AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 23924–23944.
DOI: 10.3934/math.20241163
Received: 20 March 2024
Revised: 19 July 2024
Accepted: 25 July 2024
Published: 12 August 2024

Research article

Construction of new fractional inequalities via generalized n-fractional polynomial s-type convexity

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Abstract: This paper focuses on introducing and investigating the class of generalized n-fractional polynomial s-type convex functions within the framework of fractional calculus. Relationships between the novel class of functions and other kinds of convex functions are given. New integral inequalities of Hermite-Hadamard and Ostrowski-type are established for our novel generalized class of convex functions. Using some identities and fractional operators, new refinements of Ostrowski-type inequalities are presented for generalized n-fractional polynomial s-type convex functions. Some special cases of the newly obtained results are discussed. It has been presented that, under some certain conditions, the class of generalized n-fractional polynomial s-type convex functions reduces to a novel class of convex functions. It is interesting that, our results for particular cases recaptures the Riemann-Liouville fractional integral inequalities and quadrature rules. By extending these particular types of inequalities, the objective is to unveil fresh mathematical perspectives, attributes, and connections that can enhance the evolution of more resilient mathematical methodologies. This study aids in the progression of mathematical instruments across diverse scientific fields.

Keywords: convex function; n-fractional polynomial convex function; Hermite-Hadamard inequality; Ostrowski inequality; integral inequalities
Mathematics Subject Classification: 26D10, 26D15, 26E60, 90C23

1. Introduction

Several decades ago, classical calculus underwent a transformative phase, propelled by remarkable innovations. Researchers unanimously acknowledge the remarkable efficacy and accuracy of outcomes derived from fractional-order equations. Presently, fractional calculus finds widespread application across various domains, including chaos theory, simulation, and modeling. A myriad of elegant definitions and operators, such as Riemann, Caputo, Hadamard, Katugampola, Atangana-Baleanu, and many others, exemplify the beauty of fractional calculus [1–7]. For a comprehensive overview of the origins, advancements, and applications of fractional calculus, we direct the reader to the esteemed monographs [8, 9] and compelling articles [10–15]. The Hermite-Hadamard inequality stands as a cornerstone in mathematical analysis, offering profound insights into the properties of integrable functions.

Convexity stands as a cornerstone in solving several problems in general and applied mathematics. Its robustness has led to the generalization and extension of convex functions and convex sets across numerous branches of mathematics, with many inequalities stemming from convexity theory present in the literature [16–20]. Among these inequalities, the Hermite-Hadamard (H-H) inequality shines as a strikingly useful result in the realm of mathematical inequalities [21–25]. This inequality holds pivotal significance due to its close connections with other notable inequalities such as the Hölder, Opial, Hardy, Minkowski, Ostrowski, and Young inequalities.

The H-H inequality, expressed as follows [26]:

$$\wp\left(\frac{\varkappa_1+\varkappa_2}{2}\right) \leq \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\wp(\nu)\,d\nu \leq \frac{\wp(\varkappa_1)+\wp(\varkappa_2)}{2}, \tag{1.1}$$

holds for $\wp$ to be a convex function on the interval $[\varkappa_1, \varkappa_2]$. This double inequality encapsulates profound insights into the behavior of convex functions over intervals, serving as a fundamental tool in various mathematical contexts.

In recent past several generalizations, refinements, and extensions of (1.1) are developed, which attracted the attention of a wide range of researchers both in applied and pure mathematics.

Suppose $\breve{A} \subseteq \mathbb{R}$ and $\wp : \breve{A} \to \mathbb{R}$ is a differentiable function on $\breve{A}^\circ$ (the interior of $\breve{A}$) such that $\varkappa_1, \varkappa_2 \in \breve{A}^\circ$ with $\varkappa_1 < \varkappa_2$. In this case, the well-known Ostrowski inequality [27] states that

$$\left|\wp(\nu) - \frac{1}{\varkappa_2-\varkappa_1}\int_{\varkappa_1}^{\varkappa_2}\wp(\nu)\,d\nu\right| \leq \left[\frac{1}{4} + \frac{\left(\nu - \frac{\varkappa_1+\varkappa_2}{2}\right)^2}{(\varkappa_2-\varkappa_1)^2}\right](\varkappa_2-\varkappa_1)\,S, \tag{1.2}$$

for all $\nu \in [\varkappa_1, \varkappa_2]$ if $|\wp'(\mu)| \leq S$ for all $\mu \in [\varkappa_1, \varkappa_2]$.

Ostrowski-type inequalities, which give error estimates for numerous quadrature rules, have important applications in numerical analysis. These disparities have been widened and applied to a wider range of disciplines in recent years.

Researchers in this intriguing field of study explore the applications of these variations in applied sciences and also examine the existence and uniqueness of solutions to fractional differential equations. By employing K-fractional integrals, the authors in [28], proposed several generalizations of Ostrowski-type estimations.

Definition 1.1. *[29] Let $s \in [0,1]$. A real valued function $\wp : \breve{A} \to \mathbb{R}$ is called s-type convex on $\breve{A}$ if*

$$\wp(\mu\varkappa_1 + (1-\mu)\varkappa_2) \leq (1 - s(1-\mu))\wp(\varkappa_1) + (1 - s\mu)\wp(\varkappa_2),$$

for all $\varkappa_1, \varkappa_2 \in \breve{A}$ and $\mu \in [0,1]$.

In [30], İşcan gave the definition of n-fractional polynomial convex functions as follows.

Definition 1.2. *Let $n \in \mathbb{N}$. A non-negative function $\wp : \breve{A} \subseteq \mathbb{R} \to \mathbb{R}$ is called n-fractional polynomial convex (FPC) function if*

$$\wp(\mu\varkappa_1 + (1-\mu)\varkappa_2) \leq \frac{1}{n}\sum_{u=1}^{n} \mu^{\frac{1}{u}}\wp(\varkappa_1) + \frac{1}{n}\sum_{u=1}^{n}(1-\mu)^{\frac{1}{u}}\wp(\varkappa_2),$$

for all $\varkappa_1, \varkappa_2 \in \breve{A}$ and $\mu \in [0,1]$.

Note that, every non-negative convex function is an FPC function [30].

Now, we demonstrate key concepts related to the fractional integral, primarily originating from the work of Mubeen et al. [31].

Let $\alpha, R > 0$, $\varkappa_1 < \varkappa_2$, and $\wp \in L[\varkappa_1, \varkappa_2]$. Then, the K-fractional integrals of order α are given by

$$\mathfrak{I}_{\varkappa_1}^{\alpha,K}\wp(z) = \frac{1}{K\Gamma_K(\alpha)}\int_{\varkappa_1}^{z}(z-\theta)^{\frac{\alpha}{K}-1}\wp(\theta)\,d\theta \quad (z > \theta)$$

and

$$\mathfrak{I}_{\varkappa_2}^{\alpha,K}\wp(z) = \frac{1}{K\Gamma_K(\alpha)}\int_{z}^{\varkappa_2}(\varkappa_2 - z)^{\frac{\alpha}{K}-1}\wp(\theta)\,d\theta \quad (z < \theta),$$

where $\Gamma_K(\alpha)$ is the K-Gamma function [32] given by

$$\Gamma_K(\alpha) = \int_0^{\infty}\mu^{\alpha-1}e^{-\frac{\mu^K}{K}}\,d\mu.$$

Recall that

$$\Gamma_K(K+\alpha) = \alpha\Gamma_K(\alpha)$$

and for $K = 1$, the K-fractional integrals coincide with the RL-fractional integrals.

Now, we recall the concepts of the Euler's Beta function β and hypergeometric function ${}_2F_1$, respectively.

$$\beta(\varkappa, y) = \frac{\Gamma(\varkappa)\Gamma(y)}{\Gamma(\varkappa + y)} = \int_0^1 \mu^{\varkappa-1}(1-\mu)^{y-1}\,d\mu$$

and

$${}_2F_1(a, b; c; \tau) = \frac{1}{\beta(b, c-b)}\mu^{b-1}(1-\mu)^{c-b-1}(1-\tau\mu)^{-a}\,d\mu,$$

where $\Gamma(\varkappa) = \int_0^{\infty}\mu^{\alpha-1}e^{-\mu}d\mu$ is the Euler Gamma function [33, 34].

Motivated by the aforementioned findings and existing literature, in Section 2, we will initially introduce the concept of a generalized n-fractional polynomial s-type convex function. Subsequently, in Section 3, we will establish a novel generalization of the H-H type inequality for the new class of functions. Moving forward to Section 4, we will obtain novel estimates for differentiable generalized n-fractional polynomial s-type convex functions. Notably, the results presented herein encompass RL-fractional integral inequalities and quadrature rules as special cases. The findings of our study prove beneficial in crafting fractals through iterative methodologies, an engaging research domain with implications for refining machine learning algorithms. Finally, Section 5 concludes with a brief conclusion.

