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Research article

Perfect directed codes in Cayley digraphs

Yan Wang, Kai Yuan* **and Ying Zhao**

School of Mathematics and Information Science, Yantai University, Yan Tai, China

* **Correspondence:** Email: pktide@163.com.

Abstract: A perfect directed code (or an efficient twin domination) of a digraph is a vertex subset where every other vertex in the digraph has a unique in- and a unique out-neighbor in the subset. In this paper, we show that a digraph covers a complete digraph if and only if the vertex set of this digraph can be partitioned into perfect directed codes. Equivalent conditions for subsets in Cayley digraphs to be perfect directed codes are given. Especially, equivalent conditions for normal subsets, normal subgroups, and subgroups to be perfect directed codes in Cayley digraphs are given. Moreover, we show that every subgroup of a finite group is a perfect directed code for a transversal Cayley digraph.

Keywords: Cayley digraph; perfect directed code; efficient twin domination
Mathematics Subject Classification: 05C20, 05C25, 05C69

1. Introduction

In this paper, we assume all digraphs considered here are simple digraphs without loops and multiple arcs between any two vertices. Moreover, the digraphs may not be connected. One may refer to [8] for basic definitions and well-known terminologies. Let Γ be a connected finite digraph with a vertex set $V(\Gamma)$ and an arc set $A(\Gamma)$. The ends u and v of an arc (u, v) are called the *tail* and the *head* of this arc, respectively. A subset S of $V(\Gamma)$ is called *independent* if, for any $u, v \in S$, $(u, v) \notin A(\Gamma)$. An independent subset S of $V(\Gamma)$ is called a *perfect kernel* of Γ if for each $v \in V(\Gamma) \setminus S$ there exists a unique element $x \in S$ such that $(v, x) \in A(\Gamma)$, while S is called a *perfect solution* of Γ if for each $w \in V(\Gamma) \setminus S$ there exists a unique element $y \in S$ such that $(y, w) \in A(\Gamma)$. With these terminologies, a *perfect directed code* (also known as efficient twin domination) of Γ is a subset S of $V(\Gamma)$, which is both a perfect kernel and a perfect solution.

The results about perfect directed codes (or twin domination sets) are not as fruitful as those in graphs. One may refer to [1–5, 8–10, 16, 17] for some general results. The perfect directed codes of a digraph play an important role in solving the resource location problem and the facility location problem [12, 14]. This paper is also motivated by [13].

Let G be a group and $X = \{x_1, x_2, \dots, x_n\}$ with $1_G \notin X$ be a subset of G. The *Cayley digraph* $\Gamma = \mathrm{Cay}\,(G, X)$ is defined to have a vertex set G and arcs (g, xg) for each $g \in G$ and $x \in X$. If $X = X^{-1}$, then it is a Cayley graph. The identity is usually excluded from X in order to avoid loops. The digraph $\mathrm{Cay}\,(G, X)$ is connected if and only if X generates G. An *automorphism* of $\mathrm{Cay}\,(G, X)$ refers to a bijection σ on G such that (u, v) is an arc if and only if $(\sigma(u), \sigma(v))$ is an arc. Take an element $g \in G$ and define $R_g : G \to G$ as $R_g(u) = ug$ for each element $u \in G$. Then, R_g is an automorphism of $\mathrm{Cay}\,(G, X)$, because R_g is clearly a bijection on G, and (u, v) is an arc if and only if (ug, vg) is an arc. Some useful digraphs are Cayley digraphs; see [18, 19].

Let Γ and $\overline{\Gamma}$ be digraphs, and $\gamma : \Gamma \to \overline{\Gamma}$ be a homomorphism that maps vertices to vertices, arcs to arcs, and preserves incidences (heads to heads and tails to tails). The homomorphism γ is called a *k-fold covering map* and Γ is called a *k-fold cover* of $\overline{\Gamma}$ (or Γ covers $\overline{\Gamma}$) if every vertex and arc of $\overline{\Gamma}$ has precisely k preimages, and for every $x \in V(\Gamma)$, both the out-degree and the in-degree of x are equal to the corresponding out and in degrees of $\gamma(x)$. Moreover, γ is also called a covering map. Usually, the preimage set in Γ of a vertex or an arc of $\overline{\Gamma}$ is called the fibre of this vertex or this arc, respectively.

In Section 2, we show that a digraph covers a complete digraph if and only if the vertex set of this digraph can be partitioned into perfect directed codes. Equivalent conditions for a subset to be a perfect directed code in Cayley digraphs are given in Section 3. And in Section 4, we give equivalent conditions for normal subsets, normal subgroups, and subgroups to be perfect directed codes of Cayley digraphs.

