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Research article

Prescribed-time stabilization of nonlinear systems with uncertainties/disturbances by improved time-varying feedback control

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Abstract: We address the prescribed-time stability of a class of nonlinear system with uncertainty/disturbance. With the help of the parametric Lyapunov equation (PLE), we designed a state feedback control to regulate the full-state of a controlled system within prescribed time, independent of initial conditions. The result illustrated that the controlled state converges to zero as t approaches the settling time and remains zero thereafter. It was further proved that the controller is bounded by a constant that depends on the system state. A numerical example is presented to verify the validity of the theoretical results.

Keywords: nonlinear system; prescribed-time stability; state feedback; time-varying feedback control; parametric Lyapunov equation

1. Introduction

The stability and stabilization control of nonlinear systems have always been a widely concerned issue in the control field. The design of controllers usually involves asymptotic stability [1,2]. In practical applications, the implementation of control shames and observation, the operation of optimization algorithms, must be completed in a certain time; thus, the finite-time stability (FTS) has been carried out [3–7]. FTS requires convergence to occur within a finite-time instant, that is $t \to T(x_0, u_0)$, where the settling time $T(x_0, u_0)$ is a finite function of initial conditions about state $x(t)$ and controller $u(t)$. There are many methods available for FTS controller design, such as sliding mode control [8,9], the Lyapunov inequality method and its variants [10,11], the homogeneous approach [12,13], the implicit Lyapunov function approach [14], and the additive power integrator method [15].

Although convergence occurs in a finite time for FTS, the settling time is dependent on the system conditions. Applications of results on FTS may be limited when little is known about the initial states. This leads to a stronger notion of stability called fixed-time stability (FxTS), where $T(x_0, u_0)$ is bounded [16–18]. For FxTS control, odd-order plus fractional-order feedback is always designed for various closed-loop systems [19–22], which can estimate the upper bound of the stable time without any initial condition information. Nevertheless, there is no clear relationship between the settling time and the design of the parameters, resulting in an overestimation on the settling time for the fixed time control. To alleviate the above concern, the classical idea derived from strategic and tactical missile guidance applications [23] has been reemphasized. That is, prescribed-time stability (PTS), which inherits the advantages of finite-time control and fixed-time control. Song [24] and Song [25] established a systematic method with time-varying feedback law for PTS first. After that, a series of studies were carried out [25–31].

In this work, we consider the prescribed-time stability of a class of nonlinear systems with uncertainties/disturbances. The stabilization of this problem was first studied in [32], in which it was proved that, by linear state feedback, the exponential global stabilization of the controlled system could be achieved. Then, some control schemes have been introduced. For example, a state scaling method was applied for the stabilization of the original system with uncertainties within a specified time [24]. Li [29] proposed a new backstepping design scheme to solve the prescribed-time mean-square stabilization and inverse optimality control problems of stochastic strict-feedback nonlinear systems. By adding exponential state feedback and taking fractional power integral as the Lyapunov candidate function, a pre-defined time global stability control strategy for a class of uncertain nonlinear systems with strict feedback form was established [33]. Based on the dynamic high-gain scaling technique, Krishnamurthy [34] addressed the prescribed-time stability for a class of generalized rigid-class feedback structures with state-dependent nonlinear uncertainties.

It should be noted that the Lyapunov method, as an effective approach, has been applied to systems with prescribed time stability properties and has developed various forms. For example, Jiménez-Rodríguez [35] studied the sufficient Lyapunov-like conditions for a class of dynamical systems to exhibit predefined-time stability. By parametric Lyapunov equation (PLE), a bounded linear time-varying (LTV) controller was designed to guarantee the FTS for the closed-loop system [36]. Then, Zhou [37] used the same method as [36] to investigate the PTS for a class of nonlinear systems in which the LTV controller is somewhat intricate due to multiple parameters. Later, work [37] was extended to the prescribed-time input-to-state stabilization (PT-TSS) of a class of nonlinear systems [38].

For the drawback of the controller in [37], a spontaneous question is proposed: If the LTV controller is designed with fewer parameters than [37], does it still realize the PTS? This is the first motivation of this paper. What is more important, in all these schemes on FTS and PTS by PLE in [36–38], the controlled system reaches the equilibrium state at $\boldsymbol{T}$ but that does not mean it can remain in the equilibrium state after $\boldsymbol{T}$. Then, one always asks another question: By PLE, can a LTV controller be designed to guarantee the controlled system to achieve PTS at $\boldsymbol{T}$ and remain in the equilibrium state after $\boldsymbol{T}$? This is the second and most important motivation of this paper.

Based on the above considerations, we use the approach of PLE to design a LTV controller on $[0, \infty)$ to achieve the PTS for nonlinear system with uncertainty or disturbance at prescribed time T and remain in zero after T. Hereinto, different from [36–38], the LTV controller on $[0, T)$ is improved with less parameters, and LTV controller on $[T,\infty)$ is first proposed by PLE. Moreover, it is proved that the controller is uniformly defined by constants that depend on the system states. The contribution of this article has three aspects:

1) Unlike the already-existing prescribed control method which is only for $t \in [0,T)$, our control scheme is valid for $[T,\infty)$. The result shows that the states of controlled system converge to zero at the prescribed settling time T, and remains zero thereafter, allowing controlled system to operate uninterrupted after T.

2) On $[T,\infty)$, with the help of the parameter Lyapunov equation, we construct the time-varying high-gain state feedback controller. Especially, the method PLE is utilized to design the controller after the prescribed time.

3) On $[0,T)$, we make some improvements to the LTV controller compared with [37]. Especially, the number of parameters of our controller has been reduced to form a simpler controller.

