AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 23909–23923.
DOI: 10.3934/math.20241162
Received: 12 June 2024
Revised: 24 July 2024
Accepted: 31 July 2024
Published: 12 August 2024

Research article

Generalized conditional spacings and their stochastic properties

Tie Li[1,2] **and Zhengcheng Zhang**[1,*]

[1] School of Mathematics and Statistics, Hainan Normal University, 99 South Longkun Road, Haikou, 571158, China

[2] Faculty of Mathematics, Baotou Teachers' College, 3 Science Road, Baotou, 014030, China

* **Correspondence:** Email: zhzhcheng004@163.com.

Abstract: This paper introduces the concept of generalized conditional spacings and establishes partial order relations between different generalized spacings. First, we derive the survival function of the generalized conditional spacings. Second, we construct the stochastic and hazard rate order relationships between different generalized conditional spacings and generalized normal conditional spacings, considering parent distributions that belong to the decreasing failure rate (DFR) and increasing likelihood rate (ILR) classes. Finally, for parent distributions within the DFR class, we obtain the dispersive order between different conditional spacings, along with an inequality for the variance. Additionally, we present illustrative examples involving Pareto and Gamma distributions.

Keywords: generalized conditional spacings; k-out-of-n systems; DFR; ILR; stochastic order; hazard rate order
Mathematics Subject Classification: 60E15, 62G30

Acronyms	Nomenclature
IID	independent and identically distributed
IFR	increasing failure rate
DFR	decreasing failure rate
ILR	increasing likelihood rate
DLR	decreasing likelihood rate

1. Introduction

Spacings play a crucial role in both reliability theory and statistics. They underpin nearly all well-known measures of dispersion, including sample variance, sample range, and Gini's mean difference, which are functions of sample spacings. The term "interval analysis", now known as spacings, was first introduced by Sukhatme in his seminal paper [1]. Since then, the study of spacings has captivated statisticians, particularly following Greenwood's influential 1946 paper presented to the Royal Statistical Society, where he proposed that "it is at least worth considering whether by a study of the distribution of intervals the statistician can give the epidemiologist any help". For more comprehensive results on the early stages of spacings, interested readers are referred to [2–9]. These references provide detailed studies on the construction of spacings, their applications, the limiting distributions of spacing functions, and stochastic comparisons of various spacings. For a class of goodness-of-fit tests based symmetrically on spacings, Rao et al. [10] established that a goodness-of-fit test based on m-spacings ($D^m_{i:n} \equiv W_{i+m-1:n} - W_{i-1:n}$, $m \geq 2$) is always asymptotically superior to its analogue based on simple spacings ($m = 1$). Hence, the stochastic properties as well as the log-convexity (concavity) of m-spacings and generalized spacings ($D_{i,j:n} \equiv W_{j:n} - W_{i:n}$, $0 \leq i < j \leq n$) were investigated by Misra, Hu, and Alimohammadi et al., see [11–14] for more. Recently, Zhang and Balakrishnan et al. [15–17] proposed the concept of conditional spacings based on the residual lifetimes of surviving components in a failed k-out-of-n system, and the stochastic properties of the conditional spacings of independently heteroexponentially distributed series-failure systems and independent and identically distributed (IID) failure-coherent systems are also explored in detail. To study the lifetime behavior of the k-out-of-n systems before failure, many scholars [18, 19] have investigated the conditional random variable $RLS_{k,n,t} \equiv (W_{n-k+1:n} - t \mid W_{n-k:n} = t)$. That is, for the given condition that there are $n - k$ failures at time t, the residual life of the k-out-of-n system. However, in practice, the exact time at which a component fails in a system is difficult to observe, and only the number of components failing in the system at time t is easy to observe. Therefore, Bairamov et al. in [20] were the first to study the reliability and stochastic properties of the conditional order statistic $(W_{k:n} - t \mid W_{l:n} > t)$. Since then, many researchers have used conditional order statistics as a tool to investigate the residual life and conditional distribution of systems, enhancing the understanding of system reliability and lifetime distribution. For detailed discussions, see [21–23]. The primary motivation of this article is to address the inadequacy of existing spacing theories for studying used (but not necessarily failed) k-out-of-n systems due to changes in the lifetime distribution of components compared to new systems. To broaden the applicability of spacing theories, we introduce the condition $S_n(t) = s$, which links the spacings formed by the remaining lifetimes of surviving components in used k-out-of-n systems (termed as conditional spacings $U^t_{i:n|s}$) to the system's operating time t and the number of failed components s. This adjustment broadens the applicability of the stochastic properties of spacing theories to used k-out-of-n systems. Specifically, generalized conditional spacings facilitate the analysis of the remaining lifetimes of non-failed components over a given period, which is essential for predicting system life and developing maintenance plans. Additionally, generalized conditional spacings are significant in statistical inference, including Bayesian inference and empirical distribution estimation. They provide additional statistical information, enhancing the precision of estimation and hypothesis testing [24].

Given a k-out-of-n system consisting of n IID components, let W_i represent the lifetime of the ith component for $i \in \{1, 2, \cdots, n\}$, and let $W_{i:n}$ denote the ith smallest order statistic of $W_1, W_2, \cdots W_n$. To preserve the state of the data at time t for the used k-out-of-n system, we define the statistic $S_n(t)$, which represents the number of observations in the sample $\{W_1, W_2, \cdots W_n\}$ that do not exceed the running time t of the k-out-of-n system. Clearly, $S_n(t) \leq n - k + 1$. For $s + 1 \leq i < j \leq n$, let

$$U^t_{i,j:n|s} \equiv \left(W_{j:n} - W_{i:n} \mid S_n(t) = s\right),\ U^{*,t}_{i,j:n|s} \equiv (n-i)\left(W_{j:n} - W_{i:n} \mid S_n(t) = s\right), \tag{1.1}$$

be the corresponding generalized conditional spacings and generalized normal conditional spacings based on the residual life of $n - s$ components in the used k-out-of-n system that are still alive at time t. In particular, when $j = i + 1$, we use the shorthand $U^t_{i,j:n|s}$ and $U^{*,t}_{i,j:n|s}$ for $U^t_{i:n|s}$ and $U^{*,t}_{i:n|s}$, respectively, i.e.,

$$U^t_{i:n|s} \equiv (W_{i+1:n} - W_{i:n} \mid S_n(t) = s), \quad U^{*,t}_{i:n|s} \equiv (n - i)(W_{i+1:n} - W_{i:n} \mid S_n(t) = s), \tag{1.2}$$

to represent the corresponding conditional spacings and normalized conditional spacings in the used k-out-of-n system. If we assume that n components are placed on test at time 0, then $U^t_{i:n|s}$ may be regarded as the conditional differences between consecutive observations, while $U^{*,t}_{i:n|s}$ represents the total test time observed between $W_{i+1:n}$ and $W_{i:n}$ under the condition that s components have already failed.