2. Definition of generalized n-fractional polynomial s-type convex functions

In this section, we introduce a new concept called the generalized n-fractional polynomial s-type convex function and explore its fundamental algebraic properties.

Definition 2.1. *Let* $n \in \mathbb{N}$, $s \in [0,1]$, $a_u \geq 0$ $\left(u = \overline{1,n}\right)$ *such that* $\sum_{u=1}^{n} a_u > 0$, $\breve{A} \subset \mathbb{R}$ *be an interval. A non-negative function* $\wp : \breve{A} \subset \mathbb{R} \to \mathbb{R}$ *is called a generalized n-fractional polynomial s-type convex function if for every* $\varkappa_1, \varkappa_2 \in \breve{A}$ *and* $\mu \in [0,1]$,

$$\begin{aligned} &\wp\left(\mu\varkappa_1 + (1-\mu)\varkappa_2\right) \\ \leq\ & \frac{\sum_{u=1}^{n} a_u \left(1 - s\left(1-\mu\right)\right)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u} \wp\left(\varkappa_1\right) + \frac{\sum_{u=1}^{n} a_u \left(1 - s\mu\right)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u} \wp\left(\varkappa_2\right). \end{aligned} \tag{2.1}$$

We denote the class of all generalized n-fractional polynomial s-type convex functions by $GFPC-s$.

Example 2.2. *Consider the function* $\wp(x) = x^2$, *and the parameters* $s = 0.5$, $n = 2$, $a_1 = 1$, $a_2 = 2$, *and* $\mu = 0.5$. *According to Definition 2.1,*

$$\wp\left(\mu\kappa_1 + (1-\mu)\kappa_2\right) \leq \frac{\sum_{u=1}^{n} a_u \left(1 - s(1-\mu)\right)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u} \wp(\kappa_1) + \frac{\sum_{u=1}^{n} a_u \left(1 - s\mu\right)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u} \wp(\kappa_2).$$

For $\kappa_1 = 1$ *and* $\kappa_2 = 3$,

$$\wp\left(0.5 \cdot 1 + 0.5 \cdot 3\right) = \wp(2) = 2^2 = 4,$$

$$\begin{aligned} & \frac{\sum_{u=1}^{n} a_u \left(1 - 0.5(1-0.5)\right)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u} \wp(\kappa_1) + \frac{\sum_{u=1}^{n} a_u \left(1 - 0.5 \cdot 0.5\right)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u} \wp(\kappa_2) \\ =\ & \frac{1\left(1-0.25\right)^{1/1} + 2\left(1-0.25\right)^{1/2}}{3} \cdot 1^2 + \frac{1\left(1-0.25\right)^{1/1} + 2\left(1-0.25\right)^{1/2}}{3} \cdot 3^2 \\ =\ & \frac{0.75 + 2 \cdot 0.8660}{3} \cdot 1 + \frac{0.75 + 2 \cdot 0.8660}{3} \cdot 9 \\ =\ & \frac{0.75 + 1.732}{3} \cdot 1 + \frac{0.75 + 1.732}{3} \cdot 9 \\ =\ & \frac{2.482}{3} \cdot 1 + \frac{2.482}{3} \cdot 9 \\ =\ & 0.8273 \cdot 1 + 0.8273 \cdot 9 = 0.8273 + 7.4457 = 8.273. \end{aligned}$$

So, one has

$$4 \leq 8.273.$$

According to Definition 1.2,

$$\wp(\mu\kappa_1 + (1-\mu)\kappa_2) \leq \frac{1}{n}\sum_{u=1}^{n}\mu^{\frac{1}{u}}\wp(\kappa_1) + \frac{1}{n}\sum_{u=1}^{n}(1-\mu)^{\frac{1}{u}}\wp(\kappa_2).$$

For $\kappa_1 = 1$ *and* $\kappa_2 = 3$,

$$\wp(0.5\cdot 1 + 0.5\cdot 3) = \wp(2) = 2^2 = 4,$$

$$\begin{aligned}
& \frac{1}{2}\left(0.5^1\wp(1) + 0.5^{\frac{1}{2}}\wp(1)\right) + \frac{1}{2}\left(0.5^1\wp(3) + 0.5^{\frac{1}{2}}\wp(3)\right) \\
= & \frac{1}{2}\left(0.5\cdot 1^2 + 0.7071\cdot 1^2\right) + \frac{1}{2}\left(0.5\cdot 3^2 + 0.7071\cdot 3^2\right) \\
= & \frac{1}{2}(0.5\cdot 1 + 0.7071\cdot 1) + \frac{1}{2}(0.5\cdot 9 + 0.7071\cdot 9) \\
= & \frac{1}{2}(0.5 + 0.7071) + \frac{1}{2}(4.5 + 6.3639) \\
= & \frac{1}{2}\cdot 1.2071 + \frac{1}{2}\cdot 10.8639 \\
= & 0.60355 + 5.43195 = 6.0355.
\end{aligned}$$

So, one gets

$$4 \leq 6.0355.$$

Because the generalized n-fractional polynomial s-type convexity ($GFPC - s$) is a generalization of the n-fractional polynomial convexity (FPC), the resulting bounds extend those obtained for FPC.

Note that every $GFPC - s$ is an h-convex function with

$$h(\mu) = \frac{\sum_{u=1}^{n} a_u (1 - s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}.$$

Remark 2.3. *For* $s = 1$, *Definition 2.1 reduces to the definition of generalized n-fractional polynomial convex* ($GFPC$) *functions.*

Remark 2.4. *For* $s = 1$ *and* $a_u = 1$ $\left(u = \overline{1,n}\right)$, *Definition 2.1 coincides with Definition 1.2.*

Remark 2.5. *For* $s = 1$ *and* $n = 1$, *Definition 2.1 coincides with the definition of classical convexity.*

Remark 2.6. *Every non-negative n-fractional polynomial convex function is a* $GFPC - s$ *function. It is clear from the inequalities*

$$\frac{1}{n}\sum_{u=1}^{n}\mu^{\frac{1}{u}} \leq \frac{\sum_{u=1}^{n} a_u (1 - s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}$$

and

$$\frac{1}{n}\sum_{u=1}^{n}(1-\mu)^{\frac{1}{u}} \leq \frac{\sum_{u=1}^{n} a_u (1 - s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u},$$

for all $n \in \mathbb{N}$, $s \in [0,1]$, *and* $\mu \in [0,1]$.

Note that not every $GFPC-s$ function needs to be an FPC function.

Remark 2.7. *If $\wp$ is a $GFPC-s$ function, then $\wp$ is a non-negative function. Indeed, from the definition of $GFPC-s$ function, one can write*

$$\wp(\nu)=\wp(\mu\nu+(1-\mu)\nu)\leq\left[\frac{\sum_{u=1}^{n}a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}+\frac{\sum_{u=1}^{n}a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}\right]\wp(\nu),$$

for all $\nu\in\breve{A}$ and $\mu\in[0,1]$. Therefore, one has

$$\left[\frac{\sum_{u=1}^{n}a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}+\frac{\sum_{u=1}^{n}a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}-1\right]\wp(\nu)\geq 0.$$

Since

$$\begin{aligned}\frac{\sum_{u=1}^{n}a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}+\frac{\sum_{u=1}^{n}a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u} &\geq \frac{1}{n}\sum_{u=1}^{n}\mu^{\frac{1}{u}}+\frac{1}{n}\sum_{u=1}^{n}(1-\mu)^{\frac{1}{u}}\\ &\geq \frac{1}{n}\sum_{u=1}^{n}\mu+\frac{1}{n}\sum_{u=1}^{n}(1-\mu)\\ &= \mu+(1-\mu)=1,\end{aligned}$$

for all $\mu\in[0,1]$, one obtains $\wp(\nu)\geq 0$ for all $\nu\in\breve{A}$.

3. New generalizations of H-H type inequalities using generalized n-fractional polynomial s-type convex functions

Now, we obtain a new generalization of H-H inequality for the $GFPC-s$ function $\wp$.