The rest of the paper is arranged as follows: The problem description and some preparatory works are given in the 2nd section; the design of the controller and the main result is shown in the 3rd section; and the numerical simulation and conclusion are in the 4th and 5th sections, respectively.

Notation: For matrix A, we denote its transpose by A^T, $\|A\| = \sqrt{A^T A}$ denotes the second norm, $tr(A)$ denotes the trace, and if A is square, $\lambda_i(A), \lambda_{max}(P), \lambda_{min}(P)$ denotes its ith eigenvalue, maximal and minimal eigenvalues, respectively. Moreover, $A > 0 (\geq 0)$ means that A is positive definite (semidefinite). $diag\{a_1, a_2, \cdots, a_n\}$ denote a diagonal matrix whose ith diagonal element is a_i. $I_n \in R^{n\times n}$ represents the identity matrix.

2. Problem preparation

2.1. Problem description

The system studied in this paper is as follows:

$$\begin{cases} \dot{x}_1(t) = x_2(t) + f_1(x,u,t) \\ \vdots \\ \dot{x}_{n-1}(t) = x_n(t) + f_{n-1}(x,u,t), \\ \dot{x}_n(t) = u(t) + f_n(x,u,t) \\ y(t) = x_1(t) \end{cases} \tag{1}$$

where $t \geq 0$, $x = [x_1, x_2, \cdots, x_n]^T \in R^n$ is state vector, $u \in R$ is control input, $y \in R$ is output, the non-linear terms $f_i(x,u,t), i = 1,2,\cdots,n$ are uncertain continuous vector functions caused by external disturbance and internal modeling error or uncertainty.

In order to facilitate the following use, the system (1) is simplified as:

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) + F(x(t),u(t),t) \\ y(t) = Cx(t) \end{cases}, \tag{2}$$

where

$$F(x(t),u(t),t) = [f_1(x(t),u(t),t), f_2(x(t),u(t),t), \cdots, f_n(x(t),u(t),t)]^T,$$

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}^T. \tag{3}$$

Assumption 1. For $i = 1,2,\cdots,n$, there exists some known positive constants c_i such that,

$$|f_i(x,u,t)| \le c_i(|x_1| + |x_2| + \cdots + |x_i|). \tag{4}$$

Remark 1. In our opinion, it is a common strategy to make some structural assumptions, i.e., the lower triangular linear growth condition (4) here, on uncertain continuous functions $f_i(x,u,t), i = 1,2,\cdots,n$. For example, in [37], $f_i(\cdot)$ satisfies

$$|f_i(t,x,u)| \le c_{i1}|x_1| + c_{i2}|x_2| + \cdots + c_{ii}|x_i|, c_{ij} \ge 0,$$

which is similar to our condition (4) here; In [39], the upper-triangular linear growth condition holds for $f_i(t,x,u)$, i.e.,

$$|f_i(t,x,u)| \le c(|x_{i+2}| + \cdots + |x_n|), c \ge 0.$$

In [40],

$$|f_i(\cdot)| \le \gamma_i\left(|x_1|^{\frac{1}{\sigma_1\cdots\sigma_{i-1}}} + \cdots + |x_{i-1}|^{\frac{1}{\sigma_{i-1}}} + |x_i|\right)$$

holds with $\gamma_i \ge 0$, $\sigma_i \ge 1$. In addition, it is necessary to give some assumptions on $f_i(x,u,t)$, otherwise it is impossible to conduct meaningful analysis.

In this work, we focus on the prescribed-time stability of system (1). We consider not only the prescribed-time stability at $[0,T)$, but also the stability after T. The following concept is introduced first.

Definition 1. [37] (PTS) The nonlinear system $\dot{z} = G(t,z), t \ge 0$ is said to be prescribed-time stable if it is Lyapunov stable, and the settling time $T > 0$ is a prescribed constant independent of any initial condition. That is, for any $z(0) \in R^n$, there exists a constant T, such that

$$\lim_{t\to T}\|z(t)\| = 0.$$

Definition 2. [41] (PTS-RS) The nonlinear system $\dot{z} = G(t,z), t \ge 0$ is said to be prescribed-time stable and remains stable on $[0,\infty)$ if it is PTS on $[0,T)$ and remain stable after T. That is, for any $z(0) \in R^n$, there exists a constant T, any $t' \ge T$, such that

$$\lim_{t\to T}\|z(t)\| = 0,\ \lim_{t\to t'}\|z(t)\| = 0.$$

Remark 2. In fact, Definition 2 is a special case of Definition 1 in [41] and a further definition of prescribed-time stability. A controlled system is PTS-RS, meaning that it is not only prescribed-time stable at $[0,T)$ but also continues to operate and remain stable on $[T,\infty)$.

In the sense of Definition 2, the PTS-RS control objective, to be studied in this paper, is to design a suitable control protocol $u(t)$ such that

1) $x(t)$ converges to zero when $t \to T$ and remains at zero for $t \ge T$;

2) For $t \in [0, \infty)$, the state $x(t)$ of the system is continuous, and the controller $u(t)$ is bounded by the initial condition.

2.2. Preliminaries

To accomplish the PTS-RS of the state feedback for system (1), similarly to [36], the state feedback is designed dependently on the following PLE:

$$A^T P + PA - Pbb^T P = -\mu(t)P, \tag{5}$$

where $\mu(t) > 0$ is a time-varying function to be designed. In the sequel, some lemmas on PLE (5) are listed.

