln\left(\frac{x_1}{x_1^*}\right)\right] + l_2\left[x_2 - x_2^* - x_2^*\ln\left(\frac{x_2}{x_2^*}\right)\right] + y - y^* - y^*\ln\left(\frac{y}{y^*}\right). \tag{4.14}$$

Obviously, $x_i - x_i^* - x_i^*\ln(\frac{x_i}{x_i^*}) \geq 0$ $(i = 1, 2)$ and $y - y^* - y^*\ln(\frac{y}{y^*}) \geq 0$, thus $V \geq 0$. Taking the derivative of $V(x_1, x_2, y)$ over t, one has

$$\frac{dV}{dt} = l_1\left(\frac{x_1 - x_1^*}{x_1}\right)\frac{dx_1}{dt} + l_2\left(\frac{x_2 - x_2^*}{x_2}\right)\frac{dx_2}{dt} + \frac{y - y^*}{y}\frac{dy}{dt}, \tag{4.15}$$

where

$$\begin{aligned}
\frac{x_1 - x_1^*}{x_1}\frac{dx_1}{dt} =& -\frac{k_1A_1}{(1 + k_1y)(1 + k_1y^*)}(x_1 - x_1^*)(y - y^*) - \frac{A_2(x_1 - x_1^*)^2}{K_1(1 + k_1y)(1 + k_1y^*)}\\
&- \frac{A_2k_1(x_1y^* - x_1^*y)}{K_1(1 + k_1y)(1 + k_1y^*)}(x_1 - x_1^*) - A_3(x_1 - x_1^*)(x_2 - x_2^*)\\
&- A_4(1 - m_1)(x_1 - x_1^*)(y - y^*),\\
\frac{x_2 - x_2^*}{x_2}\frac{dx_2}{dt} =& -\frac{k_2B_1}{(1 + k_2y)(1 + k_2y^*)}(x_2 - x_2^*)(y - y^*) - \frac{B_2(x_2 - x_2^*)^2}{K_2(1 + k_2y)(1 + k_2y^*)}\\
&- \frac{B_2k_2(x_2y^* - x_2^*y)}{K_2(1 + k_2y)(1 + k_2y^*)}(x_2 - x_2^*) - B_3(x_1 - x_1^*)(x_2 - x_2^*)\\
&- B_4(1 - m_2)(x_2 - x_2^*)(y - y^*),\\
\frac{y - y^*}{y}\frac{dy}{dt} =& C_1(1 - m_1)(x_1 - x_1^*)(y - y^*) + C_2(1 - m_2)(x_2 - x_2^*)(y - y^*).
\end{aligned} \tag{4.16}$$

To simplify the calculation, let

$$\begin{aligned}
&x_1y^* - x_1^*y = y(x_1 - x_1^*) - x_1(y - y^*), \ \ x_2y^* - x_2^*y = y(x_2 - x_2^*) - x_2(y - y^*),\\
&\Gamma_1 = \frac{1}{K_1(1 + k_1y)(1 + k_1y^*)}, \ \ \Gamma_2 = \frac{1}{K_2(1 + k_2y)(1 + k_2y^*)},\\
&l_1 = \frac{C_1(1 - m_1)}{\Gamma_1k_1(K_1A_1 + A_2x_1) + A_4(1 - m_1)}, \ \ l_2 = \frac{C_2(1 - m_2)}{\Gamma_2k_2(K_2B_1 + B_2x_2) + A_4(1 - m_1)}.
\end{aligned} \tag{4.17}$$

We obtain

$$\begin{aligned}
\frac{dV}{dt} &= -\{\Gamma_1l_1A_2(1 + k_1y)(x_1 - x_1^*)^2 + (l_1A_3 + l_2B_3)(x_1 - x_1^*)(x_2 - x_2^*) + \Gamma_2l_2B_2(1 + k_2y)(x_2 - x_2^*)^2\}\\
&= -Y^{'}GY,
\end{aligned} \tag{4.18}$$

where

$$Y^{'} = [(x_1 - x_1^*), (x_2 - x_2^*)], \ \ G = \begin{pmatrix} \Gamma_1l_1A_2(1 + k_1y) & \frac{l_1A_3+l_2B_3}{2} \\ \frac{l_1A_3+l_2B_3}{2} & \Gamma_2l_2B_2(1 + k_2y) \end{pmatrix}.$$

Therefore, $\frac{dV}{dt} < 0$ if and only if $4\Gamma_1\Gamma_2l_1l_2A_2B_2(1 + k_1y)(1 + k_2y) > (l_1A_3 + l_2B_3)^2$. □

5. Hopf bifurcation

In this section, we will use the normal form theory introduced by Hassard et al. [40] and the central manifold theory [41] to study the Hopf bifurcation of the system (1.7). When the system (1.7) undergoes Hopf bifurcation, the corresponding characteristic equation must have a pair of conjugate pure imaginary roots, that is,

$$\eta_{1,2} = \pm i\omega, \qquad i = \sqrt{-1}. \tag{5.1}$$

Consider the parameter k_1 as a bifurcation parameter. When the value of parameter k_1 changes near the critical point k_1^{Ξ} of Hopf bifurcation, the pure imaginary roots $\pm i\omega$ will become a complex eigenvalue $\eta = \rho + i\widetilde{\omega}$. Substituting $\eta = \rho + i\widetilde{\omega}$ into Eq (4.12), we need to separate the imaginary and real parts to get

$$\rho^3 + \psi_3 + \rho\psi_2 + \rho^2\psi_1 - 3\rho\widetilde{\omega}^2 - \psi_1\widetilde{\omega}^2 = 0, \tag{5.2}$$

$$3\rho^2\widetilde{\omega} + \psi_2\widetilde{\omega} + 2\rho\psi_1\widetilde{\omega} - \widetilde{\omega}^3 = 0. \tag{5.3}$$

By simplifying Eqs (5.2) and (5.3), we obtain

$$\psi_3 - 8\rho^3 - 2\rho\psi_2 - 8\rho^2\psi_1 - \psi_1\psi_2 - 2\rho\psi_1^2 = 0, \tag{5.4}$$

at $k_1 = k_1^{\Xi}$, taking the derivative of Eq (5.4) over k_1 yields

$$\left.\frac{d\rho}{dk_1}\right|_{k_1=k_1^{\Xi}} = \frac{1}{2}\left(\frac{d\psi_3}{dk_1} - \psi_1\frac{d\psi_2}{dk_1} - \psi_2\frac{d\psi_1}{dk_1}\right)/(\psi_2 + \psi_1^2). \tag{5.5}$$

If it satisfies $\left.\frac{d\rho}{dk_1}\right|_{k_1=k_1^{\Xi}} \neq 0$, the system (1.7) will generate Hopf bifurcation, which indicates that when parameter k_1 crosses the bifurcation critical point k_1^{Ξ}, the population state evolves from stable equilibrium to periodic oscillation over time.