2. Perfect directed codes and coverings

Let Γ be a digraph, and S_1 and S_2 be two disjoint independent subsets of $V(\Gamma)$ with $|S_1| = |S_2|$. The subgraph of Γ induced by $S_1 \cup S_2$ is called a *dimatching* if for every vertex $u_1 \in S_1$ there is exactly one vertex $v_2 \in S_2$ and one vertex $w_2 \in S_2$ (v_2 and w_2 may be the same vertex) such that (u_1, v_2) and (w_2, u_1) belong to $A(\Gamma)$, and vice versa for every vertex in S_2. From its definition, a dimatching includes several directed cycles of even lengths. The set of arcs *from S_1 to S_2* involves arcs whose tails and heads are in S_1 and S_2, respectively. A *complete digraph* on n vertices, denoted by DK_n, is a digraph on n vertices such that there are exactly two arcs (u, v) and (v, u) between any two vertices u and v.

Lemma 2.1. (1) *The perfect directed codes of a digraph have equal size. Moreover, the subgraph induced by two disjoint perfect directed codes is a dimatching.*

(2) *Let $S_1, S_2, \dots, S_n$ be n perfect directed codes of a digraph Γ that are pairwise mutually disjoint. Then, the subgraph induced by $\cup_{i=1}^{n} S_i$ is an m-fold cover of the complete digraph DK_n, where $m = |S_i|$ for each $1 \le i \le n$.*

Proof: (1) Let S_1 and S_2 be two perfect directed codes of a digraph Γ, and let $S = S_1 \cap S_2$. Let $|S_1| = 1$. Assume that $|S_2| \ge 2$. Let $u \in S_1$ and $v, w \in S_2$, $v \ne w$. Then we have (u, v) and (u, w) are arcs of these digraphs (S_1 is a perfect solution), which contradicts that S_2 is a perfect kernel. So $|S_2| = 1$ and $|S_1| = |S_2|$. Now suppose that $|S_1| \ge 2$. Note that a perfect directed code is an independent subset. For every two distinct vertices u_1 and u_2 in S_1, there are vertices v_1 and v_2 in S_2 such that (u_1, v_1) and (u_2, v_2) belong to $A(\Gamma)$ because S_2 is a perfect directed code. Because S_1 is also a perfect directed code, v_1 and v_2 are distinct as well. Thus, $|S_1| = |S_2|$. Moreover, the subgraph induced by $(S_1 \cup S_2) \setminus S$ is a dimatching because a perfect directed code is both a perfect kernel and a perfect solution. As a result, the subgraph induced by two disjoint perfect directed codes is a dimatching.

(2) Let Σ be the induced subgraph in Γ with $V(\Sigma) = \cup_{i=1}^{n} S_i$. A dimatching is clearly a covering of DK_2, with each part of the vertices covering a vertex in DK_2. Denote the vertices of DK_n by $1, 2, \ldots, n$. According to the result in (1), the subgraph induced by any two sets in $\{S_i \mid 1 \leq i \leq n\}$ is a dimatching. As a result, Σ is an m-fold cover of DK_n, with the vertices in S_i covering the vertex i of DK_n and the arcs from S_i to S_j covering the arc from i to j. □

Lemma 2.2. *Let Γ and $\overline{\Gamma}$ be digraphs, $\gamma : \Gamma \to \overline{\Gamma}$ be a k-fold covering map, $\overline{S} \subseteq V(\overline{\Gamma})$ and $S = \gamma^{-1}(\overline{S})$.*

(1) *If $\overline{S}$ is a perfect kernel, then S is a perfect kernel;*
(2) *If $\overline{S}$ is a perfect solution, then S is a perfect solution;*
(3) *If $\overline{S}$ is a perfect directed code, then S is a perfect directed code.*

Proof: Because an arc $(u, v) \in A(\Gamma)$ will be mapped by γ to an arc $(\gamma(u), \gamma(v)) \in A(\overline{\Gamma})$, the fact that S is independent, which follows from $\overline{S}$ being independent. Take an element $v \in V(\Gamma) \setminus S$ and set $\overline{v} = \gamma(v)$, then $\overline{v} \in V(\overline{\Gamma}) \setminus \overline{S}$.

(1) If $\overline{S}$ is a perfect kernel, then there exists exactly one $\overline{s} \in \overline{S}$ such that $(\overline{v}, \overline{s}) \in A(\overline{\Gamma})$. Since γ is a covering map, there exists a $s \in \gamma^{-1}(\overline{s}) \subseteq S$ such that $(v, s) \in A(\Gamma)$. Now we show that this element s is unique from the definition of the covering. Suppose $s' \in S$ is an element satisfying $(v, s') \in A(\Gamma)$, then $(\overline{v}, \gamma(s')) \in A(\overline{\Gamma})$. So, $\gamma(s) = \overline{s} = \gamma(s')$, and there will be two parallel arcs from $\overline{v}$ to $\overline{s}$ if $s \neq s'$, which contradicts to the assumption of $\overline{\Gamma}$ being simple. Consequently, S is a perfect kernel of Γ.