Lemma 1. [37] For controllable $(A, b) \in (R^{n\times n}, R^{n\times 1})$ given by (2), the PLE (5) has a unique positive definite solution

$$P(\mu) = \mu L(\mu) P_1 L(\mu), \tag{6}$$

where $P_1 = P(1)$ and

$$L(\mu) = diag\{\mu^{n-1}, \mu^{n-2}, \cdots, 1\}. \tag{7}$$

Lemma 2. If Assumption 1 holds, $L(\mu)$ is defined as (7), $\tilde{\mu} > 0$ is a given constant. Then, for any $\mu \geq \tilde{\mu} > 0$, any $x \in R^n, u \in R, t \geq 0$

$$\big(L(\mu)F(x,u,t)\big)^T \big(L(\mu)F(x,u,t)\big) \leq g^2(\tilde{\mu})\big(L(\mu)x(t)\big)^T \big(L(\mu)x(t)\big), \tag{8}$$

where

$$g^2(\tilde{\mu}) = \max\left\{\sum_{i=1}^{n} \frac{c_i^2 i}{\tilde{\mu}^{2(i-1)}}, \sum_{i=2}^{n} \frac{c_i^2 i}{\tilde{\mu}^{2(i-2)}}, \cdots, \sum_{i=n}^{n} \frac{c_i^2 i}{\tilde{\mu}^{2(i-n)}}\right\} \geq 0. \tag{9}$$

Remark 3. Lemma 2 here is similar to Lemma 1 in [37]. Hence, the proof process is omitted here.

Next, we give some properties of the general parameter Lyapunov equation. Consider the following general parameter Lyapunov equation:

$$A^T P + PA - PBB^T P = -\mu P. \tag{10}$$

Lemma 3. [36] Let $(A, B) \in (R^{n\times n}, R^{n\times m})$ be controllable.

1) The PLE (10) has a unique positive definite solution $P(\mu)$ if and only if

$$\mu > -2 \min_{i=1,2,\cdots,n} \{Re\{\lambda_i(A)\}\}. \tag{11}$$

In this case, $P(\mu)$ is given by $P(\mu) = H^{-1}(\mu)$, where $H(\mu)$ satisfies the following Lyapunov equation:

$$\left(A + \frac{\mu}{2} I_n\right) H + H \left(A + \frac{\mu}{2} I_n\right)^T = BB^T. \tag{12}$$

2) Under (11), there are

$$\frac{dP(\mu)}{d\mu} > 0, \tag{13}$$

$$tr(B^T PB) = 2tr(A) + n\mu. \tag{14}$$

Lemma 4. [36] Under PLE (5) and the controllable $(A, b) \in (R^{n\times n}, R^{n\times 1})$ given by (2), if all the eigenvalues of A are zero, then there is a constant $\delta \geq 1$, such that

$$\frac{1}{n\mu}P(\mu) \leq \frac{dP(\mu)}{d\mu} \leq \frac{\delta}{n\mu}P(\mu), \tag{15}$$

where

$$\begin{aligned} &\delta = n\big(1 + \lambda_{max}(E + P_1 E P_1^{-1})\big),\\ &E = diag\{n-1, n-2, \cdots, 1, 0\},\\ &P_1 = P(1). \end{aligned} \tag{16}$$

Proof. The left inequality in (15) can be found in [36], we just provide proof for the right side in (15). Taking the derivative of (6) with respect to μ on both sides gives

$$\begin{aligned} \frac{dP(\mu)}{d\mu} &= L(\mu)P_1 L(\mu) + \mu\frac{dL(\mu)}{d\mu}P_1 L(\mu) + \mu L(\mu)P_1\frac{dL(\mu)}{d\mu}\\ &= \frac{P(\mu)}{\mu} + \frac{dL(\mu)}{d\mu}L(\mu)^{-1}P(\mu) + P(\mu)L(\mu)^{-1}\frac{dL(\mu)}{d\mu}\\ &= \frac{P(\mu)}{\mu} + \frac{E}{\mu}P(\mu) + P(\mu)\frac{E}{\mu}\\ &= \frac{P(\mu)}{\mu} + \frac{E}{\mu}\mu L(\mu)P_1 L(\mu) + \mu L(\mu)P_1 L(\mu)\frac{E}{\mu}\\ &= \frac{P(\mu)}{\mu} + L(\mu)(EP_1 + P_1 E)L(\mu), \end{aligned} \tag{17}$$

where, with a view to the structure of L, we have used

$$L^{-1}\frac{dL}{d\mu} = diag\left(\frac{n-1}{\mu}, \frac{n-2}{\mu}, \cdots, 0\right) = \frac{E}{\mu}. \tag{18}$$

According to the definition of δ in (16), we have

$$\begin{aligned} \frac{\delta}{n} - 1 &= \lambda_{max}\big(E + P_1 E P_1{}^{-1}\big)\\ &= \lambda_{max}\left(P_1^{-\frac{1}{2}}(EP_1 + P_1 E)P_1^{-\frac{1}{2}}\right), \end{aligned} \tag{19}$$

which implies $EP_1 + P_1 E \leq (\frac{\delta}{n} - 1)P_1$. Therefore, we can obtain that

$$\frac{dP}{d\mu} \leq \frac{P}{\mu} + \left(\frac{\delta}{n} - 1\right) L P_1 L = \frac{\delta P}{n\mu}, \tag{20}$$

which is the desired inequality.