Theorem 3.1. *Let $n\in\mathbb{N}$, $a_u\geq 0\left(u=\overline{1,n}\right)$, $s\in[0,1]$, $\alpha\in[0,1]$, $K>0$, and $\wp:\breve{A}=[\varkappa_1,\varkappa_2]\to\mathbb{R}$ be a $GFPC-s$ function such that $\wp\in L[\varkappa_1,\varkappa_2]$. Then,*

$$\begin{aligned}&\frac{\sum_{u=1}^{n}a_u}{\sum_{u=1}^{n}a_u\left(1-\frac{s}{2}\right)^{\frac{1}{u}}}\wp\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\\ \leq\ &\frac{\Gamma_K(K+\alpha)}{(\varkappa_2-\varkappa_1)^{\frac{\alpha}{K}}}\left[\Im_{\varkappa_1^+}^{\alpha,K}\wp(\varkappa_2)+\Im_{\varkappa_2^-}^{\alpha,K}\wp(\varkappa_1)\right]\\ \leq\ &\frac{\wp(\varkappa_2)+\wp(\varkappa_1)}{\sum_{u=1}^{n}a_u}\int_0^1\sum_{u=1}^{n}a_u\mu^{\frac{\alpha}{K}-1}\left[(1-s(1-\mu))^{\frac{1}{u}}+(1-s\mu)^{\frac{1}{u}}\right]d\mu.\end{aligned}\tag{3.1}$$

Proof. From the definition of the $GFPC-s$ function, one obtains

$$\begin{aligned}&\wp\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\\ =\ &\wp\left(\frac{(\mu\varkappa_1+(1-\mu)\varkappa_2)+\left[(1-\mu)\varkappa_1+\mu\varkappa_2\right]}{2}\right)\end{aligned}$$

$$
\begin{aligned}
&= \wp\left(\frac{1}{2}(\mu\varkappa_1+(1-\mu)\varkappa_2)+\frac{1}{2}((1-\mu)\varkappa_1+\mu\varkappa_2)\right)\\
&\leq \frac{\sum_{u=1}^{n} a_u\left(1-\frac{s}{2}\right)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}\left[\wp(\mu\varkappa_1+(1-\mu)\varkappa_2)+\wp((1-\mu)\varkappa_1+\mu\varkappa_2)\right].
\end{aligned}
$$

Multiplying both sides of the above inequality by $\mu^{\frac{\alpha}{K}-1}$ and taking the integral with respect to $\mu \in [0,1]$, one gets

$$
\begin{aligned}
&\wp\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\int_0^1 \mu^{\frac{\alpha}{K}-1}d\mu\\
\leq\ & \frac{\sum_{u=1}^{n} a_u\left(1-\frac{s}{2}\right)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}\left[\int_0^1 \mu^{\frac{\alpha}{K}-1}\wp(\mu\varkappa_1+(1-\mu)\varkappa_2)\,d\mu+\int_0^1 \mu^{\frac{\alpha}{K}-1}\wp((1-\mu)\varkappa_1+\mu\varkappa_2)\,d\mu\right]\\
\leq\ & \frac{\sum_{u=1}^{n} a_u\left(1-\frac{s}{2}\right)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}\frac{1}{(\varkappa_2-\varkappa_1)^{\frac{\alpha}{K}}}\left[\int_{\varkappa_1}^{\varkappa_2}\left(\frac{\omega-\varkappa_1}{\varkappa_2-\varkappa_1}\right)^{\frac{\alpha}{K}-1}\wp(\omega)\,d\omega+\int_{\varkappa_1}^{\varkappa_2}\left(\frac{\varkappa_2-\omega}{\varkappa_2-\varkappa_1}\right)^{\frac{\alpha}{K}-1}\wp(\omega)\,d\omega\right]\\
\leq\ & \frac{\sum_{u=1}^{n} a_u\left(1-\frac{s}{2}\right)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}\frac{K\Gamma_K(\alpha)}{(\varkappa_2-\varkappa_1)^{\frac{\alpha}{K}}}\left[\mathfrak{I}_{\varkappa_1^+}^{\alpha,K}\wp(\varkappa_2)+\mathfrak{I}_{\varkappa_2^-}^{\alpha,K}\wp(\varkappa_1)\right].
\end{aligned}
$$

So, one has

$$
\frac{\sum_{u=1}^{n} a_u}{\sum_{u=1}^{n} a_u\left(1-\frac{s}{2}\right)^{\frac{1}{u}}}\wp\left(\frac{\varkappa_1+\varkappa_2}{2}\right)\leq\frac{\Gamma_R(R+\alpha)}{(\varkappa_2-\varkappa_1)^{\frac{\alpha}{R}}}\left[\mathfrak{I}_{\varkappa_1^+}^{\alpha,R}\wp(\varkappa_2)+\mathfrak{I}_{\varkappa_2^-}^{\alpha,R}\wp(\varkappa_1)\right],
$$

which completes the left-hand side of inequality (3.1). Now, we prove the right-hand side of inequality (3.1). Let $\mu \in [0,1]$. From the definition of the $GFPC-s$ function, one obtains

$$
\wp(\mu\varkappa_1+(1-\mu)\varkappa_2)\leq\frac{\sum_{u=1}^{n} a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}\wp(\varkappa_1)+\frac{\sum_{u=1}^{n} a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}\wp(\varkappa_2)
$$

and

$$
\wp((1-\mu)\varkappa_1+\mu\varkappa_2)\leq\frac{\sum_{u=1}^{n} a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}\wp(\varkappa_2)+\frac{\sum_{u=1}^{n} a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}\wp(\varkappa_1).
$$

Adding the above inequalities, one gets

$$
\begin{aligned}
&\wp(\mu\varkappa_1+(1-\mu)\varkappa_2)+\wp((1-\mu)\varkappa_1+\mu\varkappa_2)\\
\leq\ & [\wp(\varkappa_1)+\wp(\varkappa_2)]\left(\frac{\sum_{u=1}^{n} a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}+\frac{\sum_{u=1}^{n} a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n} a_u}\right).
\end{aligned}
$$

Multiplying both sides of the above inequality by $\mu^{\frac{\alpha}{K}-1}$, taking the integral with respect to $\mu \in [0,1]$, and changing the variable of integration, one obtains

$$
\begin{aligned}
&\frac{\Gamma_K(K+\alpha)}{(\varkappa_2-\varkappa_1)^{\frac{\alpha}{R}}}\left[\mathfrak{I}_{\varkappa_1^+}^{\alpha,K}\wp(\varkappa_2)+\mathfrak{I}_{\varkappa_2^-}^{\alpha,K}\wp(\varkappa_1)\right]\\
\leq\ & \frac{\wp(\varkappa_2)+\wp(\varkappa_1)}{\sum_{u=1}^{n} a_u}\int_0^1\sum_{u=1}^{n} a_u\mu^{\frac{\alpha}{K}-1}\left[(1-s(1-\mu))^{\frac{1}{u}}+(1-s\mu)^{\frac{1}{u}}\right]d\mu.
\end{aligned}
$$

This completes the proof. □

Corollary 3.2. *If one takes $s = 1$ in Theorem 3.1, one gets the H-H inequality for GFPC functions with K-fractional integral operators:*

$$\begin{aligned} &\frac{\sum_{u=1}^{n} a_u}{\sum_{u=1}^{n} a_u \left(\frac{1}{2}\right)^{\frac{1}{u}}} \wp\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) \\ \leq\ & \frac{\Gamma_K(K+\alpha)}{(\varkappa_2 - \varkappa_1)^{\frac{\alpha}{K}}} \left[\Im_{\varkappa_1^+}^{\alpha,K} \wp(\varkappa_2) + \Im_{\varkappa_2^-}^{\alpha,K} \wp(\varkappa_1)\right] \\ \leq\ & \frac{\wp(\varkappa_2) + \wp(\varkappa_1)}{\sum_{u=1}^{n} a_u} \sum_{u=1}^{n} a_u \int_0^1 \mu^{\frac{\alpha}{K}-1} \left[\mu^{\frac{1}{u}} + (1-\mu)^{\frac{1}{u}}\right] d\mu. \end{aligned}$$

Corollary 3.3. *If one takes $K = 1$ in Corollary 3.2, one gets H-H inequality for GFPC functions with RL-fractional integral operators:*

$$\begin{aligned} \frac{\sum_{u=1}^{n} a_u}{\sum_{u=1}^{n} a_u \left(\frac{1}{2}\right)^{\frac{1}{u}}} \wp\left(\frac{\varkappa_1 + \varkappa_2}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{(\varkappa_2 - \varkappa_1)^{\alpha}} \left[\Im_{\varkappa_1^+}^{\alpha} \wp(\varkappa_2) + \Im_{\varkappa_2^-}^{\alpha} \wp(\varkappa_1)\right] \\ &\leq \frac{\wp(\varkappa_2) + \wp(\varkappa_1)}{\sum_{u=1}^{n} a_u} \sum_{u=1}^{n} a_u \int_0^1 \mu^{\alpha-1} \left[\mu^{\frac{1}{u}} + (1-\mu)^{\frac{1}{u}}\right] d\mu. \end{aligned}$$

Remark 3.4. *If one takes $\alpha = 1$ and $n = 1$ in Corollary 3.3, then one gets the inequality (1.1).*

4. New generalizations of Ostrowski type inequalities using generalized n-fractional polynomial s-type convex functions

In this section, we find novel estimates that refine the Ostrowski-type inequalities for the functions whose first and second derivatives in absolute value at certain powers are $GFPC - s$. First, we give the following crucial lemma [35]:

Lemma 4.1. *Let $\alpha \in [0,1]$, $K > 0$, $\varkappa_1 < \varkappa_2$, and $\wp : [0,1] \to \mathbb{R}$ be a differentiable function on $\breve{A}^{\circ}$ such that $\wp' \in L[\varkappa_1, \varkappa_2]$. Then,*

$$\begin{aligned} &\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}} + (\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1} \wp(\nu) - \frac{\Gamma_K(K+\alpha)}{\varkappa_2-\varkappa_1} \left[\Im_{\nu^-}^{\alpha,K} \wp(\varkappa_1) + \Im_{\nu^+}^{\alpha,K} \wp(\varkappa_2)\right] \\ =\ & \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1} \int_0^1 \mu^{\frac{\alpha}{K}} \wp'(\mu\nu + (1-\mu)\varkappa_1)\, d\mu - \frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1} \int_0^1 \mu^{\frac{\alpha}{K}} \wp'(\mu\nu + (1-\mu)\varkappa_2)\, d\mu. \end{aligned}$$

Theorem 4.2. *Let $n \in \mathbb{N}$, $a_u \geq 0$ $\left(u = \overline{1,n}\right)$, $s \in [0,1]$, $\alpha, K > 0$, $\varkappa_1 < \varkappa_2$, and $\wp : \breve{A} = [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a differentiable function on $\breve{A}^{\circ}$ such that $\wp' \in L[\varkappa_1, \varkappa_2]$. Let $|\wp'(\nu)|$ be a $GFPC - s$ function on $\breve{A}$ with $|\wp'(\nu)| \leq S$ for all $\nu \in [\varkappa_1, \varkappa_2]$. Then,*

$$\begin{aligned} &\left|\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}} + (\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1} \wp(\nu) - \frac{\Gamma_K(K+\alpha)}{\varkappa_2-\varkappa_1} \left[\Im_{\nu^-}^{\alpha,K} \wp(\varkappa_1) + \Im_{\nu^+}^{\alpha,K} \wp(\varkappa_2)\right]\right| \\ \leq\ & \left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1} + (\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\right] \frac{S}{\sum_{u=1}^{n} a_u} \sum_{u=1}^{n} a_u \left[\int_0^1 \mu^{\frac{\alpha}{K}} (1 - s(1-\mu))^{\frac{1}{u}}\, d\mu + \int_0^1 \mu^{\frac{\alpha}{K}} (1 - s\mu)^{\frac{1}{u}}\, d\mu\right]. \end{aligned}$$

Proof. Using Lemma 4.1 and a property of the $GFPC-s$ function $|\wp'|$, one has

$$\begin{aligned}
&\left|\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma_K(K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im_{\nu^-}^{\alpha,K}\wp(\varkappa_1)+\Im_{\nu^+}^{\alpha,K}\wp(\varkappa_2)\right]\right| \\
\leq\ & \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\int_0^1\mu^{\frac{\alpha}{K}}|\wp'(\mu\nu+(1-\mu)\varkappa_1)|\,d\mu+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\int_0^1\mu^{\frac{\alpha}{K}}|\wp'(\mu\nu+(1-\mu)\varkappa_2)|\,d\mu \\
\leq\ & \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\int_0^1\mu^{\frac{\alpha}{K}}\left[\frac{\sum_{u=1}^n a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp'(\nu)|+\frac{\sum_{u=1}^n a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp'(\varkappa_1)|\right]d\mu \\
&+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\int_0^1\mu^{\frac{\alpha}{K}}\left[\frac{\sum_{u=1}^n a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp'(\nu)|+\frac{\sum_{u=1}^n a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp'(\varkappa_2)|\right]d\mu \\
\leq\ & \left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\right]\frac{S}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left[\int_0^1\mu^{\frac{\alpha}{K}}(1-s(1-\mu))^{\frac{1}{u}}\,d\mu+\int_0^1\mu^{\frac{\alpha}{K}}(1-s\mu)^{\frac{1}{u}}\,d\mu\right].
\end{aligned}$$

□

Corollary 4.3. *If one takes $s=1$ in Theorem 3.1, one gets the following inequality for GFPC functions with K-fractional integral operators:*

$$\begin{aligned}
&\left|\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma_K(K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im_{\nu^-}^{\alpha,K}\wp(\varkappa_1)+\Im_{\nu^+}^{\alpha,K}\wp(\varkappa_2)\right]\right| \\
\leq\ & \left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\right]\frac{S}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left[\int_0^1\mu^{\frac{\alpha}{K}}\mu^{\frac{1}{u}}d\mu+\int_0^1\mu^{\frac{\alpha}{K}}(1-\mu)^{\frac{1}{u}}\,d\mu\right].
\end{aligned}$$

Corollary 4.4. *If one takes $K=1$ in Corollary 4.3, one gets the following inequality for GFPC functions with RL-integral operators:*

$$\begin{aligned}
&\left|\frac{(\nu-\varkappa_1)^{\alpha}+(\varkappa_2-\nu)^{\alpha}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma(\alpha+1)}{\varkappa_2-\varkappa_1}\left[\Im_{\nu^-}^{\alpha}\wp(\varkappa_1)+\Im_{\nu^+}^{\alpha}\wp(\varkappa_2)\right]\right| \\
\leq\ & \left[\frac{(\nu-\varkappa_1)^{\alpha+1}+(\varkappa_2-\nu)^{\alpha+1}}{\varkappa_2-\varkappa_1}\right]\frac{S}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left[\int_0^1\mu^{\alpha}\mu^{\frac{1}{u}}d\mu+\int_0^1\mu^{\alpha}(1-\mu)^{\frac{1}{u}}\,d\mu\right].
\end{aligned}$$

Remark 4.5. *If one takes $\alpha=1$ and $n=1$ in Corollary 4.4, then one gets inequality (1.2).*

Theorem 4.6. *Let $n\in\mathbb{N}$, $a_u\geq 0\left(u=\overline{1,n}\right)$, $s\in[0,1]$, $\alpha,K>0$, $\varkappa_1<\varkappa_2$, $q>1$, and $\wp:\breve{A}=[\varkappa_1,\varkappa_2]\to\mathbb{R}$ be a differentiable function on $\breve{A}^\circ$ such that $\wp'\in L[\varkappa_1,\varkappa_2]$. Let $|\wp'(\nu)|^q$ be a $GFPC-s$ function on $\breve{A}$ with $|\wp'(\nu)|\leq S$ for all $\nu\in[\varkappa_1,\varkappa_2]$. Then,*

$$\begin{aligned}
&\left|\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma_K(K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im_{\nu^-}^{\alpha,K}\wp(\varkappa_1)+\Im_{\nu^+}^{\alpha,K}\wp(\varkappa_2)\right]\right| \\
\leq\ & \left(\frac{K}{K+\alpha}\right)^{1-\frac{1}{q}}\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\right] \\
&\times\left(\frac{S^q}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left[\int_0^1\mu^{\frac{\alpha}{K}}(1-s(1-\mu))^{\frac{1}{u}}\,d\mu+\int_0^1\mu^{\frac{\alpha}{K}}(1-s\mu)^{\frac{1}{u}}\,d\mu\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

Proof. Using Lemma 4.1, a property of the $GFPC-s$ function $|\wp'|^q$, and the power mean inequality, one has

$$\begin{aligned}
&\left|\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma_K(K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im_{\nu^-}^{\alpha,K}\wp(\varkappa_1)+\Im_{\nu^+}^{\alpha,K}\wp(\varkappa_2)\right]\right| \\
\leq\ & \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\int_0^1\mu^{\frac{\alpha}{K}}\left|\wp'(\mu\nu+(1-\mu)\varkappa_1)\right|d\mu+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\int_0^1\mu^{\frac{\alpha}{K}}\left|\wp'(\mu\nu+(1-\mu)\varkappa_2)\right|d\mu \\
\leq\ & \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\left(\int_0^1\mu^{\frac{\alpha}{K}}d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\mu^{\frac{\alpha}{K}}\left|\wp'(\mu\nu+(1-\mu)\varkappa_1)\right|^q d\mu\right)^{\frac{1}{q}} \\
&+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\left(\int_0^1\mu^{\frac{\alpha}{K}}d\mu\right)^{1-\frac{1}{q}}\left(\int_0^1\mu^{\frac{\alpha}{K}}\left|\wp'(\mu\nu+(1-\mu)\varkappa_2)\right|^q d\mu\right)^{\frac{1}{q}} \\
\leq\ & \left(\frac{K}{K+\alpha}\right)^{1-\frac{1}{q}}\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\left(\frac{\sum_{u=1}^n a_u\int_0^1\mu^{\frac{\alpha}{K}}(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp'(\nu)\right|^q d\mu\right.\right. \\
&\left.+\ \frac{\sum_{u=1}^n a_u\int_0^1\mu^{\frac{\alpha}{K}}(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp'(\varkappa_1)\right|^q d\mu\right)^{\frac{1}{q}} \\
&+\ \frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\left(\frac{\sum_{u=1}^n a_u\int_0^1\mu^{\frac{\alpha}{K}}(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp'(\nu)\right|^q d\mu\right. \\
&\left.\left.+\ \frac{\sum_{u=1}^n a_u\int_0^1\mu^{\frac{\alpha}{K}}(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp'(\varkappa_2)\right|^q d\mu\right)^{\frac{1}{q}}\right] \\
\leq\ & \left(\frac{K}{K+\alpha}\right)^{1-\frac{1}{q}}\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\right] \\
&\times\left(\frac{S^q}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left[\int_0^1\mu^{\frac{\alpha}{K}}(1-s(1-\mu))^{\frac{1}{u}}d\mu+\int_0^1\mu^{\frac{\alpha}{K}}(1-s\mu)^{\frac{1}{u}}d\mu\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, the proof is completed. □