(2) can be proved similarly, and (3) follows directly from (1) and (2). □

Theorem 2.3. *A digraph Γ covers the complete digraph DK_n if and only if Γ has a vertex partition $S_1, S_2, \ldots, S_n$ such that $S_i, 1 \leq i \leq n$, are perfect directed codes of Γ.*

Proof: The sufficiency follows from Lemma 2.1. For the necessity, let $\gamma : \Gamma \to DK_n$ be a covering map, and let $V(DK_n) = \{1, 2, \ldots, n\}$. Set $S_i = \gamma^{-1}(i)$. Clearly, the n-subsets $S_1, S_2, \ldots, S_n$ form a vertex partition of Γ. Note that $\{i\}$ is a perfect directed code of DK_n for each $1 \leq i \leq n$, so S_i is a perfect directed code of Γ according to Lemma 2.2. □

Example 2.4. *Let Γ be the digraph* (*left*) *in* Figure 1. *Set $S_1 = \{1, 2\}, S_2 = \{3, 4\}$ and $S_3 = \{5, 6\}$. Then S_i is a perfect directed code of Γ for each $i \in \{1, 2, 3\}$. Moreover, Γ is a cover of the complete digraph DK_3* (*right*).

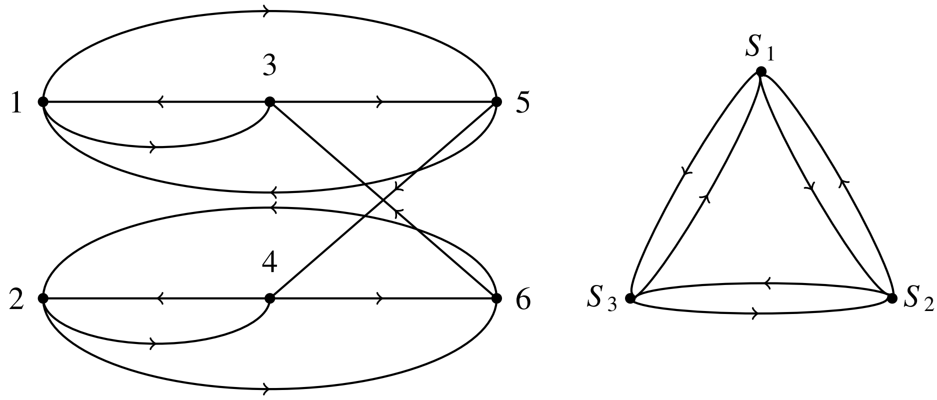

Figure 1. Digraph Γ (left) covers DK_3 (right).

3. Perfect directed codes in Cayley digraphs

In this section, we characterize perfect directed codes in Cayley digraphs. For any two subsets U, V of G, set $UV = \{uv \mid u \in U, v \in V\}$. If $U = \emptyset$ or $V = \emptyset$, then set $UV = \emptyset$. In particular, if $U = \{u\}$, then we denote UV by uV; if $V = \{v\}$, then we denote UV by Uv, .

Lemma 3.1. *Let* Cay (G, X) *be a Cayley digraph of a group G for some $X = \{x_1, x_2, \dots, x_n\}$ with $1_G \notin X$, and let S be a subset of G. Suppose that $g \in G$.*

(1) *The following are equivalent:*
   (a) *S is a perfect solution;*
   (b) *Sg is a perfect solution;*
   (c) *The $|X| + 1$ subsets, $xS, x \in X \cup \{1_G\}$, form a partition of G.*

(2) *The following are equivalent:*
   (i) *S is a perfect kernel;*
   (ii) *Sg is a perfect kernel;*
   (iii) *The $|X| + 1$ subsets, $x^{-1}S, x \in X \cup \{1_G\}$, form a partition of G.*

(3) *S is a perfect directed code of* Cay (G, X) *if and only if S is a perfect directed code of* Cay (G, X^{-1}).

Proof: If S is a perfect solution (a perfect kernel), then $\varphi(S)$ is obviously a perfect solution (a perfect kernel) for each automorphism φ of Cay (G, X). Recall that R_g is an automorphism of Γ for each $g \in G$, so $R_g(S) = Sg$ is a perfect solution (a perfect kernel) under the condition of S being a perfect solution (a perfect kernel). Similarly, $R_{g^{-1}}(Sg) = S$ is a perfect solution (a perfect kernel) under the condition of Sg being a perfect solution (a perfect kernel).

(1) Assume that S is a perfect solution, then $G = S \cup x_1S \cup \cdots \cup x_nS$. Since S is an independent subset, $S \cap x_iS = \emptyset$ for each $1 \le i \le n$. If $x_iS \cap x_jS \ne \emptyset$ for x_i and x_j, then $x_is = x_js'$ for some vertices s and s' in S. This implies that there are two arcs with tails s and s' that have the same head. Therefore, $s = s'$ because S is a solution. As a result, $x_i = x_j$ and the $|X| + 1$ subsets, $xS, x \in X \cup \{1_G\}$, form a partition of G.

For the other direction, assume that the $|X|+1$ subsets, $xS, x \in X \cup \{1_G\}$, form a partition of G. As $xS \cap S = \emptyset$ for every $x \in X$, S is an independent subset. Furthermore, for every $y \in G \setminus S$, there exists a unique $x \in X$ such that $y \in xS$. That is to say, there is a unique $s \in S$ such that $(s, y) \in A(\Gamma)$. So, S is a perfect solution.