3. Main results

Theorem 1. Under Assumption 1, if the following control scheme is applied

$$u(t) = -\frac{1}{2} b^T P\big(\mu(t)\big) x(t), \tag{21}$$

$$\mu(t) = \begin{cases} \dfrac{e^{\alpha T} - 1}{e^{\alpha T} - e^{\alpha t}} \mu_0, & t \in [0, T) \\ e^{\alpha(t-T)} \mu_T, & t \in [T, \infty) \end{cases} \tag{22}$$

with

$$\mu_0 \geq \frac{2g(\mu_0)}{1 - e^{-\alpha T}}, \alpha = \alpha(\mu_0) \triangleq \frac{2ng(\mu_0)}{n+\delta}, \tag{23}$$

$$\mu_T \geq 4g(\mu_T), \tag{24}$$

then the PTS-RS of system (1) can be achieved. Particularly, the control for the closed-loop system (1) under (21)–(24) satisfies, for $\forall\, x(0), T$,

$$\|u(t)\| \leq \frac{1}{2}\sqrt{n\mu_0 x(0)^T P(\mu_0) x(0)}, t \in [0, T), \tag{25}$$

$$\|u(t)\| \leq \frac{1}{2}\sqrt{n\mu_T x(T)^T P(\mu_T) x(T)}, t \in [T, \infty), \tag{26}$$

in which $T > 0$ is the prescribed stable time constant, $g(\cdot)$ is defined as (9), $P(\mu)$ is the only positive definite solution of the PLE (5).

Proof. It is evident that if $g(\cdot) = 0$, i.e., $c_i = 0$ in (9), then $f_i = 0$ in (4), system (1) is simplified to the form of an integrator chain, and Theorem 1 reduces to Theorem 2 in [36]. Therefore, we assume $g(\cdot) > 0$.

At phase $[0, T)$, denote

$$\theta_1(\mu_0) = \frac{2g(\mu_0)}{1 - e^{-\alpha(\mu_0)T}}, \mu_0 \in (0, +\infty), \tag{27}$$

it follows from $\frac{d(g(\mu_0))}{d\mu_0} \leq 0$ that

$$\frac{d(\theta_1(\mu_0))}{d\mu_0} = \frac{2e^{\frac{2nTg(\mu_0)}{n+\delta}}\left(e^{\frac{2nTg(\mu_0)}{n+\delta}}-\left(\frac{2nTg(\mu_0)}{n+\delta}+1\right)\right)}{\left(e^{\frac{2nTg(\mu_0)}{n+\delta}}-1\right)^2}\frac{d(g(\mu_0))}{d\mu_0} \le 0. \tag{28}$$

Notice that $\lim_{\mu_0\to 0} g(\mu_0) = +\infty$, $\lim_{\mu_0\to\infty} g(\mu_0) < \infty$, which implies that

$$\lim_{\mu_0\to 0}\frac{2g(\mu_0)}{1-e^{-\alpha(\mu_0)T}} = \frac{2}{1-e^{-\frac{2ng(\mu_0)}{n+\delta}T}} \to +\infty, \tag{29}$$

$$\lim_{\mu_0\to\infty}\frac{2g(\mu_0)}{1-e^{-\alpha(\mu_0)T}} = \frac{2g(\mu_0)}{1-e^{-\frac{2ng(\mu_0)}{n+\delta}T}} < \infty. \tag{30}$$

Therefore, the existence of $\mu_* > 0$ can be guaranteed, then (23) is satisfied for all $\mu_0 \ge \mu_*$, and μ_* can be obtained by solving the following equation:

$$\mu_* = \frac{2g(\mu_*)}{1-e^{-\alpha(\mu_*)T}}. \tag{31}$$

At phase $[T,\infty)$, define

$$\theta_2(\mu_T) = 4g(\mu_T), \mu_T \in (0,+\infty), \tag{32}$$

then we have

$$\frac{d(\theta_2(\mu_T))}{d\mu_T} \le 0. \tag{33}$$

Notice that

$$\lim_{\mu_T\to 0} 4g(\mu_T) = \infty, \tag{34}$$

$$\lim_{\mu_T\to\infty} 4g(\mu_T) < \infty. \tag{35}$$

Therefore, the existence of $\mu_{**} > 0$ can be guaranteed, then (24) is satisfied for all $\mu_T \ge \mu_{**}$, and μ_{**} be obtained by solving the following equation:

$$\mu_{**} = 4g(\mu_{**}). \tag{36}$$

In the proof below, we omit the independent variable t for all variables for the sake of simplicity. Moreover, we denote $P = P(\mu)$ for short. The controlled system follows (2) and (21) that

$$\dot{x} = \left(A - \frac{1}{2}bb^TP\right)x + F(x,u,t). \tag{37}$$

Consider the Lyapunov-like function

$$V(t,x) = n\mu x^TPx. \tag{38}$$

By (5) and (15), along the trajectory of (37), it can be evaluated as

$$\begin{aligned}
\dot{V}(t,x) &= n\dot{\mu}x^TPx + n\mu\left(\dot{x}^TPx + x^T\frac{dP}{d\mu}\dot{\mu}x + x^TP\dot{x}\right) \\
&= n\dot{\mu}x^TPx + n\mu\left[(x^TA^T + u^Tb^T + F^T)Px + x^T\frac{dP}{d\mu}\dot{\mu}x + x^TP(Ax + bu + F)\right] \\
&= n\dot{\mu}x^TPx + n\mu\left[x^T(A^TP + PA)x + 2x^TPbu + 2x^TPF + x^T\frac{dP}{d\mu}\dot{\mu}x\right] \\
&= n\dot{\mu}x^TPx + n\mu\left[x^T(A^TP + PA)x - x^TPbb^TPx + 2x^TPF + x^T\frac{dP}{d\mu}\dot{\mu}x\right] \\
&= n\dot{\mu}x^TPx + n\mu\left[x^T(A^TP + PA - Pbb^TP)x + 2x^TPF + x^T\frac{dP}{d\mu}\dot{\mu}x\right] \\
&\leq n\dot{\mu}x^TPx - n\mu^2x^TPx + 2n\gamma x^TPF + n\mu\frac{\delta}{n\mu}\dot{\mu}x^TPx \\
&= \left((n+\delta)\dot{\mu} - n\mu^2\right)x^TPx + 2n\mu x^T \\
&\leq \left((n+\delta)\dot{\mu} - n\mu^2\right)x^TPx + n\mu s x^TPx + \frac{n\mu F^TPF}{s},
\end{aligned} \tag{39}$$

where $F = F(x,u,t)$, $s > 0$ is a scalar to be determined.