Corollary 4.7. *If one takes $s=1$ in Theorem 4.6, one gets the following inequality for GFPC functions with K-fractional integral operators:*

$$\begin{aligned}
&\left|\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma_K(K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im_{\nu^-}^{\alpha,K}\wp(\varkappa_1)+\Im_{\nu^+}^{\alpha,K}\wp(\varkappa_2)\right]\right| \\
\leq\ & \left(\frac{K}{K+\alpha}\right)^{1-\frac{1}{q}}\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\right]\left(\frac{S^q}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left[\int_0^1\mu^{\frac{\alpha}{K}}\mu^{\frac{1}{u}}d\mu+\int_0^1\mu^{\frac{\alpha}{K}}(1-\mu)^{\frac{1}{u}}d\mu\right]\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 4.8. *If one takes $K=1$ in Corollary 4.7, one gets the following inequality for GFPC functions with RL-integral operators:*

$$\left|\frac{(\nu-\varkappa_1)^{\alpha}+(\varkappa_2-\nu)^{\alpha}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma(\alpha)}{\varkappa_2-\varkappa_1}\left[\mathfrak{I}^{\alpha}_{\nu^-}\wp(\varkappa_1)+\mathfrak{I}^{\alpha}_{\nu^+}\wp(\varkappa_2)\right]\right|$$
$$\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left[\frac{(\nu-\varkappa_1)^{\alpha+1}+(\varkappa_2-\nu)^{\alpha+1}}{\varkappa_2-\varkappa_1}\right]\left(\frac{S^q}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left[\int_0^1\mu^{\alpha}\mu^{\frac{1}{u}}d\mu+\int_0^1\mu^{\alpha}(1-\mu)^{\frac{1}{u}}d\mu\right]\right)^{\frac{1}{q}}.$$

Theorem 4.9. *Let $n\in\mathbb{N}$, $a_u\geq 0\left(u=\overline{1,n}\right)$, $s\in[0,1]$, $\alpha,K>0$, $\varkappa_1<\varkappa_2$, $t,q>1$ with $\frac{1}{t}+\frac{1}{q}=1$, and $\wp:\breve{A}=[\varkappa_1,\varkappa_2]\to\mathbb{R}$ be a differentiable function on $\breve{A}^{\circ}$ such that $\wp'\in L[\varkappa_1,\varkappa_2]$. Let $|\wp'(\nu)|^q$ be a $GFPC-s$ function on $\breve{A}$ with $|\wp'(\nu)|\leq S$ for all $\nu\in[\varkappa_1,\varkappa_2]$. Then,*

$$\left|\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma_K(K+\alpha)}{\varkappa_2-\varkappa_1}\left[\mathfrak{I}^{\alpha,K}_{\nu^-}\wp(\varkappa_1)+\mathfrak{I}^{\alpha,K}_{\nu^+}\wp(\varkappa_2)\right]\right|$$
$$\leq \left(\frac{K}{K+\alpha t}\right)^{\frac{1}{t}}\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\right]$$
$$\times\left(\frac{S^q}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left[\int_0^1(1-s(1-\mu))^{\frac{1}{u}}d\mu+\int_0^1(1-s\mu)^{\frac{1}{u}}d\mu\right]\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 4.1, a property of the $GFPC-s$ function $|\wp'|^q$, and the Hölder inequality, one has

$$\left|\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma_K(K+\alpha)}{\varkappa_2-\varkappa_1}\left[\mathfrak{I}^{\alpha,K}_{\nu^-}\wp(\varkappa_1)+\mathfrak{I}^{\alpha,K}_{\nu^+}\wp(\varkappa_2)\right]\right|$$
$$\leq \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\int_0^1\mu^{\frac{\alpha}{K}}|\wp'(\mu\nu+(1-\mu)\varkappa_1)|d\mu+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\int_0^1\mu^{\frac{\alpha}{K}}|\wp'(\mu\nu+(1-\mu)\varkappa_2)|d\mu$$
$$\leq \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\left(\int_0^1\mu^{\frac{t\alpha}{K}}d\mu\right)^{\frac{1}{t}}\left(\int_0^1|\wp'(\mu\nu+(1-\mu)\varkappa_1)|^q d\mu\right)^{\frac{1}{q}}$$
$$+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\left(\int_0^1\mu^{\frac{t\alpha}{K}}d\mu\right)^{\frac{1}{t}}\left(\int_0^1|\wp'(\mu\nu+(1-\mu)\varkappa_2)|^q d\mu\right)^{\frac{1}{q}}$$
$$\leq \left(\frac{K}{K+\alpha t}\right)^{\frac{1}{t}}\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\left(\frac{\sum_{u=1}^n a_u\int_0^1(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp'(\nu)|^q d\mu\right.\right.$$
$$\left.+\frac{\sum_{u=1}^n a_u\int_0^1(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp'(\varkappa_1)|^q d\mu\right)^{\frac{1}{q}}$$
$$\left.+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\left(\frac{\sum_{u=1}^n \alpha_u\int_0^1(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp'(\nu)|^q d\mu+\frac{\sum_{u=1}^n a_u\int_0^1(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp'(\varkappa_2)|^q d\mu\right)^{\frac{1}{q}}\right]$$
$$\leq \left(\frac{K}{K+\alpha t}\right)^{\frac{1}{t}}\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\right]$$
$$\times\left(\frac{S^q}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left[\int_0^1(1-s(1-\mu))^{\frac{1}{u}}d\mu+\int_0^1(1-s\mu)^{\frac{1}{u}}d\mu\right]\right)^{\frac{1}{q}}.$$

□

Corollary 4.10. *If one takes $s = 1$ in Theorem 4.9, one gets the following inequality for GFPC functions with K-fractional integral operators:*

$$\begin{aligned}&\left|\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma_K(K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im^{\alpha,K}_{\nu^-}\wp(\varkappa_1)+\Im^{\alpha,K}_{\nu^+}\wp(\varkappa_2)\right]\right|\\ \leq\;&\left(\frac{K}{K+\alpha t}\right)^{\frac{1}{t}}\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+1}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+1}}{\varkappa_2-\varkappa_1}\right]\left(\frac{S^q}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left(\frac{2u}{u+1}\right)\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 4.11. *If one takes $K = 1$ in Corollary 4.10, one gets the following inequality for GFPC functions with RL-integral operators:*

$$\begin{aligned}&\left|\frac{(\nu-\varkappa_1)^{\alpha}+(\varkappa_2-\nu)^{\alpha}}{\varkappa_2-\varkappa_1}\wp(\nu)-\frac{\Gamma(\alpha)}{\varkappa_2-\varkappa_1}\left[\Im^{\alpha}_{\nu^-}\wp(\varkappa_1)+\Im^{\alpha}_{\nu^+}\wp(\varkappa_2)\right]\right|\\ \leq\;&\left(\frac{1}{\alpha t+1}\right)^{\frac{1}{t}}\left[\frac{(\nu-\varkappa_1)^{\alpha+1}+(\varkappa_2-\nu)^{\alpha+1}}{\varkappa_2-\varkappa_1}\right]\left(\frac{S^q}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\left(\frac{2u}{u+1}\right)\right)^{\frac{1}{q}}.\end{aligned}$$

Now, we establish new Ostrowski type inequalities for twice differentiable functions. First, we give the following lemma, which will be used in what follows [36].