When the system (1.7) undergoes Hopf bifurcation at $k_1 = k_1^{\Xi}$, the final decision condition is also met. Considering that the characteristic roots of Eq (4.12) are $\eta_{1,2} = \pm i\omega$ and $\eta_3 = -\psi_1$, in order to obtain this condition, we introduce

$$z_1 = x_1 - x_1^*, \quad z_2 = x_2 - x_2^*, \quad z_3 = y - y^*. \tag{5.6}$$

Substituting (5.6) into the system (1.7) and separating the linear and nonlinear parts, it can be obtained that

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = J(P_7)\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} F_1(z_1,z_2,z_3) \\ F_2(z_1,z_2,z_3) \\ F_3(z_1,z_2,z_3) \end{pmatrix}, \tag{5.7}$$

where

$$\begin{aligned} F_1(z_1,z_2,z_3) &= \sum_{2\le j_1+j_2+j_3\le 3} t_{j_1j_2j_3} z_1^{j_1} z_2^{j_2} z_3^{j_3} + O((|z_1|+|z_2|+|z_3|)^4), \\ F_2(z_1,z_2,z_3) &= \sum_{2\le j_1+j_2+j_3\le 3} n_{j_1j_2j_3} z_1^{j_1} z_2^{j_2} z_3^{j_3} + O((|z_1|+|z_2|+|z_3|)^4), \\ F_3(z_1,z_2,z_3) &= \sum_{2\le j_1+j_2+j_3\le 3} l_{j_1j_2j_3} z_1^{j_1} z_2^{j_2} z_3^{j_3} + O((|z_1|+|z_2|+|z_3|)^4), \end{aligned} \tag{5.8}$$

where $O((|z_1| + |z_2| + |z_3|)^4)$ is a fourth-order polynomial function about variables $(|z_1|, |z_2|, |z_3|)$, while $t_{j_1j_2j_3}$,$n_{j_1j_2j_3}$, and $l_{j_1j_2j_3}$ can be obtained through calculation:

$$\begin{aligned} &t_{101} = -\frac{A_1k_1}{2(1+k_1y^*)^2} + \frac{A_2k_1}{(1+k_1y^*)^2}\frac{x_1^*}{K_1} - \frac{A_4(1-m_1)}{2}, \quad t_{002} = \frac{A_1x_1^*k_1^2}{(1+k_1y^*)^3} - \frac{A_2k_1^2}{(1+k_1y^*)^3}\frac{(x_1^*)^2}{K_1}, \\ &t_{102} = \frac{A_1k_1^2}{3(1+k_1y^*)^3} - \frac{A_2k_1^2}{3(1+k_1y^*)^3}\frac{2x_1^*}{K_1}, \quad t_{003} = -\frac{A_1x_1^*k_1^3}{(1+k_1y^*)^4} + \frac{A_2k_1^3}{(1+k_1y^*)^4}\frac{(x_1^*)^2}{K_1}, \\ &t_{110} = -\frac{A_3}{2}, \quad t_{200} = -\frac{A_2}{1+k_1y^*}\frac{1}{K_1}, \quad t_{201} = \frac{A_2k_1}{3(1+k_1y^*)^2}\frac{1}{K_1}, \\ &t_{011} = t_{020} = t_{030} = t_{012} = t_{021} = t_{111} = t_{120} = t_{210} = t_{300} = 0, \\ &n_{011} = -\frac{B_1k_2}{2(1+k_2y^*)^2} + \frac{B_2k_2}{(1+k_2y^*)^2}\frac{x_2^*}{K_2} - \frac{B_4(1-m_2)}{2}, \quad n_{002} = \frac{B_1x_2^*k_2^2}{(1+k_2y^*)^3} - \frac{B_2k_2^2}{(1+k_2y^*)^3}\frac{(x_2^*)^2}{K_2}, \\ &n_{012} = \frac{B_1k_2^2}{3(1+k_2y^*)^3} - \frac{B_2k_2^2}{3(1+k_2y^*)^3}\frac{2x_2^*}{K_2}, \quad n_{003} = -\frac{B_1x_2^*k_2^3}{(1+k_2y^*)^4} + \frac{B_2k_2^3}{(1+k_2y^*)^4}\frac{(x_2^*)^2}{K_2}, \\ &n_{110} = -\frac{B_3}{2}, \quad n_{020} = -\frac{B_2}{1+k_2y^*}\frac{1}{K_2}, \quad n_{021} = \frac{B_2k_2}{3(1+k_2y^*)^2}\frac{1}{K_2}, \\ &n_{101} = n_{200} = n_{030} = n_{111} = n_{120} = n_{102} = n_{210} = n_{201} = n_{300} = 0, \\ &l_{101} = \frac{C_1(1-m_1)}{2}, \quad l_{011} = \frac{C_2(1-m_2)}{2}, \\ &l_{110} = l_{200} = l_{020} = l_{002} = l_{030} = l_{003} = l_{012} = l_{021} = l_{111} = l_{120} = l_{102} = l_{210} = l_{201} = l_{300} = 0. \end{aligned} \tag{5.9}$$

By introducing a reversible transformation

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \tag{5.10}$$

which $q_{21}q_{32} - q_{22}q_{31} + q_{22}q_{33} - q_{23}q_{32} \neq 0$, the expression for the coefficient $q_{ij}(i = 2, 3; j = 1, 2, 3)$ is

$$\begin{aligned} q_{21} &= \frac{L_2L_3L_4L_6 + L_1L_3L_5L_6 - L_1L_2L_6^2 + L_3L_6\omega^2 - L_3^2L_4L_5}{L_3^2L_5^2 - 2L_2L_3L_5L_6 + L_2^2L_6^2 + L_3^2\omega^2}, \\ q_{22} &= \frac{(L_3^2L_4 - L_1L_3L_6 + L_3L_5L_6 - L_2L_6)\omega}{L_3^2L_5^2 - 2L_2L_3L_5L_6 + L_2^2L_6^2 + L_3^2\omega^2}, \\ q_{23} &= \frac{L_1L_6 - L_6\eta_3 - L_3L_4}{L_3L_5 - L_2L_6 - L_3\eta_3}, \\ q_{31} &= \frac{L_2L_3L_4L_5 - L_1L_3L_5^2 - L_2^2L_4L_6 + L_1L_2L_5L_6 - L_1L_3\omega^2 - L_2L_6\omega^2}{L_3^2L_5^2 - 2L_2L_3L_5L_6 + L_2^2L_6^2 + L_3^2\omega^2}, \\ q_{32} &= \frac{\omega(L_1L_2L_6 + L_2L_5L_6 - L_2L_3L_4 - L_3L_5^2 - L_3\omega^2)}{L_3^2L_5^2 - 2L_2L_3L_5L_6 + L_2^2L_6^2 + L_3^2\omega^2}, \\ q_{33} &= \frac{L_2L_4 - L_1L_5 + L_1\eta_3 + L_5\eta_3 - \eta_3^2}{L_3L_5 - L_2L_6 - L_3\eta_3}, \end{aligned} \tag{5.11}$$

$L_i(i = 1, 2, \cdots, 9)$ is defined in (4.11). So the standard type of system (5.7) can be written as

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & \eta_3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} \widetilde{F_1}(y_1, y_2, y_3) \\ \widetilde{F_2}(y_1, y_2, y_3) \\ \widetilde{F_3}(y_1, y_2, y_3) \end{pmatrix}, \tag{5.12}$$

where

$$\begin{pmatrix} \widetilde{F_1}(y_1, y_2, y_3) \\ \widetilde{F_2}(y_1, y_2, y_3) \\ \widetilde{F_3}(y_1, y_2, y_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}^{-1} \begin{pmatrix} F_1(z_1, z_2, z_3) \\ F_2(z_1, z_2, z_3) \\ F_3(z_1, z_2, z_3) \end{pmatrix}. \tag{5.13}$$

In Eq (5.13), the coefficients of polynomial $\widetilde{F}(y_1, y_2, y_3)$ are $\widetilde{t}_{j_1j_2j_3}$, $\widetilde{n}_{j_1j_2j_3}$ and $\widetilde{l}_{j_1j_2j_3}$.