(2) Note that if S is a perfect kernel, then $G = S \cup x_1^{-1}S \cup \cdots \cup x_n^{-1}S$. The equivalence of S being a perfect kernel and the partition of G by the $|X|+1$ subsets, $x^{-1}S, x \in X \cup \{1_G\}$, can be shown in a quite similar way. It is clear that (3) can be inferred directly from (1) and (2). □

Remark 3.2. Cay (G, X) *may not be connected in Lemma 3.1.*

Note that a perfect solution of a digraph may not be the perfect kernel of this digraph, and vice versa.

Example 3.3. *Let* $D_6 = \{a, b \mid a^3 = b^2 = 1, b^{-1}ab = a^{-1}\}$ *be the dihedral group of order* 6. *Take* $X = S_1 = \{a, b\}, S_2 = \{a, ba^2\}$. *Then,* S_1 *is a perfect solution but not a perfect kernel of* $\Gamma =$ Cay (D_6, X), *while* S_2 *is a perfect kernel of* Γ *but not a perfect solution. In fact, it is easy to see in Figure 2 that* Γ *does not have perfect directed codes.*

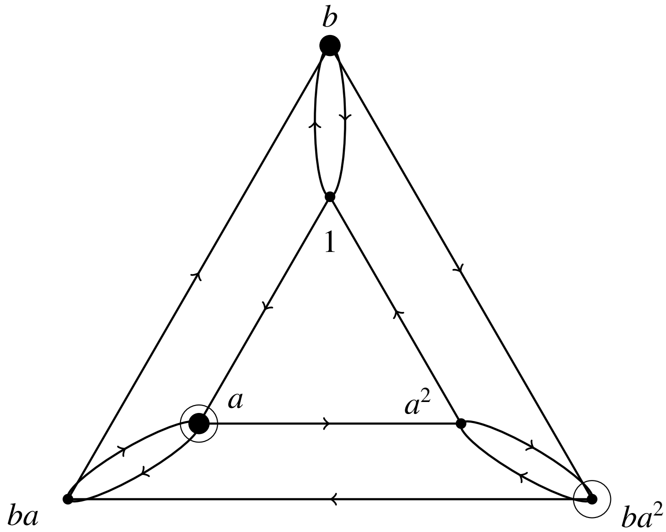

Figure 2. A perfect solution $\{a, b\}$ and a perfect kernel $\{a, ba^2\}$.

Let G be a group and S be a subset of G. If $Sg = gS$ for every $g \in G$, then S is called a *normal subset* of G. It is obvious that S is normal if and only if $Sx = xS$ for every $x \in X$ when X generates G.

Corollary 3.4. *Let* $\Gamma =$ Cay (G, X) *be a Cayley digraph of a group G, and let S be a perfect directed code of* Γ. *If S is a normal subset of G, then there exists a covering map* $\gamma : \Gamma \to DK_{|X|+1}$ *such that* $\{Sx \mid x \in X \cup \{1_G\}\}$ *are the fibres of vertices in* $DK_{|X|+1}$.

Proof: According to Lemma 3.1, the $|X|+1$ subsets, $S, xS = Sx, x \in X$, are mutually disjoint perfect directed codes. So, according to Theorem 2.3, Γ covers $DK_{|X|+1}$ with $Sx, x \in X \cup \{1_G\}$, as the fibres of vertices in $DK_{|X|+1}$. □

Theorem 3.5. *Let G be a finite group, Γ = Cay (G, X) be a Cayley digraph with $X = \{x_1, x_2, \ldots, x_n\}$, and S be a normal subset of G. The following items are equivalent.*

(1) *S is a perfect directed code of Γ;*
(2) *There exists a covering map $\gamma : \Gamma \to DK_{n+1}$ such that $\gamma^{-1}(v) = S$ for some $v \in V(DK_{n+1})$;*
(3) *$|S| = \frac{|G|}{n+1}$ and $S(X \cup ((X^{-1}X) \setminus \{1_G\})) \cap S = \emptyset$;*
(4) *The $n + 1$ subsets, $Sx, x \in X \cup \{1_G\}$, form a partition of $V(\Gamma)$.*

Proof: Because S is a normal subset of G, $Sg = gS$ for every $g \in G$, and the fact that (1) and (2) are equivalent is obvious according to Theorem 2.3 and Corollary 3.4.

(1) ⇒ (3) By Lemma 3.1, the subsets $xS, x \in X \cup \{1_G\}$, form a vertex partition of Γ. So, $|S| = \frac{|G|}{n+1}$ and $Sx_i \cap S = \emptyset$ for each $x_i \in X$.

Suppose $S(X^{-1}X \setminus \{1_G\}) \cap S \neq \emptyset$, then there exist $s_1, s_2 \in S$, $x_1, x_2 \in X$ and $x_1 \neq x_2$ such that $x_2^{-1}x_1s_1 = s_2$, i.e., $x_1s_1 = x_2s_2$. This contradicts $x_1S \cap x_2S = \emptyset$.