Lemma 4.12. *Let $\alpha, K > 0$, $\varkappa_1 < \varkappa_2$, $\breve{A} = [\varkappa_1, \varkappa_2]$, and $\wp : \breve{A} \to \mathbb{R}$ be a twice differentiable function on $\breve{A}^\circ$ such that $\wp'' \in L[\varkappa_1, \varkappa_2]$. Then,*

$$\begin{aligned}&(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+\left(1+\frac{\alpha}{K}-\kappa\right)\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp(\nu)\\ &+\kappa\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}\wp(\varkappa_2)+(\varkappa_2-\nu)^{\frac{\alpha}{K}}\wp(\varkappa_1)}{\varkappa_2-\varkappa_1}\right]-\frac{\Gamma_K(2K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im^{\alpha,K}_{\nu^-}\wp(\varkappa_1)+\Im^{\alpha,K}_{\nu^+}\wp(\varkappa_2)\right]\\ =\;&\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\wp''(\mu\nu+(1-\mu)\varkappa_1)\,d\mu\\ &+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\wp''(\mu\nu+(1-\mu)\varkappa_2)\,d\mu,\end{aligned}$$

holds for all $\nu \in [\varkappa_1, \varkappa_2]$ and $\kappa \in [0, 1]$.

Theorem 4.13. *Let $n \in \mathbb{N}$, $a_u \geq 0$ $\left(u = \overline{1, n}\right)$, $s \in [0, 1]$, $\alpha, K > 0$, $\varkappa_1 < \varkappa_2$, and $\wp : \breve{A} = [\varkappa_1, \varkappa_2] \to \mathbb{R}$ be a twice differentiable function on $\breve{A}^\circ$ such that $\wp'' \in L[\varkappa_1, \varkappa_2]$. Let $|\wp''(\nu)|$ be a $GFPC - s$ function on $\breve{A}$. Then,*

$$\begin{aligned}&\left|(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+\left(1+\frac{\alpha}{K}-\kappa\right)\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp(\nu)\right.\\ &\left.+\kappa\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}\wp(\varkappa_1)+(\varkappa_2-\nu)^{\frac{\alpha}{K}}\wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right]-\frac{\Gamma_K(2K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im^{\alpha,K}_{\nu^-}\wp(\varkappa_1)+\Im^{\alpha,K}_{\nu^+}\wp(\varkappa_2)\right]\right|\\ \leq\;&\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}|\wp''(\varkappa_1)|+(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}|\wp''(\varkappa_2)|}{\varkappa_2-\varkappa_1}\right]\frac{1}{\sum_{u=1}^n a_u}\sum_{u=1}^n\alpha_u\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)(1-s\mu)^{\frac{1}{u}}\,d\mu\\ &+\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\sum_{u=1}^n a_u(\varkappa_2-\varkappa_1)}|\wp''(\nu)|\sum_{u=1}^n a_u\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)(1-s(1-\mu))^{\frac{1}{u}}\,d\mu.\end{aligned}$$

Proof. Using Lemma 4.12, a property of the $GFPC - s$ function $|\wp''|$, and the property of modulus, one has

$$\begin{aligned}
&\Bigg|(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+\left(1+\frac{\alpha}{K}-\kappa\right)\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp(\nu) \\
&+\kappa\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}\wp(\varkappa_1)+(\varkappa_2-\nu)^{\frac{\alpha}{K}}\wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right]-\frac{\Gamma_K(2K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im^{\alpha,K}_{\nu^-}\wp(\varkappa_1)+\Im^{\alpha,K}_{\nu^+}\wp(\varkappa_2)\right]\Bigg| \\
\leq\ & \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\wp''\left|(\mu\nu+(1-\mu)\varkappa_1)\right|d\mu \\
&+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\wp''\left|(\mu\nu+(1-\mu)\varkappa_2)\right|d\mu \\
\leq\ & \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left[\frac{\sum_{u=1}^n a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\varkappa_1)\right|+\frac{\sum_{u=1}^n a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\nu)\right|\right]d\mu \\
&+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left[\frac{\sum_{u=1}^n a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\varkappa_2)\right|+\frac{\sum_{u=1}^n a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\nu)\right|\right]d\mu \\
=\ & \left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}\left|\wp''(\varkappa_1)\right|+(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}\left|\wp''(\varkappa_2)\right|}{\varkappa_2-\varkappa_1}\right]\frac{1}{\sum_{u=1}^n a_u}\sum_{u=1}^n a_u\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)(1-s\mu)^{\frac{1}{u}}d\mu \\
&+\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\sum_{u=1}^n a_u(\varkappa_2-\varkappa_1)}\left|\wp''(\nu)\right|\sum_{u=1}^n a_u\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)(1-s(1-\mu))^{\frac{1}{u}}d\mu.
\end{aligned}$$

So, the proof is completed. □

Corollary 4.14. *If one takes $s = 1$ in Theorem 4.13, one gets the following inequality for GFPC functions with K-fractional integral operators:*

$$\begin{aligned}
&\Bigg|(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+\left(1+\frac{\alpha}{K}-\kappa\right)\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp(\nu) \\
&+\kappa\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}\wp(\varkappa_1)+(\varkappa_2-\nu)^{\frac{\alpha}{K}}\wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right]-\frac{\Gamma_K(2K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im^{\alpha,K}_{\nu^-}\wp(\varkappa_1)+\Im^{\alpha,K}_{\nu^+}\wp(\varkappa_2)\right]\Bigg| \\
\leq\ & \left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}\left|\wp''(\varkappa_1)\right|+(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}\left|\wp''(\varkappa_2)\right|}{\varkappa_2-\varkappa_1}\right]\frac{1}{\sum_{u=1}^n a_u}\sum_{u=1}^n \alpha_u\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)(1-\mu)^{\frac{1}{u}}d\mu \\
&+\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}+(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\sum_{u=1}^n a_u(\varkappa_2-\varkappa_1)}\left|\wp''(\nu)\right|\sum_{u=1}^n a_u\int_0^1\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\mu^{1+\frac{1}{u}}d\mu.
\end{aligned}$$

Corollary 4.15. *If one takes $K = 1$ in Corollary 4.14, one gets the following inequality for GFPC functions with RL-integral operators:*

$$\begin{aligned}
&\Bigg|(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\alpha}-(\nu-\varkappa_1)^{\alpha}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+(1+\alpha-\kappa)\left[\frac{(\nu-\varkappa_1)^{\alpha}+(\varkappa_2-\nu)^{\alpha}}{\varkappa_2-\varkappa_1}\right]\wp(\nu) \\
&+\kappa\left[\frac{(\nu-\varkappa_1)^{\alpha}\wp(\varkappa_1)+(\varkappa_2-\nu)^{\alpha}\wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right]-\frac{\Gamma(2+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im^{\alpha}_{\nu^-}\wp(\varkappa_1)+\Im^{\alpha}_{\nu^+}\wp(\varkappa_2)\right]\Bigg|
\end{aligned}$$

$$\leq \left[\frac{(\nu-\varkappa_1)^{\alpha+2}|\wp''(\varkappa_1)|+(\varkappa_2-\nu)^{\alpha+2}|\wp''(\varkappa_2)|}{\varkappa_2-\varkappa_1}\right]\frac{1}{\sum_{u=1}^{n}a_u}\sum_{u=1}^{n}a_u\int_0^1\mu(\kappa-\mu^{\alpha})(1-\mu)^{\frac{1}{u}}d\mu$$
$$+\frac{(\nu-\varkappa_1)^{\alpha+2}+(\varkappa_2-\nu)^{\alpha+2}}{\sum_{u=1}^{n}a_u(\varkappa_2-\varkappa_1)}|\wp''(\nu)|\sum_{u=1}^{n}a_u\int_0^1(\kappa-\mu^{\alpha})\mu^{1+\frac{1}{u}}d\mu.$$

Theorem 4.16. *Let $n\in\mathbb{N}$, $a_u\geq 0\left(u=\overline{1,n}\right)$, $s\in[0,1]$, $\alpha,K>0$, $\varkappa_1<\varkappa_2$, $q>1$, and $\wp:\breve{A}=[\varkappa_1,\varkappa_2]\to\mathbb{R}$ be a twice differentiable function on $\breve{A}^\circ$ such that $\wp''\in L[\varkappa_1,\varkappa_2]$. Let $|\wp''(\nu)|^q$ be a $GFPC-s$ function on $\breve{A}$. Then for every $\nu\in[\varkappa_1,\varkappa_2]$, one has*

$$(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+\left(1+\frac{\alpha}{K}-\kappa\right)\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp(\nu)$$
$$+\kappa\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}\wp(\varkappa_1)+(\varkappa_2-\nu)^{\frac{\alpha}{K}}\wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right]-\frac{\Gamma_K(2K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im_{\nu^-}^{\alpha,K}\wp(\varkappa_1)+\Im_{\nu^+}^{\alpha,K}\wp(\varkappa_2)\right]$$
$$\leq M^{1-\frac{1}{q}}(\alpha,K,\kappa)$$
$$\times\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left[\frac{\sum_{u=1}^{n}a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}|\wp''(\varkappa_1)|^q\right.\right.\right.$$
$$\left.\left.+\frac{\sum_{u=1}^{n}a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}|\wp''(\nu)|^q\right]d\mu\right)^{\frac{1}{q}}$$
$$+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left[\frac{\sum_{u=1}^{n}a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}|\wp''(\varkappa_2)|^q\right.\right.$$
$$\left.\left.\left.+\frac{\sum_{u=1}^{n}a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}|\wp''(\nu)|^q\right]d\mu\right)^{\frac{1}{q}}\right],$$

where

$$M(\alpha,K,\kappa) = \int_0^1\left[\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\right]^q d\mu$$
$$= \frac{K\kappa^{\frac{K(1+q)+\alpha q}{\alpha}}}{\alpha}\left[\Gamma(1+q)\Gamma\left(\frac{R(1+q)+\alpha}{\alpha}\right)_2F_1\left(1,1+q,2+q+\frac{K(1+q)}{\alpha},1\right)\right)$$
$$+\beta\left(1+q,-\frac{R(1+q)+\alpha q}{\alpha}\right)-\beta\left(\kappa,1+q,-\frac{K(1+q)+\alpha q}{\alpha}\right).$$