Based on the central manifold theory, use the center manifold $W^c(0, 0, 0)$ existing at the origin to reduce the dimension of the system (5.12), that is

$$W^c(0, 0, 0) = \left\{(y_1, y_2, y_3) \in R^3 \middle| y_3 = h(y_1, y_2), h(0, 0) = 0, Dh(0, 0) = 0\right\}, \tag{5.14}$$

where

$$h(y_1, y_2) = \sum_{1 \le v_1 + v_2 \le 3} h_{v_1v_2} y_1 y_2 + O((|y_1| + |y_2|)^4). \tag{5.15}$$

Through calculation, we can obtain

$$
\begin{aligned}
&h_{10} = h_{01} = h_{11} = 0, \quad h_{20} = -\frac{\widetilde{l}_{200} + \widetilde{l}_{020}\omega^2}{\eta_3(1+\omega^4)}, \quad h_{02} = -\frac{\widetilde{l}_{020} - \widetilde{l}_{200}\omega^2}{1+\omega^4},\\
&h_{30} = \frac{1}{(1+\omega^4)(\eta_3^2+\omega^6)}(\widetilde{l}_{200}\widetilde{n}_{101} - \widetilde{l}_{300}\eta_3 - 2\widetilde{l}_{020}\widetilde{n}_{200}\eta_3\omega + \widetilde{l}_{020}\widetilde{n}_{101}\omega^2 + \widetilde{l}_{011}\widetilde{l}_{020}\omega^3\\
&\qquad - \widetilde{l}_{030}\omega^3 + 2\widetilde{l}_{200}\widetilde{n}_{200}\eta_3\omega^3 - \widetilde{l}_{300}\eta_3\omega^4 - \widetilde{l}_{011}\widetilde{l}_{200}\omega^5 - \widetilde{l}_{030}\omega^7),\\
&h_{21} = \frac{(\widetilde{l}_{200} + \widetilde{l}_{020}\omega^2)(\widetilde{l}_{011}\eta_3 + 2\widetilde{t}_{200}\eta_3\omega + \widetilde{n}_{110}\eta_3^2\omega^3 - 2\widetilde{t}_{110}\omega^4 + \widetilde{n}_{101}\eta_3\omega^5 - 2\widetilde{n}_{020}\eta_3\omega^6)}{\eta_3(1+\omega^4)(\eta_3^2+\omega^6)}\\
&\qquad - \frac{\widetilde{l}_{020}(2\widetilde{n}_{110}\eta_3\omega + \widetilde{n}_{101}\omega^3 - 2\widetilde{n}_{020}\omega^4)}{\eta_3^2+\omega^6},\\
&h_{12} = \frac{(\widetilde{l}_{200} + \widetilde{l}_{020}\omega^2)(2\widetilde{t}_{110}\eta_3\omega - \widetilde{n}_{101}\eta_3^2\omega^2 + \widetilde{l}_{011}\omega^3 + 2\widetilde{n}_{020}\eta_3^2\omega^3 + 2\widetilde{t}_{200}\omega^4 + 2\widetilde{n}_{110}\eta_3\omega^6)}{\eta_3(1+\omega^4)(\eta_3^2+\omega^6)}\\
&\qquad - \frac{\widetilde{l}_{020}(2\widetilde{n}_{020}\eta_3\omega + 2\widetilde{n}_{110}\omega^4 - \widetilde{n}_{101}\eta_3)}{\eta_3^2+\omega^6},\\
&h_{03} = -\frac{(\widetilde{l}_{200} + \widetilde{l}_{020}\omega^2)(\widetilde{l}_{011}\eta_3^2 + \omega^2 + \widetilde{n}_{101}\omega^3 + 2\widetilde{n}_{200}\eta_3\omega^6)}{\eta_3(1+\omega^4)(\eta_3^2+\omega^6)}\\
&\qquad + \frac{\widetilde{l}_{011}\widetilde{l}_{020}\eta_3 - \widetilde{l}_{030}\eta_3 + \widetilde{l}_{300}\omega^3 + 2\widetilde{l}_{020}\widetilde{n}_{200}\omega^4}{\eta_3^2+\omega^6}.
\end{aligned}
\tag{5.16}
$$

Correspondingly, the dynamic properties of the system are limited to the central flow $W^c(0,0,0)$, and in conjunction with Eq (5.14), system (5.12) can be simplified as

$$
\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix} = \begin{pmatrix}U(y_1,y_2)\\ N(y_1,y_2)\end{pmatrix},
\tag{5.17}
$$

where

$$
\begin{aligned}
U(y_1,y_2) &= -\omega y_2 + \ell_1 y_1^2 + \ell_2 y_1 y_2 + \ell_3 y_2^2 + \ell_4 y_1^3 + \ell_5 y_1^2 y_2 + \ell_6 y_1 y_2^2 + \ell_7 y_2^3,\\
N(y_1,y_2) &= \omega y_1 + \jmath_1 y_1^2 + \jmath_2 y_1 y_2 + \jmath_3 y_2^2 + \jmath_4 y_1^3 + \jmath_5 y_1^2 y_2 + \jmath_6 y_1 y_2^2 + \jmath_7 y_2^3,
\end{aligned}
\tag{5.18}
$$

in the formula, we have

$$
\begin{aligned}
&\ell_1 = \widetilde{t}_{200},\ \ \ell_2 = \widetilde{t}_{110},\ \ \ell_3 = \widetilde{t}_{020},\ \ \ell_4 = h_{20}\widetilde{t}_{101} + \widetilde{t}_{300},\ \ \ell_5 = \widetilde{t}_{210},\ \ \ell_6 = h_{02}\widetilde{t}_{101},\ \ \ell_7 = 0,\\
&\jmath_1 = \widetilde{n}_{200},\ \ \jmath_2 = \widetilde{n}_{110},\ \ \jmath_3 = \widetilde{n}_{020},\ \ \jmath_4 = \widetilde{n}_{300},\ \ \jmath_5 = h_{20}\widetilde{n}_{011},\ \ \jmath_6 = 0,\ \ \jmath_7 = h_{02}\widetilde{n}_{011} + \widetilde{n}_{030}.
\end{aligned}
\tag{5.19}
$$

We introduce the partial derivative sign

$$
\frac{\partial U}{\partial y_1}(y_{k_1^\Xi}) = U_{y_1},\ \ \frac{\partial^3 U}{\partial y_1^2\partial y_2}(y_{k_1^\Xi}) = U_{y_1y_1y_2},\ \ \frac{\partial^2 N}{\partial y_2^2}(y_{k_1^\Xi}) = N_{y_2y_2},\ \ \cdots,
\tag{5.20}
$$

where subscripts y_1 and y_2 indicate partial derivatives for the first and second variable, respectively. Based on Eq (5.18), it can be obtained that $U_{y_1} = 0$, $U_{y_2} \neq 0$, $N_{y_1} \neq 0$, $N_{y_2} = 0$, and $U_{y_2}N_{y_1} \neq 0$. In addition, it ensures that the system (5.18) has pure virtual feature roots $\pm i\sqrt{|U_{y_2}N_{y_1}|}$. Thus, it can be determined that system (1.7) produces Hopf bifurcation; the direction of the bifurcation is determined by the following equation:

$$
Q_{k_1^\Xi} = \frac{1}{16\omega}(\ell_3 + \ell_5 + \jmath_5 + \jmath_7) + \frac{1}{16\omega}(\ell_1\ell_3 - \jmath_2\jmath_3 - \jmath_1\jmath_2 - \jmath_1\ell_1).
\tag{5.21}
$$