(3) ⇒ (4) Since $(X \cup X^{-1}X \setminus \{1_G\})S \cap S = \emptyset$, for each $x_i \in X$, $x_iS \cap S = \emptyset$, i.e., $Sx_i \cap S = \emptyset$, and for any two different elements $x_i, x_j \in X$, $x_i^{-1}x_jS \cap S = \emptyset$, i.e., $x_jS \cap x_iS = \emptyset$, that is $Sx_j \cap Sx_i = \emptyset$. Furthermore, the condition $|S| = \frac{|G|}{n+1}$ implies $G = S \cup Sx_1 \cup \cdots \cup Sx_n$.

(4) ⇒ (1) Since the $|X| + 1$ subsets, $xS, x \in X \cup \{1_G\}$, form a vertex partition of Γ, S is a perfect solution according to Lemma 3.1.

For any two different elements x_i and x_j in X, $x_iS \cap x_jS = \emptyset$ is equivalent to $x_j^{-1}x_iS \cap S = \emptyset$, which is $Sx_j^{-1}x_i \cap S = \emptyset$, and so $Sx_j^{-1} \cap Sx_i^{-1} = \emptyset$. Hence, $Sx^{-1} = x^{-1}S, x \in X \cup \{1_G\}$, also form a vertex partition of Γ. So, S is a perfect kernel of Γ. □

Example 3.6. *Let $G = \mathbb{Z}_p^n$ be the n-times direct product group of $\mathbb{Z}_p$, where p is an odd prime number. Let $X = \{e_1, e_2, \ldots, e_n\}$, where $e_i = (0, \ldots, 1, \ldots, 0) \in G$ has exactly a '1' at the i-th coordinate. Then,* Cay (G, X) *has a perfect directed code if and only if $n = p^m - 1$ for some positive integer m.*

Proof: Clearly, Cay (G, X) is a connected Cayley digraph, and each vertex has both out and in degree n.

On the one side, by Theorem 3.5, if Cay (G, X) has a perfect directed code H, then the subsets, $H, H + e_1, \cdots, H + e_n$, form a partition of G. So, $p^n = |G| = |H|(n + 1)$, and $n = p^m - 1$ for some positive integer m.

On the other side, assume $n = p^m - 1$ for some positive integer m. Denote the vector space of dimension m on the finite field F_p with p elements as F_p^m. Construct an $n \times m$-matrix A: the rows of A are all non-zero vectors in F_p^m. Because the rank of A is m, all solutions of $\zeta A = \mathbf{0}$ where ζ is a vector constitute a $(n - m)$-dimension subspace T of F_p^n.

We claim that T is a perfect directed code of Cay (G, X).

Now we show that T is an independent subset. Otherwise, there exist two different vectors α and β in T such that $e_i = \alpha - \beta \in X$ for some $1 \leq i \leq n$. It follows that $e_iA = \mathbf{0}$. Note that $\mathbf{0} = e_iA$ is the i-th row of A. Then the i-th row of A is a zero vector, contradicting the construction of A. So, T is an independent subset.

Note that the elements of G can be identified with the vectors in F_p^n. For each $\delta \in G \setminus T$, suppose there are two different vectors α and β in T such that $\delta = \alpha + e_i = \beta + e_j$ for $1 \leq i \neq j \leq n$, then

$(e_i - e_j)A = \mathbf{0}$, which implies that A has two equal rows, also a contradiction. Similarly, there is at most one vector η in T satisfying $\delta + e_k = \eta$ for some $1 \leq k \leq n$. Considering $|T|(n+1) = p^n$, both $T, T+e_1, \ldots, T+e_n$ and $T, T-e_1, \ldots, T-e_n$ are partitions of G. By Theorem 3.5, T is a perfect directed code. □

Remark 3.7. *For $p = 2$ in Example* 3.6, Cay (G, X) *is the hypercube Q_n. Moreover, Q_n has a Hamming code (or perfect directed code) if and only if $n = 2^m - 1$ for a positive integer m, see* [13].

4. Perfect directed coding subgroup

Let G be a finite group and Cay (G, X) be a Cayley digraph of degree n, which admits a perfect directed code C. Assume $X = \{x_1, \ldots, x_n\}$. Then, according to Lemma 3.1, both the $n+1$ subsets, $C, x_rC, 1 \leq r \leq n$, and $C, x_s^{-1}C, 1 \leq s \leq n$, form partitions of G. So, $|G| = (n+1)|C|$, and $|C|$ is a factor of $|G|$. Let H be a subgroup of G. Because $|H|$ is a factor of $|G|$, a natural question is: under what conditions can H be a perfect directed code?