Proof. Using Lemma 4.12, a property of the $GFPC-s$ function $|\wp''|^q$, and the power mean inequality, one has

$$\left|(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+\left(1+\frac{\alpha}{K}-\kappa\right)\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}}+(\varkappa_2-\nu)^{\frac{\alpha}{K}}-}{\varkappa_2-\varkappa_1}\right]\wp(\nu)\right.$$
$$\left.+\kappa\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}\wp(\varkappa_1)-(\nu-\varkappa_1)^{\frac{\alpha}{K}}\wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right]-\frac{\Gamma_K(2K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im_{\nu^-}^{\alpha,K}\wp(\varkappa_1)+\Im_{\nu^+}^{\alpha,K}\wp(\varkappa_2)\right]\right|$$
$$\leq \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\int_0^1\left|\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\right||\wp''(\mu\nu+(1-\mu)\varkappa_1)|d\mu$$

$$
\begin{aligned}
&+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\int_0^1 \left|\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\right| \left|\wp''\left(\mu\nu+(1-\mu)\varkappa_2\right)\right| d\mu \\
\leq\ & \left(\int_0^1 \mu^q\left(\kappa-\mu^{\frac{\alpha}{K}}\right)^q d\mu\right)^{\frac{1}{q}} \\
&\times\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1 \mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left[\frac{\sum_{u=1}^n a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\varkappa_1)\right|^q\right.\right.\right. \\
&+\left.\left.\frac{\sum_{u=1}^n a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}} \\
&+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1 \mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left[\frac{\sum_{u=1}^n a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\varkappa_2)\right|^q\right.\right. \\
&+\left.\left.\left.\frac{\sum_{u=1}^n a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right] \\
=\ & M^{1-\frac{1}{q}}(\alpha,K,\kappa) \\
&\times\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1 \mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left[\frac{\sum_{u=1}^n a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\varkappa_1)\right|^q\right.\right.\right. \\
&+\left.\left.\frac{\sum_{u=1}^n a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}} \\
&+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1 \mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left[\frac{\sum_{u=1}^n a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\varkappa_2)\right|^q\right.\right. \\
&+\left.\left.\left.\frac{\sum_{u=1}^n a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

□

Corollary 4.17. *If one takes $s=1$ in Theorem 4.16, one gets the following inequality for generalized n-fractional polynomial convex functions with K-fractional integral operators:*

$$
\begin{aligned}
&(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+\left(1+\frac{\alpha}{K}-\kappa\right)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp(\nu) \\
&+\kappa\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}\wp(\varkappa_1)-(\nu-\varkappa_1)^{\frac{\alpha}{K}}\wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right]+\frac{\Gamma_K(2K+\alpha)}{\varkappa_2-\varkappa_1}\left[\mathfrak{I}_{\nu^-}^{\alpha,K}\wp(\varkappa_1)+\mathfrak{I}_{\nu^+}^{\alpha,K}\wp(\varkappa_2)\right] \\
\leq\ & M^{1-\frac{1}{q}}(\alpha,K,\kappa) \\
&\times\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1 \mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left[\frac{\sum_{u=1}^n a_u(1-\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\varkappa_1)\right|^q+\frac{\sum_{u=1}^n a_u^{\frac{1}{u}}\mu^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right. \\
&+\left.\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1 \mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left[\frac{\sum_{u=1}^n a_u(1-\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\varkappa_2)\right|^q+\frac{\sum_{u=1}^n a_u\mu^{\frac{1}{u}}}{\sum_{u=1}^n a_u}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

for all $\nu \in [\varkappa_1, \varkappa_2]$, *where*

$$\begin{aligned} M(\alpha, K, \kappa) &= \int_0^1 \left[\mu\left(\kappa - \mu^{\frac{\alpha}{K}}\right)\right]^q d\mu \\ &= \frac{K\kappa^{\frac{K(1+q)+\alpha q}{\alpha}}}{\alpha}\left[\Gamma(1+q)\Gamma\left(\frac{R(1+q)+\alpha}{\alpha}\right)_2 F_1\left(1, 1+q, 2+q+\frac{R(1+q)}{\alpha}, 1\right)\right. \\ &\quad \left. +\beta\left(1+q, -\frac{R(1+q)+\alpha}{\alpha q}\right) - \beta\left(\kappa, 1+q, -\frac{R(1+q)+\alpha}{\alpha q}\right)\right]. \end{aligned}$$

Corollary 4.18. *If one takes* $K = 1$ *in Corollary 4.17, one gets the following inequality for GFPC functions with RL-integral operators:*

$$\begin{aligned} &(1-\kappa)\left[\frac{(\varkappa_2-\nu)^\alpha - (\nu-\varkappa_1)^\alpha}{\varkappa_2-\varkappa_1}\right]\wp'(\nu) + (1+\alpha-\kappa)\left[\frac{(\varkappa_2-\nu)^\alpha - (\nu-\varkappa_1)^\alpha}{\varkappa_2-\varkappa_1}\right]\wp(\nu) \\ &+\kappa\left[\frac{(\varkappa_2-\nu)^\alpha \wp(\varkappa_1) - (\nu-\varkappa_1)^\alpha \wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right] + \frac{\Gamma(2+\alpha)}{\varkappa_2-\varkappa_1}\left[\mathfrak{I}^\alpha_{\nu^-}\wp(\varkappa_1) + \mathfrak{I}^\alpha_{\nu^+}\wp(\varkappa_2)\right] \\ \leq\ & M^{1-\frac{1}{q}}(\alpha, \kappa) \\ &\times\left[\frac{(\nu-\varkappa_1)^{\alpha+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1 \mu(\kappa-\mu^\alpha)\left[\frac{\sum_{u=1}^n a_u(1-\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp''(\varkappa_1)|^q + \frac{\sum_{u=1}^n a_u^{\frac{1}{u}}\mu^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp''(\nu)|^q\right]d\mu\right)^{\frac{1}{q}}\right. \\ &\left.+\frac{(\varkappa_2-\nu)^{\alpha+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1 \mu(\kappa-\mu^\alpha)\left[\frac{\sum_{u=1}^n a_u(1-\mu)^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp''(\varkappa_2)|^q + \frac{\sum_{u=1}^n a_u\mu^{\frac{1}{u}}}{\sum_{u=1}^n a_u}|\wp''(\nu)|^q\right]d\mu\right)^{\frac{1}{q}}\right], \end{aligned}$$

for all $\nu \in [\varkappa_1, \varkappa_2]$, *where*

$$\begin{aligned} M(\alpha, \kappa) &= \int_0^1 \left[\mu(\kappa-\mu^\alpha)\right]^q d\mu \\ &= \frac{\kappa^{\frac{(1+q)+\alpha q}{\alpha}}}{\alpha}\left[\Gamma(1+q)\Gamma\left(\frac{1+q+\alpha}{\alpha}\right)_2 F_1\left(1, 1+q, 2+q+\frac{1+q}{\alpha}, 1\right)\right. \\ &\quad \left. +\beta\left(1+q, -\frac{1+q+\alpha}{\alpha q}\right) - \beta\left(\kappa, 1+q, -\frac{1+q+\alpha}{\alpha q}\right)\right]. \end{aligned}$$