This paper focuses on the stochastic properties of generalized conditional spacings and generalized normal conditional spacings. The rest of this paper is organized as follows. In Section 2, we review some definitions as well as lemmas. In Section 3, we obtain some stochastic order results for generalized conditional spacings and generalized normal conditional spacings.

2. Definitions and preliminaries

In the following definitions, let the cumulative distribution functions of the random variables R and L be $F_R(x)$ and $F_L(x)$, the probability density functions be $f_R(x)$ and $f_L(x)$, and denote their corresponding survival functions by $\bar{F}_R(x) = 1 - F_R(x)$ and $\bar{F}_L(x) = 1 - F_L(x)$.

Definition 2.1. *The random variable R is said to be smaller than L in the following ways:*

(a) stochastic order (denoted by $R \le_{st} L$) if $\bar{F}_R(x) \le \bar{F}_L(x)$ for all x;

(b) hazard(failure) rate order (denoted by $R \le_{hr} L$) if $\bar{F}_R(x)/\bar{F}_L(x)$ is decreasing *in x;*

(c) likelihood ratio order (denoted by $R \le_{lr} L$) if $f_L(x)/f_R(x)$ is increasing *in the union of their* supports*;*

(d) up shifted likelihood ratio order (denoted by $R \le_{lr\uparrow} L$), if $f_L(x)/f_R(x + t)$ is increasing *in $x \ge 0$ for all $t \ge 0$;*

(e) down shifted likelihood ratio order (denoted by $R \le_{lr\downarrow} L$), if $f_L(x + t)/f_R(x)$ is increasing *in $x \ge 0$ for all $t \ge 0$;*

(f) dispersive order (denoted by $R \le_{disp} L$), if $F_R^{-1}(\beta) - F_R^{-1}(\alpha) \le F_L^{-1}(\beta) - F_L^{-1}(\alpha)$, whenever $0 < \alpha \le \beta < 1$, where $F_R^{-1}(\cdot)$ and $F_L^{-1}(\cdot)$ denote the right-continuous inverse functions of $F_R(\cdot)$ and $F_L(\cdot)$.

It is well known that the relationship between these orderings is as shown below, see [25, 26] for details.

$$R \le_{lr} L \Longrightarrow R \le_{hr} L \Longrightarrow R \le_{st} L.$$

For convenience, we sometimes write $F_R(\cdot) \le_* F_L(\cdot)$ or $\bar{F}_R(\cdot) \le_* \bar{F}_L(\cdot)$ instead of $R \le_* L$, where $\le_*$ is one of the above stochastic orders.

Definition 2.2. *The random variable R(or $F_R(\cdot)$) is said to be:*

(a) IFR (increasing failure rate) if $\bar{F}_R(\cdot)$ is logconcave;

(b) DFR (decreasing failure rate) if $\bar{F}_R(\cdot)$ is logconvex;

(c) ILR (increasing likelihood rate) if $f_R(\cdot)$ is logconcave;

(d) DLR (decreasing likelihood rate) if $f_R(\cdot)$ is logconvex.

Obviously, $ILR \Rightarrow IFR$ and $DLR \Rightarrow DFR$. For more details on these concepts, see [27,28].

For the concise proof of theorems, we introduce some lemmas.

Lemma 2.1. *[26] Let $W_1, \cdots, W_n$ be IID random variables. Then, $W_{i:n} \leq_{lr} W_{p:m}$ whenever $p \geq i$ and $p - i \geq m - n$.*

Remark 2.1. *If W_1 strengthens to ILR, then the result of Lemma 2.1 can be strengthened to $W_{i:n} \leq_{lr\uparrow} W_{p:m}$. But, using DLR instead of ILR, the result $W_{i:n} \leq_{lr\downarrow} W_{p:m}$ does not necessarily hold.*

Let $B_{\alpha,\beta}(\cdot)$ denote the cumulative distribution function of a beta distribution with parameters $\alpha \geq 0$ and $\beta \geq 0$ with the probability density function

$$b_{\alpha,\beta}(\mu) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\mu^{\alpha-1}(1-\mu)^{\beta-1}, 0 \leq \mu \leq 1. \tag{2.1}$$

Then, the cumulative distribution function and survival function of the ith order statistic $W_{i:n}$ can be written as $Pr(W_{i:n} \leq x) = B_{i,n-i+1}(F(x))$ and $Pr(W_{i:n} > x) = B_{n-i+1,i}(\bar{F}(x))$, respectively.

Lemma 2.2. *[29] (a) If $R \leq_{hr} L$, then $R_{i:n} \leq_{hr} L_{p:m}$ whenever $p \geq i$ and $p - i \geq m - n$.*
(b) If $R \leq_{hr} L$, then $B_{\alpha,\beta}(\bar{F}_R(\cdot)) \leq_{hr} B_{p,q}(\bar{F}_L(\cdot))$ for all non-negative integers α, β, p, q such that $\alpha \geq p$ and $\beta \leq q$.

Remark 2.2. *By replacing all instances of $\leq_{hr}$ with $\leq_{lr\uparrow}$ in Lemma 2.2, the lemma still holds, as confirmed by [26].*

Lemma 2.3. *[11] If F is $DFR(IFR)$, then $F^U_{t_1} \leq_{hr} (\geq_{hr}) F^U_{t_2}$ for $t_1 \leq t_2$.*

Lemma 2.4. *[30] Let R and L be independent random variables. Then, $R \leq_{lr} L$ if and only if $E[g(L,R)] \geq E[g(R,L)]$ for all $g \in \mathcal{G}_{lr}$, where $\mathcal{G}_{lr} = \{g | g(x,y) \geq g(y,x) \text{ whenever } x \geq y\}$.*

Throughout this paper, we shall be assuming that all distributions under study are absolutely continuous with common support $(0,\infty)$ and use the terms *decreasing* and *increasing* to denote non-increasing and non-decreasing, respectively.