By Lemmas 1 and 2, as $\mu(t)$ in two stages is increasing, we have

$$\begin{aligned}
F^TPF &= F^T\mu LP_1LF = \mu P_1(LF)^T(LF) \\
&\leq \mu P_1 g(\tilde{\mu})^2(Lx)^T(Lx) = \mu g(\tilde{\mu})^2 x^TL^TP_1Lx \\
&= g(\tilde{\mu})^2x^TPx,
\end{aligned} \tag{40}$$

where

$$\tilde{\mu} = \begin{cases} \mu_0, t \in [0,T) \\ \mu_T, t \in [T,\infty) \end{cases}. \tag{41}$$

Case 1. For $t \in [0,T)$, if we choose $s = g(\mu_0)$, $\dot{V}(t,x)$ can be continued as

$$\begin{aligned}
\dot{V}(t,x) &\leq \left((n+\delta)\dot{\mu} - n\mu^2 + n\mu s + \frac{n\mu g(\mu_0)^2}{s}\right)x^TPx \\
&= (n+\delta)\left(\dot{\mu} - \frac{\alpha}{2g(\mu_0)}\mu^2 + \alpha\left(\frac{s}{2g(\mu_0)} + \frac{g(\mu_0)}{2s}\right)\mu\right)x^TPx \\
&= (n+\delta)\left(\dot{\mu} - \frac{\alpha}{2g(\mu_0)}\mu^2 + \alpha\mu\right)x^TPx \\
&\triangleq (n+\delta)h(\mu)x^TPx.
\end{aligned} \tag{42}$$

With (23),

$$
\begin{aligned}
h(\mu) &= \dot{\mu} - \frac{\alpha}{2g(\mu_0)}\mu^2 + \alpha\mu \\
&= \frac{\alpha\mu_0(e^{\alpha T}-1)^2}{(e^{\alpha T-\alpha t})^2}\left(\frac{e^{\alpha T}}{e^{\alpha T}-1} - \frac{\mu_0}{2g(\mu_0)}\right) \\
&\leq 0, t \in [0,T),
\end{aligned} \tag{43}
$$

therefore, it yields from (42) that

$$
\dot{V}(t,x) \leq 0, t \in [0,T), \tag{44}
$$

which implies

$$
V(t,x) \leq V\big(0,x(0)\big), t \in [0,T). \tag{45}
$$

Notice that

$$
V\big(0,x(0)\big) = n\mu_0 x_0^T P(\mu_0)x(0) \leq n\mu_0\lambda_{max}\big(P(\mu_0)\big)\|x(0)\|^2, \tag{46}
$$

$$
V\big(t,x(t)\big) \geq n\mu(t)x^T(t)P(\mu_0)x(t) \geq n\mu(t)\lambda_{min}\big(P(\mu_0)\big)\|x(t)\|^2, \tag{47}
$$

where we have used (13) in Lemma 3. Hence, we are able to obtain from (29) that

$$
\|x(t)\| \leq \sqrt{\frac{\lambda_{max}(P(\mu_0))}{\lambda_{min}(P(\mu_0))}}\sqrt{\frac{e^{\alpha T}-e^{\alpha t}}{e^{\alpha T}-1}}\|x(0)\|. \tag{48}
$$

It implies that, for any $\varepsilon > 0$, there exist a $\delta(\varepsilon) > 0$, such that $\|x(t)\| \leq \varepsilon, \forall t \geq 0$ for any $\|x(0)\| \leq \delta$, namely, $\lim_{t\to T^-}\|x(t)\| = 0$.

Case 2. For $t \in [T,\infty)$, if we choose $s = g(\mu_T)$, we can obtain that

$$
\begin{aligned}
\dot{V}(t,x) &\leq \left((n+\delta)\dot{\mu} - n\mu^2 + n\mu s + \frac{n\mu g(\mu_T)^2}{s}\right)x^T P x \\
&= (n+\delta)\left(\dot{\mu} - \frac{\alpha}{2g(\mu_T)}\mu^2 + \alpha\left(\frac{s}{2g(\mu_T)} + \frac{g(\mu_T)}{2s}\right)\mu\right)x^T P x \\
&= (n+\delta)\left(\dot{\mu} - \frac{\alpha}{2g(\mu_T)}\mu^2 + \alpha\mu\right)x^T P x \\
&\triangleq (n+\delta)h(\mu)x^T P x.
\end{aligned} \tag{49}
$$