Theorem 5.1. *If* $\frac{d\rho}{dk_1}\Big|_{k_1=k_1^{\Xi}} \neq 0$*, then system (1.7) will generate Hopf bifurcation at interior equilibrium* P_7*. In addition, when* $\frac{d\rho}{dk_1}\Big|_{k_1=k_1^{\Xi}} < 0$*, if* $Q_{k_1^{\Xi}} < 0$ *and* $0 < k_1 - k_1^{\Xi} \ll 1$*, then system (1.7) will generate supercritical Hopf bifurcation and form a stable periodic orbit, or if* $Q_{k_1^{\Xi}} > 0$ *and* $0 < k_1 - k_1^{\Xi} \ll 1$*, then system (1.7) will generate subcritical Hopf bifurcation and form a stable periodic orbit.*

6. Numerical simulations

In this section, we first discussed equilibria P_1 to P_7 of system (1.7) with distinct values of α, w_1, and w_2. Consider the parameter values as follows: $\widetilde{r_1} = (2.8, 3, 3.2)$, $\widetilde{r_2} = (2.8, 3, 3.2)$, $\widetilde{c_1} = (0.1, 0.2, 0.3)$, $\widetilde{c_2} = (0.5, 0.6, 0.7)$, $\widetilde{a_1} = (0.1, 0.2, 0.3)$, $\widetilde{a_2} = (0.2, 0.3, 0.4)$, $\widetilde{e_1} = (0.2, 0.3, 0.4)$, $\widetilde{e_2} = (0.3, 0.4, 0.5)$, and $\widetilde{d} = (0.1, 0.2, 0.3)$. Tables 2–8 showed that the trivial equilibrium P_1 retained constant at (0,0,0), the values of prey x_1, prey x_2, and predator y always maintained at 0; the values of prey x_1 in P_2 and prey x_2 in P_3 severally decreased with increasing w_1 under the same α; the values of prey x_1 and predator y in P_4 increased with increasing w_1, and for P_5 the value of prey x_2 and predator y rose with growing w_1; the values of prey x_1 and x_2 in P_6 decreased with growing w_1; and for the same α, considering interior equilibrium P_7, the values of prey x_1, prey x_2, and predator y decreased with growing w_1.

Table 2. The trivial equilibrium P_1 for $k_1 = 0.1, k_2 = 0.7, q_1 = 0.7, q_2 = 0.5, q_3 = 0.7, E_1 = 0.3, E_2 = 0.2, E_3 = 0.2, K_1 = 5, K_2 = 5, m_1 = 0.9, m_2 = 0.3$.

w_1	w_2	P_1 at $\alpha = 0$	P_1 at $\alpha = 0.3$	P_1 at $\alpha = 0.6$	P_1 at $\alpha = 0.9$
0	1	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
0.2	0.8	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
0.4	0.6	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
0.6	0.4	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
0.8	0.2	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)
1	0	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	(0, 0, 0)

Table 3. The axial equilibrium P_2 for $k_1 = 0.1, k_2 = 0.7, q_1 = 0.7, q_2 = 0.5, q_3 = 0.7, E_1 = 0.3, E_2 = 0.2, E_3 = 0.2, K_1 = 5, K_2 = 5, m_1 = 0.9, m_2 = 0.3$.

w_1	w_2	P_2 at $\alpha = 0$	P_2 at $\alpha = 0.3$	P_2 at $\alpha = 0.6$	P_2 at $\alpha = 0.9$
0	1	(5.3393, 0, 0)	(5.1224, 0, 0)	(4.9144, 0, 0)	(4.7148, 0, 0)
0.2	0.8	(5.0521, 0, 0)	(4.9280, 0, 0)	(4.8069, 0, 0)	(4.6888, 0, 0)
0.4	0.6	(4.7804, 0, 0)	(4.7409, 0, 0)	(4.7017, 0, 0)	(4.6629, 0, 0)
0.6	0.4	(4.5230, 0, 0)	(4.5608, 0, 0)	(4.5988, 0, 0)	(4.6372, 0, 0)
0.8	0.2	(4.2788, 0, 0)	(4.3872, 0, 0)	(4.4980, 0, 0)	(4.6116, 0, 0)
1	0	(4.0469, 0, 0)	(4.2197, 0, 0)	(4.3994, 0, 0)	(4.5861, 0, 0)

Table 4. The axial equilibrium P_3 for $k_1 = 0.7, k_2 = 0.1, q_1 = 0.7, q_2 = 0.5, q_3 = 0.7, E_1 = 0.3, E_2 = 0.2, E_3 = 0.2, K_1 = 5, K_2 = 5, m_1 = 0.3, m_2 = 0.9$.

w_1	w_2	P_3 at $\alpha = 0$	P_3 at $\alpha = 0.3$	P_3 at $\alpha = 0.6$	P_3 at $\alpha = 0.9$
0	1	$(0, 5.5357, 0)$	$(0, 5.3147, 0)$	$(0, 5.1027, 0)$	$(0, 4.8993, 0)$
0.2	0.8	$(0, 5.2431, 0)$	$(0, 5.1166, 0)$	$(0, 4.9932, 0)$	$(0, 4.8728, 0)$
0.4	0.6	$(0, 4.9662, 0)$	$(0, 4.9260, 0)$	$(0, 4.8861, 0)$	$(0, 4.8465, 0)$
0.6	0.4	$(0, 4.7039, 0)$	$(0, 4.7424, 0)$	$(0, 4.7812, 0)$	$(0, 4.8202, 0)$
0.8	0.2	$(0, 4.4551, 0)$	$(0, 4.5655, 0)$	$(0, 4.6785, 0)$	$(0, 4.7942, 0)$
1	0	$(0, 4.2187, 0)$	$(0, 4.3949, 0)$	$(0, 4.5779, 0)$	$(0, 4.7682, 0)$

Table 5. The axial equilibrium P_4 for $k_1 = 0.3, k_2 = 0.7, q_1 = 0.7, q_2 = 0.5, q_3 = 0.7, E_1 = 0.3, E_2 = 0.2, E_3 = 0.2, K_1 = 20, K_2 = 20, m_1 = 0.2, m_2 = 0.6$.

w_1	w_2	P_4 at $\alpha = 0$	P_4 at $\alpha = 0.3$	P_4 at $\alpha = 0.6$	P_4 at $\alpha = 0.9$
0	1	$(0.4800, 0, 0.1496)$	$(0.5838, 0, 0.1767)$	$(0.7059, 0, 0.2040)$	$(0.8516, 0, 0.2317)$
0.2	0.8	$(0.6222, 0, 0.1858)$	$(0.6971, 0, 0.2022)$	$(0.7802, 0, 0.2188)$	$(0.8732, 0, 0.2354)$
0.4	0.6	$(0.8000, 0, 0.2224)$	$(0.8306, 0, 0.2280)$	$(0.8623, 0, 0.2335)$	$(0.8954, 0, 0.2391)$
0.6	0.4	$(1.0286, 0, 0.2595)$	$(0.9902, 0, 0.2539)$	$(0.9534, 0, 0.2484)$	$(0.9181, 0, 0.2428)$
0.8	0.2	$(1.3333, 0, 0.2969)$	$(1.1845, 0, 0.2801)$	$(1.0551, 0, 0.2633)$	$(0.9415, 0, 0.2465)$
1	0	$(1.7600, 0, 0.3343)$	$(1.4261, 0, 0.3062)$	$(1.1692, 0, 0.2782)$	$(0.9655, 0, 0.2502)$