3. Main results

We start by defining some symbols that will be used in the following theorems. Let $W_1, \cdots, W_n$ be nonnegative independent random variables with the identical distribution as W, and $W_{1:n} \leq W_{2:n} \leq \cdots W_{n:n}$ denote the corresponding order statistics. Let $Z_{i:n-s}$ be the ith order statistics of independent random variables $Z_1, \cdots, Z_{n-s}$, which have the same distribution as $Z \equiv (W - t | W > t)$. Denote the distribution functions of random variables W and Z as $F(\cdot)$ and $F^U_t(\cdot)$, density functions $f(\cdot)$ and $f^U_t(\cdot)$, and survival functions $\bar{F}(\cdot)$ and $\bar{F}^U_t(\cdot)$, respectively. It is easy to check that $F^U_t(\cdot) = \frac{F(t+\cdot)-F(t)}{\bar{F}(t)}$. For other random variables, such as T, we use $F_T(\cdot)$ for the cumulative distribution function of T, $\bar{F}_T(\cdot)$ for the survival function of T, and $f_T(\cdot)$ for the probability density function of T as general notations.

Below we give the survival functions of $U^t_{i,j:n|s}$ and $U^{*,t}_{i:n|s}$ introduced by Eq (1.1).

Theorem 3.1. *For $s + 1 \leq i < j \leq n$, any $x > 0$, the survival function of $U^t_{i,j:n|s}$ is*

$$Pr(U^t_{i,j:n|s} > x) = E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-s:n-s}}(x))\right],$$

where $E(\cdot)$ is the expectation of a random variable and $B_{n-j+1,j-i}(\cdot)$ denotes a cumulative distribution function of the beta distribution, see Eq (2.1).

Proof. For $s + 1 \leq i < j \leq n$ and any $x > 0$,

$$\begin{aligned}
\bar{F}_{U^t_{i,j:n|s}}(x) &= Pr(U^t_{i,j:n|s} > x) \\
&= Pr\left(W_{j:n} - W_{i:n} > x \mid S_n(t) = s\right) \\
&= Pr\left(Z_{j-s:n-s} - Z_{i-s:n-s} > x\right) \\
&= \int_0^{+\infty} Pr\left(Z_{j-s:n-s} - Z_{i-s:n-s} > x \mid Z_{i-s:n-s} = y\right) dF_{Z_{i-s:n-s}}(y).
\end{aligned}$$

Recall that the conditional distribution of $Z_{j-s:n-s} - Z_{i-s:n-s}$ given $Z_{i-s:n-s} = y$ is the same as the unconditional distribution of the $(j-i)$th order statistic of a random sample of size $n-i$ from the distribution $F^U_{t+y}(\cdot)$.

$$\begin{aligned}
\bar{F}_{U^t_{i,j:n|s}}(x) &= \int_0^{+\infty} B_{n-j+1,j-i}(\bar{F}^U_{t+y}(x)) dF_{Z_{i-s:n-s}}(y), \ \forall x \geq 0 \\
&= E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-s:n-s}}(x))\right].
\end{aligned}$$

Remark 3.1. *From Theorem 3.1 it is easy to obtain that the survival function of generalized normal conditional spacing* $U^{*,t}_{i:n|s}$ *is* $Pr(U^{*,t}_{i:n|s} > x) = E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-s:n-s}}(\frac{x}{n-i}))\right]$.

In the following, we will perform some stochastic comparisons among generalized conditional spacings.

Theorem 3.2. *Let* $W_1, \cdots, W_n$ *be IID component lifetimes of a k-out-of-n system with* W_1 *being a DFR distribution, for* $\forall\ t_1 \leq t_2$, $p - h \geq i - s$ *and* $q - j \geq p - i \geq m - n$. *Then,*

$$U^{t_1}_{i,j:n|s} \leq_{st} U^{t_2}_{p,q:m|h}.$$

Proof. As shown in the above theorem, we can obtain that

$$\begin{aligned}
\bar{F}_{U^{t_1}_{i,j:n|s}}(x) &= \int_0^{+\infty} B_{n-j+1,j-i}(\bar{F}^U_{t_1+y}(x)) dF_{Z_{i-s:n-s}}(y), \ \forall x \geq 0, \\
\bar{F}_{U^{t_2}_{p,q:m|h}}(x) &= \int_0^{+\infty} B_{m-q+1,q-p}(\bar{F}^U_{t_2+y}(x)) dF_{Z_{p-h:m-h}}(y), \ \forall x \geq 0.
\end{aligned}$$

Since the parameters satisfy $q - j \geq p - i \geq m - n$, by Lemma 2.2(b),

$$B_{n-j+1,j-i}(\bar{F}^U_{t_2+y}(\cdot)) \leq_{hr} B_{m-q+1,q-p}(\bar{F}^U_{t_2+y}(\cdot)).$$

Since the hazard rate order implies the usual stochastic order, we have

$$\bar{F}_{U^{t_2}_{p,q:m|h}}(x) \geq \int_0^{+\infty} B_{n-j+1,j-i}(\bar{F}^U_{t_2+y}(x)) dF_{Z_{p-h:m-h}}(y), \ \forall x \geq 0.$$

By Lemmas 2.3 and 2.2(b), the function $B_{n-j+1,j-i}(\bar{F}^U_y(x))$ is increasing in y for each x. Hence,

$$\bar{F}_{U^{t_2}_{p,q:m|h}}(x) \geq \int_0^{+\infty} B_{n-j+1,j-i}(\bar{F}^U_{t_1+y}(x)) dF_{Z_{i-s:n-s}}(y) = \bar{F}_{U^{t_1}_{i,j:n|s}}(x), \ \forall x \geq 0,$$

since Lemma 2.1 implies $Z_{i-s:n-s} \leq_{lr} Z_{p-h:m-h}$ and hence $Z_{i-s:n-s} \leq_{st} Z_{p-h:m-h}$ for $\forall\ t_1 \leq t_2$, $p-h \geq i-s$, and $p-i \geq m-n$. We get the required result.