With (24),

$$\begin{aligned} h(\mu) &= \dot{\mu} - \frac{\alpha}{2g(\mu_T)}\mu^2 + \alpha\mu \\ &= \alpha\mu_T e^{2\alpha(t-T)}\left(1 - 2e^{\alpha(T-t)} - \frac{\mu_T}{2g(\mu_T)}\right) \\ &\leq 0, t \in [T, \infty), \end{aligned} \tag{50}$$

we can get

$$\dot{V}(t, x) \leq 0, t \in [T, \infty), \tag{51}$$

which means

$$V(t, x) \leq V(T, x(T)), t \in [T, \infty). \tag{52}$$

Notice that

$$V\big(T, x(T)\big) = n\mu_T x(T)^T P(\mu_T) x(T) \leq n\mu_T \lambda_{max}\big(P(\mu_T)\big)\|x(T)\|^2, \tag{53}$$

$$V\big(t, x(t)\big) \geq n\mu(t) x^T(t) P(\mu_T) x(t) \geq n\mu(t)\lambda_{min}\big(P(\mu_T)\big)\|x(t)\|^2. \tag{54}$$

Thus, we are able to get from (52) that

$$\begin{aligned} \|x(t)\| &\leq \sqrt{\frac{\lambda_{max}\big(P(\mu_T)\big)}{\lambda_{min}\big(P(\mu_T)\big)}}\sqrt{\frac{1}{e^{\alpha(t-T)}}}\|x(T)\| \\ &\triangleq km(t)\|x(T)\|, \end{aligned} \tag{55}$$

where $m(t) = \sqrt{\frac{1}{e^{\alpha(t-T)}}} \leq 1$ is a decreasing positive function. From $\lim_{t\to T^-}\|x(t)\| = 0$, and taking into account the continuity of $x(t)$ (since $x(t)$ is differentiable), it follows that $x(T) = 0$. Furthermore, for $\forall\, T < t' < \infty$, there is $\lim_{t\to t'}\|x(t)\| = 0$. As t goes to infinity, we have $\lim_{t\to\infty} m(t) = 0$, that is, $\lim_{t\to\infty}\|x(t)\| = 0$.

Combining Cases 1 and 2 together, the controlled system (1) is prescribed-time stable in $[0, T)$ and remains stable after T, which is actually the PTS-RS in Definition 2.

Finally, by (14), noticing $tr(A) = 0$, we have

$$\begin{aligned} \|u(t)\|^2 &= \frac{1}{4}x^T P b b^T P x \leq \frac{1}{4}x^T P^{\frac{1}{2}} tr\left(P^{\frac{1}{2}} b b^T P^{\frac{1}{2}}\right) P^{\frac{1}{2}} x \\ &= \frac{1}{4}tr(b^T P b) x^T P x = \frac{1}{4}(2tr(A) + n\mu) x^T P x = \frac{1}{4}V(t, x), \end{aligned} \tag{56}$$

then, by (45) and (52),

$$\|u(t)\|^2 \leq \tfrac{1}{4}V\big(0, x(0)\big) = \tfrac{1}{4}n\mu_0 x(0)^T P(\mu_0) x(0), t \in [0, T), \tag{57}$$

$$\|u(t)\|^2 \le \frac{1}{4}V\big(T,x(T)\big) = \frac{1}{4}n\mu_T x(T)^T P(\mu_T)x(T), t \in [T,\infty), \tag{58}$$

which shows the boundedness of controller. The proof is finished.

Remark 4. It is worth noting that $\dot{V}(t,x) \le 0$ holds on $[0,\infty)$, which means that $V(t,x) \le V\big(T,x(T)\big) \le V\big(0,x(0)\big)$. Thus, it can be obtained directly $V(t,x) \le V\big(0,x(0)\big)$ on $[0,\infty)$, that is,

$$\|u(t)\| \le \frac{1}{2}\sqrt{n\mu_0 x(0)^T P(\mu_0)x(0)}, t \in [0,\infty).$$

For structural wholeness, we give boundedness on $u(t)$ in the two stages.

Remark 5. By the parameter Lyapunov method, the design of the prescribed-time stabilizing controller has been converted into the design of the function $\mu(t)$. Actually, according to the proof of Theorem 1, we can see that $\mu(t)$ should be designed to satisfy the scalar differential inequalities (43) and (50), namely,

$$h(\mu) = \dot{\mu} - \frac{\alpha}{2g(\mu_0)}\mu^2 + \alpha\mu \le 0.$$

Therefore, for different forms of $\mu(t)$ in [42], as long as appropriate function $\mu(t)$ can be designed to satisfy $h(\mu) \le 0$, then they are feasible. For example, when $\mu_1(t) = -\frac{1}{(1-\alpha_1)(T-t)}$, for $0 < \alpha_1 < 1$, Eq (43) takes the following form:

$$\begin{aligned} h(\mu_1) &= \frac{1}{(1-\alpha_1)^2(T-t)^2}\left[-(1-\alpha_1) - \frac{\alpha}{2g} - \alpha(1-\alpha_1)(T-t)\right] \\ &= \frac{1}{(1-\alpha_1)^2(T-t)^2}\left[(\alpha_1 - 1)\left[1 + \frac{2ng(\mu_0)}{n+\delta}(T-t)\right] - \frac{n}{n+\delta}\right] \\ &\le 0. \end{aligned}$$

Then, the prescribed-time stability of controlled system on $[0,T)$ can be achieved, while the control scheme on $[T,\infty)$ needs to be further designed. We will study the stabilizing effect of $\mu(t)$ with different forms in our future work.