The elements of G can be partitioned by H into $m = \frac{|G|}{|H|}$ disjoint subsets $Hv_i, 1 \leq i \leq m$, by the equivalent relation of two elements x and y in G being equivalent if and only if $xy^{-1} \in H$. In group theory, Hv_i is called a *right coset* of H in G, $v_i \in G$ a representative element of this coset, and $\{v_1, v_2, \ldots, v_m\}$ *a right transversal* of H in G. Similarly, G can also be partitioned into m subsets, $u_iH, 1 \leq i \leq m$, by the equivalent relation of two elements x and y in G being equivalent if and only if $y^{-1}x \in H$. And, u_iH is called a *left coset* of H in G, $u_i \in G$ is a representative element of this coset; and $\{u_1, u_2, \ldots, u_m\}$ *a left transversal* of H in G. Because H itself is both a right and a left coset, each right and left transversal of H consists of exactly one element from H. A set is called a *transversal* of H if it is both a right transversal and a left transversal of H. A *normal subgroup N* of a group G refers to a subgroup N satisfying $Ny = yN$ for every $y \in G$. So, one does not need to distinguish between the left and right cosets of normal subgroups.

Corollary 4.1. *Let Γ =* Cay (G, X) *be a digraph with $X = \{x_1, x_2, \ldots, x_n\}$, and let H be a normal subgroup of G. The following items are equivalent.*

(1) *H is a perfect directed code of Γ;*
(2) *There exists a covering map $\gamma : \Gamma \to DK_{n+1}$ such that $\gamma^{-1}(v) = H$ for some $v \in V(DK_{n+1})$;*
(3) *$|H| = \frac{|G|}{n+1}$ and $(X \cup X^{-1}X) \cap H = \{1_G\}$;*
(4) *$X \cup \{1_G\}$ is a transversal of H in G.*

Proof: Firstly, we want to show that the condition $H(X \cup X^{-1}X \setminus \{1_G\}) \cap H = \emptyset$ is equivalent to that of $(X \cup X^{-1}X) \cap H = \{1_G\}$.

Assume $H(X \cup X^{-1}X \setminus \{1_G\}) \cap H = \emptyset$, then $(X \cup X^{-1}X \setminus \{1_G\}) \cap H = \emptyset$ and so $(X \cup X^{-1}X) \cap H = \{1_G\}$.

If $H(X \cup X^{-1}X \setminus \{1_G\}) \cap H \neq \emptyset$, then there exist $h_1, h_2, h_3, h_4 \in H$, $x_1, x_2, x_3 \in X$, $x_1 \neq x_2$, such that $h_1 = h_2x_1^{-1}x_2$ or $h_3x_3 = h_4$. Then, $1_G \neq x_1^{-1}x_2 = h_2^{-1}h_1 \in H$ or $x_3 = h_3^{-1}h_4 \in H$. Therefore, $(X \cup X^{-1}X) \cap H \neq \{1_G\}$.

Secondly, it is obvious that the subsets, $Hx, x \in X \cup \{1_G\}$, form a vertex partition of Γ if and only if $X \cup \{1_G\}$ is a transversal of H in G.

Thus, the four items are equivalent according to Theorem 3.5. □

Generally speaking, take a subgroup T of G, then $Ty_1, Ty_2, \cdots, Ty_m$ may not be a partition of G even if $y_1T, y_2T, \cdots, y_mT$ is a partition of G. But if T is a perfect directed code of a Cayley digraph of G, then it has some similar properties to a normal subgroup.

Theorem 4.2. *Let $\Gamma = \mathrm{Cay}\,(G, X)$ be a digraph of a group G with $X = \{x_1, x_2, \ldots, x_n\}$, and let T be a subgroup of G. The following items are equivalent.*

(1) *T is a perfect directed code of Γ;*
(2) *There exists a covering map $\gamma : \Gamma \to DK_{n+1}$ such that $\gamma^{-1}(v) = T$ for some $v \in V(DK_{n+1})$;*
(3) *$|T| = \frac{|G|}{n+1}$, $(X \cup X^{-1}X \cup XX^{-1}) \cap T = \{1_G\}$;*
(4) *$X \cup \{1_G\}$ is a transversal of T.*

Proof: Under the condition of T being a perfect directed code, the $n + 1$ subsets, $T, x_1T, \ldots, x_nT$, form a partition of G according to Lemma 3.1. So, T has $n + 1$ left or right cosets in G, $X \cup \{1_G\}$ is a left transversal of T, and $|G| = |T|(n + 1)$. Because Tg is a perfect directed code of Γ for every $g \in G$, the $n + 1$ right cosets of T are mutually disjoint perfect directed codes of Γ. According to Theorem 2.3, (1) and (2) are equivalent.

(1)⇒(3): Take an element $a \in (X \cup X^{-1}X \cup XX^{-1}) \cap T$, then $a \in X \cap T$ or $a = x_i^{-1}x_j \in T$ or $a = x_r x_t^{-1} \in T$ for x_i, x_j, x_r, x_t in X. Under the assumption of T being a perfect directed code, for any $x, x_i, x_j \in X$ and $x_i \neq x_j$,

$$T \cap xT = x_iT \cap x_jT = T \cap x^{-1}T = x_i^{-1}T \cap x_j^{-1}T = \emptyset,$$

so $a = 1_G$.