For two random variables, we typically compare their expectations and variances to measure their stochastic properties. However, sometimes the comparison based solely on expectations and variances is insufficient, and in some cases expectations and variances may not even exist. Therefore, utilizing stochastic orders allows for a more comprehensive and detailed comparison of certain stochastic properties of random variables. Roughly speaking, $U^{t_1}_{i,j:n|s} \leq_{st} U^{t_2}_{p,q:m|h}$ says that $U^{t_1}_{i,j:n|s}$ is less likely than $U^{t_2}_{p,q:m|h}$ to take on large values, where "large" means any value greater than x, and that this is the case for all x's. In risk management, stochastic orders enable the comparison of different risk profiles. They help in determining which of the two risks is greater (or smaller) in a stochastic sense, aiding in better risk assessment and mitigation strategies.

Remark 3.2. *The corner labels in Theorem 3.2 have many constraints, but the conclusion holds when the conditions take an equal sign. For ease of application, we list some special cases.*

(1) If $p \geq i$, then $U^t_{i,i+m:n|s} \leq_{st} U^t_{p,p+m:n|s}$(Conditional m-spacings).
(2) If $q \geq j$, then $U^t_{i,j:n|s} \leq_{st} U^t_{i,q:n|s}$.
(3) If $n \geq m$, then $U^t_{i,j:n|s} \leq_{st} U^t_{i,j:m|s}$.
(4) If $s \geq h$, then $U^t_{i,j:n|s} \leq_{st} U^t_{i,j:n|h}$.
(5) If $t_1 \leq t_2$, then $U^{t_1}_{i,j:n|s} \leq_{st} U^{t_2}_{i,j:n|s}$.

Example 3.1. *Suppose a k-out-of-n system with n components whose lifetimes are IID follows the first Pareto distribution, i.e., $f(x) = \frac{1}{x^2}, (x \geq 1)$. Then, W_1 is clearly DFR. From Theorem 3.1, we can obtain generalized conditional spacings and survival functions, respectively.*

$$U^t_{3,6:6|1} = (W_{6:6} - W_{3:6} \mid S_6(t) = 1)\,, \quad \bar{F}_{U^t_{3,6:6|1}}(x) = \int_0^{+\infty} \frac{20t^4 y}{(t+y)^6} \cdot [1 - \frac{x^3}{(t+x+y)^3}]dy,$$
$$U^t_{3,5:5|2} = (W_{5:5} - W_{3:5} \mid S_5(t) = 2)\,, \quad \bar{F}_{U^t_{3,5:5|2}}(x) = \int_0^{+\infty} \frac{3t^3}{(t+y)^3} \cdot \frac{t+2x+y}{(t+x+y)^2}dy,$$
$$U^t_{2,5:6|1} = (W_{5:6} - W_{2:6} \mid S_6(t) = 1)\,, \quad \bar{F}_{U^t_{2,5:6|1}}(x) = \int_0^{+\infty} \frac{5t^5}{(t+y)^6} \cdot [1 - \frac{x^3(4t+x+4y)}{(t+x+y)^4}]dy,$$
$$U^t_{3,5:6|1} = (W_{5:6} - W_{3:6} \mid S_6(t) = 1)\,, \quad \bar{F}_{U^t_{3,5:6|1}}(x) = \int_0^{+\infty} \frac{20t^4 y(t+3x+y)}{(t+x+y)^3(t+y)^4}dy,$$
$$U^t_{3,5:6|2} = (W_{5:6} - W_{3:6} \mid S_6(t) = 2)\,, \quad \bar{F}_{U^t_{3,5:6|2}}(x) = \int_0^{+\infty} \frac{4t^4(t+3x+y)}{(t+x+y)^3(t+y)^3}dy.$$

Suppose $t = 1$, we can obtain the curves $g_1(x) = \bar{F}_{U^1_{3,6:6|1}}(x)$, $g_2(x) = \bar{F}_{U^1_{3,5:5|2}}(x)$, $g_3(x) = \bar{F}_{U^1_{2,5:6|1}}(x)$, $g_4(x) = \bar{F}_{U^1_{3,5:6|1}}(x)$, and $g_5(x) = \bar{F}_{U^1_{3,5:6|2}}(x)$ as shown in Figure 1. From the positional relationships of the function curves, we can determine conclusions corresponding to Remark 3.2:
(1) $U^1_{2,5:6|1} \leq_{st} U^1_{3,6:6|1}$,
(2) $U^1_{3,5:6|1} \leq_{st} U^1_{3,6:6|1}$,
(3) $U^1_{3,5:6|2} \leq_{st} U^1_{3,5:5|2}$,
(4) $U^1_{3,5:6|2} \leq_{st} U^1_{3,5:6|1}$.

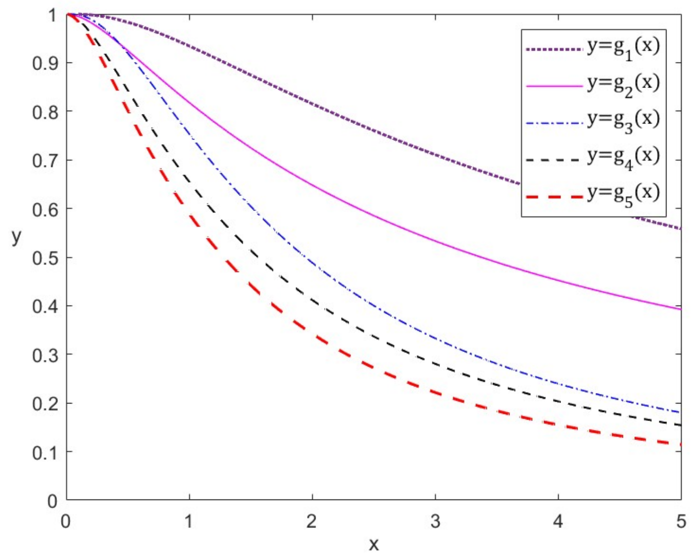

Figure 1. $g_3(x) \le g_1(x)$, $g_4(x) \le g_1(x)$, $g_5(x) \le g_2(x)$,$g_5(x) \le g_4(x)$.