Remark 6. We resort to the key properties of the solution of PLE (5) to solve the prescribed-time stability of a class of nonlinear systems with uncertainty/disturbance. The design is inspired by work [37], but there are some significant differences. 1) The controller in [37] is a bit complicated, and it works only in the prescribed time instant. Specifically, on $[0,T)$, the controller in [37] is designed as

$$u(t) = -\frac{1}{2}b^T P\big(\gamma(t)\big)x(t),\ \gamma(t) = \frac{e^{\alpha_1\beta T}-1}{e^{\alpha_1\beta T}-e^{\alpha_1\beta t}}\gamma_0,$$

with four parameters

$$\alpha_1 \triangleq \frac{n}{n+\delta},\ \beta = \beta(\gamma_0) \triangleq 2g(\gamma_0)\Lambda,\ \Lambda \triangleq \lambda_{max}(P_n),\ \gamma_0.$$

However, in our work, on $[0,T)$, we improve the controller by simplifying four parameters in to two parameters, i.e., α, μ_0, resulting in a more concise control scheme; 2) [37] considers only the stability of $[0, T)$ theoretically, but the stability after T cannot be guaranteed. We design the controller for $[0, T)$ and $[T, \infty)$ to ensure that the controlled system can achieve the prescribed time stability within T and remain stable after T.

Remark 7. Theorem 1 is carried out under Assumption 1, i.e., the lower triangular linear growth condition (4) without considering other cases, such as the upper-triangular linear growth condition [39] and triangular nonlinear growth condition [40], which is the limitation of our result.

It follows from (21) and (22) that when t approaches the settling time T, $\mu(t)$ approaches to infinity, that is, "$\infty \times 0$" is included in $u(t)$, which may lead to singularity problems in the calculation. In order to avoid it, (22) can be replaced by

$$\mu(t) = \begin{cases} \frac{e^{\alpha T}-1}{e^{\alpha T}-e^{\alpha t}}\mu_0, t \in [0, T_*) \\ \frac{e^{\alpha T}-1}{e^{\alpha T}-e^{\alpha T_*}}\mu_0, t \in [T_*, T) \\ e^{\alpha(t-T)}\mu_T,\ t \in [T, \infty) \end{cases}, \tag{59}$$

where $T_* = T - \varepsilon_T$, with ε_T being a sufficiently small number [36].

4. Numerical example

We illustrate the functionality of the controller with the following second-order nonlinear system

$$\begin{cases} \dot{x}_1 = x_2 + x_1 \sin(x_2^2) \\ \dot{x}_2 = u + x_1 \sin(u x_2) + x_2 \\ y = x_1 \end{cases}. \tag{60}$$

Obviously,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, b = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{61}$$

According to (12) in Lemma 3, we have

$$P(\mu) = H^{-1}(\mu) = \begin{bmatrix} \mu^3 & \mu^2 \\ \mu^2 & 2\mu \end{bmatrix}, \tag{62}$$

By Theorem 1, we design

$$u(t) = -\frac{1}{2} b^T P\big(\mu(t)\big)x(t) = -\frac{1}{2}\big(\mu(t)^2 x_1(t) + 2\mu(t) x_2(t)\big), \tag{63}$$

where, for practical calculation,

$$\mu(t) = \begin{cases} \frac{e^{\alpha T}-1}{e^{\alpha T}-e^{\alpha t}}\mu_0, & t \in [0, T - \varepsilon_T) \\ \frac{e^{\alpha T}-1}{e^{\alpha T}-e^{\alpha (T-\varepsilon_T)}}\mu_0, & t \in [T - \varepsilon_T, T) \\ e^{\alpha(t-T)}\mu_T, & t \in [T, \infty) \end{cases}. \tag{64}$$

Further, we have

$$|f_1(x,t,u)| = |x_1 \sin(x_2^2)| \le |x_1|, \tag{65}$$

$$|f_2(x,t,u)| = |x_1 \sin(u x_2) + x_2| \le |x_1| + |x_2|. \tag{66}$$

Thus, Assumption 1 holds with $c_1 = 1, c_2 = 1$. It follows from Lemma 2 that

$$g^2(\tilde{\mu}) = \max\left\{{c_1}^2 + \frac{2{c_2}^2}{\tilde{\mu}^2}, 2{c_2}^2\right\} = \max\left\{1 + \frac{2}{\tilde{\mu}^2}, 2\right\}. \tag{67}$$

Case 1. $\tilde{\mu} < \sqrt{2}$, then $g^2(\tilde{\mu}) = 1 + \frac{2}{\tilde{\mu}^2}$. By solving the nonlinear inequality (31), we get $\mu_* = 2.8850$, which results in a contradiction.

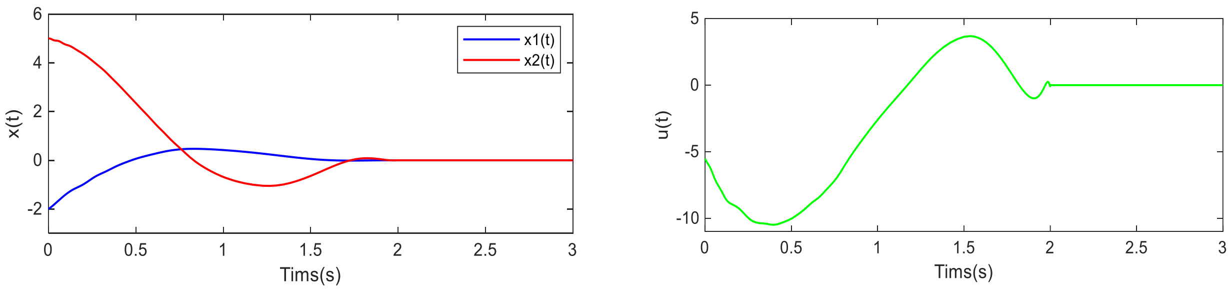
Case 2. $\tilde{\mu} \ge \sqrt{2}$, then $g^2(\tilde{\mu}) = 2$. By solving (31) we get $\mu_* = 3.3561$, thus we choose $\mu_0 = 3.3561$. With the same method, by solving the nonlinear inequality (36), we get $\mu_T = 4$. Other parameters are calculated as follows.