Table 6. The axial equilibrium P_5 for $k_1 = 0.3, k_2 = 0.7, q_1 = 0.7, q_2 = 0.5, q_3 = 0.7, E_1 = 0.3, E_2 = 0.2, E_3 = 0.2, K_1 = 20, K_2 = 20, m_1 = 0.8, m_2 = 0.6$.

w_1	w_2	P_5 at $\alpha = 0$	P_5 at $\alpha = 0.3$	P_5 at $\alpha = 0.6$	P_5 at $\alpha = 0.9$
0	1	$(0, 0.1920, 0.3638)$	$(0, 0.2298, 0.3833)$	$(0, 0.2727, 0.4029)$	$(0, 0.3220, 0.4226)$
0.2	0.8	$(0, 0.2435, 0.3899)$	$(0, 0.2697, 0.4016)$	$(0, 0.2981, 0.4134)$	$(0, 0.3291, 0.4252)$
0.4	0.6	$(0, 0.3048, 0.4160)$	$(0, 0.3150, 0.4199)$	$(0, 0.3255, 0.4239)$	$(0, 0.3363, 0.4278)$
0.6	0.4	$(0, 0.3789, 0.4422)$	$(0, 0.3668, 0.4383)$	$(0, 0.3551, 0.4343)$	$(0, 0.3437, 0.4304)$
0.8	0.2	$(0, 0.4706, 0.4683)$	$(0, 0.4268, 0.4566)$	$(0, 0.3872, 0.4448)$	$(0, 0.3513, 0.4330)$
1	0	$(0, 0.5867, 0.4942)$	$(0, 0.4970, 0.4748)$	$(0, 0.4222, 0.4553)$	$(0, 0.3590, 0.4357)$

Table 7. The axial equilibrium P_6 for $k_1 = 0.3, k_2 = 0.4, q_1 = 0.7, q_2 = 0.5, q_3 = 0.7, E_1 = 0.6, E_2 = 0.1, E_3 = 0.2, K_1 = 5, K_2 = 5, m_1 = 0.9, m_2 = 0.4$.

w_1	w_2	P_6 at $\alpha = 0$	P_6 at $\alpha = 0.3$	P_6 at $\alpha = 0.6$	P_6 at $\alpha = 0.9$
0	1	$(4.9826, 3.8455, 0)$	$(4.6419, 3.5356, 0)$	$(4.3385, 3.2568, 0)$	$(4.0671, 3.0043, 0)$
0.2	0.8	$(4.5369, 3.4395, 0)$	$(4.3577, 3.2745, 0)$	$(4.1901, 3.1191, 0)$	$(4.0331, 2.9724, 0)$
0.4	0.6	$(4.1543, 3.0858, 0)$	$(4.1016, 3.0366, 0)$	$(4.0500, 2.9883, 0)$	$(3.9995, 2.9408, 0)$
0.6	0.4	$(3.8232, 2.7740, 0)$	$(3.8700, 2.8184, 0)$	$(3.9177, 2.8636, 0)$	$(3.9665, 2.9097, 0)$
0.8	0.2	$(3.5351, 2.4958, 0)$	$(3.6599, 2.6172, 0)$	$(3.7926, 2.7447, 0)$	$(3.9338, 2.8789, 0)$
1	0	$(3.2834, 2.2448, 0)$	$(3.4690, 2.4307, 0)$	$(3.6742, 2.6311, 0)$	$(3.9017, 2.8485, 0)$

Table 8. The interior equilibrium P_7 for $k_1 = 0.4, k_2 = 0.5, q_1 = 0.6, q_2 = 0.4, q_3 = 0.2, E_1 = 0.2, E_2 = 0.3, E_3 = 0.2, K_1 = 100, K_2 = 100, m_1 = 0.9, m_2 = 0.3$.

w_1	w_2	P_7 at $\alpha = 0$	P_7 at $\alpha = 0.3$
0	1	(1.7240, 3.6276, 1.1986)	(1.5910, 3.4597, 1.0271)
0.2	0.8	(1.6168, 3.5123, 1.0583)	(1.5042, 3.3223, 0.9139)
0.4	0.6	(1.5417, 3.4296, 0.9337)	(1.4211, 3.1793, 0.8023)
0.6	0.4	(1.4309, 3.2886, 0.7930)	(1.3913, 3.1413, 0.7269)
0.8	0.2	(1.4291, 3.2451, 0.6884)	(1.3195, 3.0467, 0.6350)
1	0	(1.3764, 2.9890, 0.5245)	(1.2797, 2.9766, 0.5529)
w_1	w_2	P_7 at $\alpha = 0.6$	P_7 at $\alpha = 0.9$
0	1	(1.7066, 3.7444, 1.0113)	(1.5325, 3.4231, 0.7999)
0.2	0.8	(1.5291, 3.4203, 0.8678)	(1.4899, 3.3773, 0.7762)
0.4	0.6	(1.5273, 3.4215, 0.8306)	(1.4789, 3.3677, 0.7638)
0.6	0.4	(1.4894, 3.3737, 0.7779)	(1.4685, 3.3587, 0.7516)
0.8	0.2	(1.4594, 3.3320, 0.7279)	(1.4590, 3.3504, 0.7397)
1	0	(1.4328, 3.2552, 0.6687)	(1.4124, 3.2841, 0.7105)

Considering four sets of different initial values, it could be seen from Figure 1 that different orbits eventually converged to the same value, which concluded that the interior equilibrium of the system (1.7) fulfills the character of globally asymptotical stability. Figure 2 plotted the bifurcation graph of system (1.7) with the horizontal coordinates k_1, and the Hopf bifurcation of the system occurred with k_1 taking values in the range of $0.01 \le k_1 \le 0.7$. When $0.01 \le k_1 < 0.384$, the system oscillates periodically, while it maintains a stable steady-state when $0.384 < k_1 \le 0.7$. Therefore, based on Figure 2, it could be concluded that the fear of prey x_1 for predator y affected the stability of the system. We further observed that as k_1 increased, the prey x_1 density continued to decrease while the predator y density kept increasing. Thus, the result also suggested that greater fear of predators had a negative impact on prey populations while having a positive impact on predator populations. Correspondingly, Figures 3 and 4 showed the waveform plots and phase diagram at $k_1 = 0.1$ and $k_1 = 0.7$, respectively.

In addition, Figure 5 also plots the bifurcation graph with changing m_1. As can be seen in Figure 5, m_1 took values from 0.3 to 1, in which the system also underwent a Hopf bifurcation. When the value m_1 ranged from 0.3 to 0.657, the system (1.7) was stable; nevertheless, it would become unstable at $0.657 < m_1 \le 1$. Correspondingly, Figures 6 and 7 showed the waveform plots and phase diagram at $m_1 = 0.6$ and $m_1 = 0.9$, respectively.

Further, we find an interesting dynamic phenomenon through some numerical simulations. System (1.7) appears as a chaotic phenomenon, as shown in Figure 8.