For $t = 2$, we can obtain the curves $g_2^{t=2}(x) = \bar{F}_{U^2_{3,5:5|2}}(x)$ and $g_5^{t=2}(x) = \bar{F}_{U^2_{3,5:6|2}}(x)$ as shown in Figure 2. Based on the positional relationship of the function curves, we can determine that the conclusion corresponding to Remark 3.2: (5) $U^1_{3,5:5|2} \le_{st} U^2_{3,5:5|2}$ *and* $U^1_{3,5:6|2} \le_{st} U^2_{3,5:6|2}$.

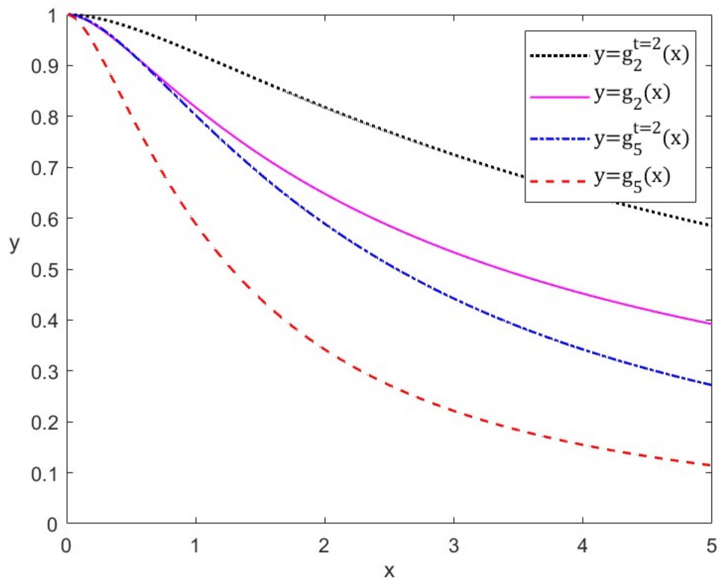

Figure 2. $g_2(x) \le g_2^{t=2}(x)$, $g_5(x) \le g_5^{t=2}(x)$.

Theorem 3.3. *Let $W_1, \cdots, W_n$ be IID component lifetimes of a k-out-of-n system with W_1 being a DFR(IFR) distribution, for $s \ge h$. Then,*

$$U^t_{i,j:n|s} \le_{hr} (\ge_{hr}) U^t_{i,j:n|h}.$$

Proof. By Theorem 3.1, the survival functions of $U^t_{i,j:n|s}$ and $U^t_{i,j:n|h}$ can be obtained as follows:

$$\bar{F}_{U^t_{i,j:n|s}}(x) = E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-s:n-s}}(x))\right],$$
$$\bar{F}_{U^t_{i,j:n|h}}(x) = E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-h:n-h}}(x))\right].$$

The conclusion $U^t_{i,j:n|s} \leq_{hr} U^t_{i,j:n|h}$ to be proved is equivalent to proving

$$\frac{E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-s:n-s}}(x_1))\right]}{E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-h:n-h}}(x_1))\right]} \geq \frac{E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-s:n-s}}(x_2))\right]}{E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-h:n-h}}(x_2))\right]},\ for\ 0 < x_1 \leq x_2. \tag{3.1}$$

Since F is DFR, by Lemmas 2.3 and 2.2(b) one obtains $B_{n-j+1,j-i}(\bar{F}^U_{y_1}) \leq_{hr} B_{n-j+1,j-i}(\bar{F}^U_{y_2})$ for $y_1 \leq y_2$. Hence, $g(y_2, y_1) = B_{n-j+1,j-i}(\bar{F}^U_{y_1}(x_1)) \cdot B_{n-j+1,j-i}(\bar{F}^U_{y_2}(x_2)) \in \mathcal{G}_{lr}$ for $x_1 \leq x_2$.

By Lemma 2.1, $Z_{i-s:n-s} \leq_{lr} Z_{i-h:n-h}$ for $s \geq h$. So, applying Lemma 2.4 yields that, for $x_1 \leq x_2$,

$$E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-s:n-s}}(x_1))\right] \cdot E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-h:n-h}}(x_2))\right]$$
$$\geq E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-h:n-h}}(x_1))\right] \cdot E\left[B_{n-j+1,j-i}(\bar{F}^U_{t+Z_{i-s:n-s}}(x_2))\right].$$

That is, inequality (3.1) holds, so that the proof of Theorem 3.3 is completed.

$R \leq_{hr} L$ is defined as $\bar{F}_R(x)/\bar{F}_L(x)$ being a reduced function of x, which is equivalent to $\frac{f_R(x)}{\bar{F}_R(x)} \geq \frac{f_L(x)}{\bar{F}_L(x)}$. The hazard rate of R can alternatively be expressed as

$$r(x) = \frac{f_R(x)}{\bar{F}_R(x)} = \lim_{\Delta x \to 0} \frac{Pr(x < R \leq x + \Delta x \mid R > x)}{\Delta x}.$$

From the limit of the above equation, the hazard rate $r(x)$ can be thought of as the intensity of failure of a device, with a random lifetime R, at time x. $U^t_{i,j:n|s} \leq_{hr} U^t_{i,j:n|h}$ provide insights into the aging properties of systems or components. They help determine whether one system tends to fail more quickly or slowly compared to another over time.

In particular, when $j = i+1$, the following result on conditional spacings holds. The proof is similar to that of Theorem 3.3 so it is omitted.

Theorem 3.4. *Let $W_1, \cdots, W_n$ be IID component lifetimes of a k-out-of-n system with W_1 being a $DFR(IFR)$ distribution. Then, $U^t_{i:n|s} \leq_{hr} (\geq_{hr}) U^t_{(i+1):(n+1)|s}$ for fixed $i \in \{s+1, s+2, \cdots, n-1\}$.*

If the conditional IFR in Theorem 3.3 is enhanced to ILR, the following stronger conclusion can be obtained.

Theorem 3.5. *Let $W_1, \cdots, W_n$ be IID component lifetimes of a k-out-of-n system with W_1 being an ILR distribution, for $\forall\ t_1 \leq t_2$, $s \geq h$. Then,*

$$U^{t_1}_{i,j:n|s} \geq_{hr} U^{t_2}_{i,j:n|h}.$$

Proof. Using the conclusions of Remarks 2.1 and 2.2, Theorem 3.5 can be proved using a similar proof procedure to that of Theorem 3.3.