(3)⇒(4): The assumption of $(X \cup X^{-1}X \cup XX^{-1}) \cap T = \{1_G\}$ implies that $T \cap x_iT = x_iT \cap x_jT = T \cap x_i^{-1}T = x_i^{-1}T \cap x_j^{-1}T = \emptyset$ for any two different elements x_i and x_j in X. Furthermore, because $|T| = \frac{|G|}{n+1}$, both $X \cup \{1_G\}$ and $X^{-1} \cup \{1_G\}$ are left transversals of T.

By taking the inverse subset of $x^{-1}T$ for every $x \in X$, $T \cap Tx_i = Tx_i \cap Tx_j = \emptyset$, which follows directly from $T \cap x_i^{-1}T = x_i^{-1}T \cap x_j^{-1}T = \emptyset$ for any two different elements x_i and x_j in X. Therefore, $X \cup \{1_G\}$ is also a right transversal of T.

(4)⇒(1): The assumption of $X \cup \{1_G\}$ being both a left and a right transversal of T in G implies that both $T, x_1T, \ldots, x_nT$ and $T, Tx_1, \ldots, Tx_n$ are partitions of G. Therefore, $T, x_1^{-1}T, \ldots, x_n^{-1}T$ form a partition of G as well. According to Lemma 3.1, T is a perfect directed code of Γ. □

Remark 4.3. *In Theorem 4.2, if $X = X^{-1}$, then clearly $X \cup \{1_G\}$ forms a left transversal if and only if it forms a right transversal. But, generally, it is common that X may not be a right transversal of T even if X is a left transversal. As in Example 3.3, no subgroup of D_6 can be a perfect directed code.*

Example 4.4. *Let p be an odd prime number and $k \geq 2$ be a factor of $p - 1$. Let $G = \langle a, b \mid a^p = b^k = 1_G, b^{-1}ab = a^i, i^k \equiv 1 \bmod p\rangle$ be a split metacyclic group, which is a group of order pk. Take $T = \langle b\rangle$ and T is not normal in G. Set $X = \{a^ib \mid 1 \leq i \leq p - 1\}$. Then, $X \cup \{1_G\}$ is both a left and a right transversal of T. So, T is a perfect directed code of* Cay (G, X).

According to Lemma 3.1, if a subset C with $|C| = m$ is a perfect directed code of a Cayley digraph of a finite group G with $|G| = n$, then $m|n$. While this condition may not be sufficient, it corresponds to a special perfect directed code if G is a cyclic group.

Lemma 4.5. *Let $G = \langle a \rangle$ be a cyclic group of order n. Then, a connected Cayley digraph* Cay (G, X) *of degree* $m - 1$ *admits a perfect directed code if m divides n and $x_i x_j^{-1} \notin \langle a^m \rangle$ for any two distinct x_i and x_j in $X \cup \{1_G\}$.*

Proof: Assume that $m \mid n$ and $x_i x_j^{-1} \notin \langle a^m \rangle$ for any two distinct x_i and x_j in $X \cup \{1_G\}$. Set $T = \langle a^m \rangle$. Then, under the condition of $x_i x_j^{-1} \notin T$, both the $|X| + 1$ subsets, $T, x_i T, x_i \in X$, and $T, x_j^{-1} T, x_j \in X$, form partitions of G. So, according to Lemma 3.1, T is a perfect directed code of Cay (G, X). □

Note that the condition $x_i x_j^{-1} \notin \langle a^m \rangle$ in Lemma 4.5 is not necessary, which is different from that in Cayley graphs [7].

Example 4.6. *Let $G = \langle a \rangle$ be the cyclic group of order* 24. *Set $C = \{1_G, a^2, a^8, a^{10}, a^{16}, a^{18}\}$ and $X = \{a, a^4, a^5\}$. It is easy to check that both the* 4 *subsets, C, aC, a^4C, a^5C and $C, a^{-1}C, a^{-4}C, a^{-5}C$ form partitions of G. So, C is a perfect directed code of* Cay (G, X).

A subgroup T of G is called a *perfect directed coding subgroup* if there exists $X \subset G$ such that T is a perfect directed code of the Cayley digraph Cay (G, X), and T is called a *connected perfect directed coding subgroup* if Cay (G, X) is connected. We will show that every subgroup of the Frattini subgroup of a finite group is a connected, perfect directed coding subgroup.

Given a group G, the intersection of all maximal subgroups of G is called the *Frattini subgroup* of G, is denoted by $\Phi(G)$. An element a of a group G is called a *non-generator* of G if, whenever the set X generates G, then the set $X \backslash \{a\}$ also generates G. The following result shows that the Frattini subgroup equals the set of non-generators.

Proposition 4.7. [15, Theorem 10.12] *For all finite groups G, the set of non-generators of G equals the Frattini subgroup of G.*

Let Γ be an undirected graph, $S \subseteq V(\Gamma)$, and $N(S)$ be a set of vertices of $V(\Gamma)$ adjacent to a vertex in S. The following Proposition 4.8 (Hall's marriage theorem) is well known.