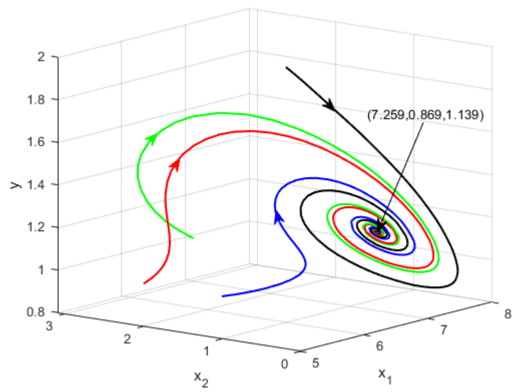

Figure 1. Global stability of the internal equilibrium P_7 = (5.665, 1.668, 2.047) of system (1.7) is given by the following parameter values: $\alpha = 1, w_1 + w_2 = 1, A_1 = 2.0, A_2 = 2.0, B_1 = 2.0, B_2 = 2.0, k_1 = 0.2, k_2 = 0.1, q_1 = 0.4, q_2 = 0.4, q_3 = 0.2, E_1 = 0.2, E_2 = 0.2, E_3 = 0.2, A_3 = 0.1, B_3 = 0.1, A_4 = 0.3, B_4 = 0.6, K_1 = 10, K_2 = 10, m_1 = 0.4, m_2 = 0.4, C_1 = 0.1, C_2 = 0.2, C_3 = 0.5$.

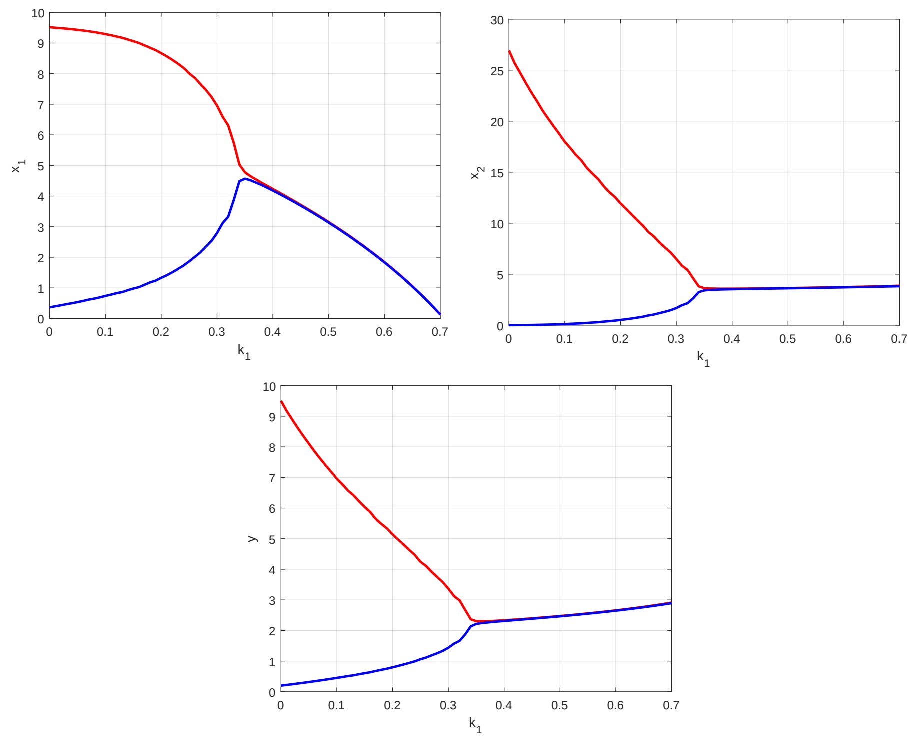

Figure 2. Hopf bifurcation occurs as a bifurcation parameter k_1, and the remaining parameters take the following values: $\alpha = 1, w_1 + w_2 = 1, A_1 = 3.0, A_2 = 3.0, B_1 = 3.0, B_2 = 3.0, k_2 = 0.4, q_1 = 0.6, q_2 = 0.4, q_3 = 0.2, E_1 = 0.2, E_2 = 0.2, E_3 = 0.2, A_3 = 0.2, B_3 = 0.1, A_4 = 0.3, B_4 = 0.6, K_1 = 10, K_2 = 70, m_1 = 0.9, m_2 = 0.3, C_1 = 0.1, C_2 = 0.2, C_3 = 0.5$.

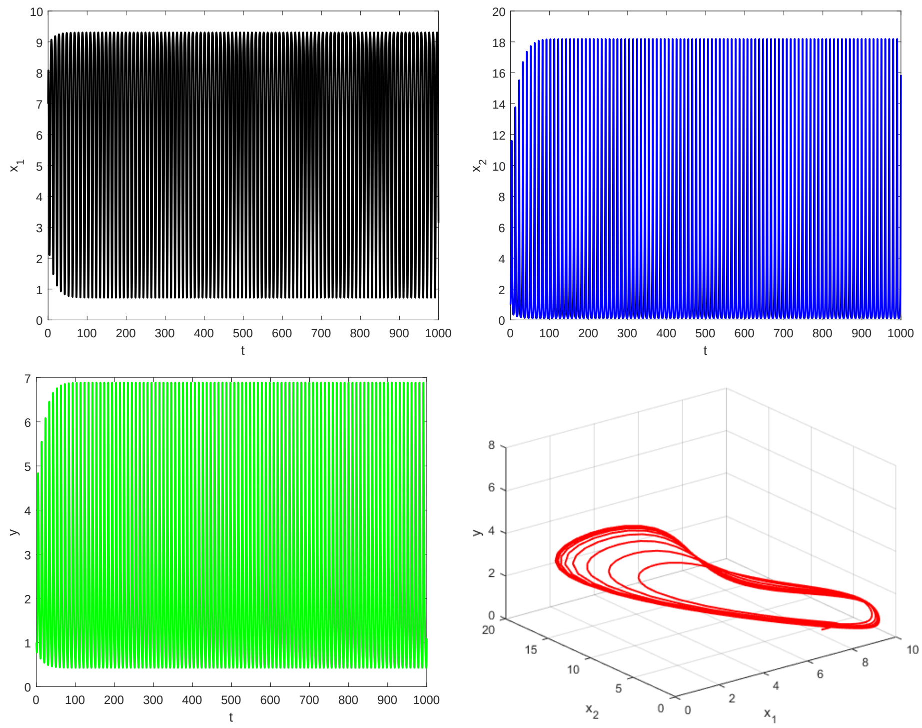

Figure 3. Waveform plots and phase diagram of system (1.7) with $k_1 = 0.1$, and $\alpha = 1, w_1 + w_2 = 1$.

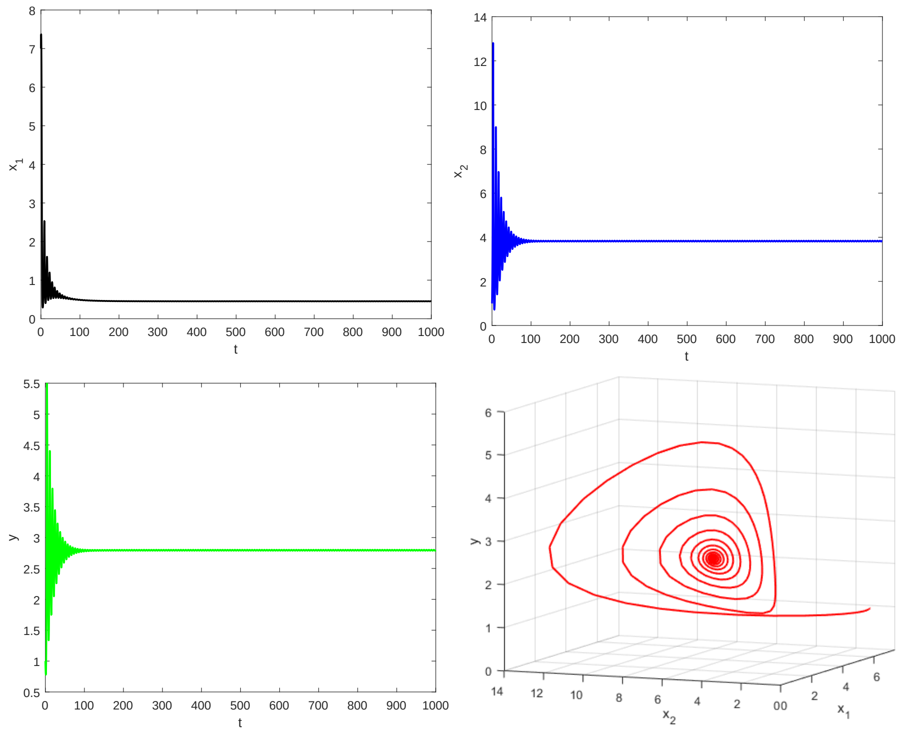

Figure 4. Waveform plots and phase diagram of system (1.7) with $k_1 = 0.7$, and $\alpha = 1, w_1 + w_2 = 1$.