Remark 3.3. *If DLR is used in Theorem 3.5 instead of ILR, it is still open whether the inverse partial order conclusion holds. Under the conditions of Theorem 3.5, it is easy to obtain*

$$U^{t_1}_{i,j:n|s} \geq_{hr} U^{t_2}_{i,j:n|h} \quad \Longleftrightarrow \quad \begin{array}{ccccc} & & U^{t_2}_{i,j:n|s} & & \\ & \overset{\geq_{hr}}{\nearrow}{\scriptstyle (1)} & & \overset{\geq_{hr}}{\searrow}{\scriptstyle (3)} & \\ U^{t_1}_{i,j:n|s} & & & & U^{t_2}_{i,j:n|h}. \\ & \underset{\geq_{hr}}{\searrow}{\scriptstyle (2)} & & \underset{\geq_{hr}}{\nearrow}{\scriptstyle (4)} & \\ & & U^{t_1}_{i,j:n|h} & & \end{array}$$

Example 3.2. *Suppose a parallel system has six components whose lifetimes are IID follows the Gamma distribution, i.e., $f(x) = xe^{-x}, (x \geq 0)$. Then W is clearly ILR. By definition $U^t_{i,j:n|s} \equiv \left(W_{j:n} - W_{i:n} \mid S_n(t) = s\right)$, and we have*

$$U^t_{3,5:6|2} = (W_{5:6} - W_{3:6} \mid S_6(t) = 2), \quad U^t_{3,5:6|1} = (W_{4:4} - W_{2:4} \mid S_6(t) = 1).$$

From Theorem 3.1, we can calculate the survival functions

$$\bar{F}_{U^t_{3,5:6|2}}(x) = \int_0^{+\infty} \frac{4(1+t+x+y)(1+t+y)(t+y)}{(1+t)^4 e^{2x+4y}} \cdot [3 - \frac{2(1+t+x+y)}{(1+t+y)e^x}]dy,$$

$$\bar{F}_{U^t_{3,5:6|1}}(x) = \int_0^{+\infty} \frac{20(1+t+x+y)^2(1+t+y)(t+y)}{(1+t)^4 e^{2x+4y}}[3 - \frac{2(1+t+x+y)}{(1+t+y)e^x}][1 - \frac{1+t+y}{(1+t)e^y}]dy.$$

To determine the hazard rate relationship of $U^1_{3,5:6|2}$, $U^2_{3,5:6|2}$, $U^1_{3,5:6|1}$ and $U^2_{3,5:6|1}$, let

$$g_6(x) = \frac{\bar{F}_{U^2_{3,5:6|2}}(x)}{\bar{F}_{U^1_{3,5:6|2}}(x)}, \; g_7(x) = \frac{\bar{F}_{U^1_{3,5:6|1}}(x)}{\bar{F}_{U^1_{3,5:6|2}}(x)}, \; g_8(x) = \frac{\bar{F}_{U^2_{3,5:6|1}}(x)}{\bar{F}_{U^2_{3,5:6|2}}(x)}, \; g_9(x) = \frac{\bar{F}_{U^2_{3,5:6|1}}(x)}{\bar{F}_{U^1_{3,5:6|1}}(x)},$$

and draw their curves as shown in Figure 3.

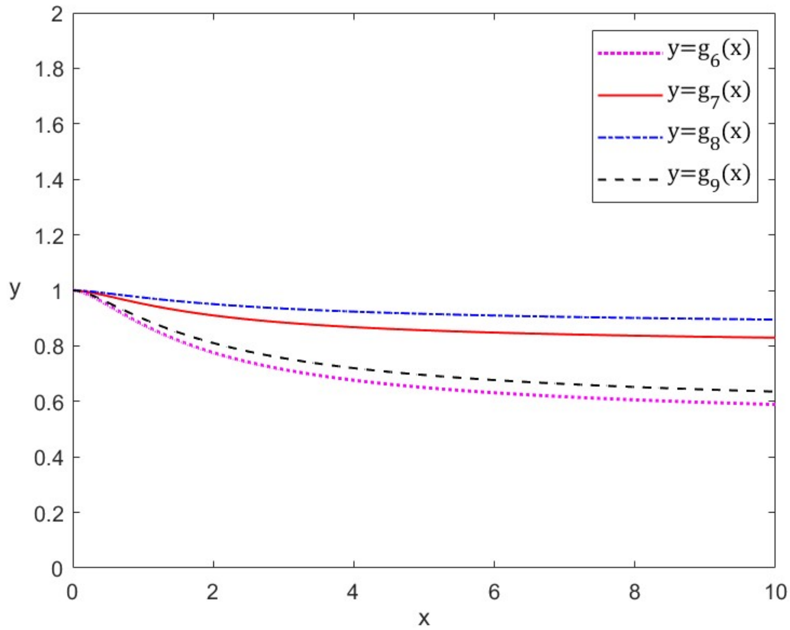

Figure 3. Curve monotonically decreasing.

From the monotonicity of curves in Figure 3, which corresponds to the conclusion of Remark 3.3,

(1)

$$U^1_{3,5:6|2} \geq_{hr} U^2_{3,5:6|2}.$$

(2)

$$U^1_{3,5:6|2} \geq_{hr} U^1_{3,5:6|1}.$$

(3)

$$U^2_{3,5:6|2} \geq_{hr} U^2_{3,5:6|1}.$$

(4)

$$U^1_{3,5:6|1} \geq_{hr} U^2_{3,5:6|1}.$$

When $j = i + 1$, we have the following hazard rate order relation for between generalized normal condition spacings.