$$P_1 = P(1) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

$$\delta = n\left(1 + \lambda_{max}(E + P_1 E P_1^{-1})\right) = 6.8284,$$

$$\alpha = \frac{2ng(\mu_0)}{n + \delta} = 0.4531.$$

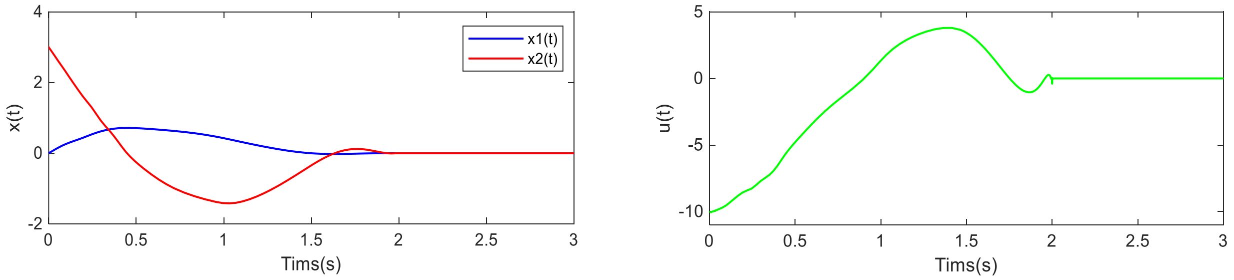
The settling time is set as $T = 2$, and the parameter ε_T in (59) is chosen as 0.0001. We use two initial conditions for simulation: (-2,5), (0,3). The states and control signal of the closed-loop system (60) are shown in Figures 1 and 2, from which we can observe the PTS-RS of a controlled system (60) under (63) and (64) and the boundedness of the control signal. To furtherly compare our method with [37], we apply the controller in [37] to the system (60). The results are shown in Figure 3.



(a) State

(b) Control signal

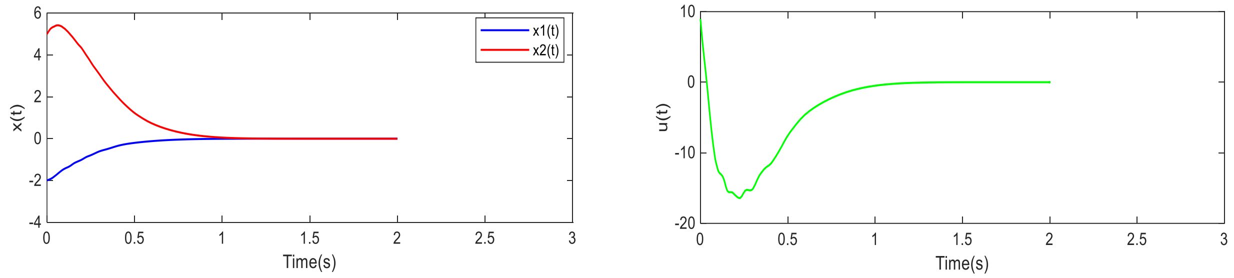
Figure 1. Response of system (60) with control law (63) under $x(0) = (-2,5)$.



(a) State

(b) Control signal

Figure 2. Response of system (60) with control law (63) under $x(0) = (0,3)$.



(a) State

(b) Control signal

Figure 3. Response of system (60) with control law in [37] under $x(0) = (-2,5)$.

On $[0, T)$, according to the design theory of [37], the controller is designed as

$$u(t) = -b^T P\big(\gamma(t)\big)x(t),\ \gamma(t) = \frac{e^{\alpha_1 \beta T} - 1}{e^{\alpha_1 \beta T} - e^{\alpha_1 \beta t}}\gamma_0,$$

in which the parameters are calculated as $\gamma_0 = 3.7971$, $\alpha_1 = 0.2265$, $\beta = 2.6587$, $\Lambda = 2.6180$. Compared with [37], our controller (63) and (64) with $\alpha = 0.4531$, $\mu_0 = 3.3561$ is not only significantly simpler, but also enables the controlled system to achieve PTS within $[0, T)$.

In addition, it is seen that the controller in [37] achieves stability on only $[0, T)$, and the subsequent states of controlled system on $[T, \infty)$ cannot be guaranteed. We design the controller for $[0, T)$ and $[T, \infty)$ to ensure that the controlled system remains stable after T, which confirms the appealing performance of the proposed PTS-RS scheme.

5. Conclusions

In this paper, the prescribed-time stability problem and its extension have been addressed for a class of nonlinear systems with uncertainty/disturbance. With the help of the parametric Lyapunov equation, based on research [37], we improve the controller for $[0, T)$ and design a new controller for $[T, \infty)$. The full state regulation is realized within the prescribed time T that is irrespective of an initial condition or any parameter, and the corresponding control is fully bounded over the whole time interval $[0, \infty)$. The efficiency of the proposed method is verified. Considering that $\mu(t)$ is feasible for a controller as long as it is designed to satisfy $h(\mu) \leq 0$, we will consider the effects of different forms of $\mu(t)$ in our future work.

Author contributions

Lichao Feng: Conceptualization, Investigation, Supervision, Funding acquisition; Mengyuan Dai: Investigation, Methodology, Writing – original draft, Validation; Nan Ji: Investigation; Yingli Zhang: Data curation and Software; Liping Du: Formal Analysis. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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