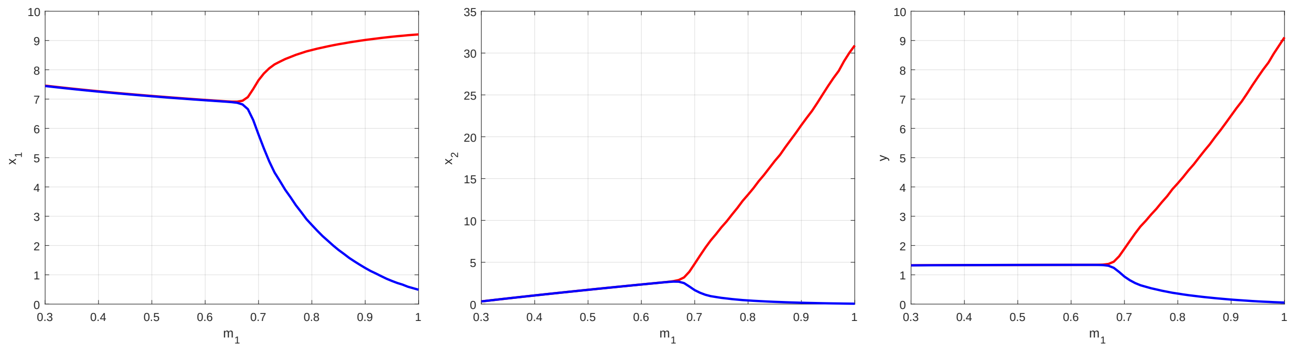

Figure 5. Hopf bifurcation occurs as a bifurcation parameter of system (1.7) parameter m_1, and the remaining parameters take the following values: $\alpha = 1, w_1 + w_2 = 1, A_1 = 2.0, A_2 = 2.0, B_1 = 2.0, B_2 = 2.0, k_1 = 0.1, k_2 = 0.4, q_1 = 0.7, q_2 = 0.4, q_3 = 0.2, E_1 = 0.2, E_2 = 0.2, E_3 = 0.3, A_3 = 0.1, B_3 = 0.1, A_4 = 0.3, B_4 = 0.6, K_1 = 10, K_2 = 70, m_2 = 0.4, C_1 = 0.1, C_2 = 0.2, C_3 = 0.5$.

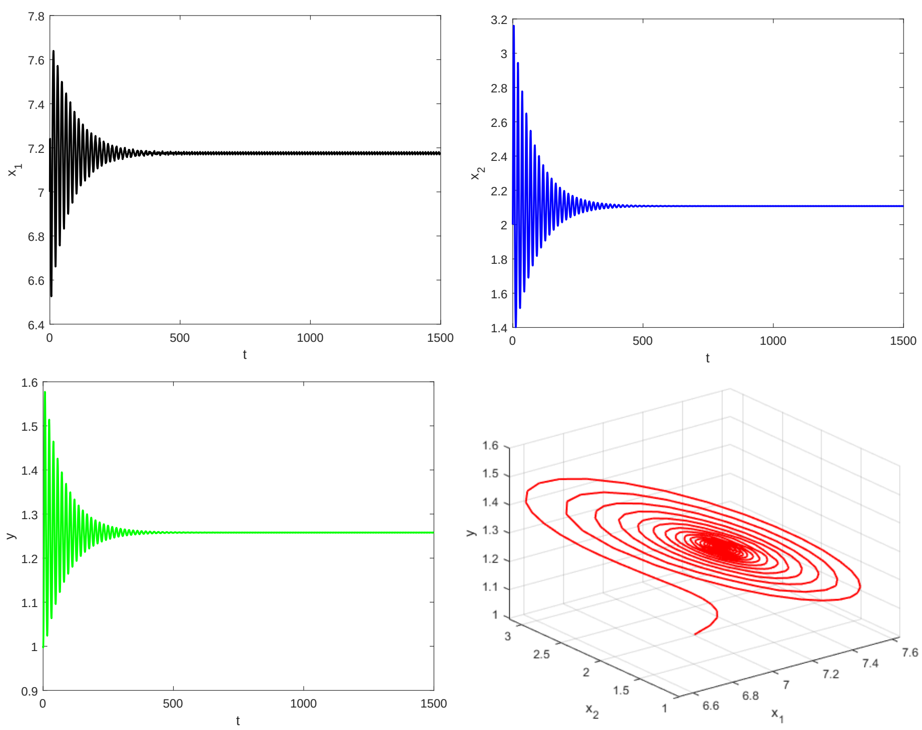

Figure 6. Waveform plots and phase diagram of system (1.7) with $m_1 = 0.6$ and $\alpha = 1, w_1 + w_2 = 1$.

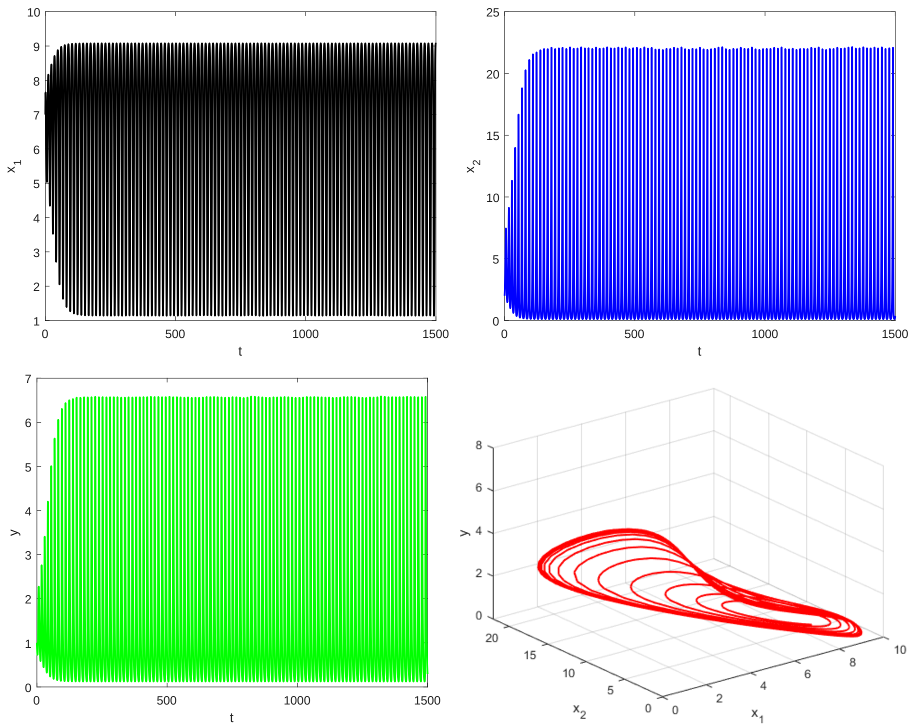

Figure 7. Waveform plots and phase diagram of system (1.7) with $m_1 = 0.9$ and $\alpha = 1, w_1 + w_2 = 1$.