Theorem 3.6. *Let $W_1, \cdots, W_n$ be IID component lifetimes of a k-out-of-n system with W_1 being a DFR distribution. Then, for fixed $t > 0$,*

$$U^{*,t}_{i:n|s} \leq_{hr} U^{*,t}_{(i+1):n|s},\ i \in \{s+1, s+2, \cdots, n-2\}.$$

Proof. For $j = i + 1$, we further simplify the survival function of $U^t_{i:n|s}$.

$$\begin{aligned}
\bar{F}_{U^t_{i:n|s}}(x) &= Pr\left(W_{i+1:n} - W_{i:n} > x \mid S_n(t) = s\right) \\
&= \int_0^{+\infty} Pr\left(Z_{i-s+1:n-s} - Z_{i-s:n-s} > x \mid Z_{i-s:n-s} = y\right) dF_{Z_{i-s:n-s}}(y) \\
&= \binom{n-s}{i-s} \int_0^{+\infty} [\bar{F}^U_t(y+x)]^{n-i}(i-s)[F^U_t(y)]^{i-s-1} f^U_t(y)dy \\
&= \binom{n-s}{i-s} \int_0^{+\infty} [\bar{F}^U_t(y+x)]^{n-i} dF_{Z_{i-s,i-s}}(y) \\
&= \binom{n-s}{i-s} E[\bar{F}^U_t(Z_{i-s,i-s}+x)]^{n-i}.
\end{aligned} \tag{3.2}$$

From Remark 3.1, the survival functions of $U^{*,t}_{i:n|s}$ and $U^{*,t}_{(i+1):n|s}$ are obtained as

$$\begin{aligned}
\bar{F}_{U^{*,t}_{i:n|s}}(x) &= \binom{n-s}{i-s} E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x}{n-i})]^{n-i}, \\
\bar{F}_{U^{*,t}_{(i+1):n|s}}(x) &= \binom{n-s}{i-s+1} E[\bar{F}^U_t(Z_{i-s+1,i-s+1} + \frac{x}{n-i-1})]^{n-i-1}.
\end{aligned}$$

The conclusion $U^{*,t}_{i:n|s} \leq_{hr} U^{*,t}_{(i+1):n|s}$ to be proved is equivalent to proving

$$\frac{E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x_1}{n-i})]^{n-i}}{E[\bar{F}^U_t(Z_{i-s+1,i-s+1} + \frac{x_1}{n-i-1})]^{n-i-1}} \geq \frac{E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x_2}{n-i})]^{n-i}}{E[\bar{F}^U_t(Z_{i-s+1,i-s+1} + \frac{x_2}{n-i-1})]^{n-i-1}},\ for\ 0 < x_1 \leq x_2. \tag{3.3}$$

Let $M(x,y) = [\bar{F}^U_t(y + \frac{x}{n-i-1})]^{n-i-1}$, $N(x,y) = [\bar{F}^U_t(y + \frac{x}{n-i})]^{n-i}$, for $0 < x_1 \leq x_2,\ 0 < y_1 \leq y_2,\ 0 < x,\ 0 < y$. Then:

(a) $\frac{M(x,y_2)}{N(x,y_1)}$ is increasing in x.
(b) $\frac{N(x,y_2)}{N(x,y_1)}$ is increasing in x.
(c) $\frac{M(x,y)}{N(x,y)}$ is increasing in y.
Next the conclusion (a) is verified.

$$ln\left(\frac{M(x,y_2)}{N(x,y_1)}\right) = (n-i-1)ln\bar{F}_t^U(\frac{x}{n-i-1}+y_2) - (n-i)ln\bar{F}_t^U(\frac{x}{n-i}+y_1).$$

Calculate the partial derivatives of the above equation with respect to x, and then get the right-hand side of the following equation to be non-negative, based on the DFR property of W_1.

$$\frac{\partial}{\partial x}ln\left(\frac{M(x,y_2)}{N(x,y_1)}\right) = -\frac{f_t^U(\frac{x}{n-i-1}+y_2)}{\bar{F}_t^U(\frac{x}{n-i-1}+y_2)} + \frac{f_t^U(\frac{x}{n-i}+y_1)}{\bar{F}_t^U(\frac{x}{n-i}+y_1)} \geq 0.$$

Thus, for $0 < y_1 \leq y_2$, $\frac{M(x,y_2)}{N(x,y_1)}$ is increasing in x. Conclusions (b) and (c) are verified in a similar way, so they are omitted.

From (a) and (c), $\frac{M(x_2,y_2)}{N(x_2,y_2)} - \frac{M(x_1,y_1)}{N(x_1,y_1)} \geq \frac{M(x_1,y_2)}{N(x_1,y_2)} - \frac{M(x_2,y_1)}{N(x_2,y_1)}$ and $\frac{M(x_2,y_2)}{N(x_2,y_2)} - \frac{M(x_1,y_1)}{N(x_1,y_1)} \geq 0$.

From (b), $N(x_2,y_2)\cdot N(x_1,y_1) \geq N(x_1,y_2)\cdot N(x_2,y_1)$. Consequently,

$$M(x_2,y_2)N(x_1,y_1) - M(x_1,y_1)N(x_2,y_2) \geq M(x_1,y_2)N(x_2,y_1) - M(x_2,y_1)N(x_1,y_2). \tag{3.4}$$

By the fact that the random variables $Z_{i-s,i-s}$ and $Z_{i-s+1,i-s+1}$ are independent, and from inequality (3.4) and $Z_{i-s,i-s} \leq_{lr} Z_{i-s+1,i-s+1}$, it follows that

$$\begin{aligned}
& E\left[N(x_2,Z_{i-s,i-s})M(x_1,Z_{i-s+1,i-s+1})\right] - E\left[N(x_1,Z_{i-s,i-s})M(x_2,Z_{i-s+1,i-s+1})\right] \\
= & \int\int_{y_1\leq y_2}[N(x_2,y_1)M(x_1,y_2) - N(x_1,y_1)M(x_2,y_2)]h(y_1,y_2)dy_1dy_2 \\
& + \int\int_{y_1>y_2}[N(x_2,y_1)M(x_1,y_2) - N(x_1,y_1)M(x_2,y_2)]h(y_1,y_2)dy_1dy_2 \\
= & \int\int_{y_1\leq y_2}[N(x_2,y_1)M(x_1,y_2) - N(x_1,y_1)M(x_2,y_2)]h(y_1,y_2)dy_1dy_2 \\
& + \int\int_{y_1<y_2}[N(x_2,y_2)M(x_1,y_1) - N(x_1,y_2)M(x_2,y_1)]h(y_2,y_1)dy_2dy_1 \\
& (Note\ that\ here\ N(x_2,y_1)M(x_1,y_2) - N(x_1,y_1)M(x_2,y_2) \leq 0) \\
\leq & \int\int_{y_1\leq y_2}[N(x_2,y_1)M(x_1,y_2) - N(x_1,y_1)M(x_2,y_2) \\
& +N(x_2,y_2)M(x_1,y_1) - N(x_1,y_2)M(x_2,y_1)]h(y_2,y_1)dy_1dy_2 \leq 0,
\end{aligned}$$

where $h(\cdot,\cdot)$ denotes the joint density function of $Z_{i-s,i-s}$ and $Z_{i-s+1,i-s+1}$. This implies that inequality (3.3) holds, so the proof of Theorem 3.6 is complete.