Theorem 4.19. *Let* $n \in \mathbb{N}$, $a_u \geq 0 \left(u = \overline{1, n}\right)$, $s \in [0, 1]$, $\alpha, K > 0$, $\varkappa_1 < \varkappa_2$, $t, q > 1$ *with* $\frac{1}{t} + \frac{1}{q} = 1$, *and* $\wp : \breve{A} = [\varkappa_1, \varkappa_2] \to \mathbb{R}$ *be a twice differentiable function on* $\breve{A}^\circ$ *such that* $\wp'' \in L[\varkappa_1, \varkappa_2]$. *Let* $|\wp''(\nu)|^q$ *be a* $GFPC - s$ *function on* $\breve{A}$. *Then*

$$\begin{aligned} &(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}} - (\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu) + \left(1+\frac{\alpha}{K}-\kappa\right)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}} - (\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp(\nu) \\ &+\kappa\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}} \wp(\varkappa_1) - (\nu-\varkappa_1)^{\frac{\alpha}{K}} \wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right] + \frac{\Gamma_K(2K+\alpha)}{\varkappa_2-\varkappa_1}\left[\mathfrak{I}^{\alpha,K}_{\nu^-}\wp(\varkappa_1) + \mathfrak{I}^{\alpha,K}_{\nu^+}\wp(\varkappa_2)\right] \\ \leq\ & M^{\frac{1}{t}}(\alpha, K, \kappa) \end{aligned}$$

$$\times\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\frac{1}{\sum_{u=1}^{n}a_u}\sum_{u=1}^{n}a_u\int_0^1\left[(1-s\mu)^{\frac{1}{u}}\left|\wp''(\varkappa_1)\right|^q+(1-s(1-\mu))^{\frac{1}{u}}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\frac{1}{\sum_{u=1}^{n}a_u}\sum_{u=1}^{n}a_u\int_0^1\left[(1-s\mu)^{\frac{1}{u}}\left|\wp''(\varkappa_2)\right|^q+(1-s(1-\mu))^{\frac{1}{u}}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right].$$

Proof. Using Lemma 4.12, a property of the $GFPC-s$ function $|\wp''|^q$, and the Hölder inequality, one has

$$\left|(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+\left(1+\frac{\alpha}{K}-\kappa\right)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp(\nu)\right.$$
$$\left.+\kappa\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}\wp(\varkappa_1)-(\nu-\varkappa_1)^{\frac{\alpha}{K}}\wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right]-\frac{\Gamma_K(2K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im^{\alpha,K}_{\nu^-}\wp(\varkappa_1)+\Im^{\alpha,K}_{\nu^+}\wp(\varkappa_2)\right]\right|$$
$$\leq \frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left|\wp''(\mu\nu+(1-\mu)\varkappa_1)\right|d\mu$$
$$+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\int_0^1\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\left|\wp''(\mu\nu+(1-\mu)\varkappa_2)\right|d\mu$$
$$\leq \left(\int_0^1\left|\mu\left(\kappa-\mu^{\frac{\alpha}{K}}\right)\right|^t d\mu\right)^{\frac{1}{t}}$$
$$\times\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1\left[\frac{\sum_{u=1}^{n}a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}\left|\wp''(\varkappa_1)\right|^q+\frac{\sum_{u=1}^{n}a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\int_0^1\left[\frac{\sum_{u=1}^{n}a_u(1-s\mu)^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}\left|\wp''(\varkappa_2)\right|^q+\frac{\sum_{u=1}^{n}a_u(1-s(1-\mu))^{\frac{1}{u}}}{\sum_{u=1}^{n}a_u}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right]$$
$$= M^{\frac{1}{t}}(\alpha,K,\kappa)$$
$$\times\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\frac{1}{\sum_{u=1}^{n}a_u}\sum_{u=1}^{n}a_u\int_0^1\left[(1-s\mu)^{\frac{1}{u}}\left|\wp''(\varkappa_1)\right|^q+(1-s(1-\mu))^{\frac{1}{u}}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\frac{1}{\sum_{u=1}^{n}a_u}\sum_{u=1}^{n}a_u\int_0^1\left[(1-s\mu)^{\frac{1}{u}}\left|\wp''(\varkappa_2)\right|^q+(1-s(1-\mu))^{\frac{1}{u}}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right].$$

□

Corollary 4.20. *If one takes $s=1$ in Theorem 4.19, one gets the following inequality for GFPC functions with K-fractional integral operators:*

$$\left|(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+\left(1+\frac{\alpha}{K}-\kappa\right)\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}-(\nu-\varkappa_1)^{\frac{\alpha}{K}}}{\varkappa_2-\varkappa_1}\right]\wp(\nu)\right.$$
$$\left.+\kappa\left[\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}}\wp(\varkappa_1)-(\nu-\varkappa_1)^{\frac{\alpha}{K}}\wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right]-\frac{\Gamma_K(2K+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im^{\alpha,K}_{\nu^-}\wp(\varkappa_1)+\Im^{\alpha,K}_{\nu^+}\wp(\varkappa_2)\right]\right|$$
$$\leq M^{\frac{1}{t}}(\alpha,R,\kappa)$$

$$\times\left[\frac{(\nu-\varkappa_1)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\frac{1}{\sum_{u=1}^{n}a_u}\sum_{u=1}^{n}a_u\int_0^1\left[(1-\mu)^{\frac{1}{u}}\left|\wp''(\varkappa_1)\right|^q+\mu^{\frac{1}{u}}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\varkappa_2-\nu)^{\frac{\alpha}{K}+2}}{\varkappa_2-\varkappa_1}\left(\frac{1}{\sum_{u=1}^{n}a_u}\sum_{u=1}^{n}a_u\int_0^1\left[(1-\mu)^{\frac{1}{u}}\left|\wp''(\varkappa_2)\right|^q+\mu^{\frac{1}{u}}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right].$$

Corollary 4.21. *If one takes $K = 1$ in Corollary 4.20, one gets the following inequality for GFPC functions with RL-integral operators:*

$$\left|(1-\kappa)\left[\frac{(\varkappa_2-\nu)^{\alpha}-(\nu-\varkappa_1)^{\alpha}}{\varkappa_2-\varkappa_1}\right]\wp'(\nu)+(1+\alpha-\kappa)\left[\frac{(\varkappa_2-\nu)^{\alpha}-(\nu-\varkappa_1)^{\alpha}}{\varkappa_2-\varkappa_1}\right]\wp(\nu)\right.$$
$$\left.+\kappa\left[\frac{(\varkappa_2-\nu)^{\alpha}\wp(\varkappa_1)-(\nu-\varkappa_1)^{\alpha}\wp(\varkappa_2)}{\varkappa_2-\varkappa_1}\right]-\frac{\Gamma(2+\alpha)}{\varkappa_2-\varkappa_1}\left[\Im_{\nu^-}^{\alpha}\wp(\varkappa_1)+\Im_{\nu^+}^{\alpha}\wp(\varkappa_2)\right]\right|$$
$$\leq M^{\frac{1}{t}}(\alpha,\kappa)$$
$$\times\left[\frac{(\nu-\varkappa_1)^{\alpha+2}}{\varkappa_2-\varkappa_1}\left(\frac{1}{\sum_{u=1}^{n}a_u}\sum_{u=1}^{n}a_u\int_0^1\left[(1-\mu)^{\frac{1}{u}}\left|\wp''(\varkappa_1)\right|^q+\mu^{\frac{1}{u}}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{(\varkappa_2-\nu)^{\alpha+2}}{\varkappa_2-\varkappa_1}\left(\frac{1}{\sum_{u=1}^{n}a_u}\sum_{u=1}^{n}a_u\int_0^1\left[(1-\mu)^{\frac{1}{u}}\left|\wp''(\varkappa_2)\right|^q+\mu^{\frac{1}{u}}\left|\wp''(\nu)\right|^q\right]d\mu\right)^{\frac{1}{q}}\right].$$

5. Conclusions

We introduced the concept of n-fractional polynomial s-type convex functions and investigated their related properties. Algebraic relationships between such functions and other kinds of convex functions were explored. Several novel variants of the well known H-H and Ostrowski-type inequalities were established using the newly defined class of functions and K-fractional integral operators. Considering the introduced class and employing fractional operators, we have derived new refinements of the Ostrowski-type inequalities. Several special cases of our results were discussed. For some special cases, the definition and results of generalized n-fractional polynomial s-type convex functions reduce to a novel definition and new results for the class of convex functions, called generalized n-fractional polynomial convex functions. The results obtained from the future plan are even more exhilarating compared to the results available in the literature.

Author contributions

Serap Özcan: Conceptualization, Formal Analysis, Investigation, Writing-Original Draft, Writing-Review & Editing, Supervision; Saad Ihsan Butt: Conceptualization, Formal Analysis, Investigation, Methodology, Writing-Original Draft, Writing-Review & Editing, Supervision; Sanja Tipurić-Spužević: Conceptualization, Formal Analysis, Methodology, Software, Writing-Review & Editing, Funding Acquisition; Bandar Bin Mohsin: Conceptualization, Formal Analysis, Software, Writing-Review & Editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is supported by Researcher Supporting Project number (RSP2024R158), King Saud University, Riyadh, Saudi Arabia.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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