Proposition 4.8. [6, Ch.3.3 Theorem 7] *A bipartite graph Γ with vertex sets V_1 and V_2 contains a complete matching from V_1 to V_2 iff*

$$|N(S)| \geq |S| \text{ for every } S \subseteq V_1.$$

The following Proposition 4.9 can be obtained from [11] or Proposition 4.8. Here is a short proof.

Proposition 4.9. *Let G be a finite group, and let T be a subgroup of G such that $|G : T| = n$. Then there exists $X = \{x_1, x_2, \ldots, x_{n-1}\} \subseteq G$ such that $X \cup \{1_G\}$ is a transversal of T in G.*

Proof Let $\{a_1, a_2, \ldots, a_{n-1}\} \cup \{1_G\}$ be a left transversal of T and $\{b_1, b_2, \ldots, b_{n-1}\} \cup \{1_G\}$ be a right transversal of T. Let Γ be a bipartite graph with vertex sets $V_1 = \{a_i T \mid 1 \leq i \leq n-1\}$ and $V_2 = \{T b_i \mid 1 \leq i \leq n-1\}$ such that $\{a_i T, T b_j\}$ is an edge if and only if $a_i T \cap T b_j \neq \emptyset$. Let S be an arbitrary subset of V_1 and $S = \{g_i T \mid 1 \le i \le m\}$ for some m. We claim that there are at least m vertices in $N(S)$. Otherwise, $|\bigcup_{x\in N(S)} x| < |\bigcup_{x\in S} x| = m|T|$, which implies that there exists a vertex $Tb \notin N(S)$ adjacent to a vertex in S, a contradiction. So $|N(S)| \ge |S|$. By Proposition 4.8, Γ contains a complete matching from V_1 to V_2. So we may assume, without loss of generality, that

$$\{\{a_i T, Tb_i\} \mid 1 \le i \le n-1\}$$

is a complete matching. Chose $x_i \in a_i T \cap Tb_i$ for $1 \le i \le n-1$ and let $X = \{x_1, x_2, \ldots, x_{n-1}\}$. Then $X \cup \{1_G\}$ forms both a left transversal and a right transversal of T in G, as required. □

A subgroup H of G is called *proper* if $H \neq G$. Since every proper subgroup has a transversal, we introduce the following transversal Cayley digraphs:

Definition 4.10. *Let G be a finite group, and let X be a subset of G. The Cayley digraph* Cay (G, X) *is called a transversal Cayley digraph if $X \cup \{1_G\}$ is a transversal of a suitable subgroup of G.*

By Theorem 4.2 and Proposition 4.9, we get the following theorem:

Theorem 4.11. *Let G be a finite group. Then every proper subgroup H of G is a perfect directed coding subgroup.*

Proof By Proposition 4.9, there exists $X = \{x_1, x_2, \ldots, x_{n-1}\} \subseteq G$ such that $X \cup \{1_G\}$ is a transversal of H in G. By Theorem 4.2, H is a perfect directed code of the transversal Cayley digraph Cay (G, X). Therefore, H is a perfect directed coding subgroup. □

Corollary 4.12. *Let G be a finite group, and let T be a subgroup of $\Phi(G)$. Then T is a connected, perfect directed coding subgroup.*

Proof By Theorem 4.11, we may assume that T is a perfect directed code for the transversal Cayley digraph Cay (G, X). And so $G = \langle X, T\rangle = \langle X, \Phi(G)\rangle$. From Proposition 4.7, we have $G = \langle X\rangle$. It follows that Cay (G, X) is connected, as desired. □

Example 4.13. *Let p be a prime. If G is a p-group and G' is the commutator group of G, then every subgroup of $\Phi(G) = G'\langle a^p \mid a \in G\rangle$ is a perfect directed coding subgroup of a connected Cayley digraph.*

Theorem 4.14. *Every proper subgroup of a cyclic group G is a perfect directed coding subgroup of a connected Cayley digraph.*

Proof Let T be a proper subgroup of $G = \langle a\rangle$ and $X \cup \{1_G\}$ be a transversal of T. Then we may assume $a \in X$. And so $G = \langle X\rangle$. Clearly, T is a perfect directed code of the connected Cayley digraph Cay (G, X). Then T is a perfect directed coding subgroup, as desired. □

Let $G = \langle a, b \mid a^n = b^2 = 1, a^b = a^{-1}\rangle$ be a dihedral group. Then $\langle a\rangle$ is not a perfect directed coding subgroup of any connected Cayley digraph, and $\langle b\rangle$ is a perfect directed coding subgroup of a connected Cayley digraph. So we propose the following open problem:

Open Problem 4.15. *Characterize finite groups such that each of their proper subgroups is a perfect directed code of a connected transversal Cayley digraph.*

Author contributions

Yan Wang: Conceptualization, Methodology, Validation, Writing-review and editing, Formal analysis, Supervision; Kai Yuan: Conceptualization, Investigation, Methodology, Formal analysis, Writing-review and editing; Ying Zhao: Methodology, Writing-original draft preparation, Visualization, Validation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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