Remark 3.4. *Under the assumptions of Theorem 3.6, we further have*

$$U^{*,t}_{i:(n+1)|s} \leq_{hr} U^{*,t}_{i:n|s} \leq_{hr} U^{*,t}_{(i+1):(n+1)|s},\ i \in \{s+1, s+2, \cdots, n-1\}.$$

Proof. Similar to the calculation of Eq (3.2), the survival functions of $U^{*,t}_{i:n|s}$, $U^{*,t}_{i:(n+1)|s}$, and $U^{*,t}_{(i+1):(n+1)|s}$ can be respectively obtained as follows:

$$\begin{aligned} \bar{F}_{U^{*,t}_{i:n|s}}(x) &= \binom{n-s}{i-s} E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x}{n-i})]^{n-i}, \\ \bar{F}_{U^{*,t}_{i:(n+1)|s}}(x) &= \binom{n-s+1}{i-s} E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x}{n-i+1})]^{n-i+1}, \\ \bar{F}_{U^{*,t}_{(i+1):(n+1)|s}}(x) &= \binom{n-s+1}{i-s+1} E[\bar{F}^U_t(Z_{i-s+1,i-s+1} + \frac{x}{n-i})]^{n-i}. \end{aligned}$$

The conclusion $U^{*,t}_{i:(n+1)|s} \leq_{hr} U^{*,t}_{i:n|s}$ to be proved is equivalent to proving

$$\frac{E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x_1}{n-i+1})]^{n-i+1}}{E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x_1}{n-i})]^{n-i}} \geq \frac{E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x_2}{n-i+1})]^{n-i+1}}{E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x_2}{n-i})]^{n-i}}, \ for\ 0 < x_1 \leq x_2. \tag{3.5}$$

The conclusion $U^{*,t}_{i:n|s} \leq_{hr} U^{*,t}_{(i+1):(n+1)|s}$ to be proved is equivalent to proving

$$\frac{E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x_1}{n-i})]^{n-i}}{E[\bar{F}^U_t(Z_{i-s+1,i-s+1} + \frac{x_1}{n-i})]^{n-i}} \geq \frac{E[\bar{F}^U_t(Z_{i-s,i-s} + \frac{x_2}{n-i})]^{n-i}}{E[\bar{F}^U_t(Z_{i-s+1,i-s+1} + \frac{x_2}{n-i})]^{n-i}}, \ for\ 0 < x_1 \leq x_2. \tag{3.6}$$

By a process similar to that used to prove inequality (3.3), it follows that inequalities (3.5) and (3.6) hold. Thus, the proof is complete.

Since W_1 is DFR if and only if $Z = (W_1 - t|W_1 > t)$ is DFR, the following results are easily obtained according to Kochar and Kirmani [4].

Theorem 3.7. *Let $W_1, \cdots, W_n$ be IID component lifetimes of a k-out-of-n system with W_1 being a DFR distribution. Then:*

(1) $U^t_{i:n|s} \leq_{disp} U^t_{i+1:n|s}$ for $i \in \{s+1, \cdots, n-2\}$;
(2) $Var(U^t_{i:n|s}) \leq Var(U^t_{i+1:n|s})$ for $i \in \{s+1, \cdots, n-2\}$;
(3) $U^t_{i:n+1|s} \leq_{disp} U^t_{i:n|s}$ for $i \in \{s+1, \cdots, n-1\}$;
(4) $Var(U^t_{i:n+1|s}) \leq Var(U^t_{i:n|s})$ for $i \in \{s+1, \cdots, n-1\}$,
where $Var(\cdot)$ is the variance of a random variable.

4. Conclusions and perspectives

The number of components that fail and the cumulative operating time of the system are closely related to the remaining life of the used k-out-of-n system. Inspired by this, this paper presents the concepts of generalized conditional spacings and generalized normal conditional spacings based on the used k-out-of-n systems, which incorporate the factors of system operating time and number of faulty components. First, survival functions of generalized conditional spacings are obtained. Then, the stochastic order $U^{t_1}_{i,j:n|s} \leq_{st} U^{t_2}_{p,q:m|h}(\ \forall\ t_1 \leq t_2,\ p-h \geq i-s$, and $q-j \geq p-i \geq m-n)$ and the hazard rate order $U^t_{i,j:n|s} \leq_{hr} U^t_{i,j:n|h}(s \geq h)$ are obtained when the parent distribution of component lifetimes is DFR. In particular, when $j = i+1$, the hazard rate ordinal relations $U^t_{i:n|s} \leq_{hr} U^t_{(i+1):(n+1)|s}$ and $U^{*,t}_{i:n|s} \leq_{hr} U^{*,t}_{(i+1):n|s}$ are obtained. Moreover, $U^{t_1}_{i,j:n|s} \geq_{hr} U^{t_2}_{i,j:n|h}(\forall\ t_1 \leq t_2,\ s \geq h)$ is obtained when the parent distribution family is strengthened to ILR.

To simplify the problem, we assumed that the components are independent and identically distributed. However, in practice, the failure of one component can affect the lifetimes of the remaining components to varying degrees. Therefore, future research could consider the following issues. First, investigate the stochastic properties of conditional spacings formed by sequential order statistics. Second, explore the properties of conditional spacings by incorporating the system's structure and the positions of the failed components. Finally, it is worth noting the replacement of the DFR in Theorems 3.2 and 3.6 by the IFR, and it is still open whether the reverse partial order of the corresponding conclusion holds.

Author contributions

Tie Li, Zhengcheng Zhang: Dealt with conceptualization, supervision, methodology, investigation, writing-original draft, formal analysis, editing and preparing the figures and the table. All authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by the Natural Science Foundation of China (No. 72161009), the Hainan Provincial Natural Science Foundation of China (No. 122MS057 and No. 124MS055), the Inner Mongolia University scientific research projects (No. NJZZ21050), and the Lucheng Talent Project. Once again, we sincerely thank the reviewers for their valuable comments and suggestions on this manuscript.

Conflict of interest

The authors declare that they have no competing interests.

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