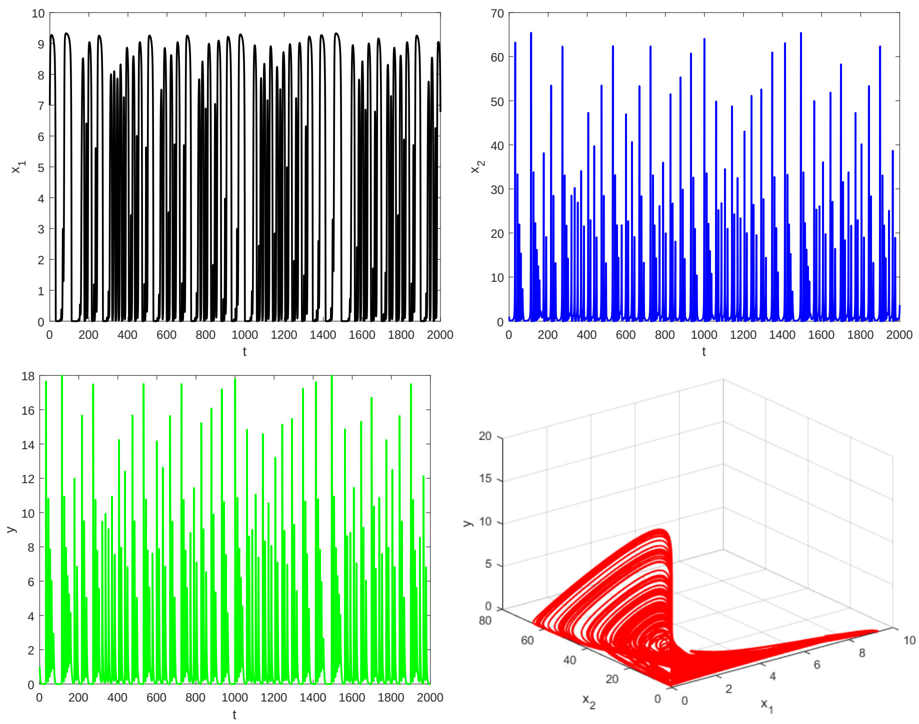

Figure 8. Waveform plots and phase diagram of chaotic phenomena with the following parameter values: $\alpha = 1, w_1 + w_2 = 1, A_1 = 2.0, A_2 = 2.0, B_1 = 3.0, B_2 = 3.0, k_1 = 0.2, k_2 = 0.5, q_1 = 0.6, q_2 = 0.4, q_3 = 0.2, E_1 = 0.2, E_2 = 0.3, E_3 = 0.2, A_3 = 0.2, B_3 = 0.3, A_4 = 0.3, B_4 = 0.6, K_1 = 10, K_2 = 70, m_1 = 0.9, m_2 = 0.3, C_1 = 0.1, C_2 = 0.2, C_3 = 0.5$.

7. Conclusions

In this work, we develop a model of one-predator and two-prey interactions in a fuzzy environment, examine the effects of fear and prey refuge on the system, and provide insight into the dynamic complexity. The proofs of the theoretical parts of this paper are based on system (1.7). It has been proven that all equilibria in system (1.7) are locally asymptotically stable, and interior equilibrium P_7 is also globally asymptotically stable. We have been further concerned about the appearance and direction of Hopf bifurcation. With the support of theoretical research, our numerical simulations have been able to display a wealth of charts and graphs.

First of all, different equilibria are displayed from Tables 2–8 with different α, w_1, w_2, respectively. Throughout Figure 1, we have verified the global asymptotical stability of interior equilibrium P_7, and find that the system is from unstable to stable with the increase of fear k_1, which demonstrates that the fear effect may be an important factor influencing the stability of the system (see Figures 2–4). Furthermore, it has also been observed that an increase in prey refuge m_1 leads to oscillatory phenomena (see Figures 5–7). Finally, through studying the Hopf bifurcation, we have discovered some interesting biological phenomena, namely that system (1.7) appears to be in a chaotic state (see Figure 8).

Author contributions

Xuyang Cao: Conceptualization, Investigation, Methodology, Validation, Writing-original draft, Formal analysis, Software; Qinglong Wang: Conceptualization, Methodology, Formal analysis, Writing-review and editing, Supervision; Jie Liu: Validation, Visualization, Data curation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Appendix A

Definition 1. *[34] Fuzzy set: A fuzzy set $\widetilde{\hbar}$ in a universe of discourse S is denoted by the set of pairs*

$$\widetilde{\hbar} = \{(s, \mu_{\widetilde{\hbar}}(s)) : s \in S\},$$

where the mapping $\mu_{\widetilde{\hbar}} : S \to [0,1]$ is the membership function of the fuzzy set $\widetilde{\hbar}$ and $\mu_{\widetilde{\hbar}}$ is the membership value or degree of membership of $s \in S$ in the fuzzy set $\widetilde{\hbar}$.

Definition 2. *[42] α-cut of fuzzy set: For any $\alpha \in (0,1]$, the α-cut of fuzzy set $\widetilde{\hbar}$ defined by $\hbar_\alpha = \{s : \mu_{\widetilde{\hbar}}(s)) \geq \alpha\}$ is a crisp set. For $\alpha = 0$ the support of $\widetilde{\hbar}$ is defined as $\hbar_0 = \mathrm{Supp}(\widetilde{\hbar}) = \overline{\{s \in \mathbb{R}, \mu_{\widetilde{\hbar}}(s) > 0\}}$.*

Definition 3. *[43] Fuzzy number: A fuzzy number satisfying the property $S = \mathbb{R}$ is called a convex fuzzy set.*

Definition 4. *[44] Triangular fuzzy number: A triangular fuzzy number (TFN) $\widetilde{\hbar} \equiv (b_1, b_2, b_3)$ represent fuzzy set of the real line $\mathbb{R}$ satisfying the property that the membership function $\mu_{\widetilde{\hbar}} : \mathbb{R} \to [0,1]$ can be espressed by*

$$\mu_{\widetilde{\hbar}} = \begin{cases} \dfrac{s - b_1}{b_2 - b_1} & if \;\; b_1 \leq s \leq b_2, \\ \dfrac{b_3 - s}{b_3 - b_2} & if \;\; b_2 \leq s \leq b_3, \\ 0 & otherwise. \end{cases}$$

Hence, the α-cut of triangular fuzzy number meets boundedness and encapsulation on $[\hbar_L(\alpha), \hbar_R(\alpha)]$, in which $\hbar_L(\alpha) = \inf s : \mu_{\widetilde{\hbar}}(s) \geq \alpha = b_1 + \alpha(b_2 - b_1)$ and $\hbar_R(\alpha) = \sup\{s : \mu_{\widetilde{\hbar}}(s) \geq \alpha\} = b_3 + \alpha(b_3 - b_2)$.

Lemma 1. *[45] In weighted sum method, w_j stands for the weight of jth objective. $w_j g_j$ represent a utility function for jth objective, and the total utility function π is represented by*

$$\pi = \sum_{j}^{l} w_j g_j, \quad j = 1, 2, \cdots, l,$$

where $w_j > 0$ and $\sum_j^l w_j = 1$ are satisfied.

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