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Research article

Security of image transfer and innovative results for (p,q)-Bernstein-Schurer operators

Nazmiye Gonul Bilgin*, Yusuf Kaya and Melis Eren

Department of Mathematics, Faculty of Science, Zonguldak Bulent Ecevit University, Zonguldak 67100, Turkey

* **Correspondence:** Email: nazmiyegonul@beun.edu.tr; Tel: +903722911602.

Abstract: With the advent of quantum computing, traditional cryptography algorithms are at risk of being broken. Post-quantum encryption algorithms, developed to include mathematical challenges to make it impossible for quantum computers to solve problems, are constantly being updated to ensure that sensitive information is protected from potential threats. In this study, a hybrid examination of a (p,q)-Bernstein-type polynomial, which is an argument that can be used for encryption algorithms with a post-quantum approach, was made from a mathematical and cryptography perspective. In addition, we have aimed to present a new useful operator that approximates functions and can be used in cases where it is not possible to work with functions in the fields of technology, medicine, and engineering. Based on this idea, a new version of the (p,q)-Bernstein-Schurer operator was introduced in our study on a variable interval and the convergence rate was calculated with two different methods. At the same time, the applications of the theoretical situation in the study were presented with the help of visual illustrations and tables related to the approach. Additionally, our operator satisfied the statistical-type Korovkin theorem and is suitable for variable interval approximation. This is the first paper to study the statistical convergence properties of (p,q)-Bernstein-Schurer operators defined on a variable bounded interval, to obtain special matrices with the help of (p,q)-basis functions, and to give an application of (p,q)-type operators for encrypted image transmission.

Keywords: (p,q)-Bernstein-Schurer operators; encryption; special matrices; statistical convergence
Mathematics Subject Classification: 41A10, 41A25, 41A36, 68W99

1. Introduction

In the digital world where we need to communicate and transfer information more and more, we have to protect the information we want to transfer, our personal data, and our interests from the dangerous actions of malicious people [1]. The elements of communication are sender, receiver, messages, channels, flashbacks, codes, encoding and decoding, noise, and status [2]. In today's rapidly advancing technology, security has become an issue that needs to be evaluated separately in order to protect data from external threats during storage and transmission. The way to protect data from destruction and prevent unauthorized access is encryption. Encryption is used for transmitted data or plaintext to produce data or password text, whereas decryption is used to regenerate the original data from the password text through the secret key shared between the receiver and the transmitter. The information security practice, known as cryptology, is related to protecting data integrity, ensuring confidentiality and authentication [3]. Cryptology, which for centuries has focused on the codes used for secret communication, today encompasses much more: Cryptology today deals with techniques for exchanging secret keys, mechanisms for ensuring data integrity, protocols for authentication, digital coin, electronic elections, and more. Modern cryptography involves the study of complex computational and mathematical techniques for protecting digital information and systems against enemy attacks [4].

Ensuring data security requires a complex and multidimensional approach. For this, precautions such as encryption of data and using strong authentication methods should be taken. Along with the studies carried out to prevent information loss, to cope with the noise problem and the need for excessive memory, research to prevent security vulnerabilities continues at a great pace, both theoretically and experimentally. Following the initiation of research on post-quantum cryptography by the National Institute of Standards and Technology in 2012, global studies began in 2016 to establish standards for post-quantum encryption. As of 2017, the National Institute of Standards and Technology has published 69 draft algorithms using four main mathematical methods: lattice-based algorithms, code-based algorithms, algorithms based on multivariate polynomials, and hash-based signature algorithms [5].

Polynomials are used in many fields other than mathematics. For example, in [6], the energy compression performance of several different signal decomposition techniques for AR(l) signal sources, where a set of specific sample points of the square function of the desired size was approximately found using Bernstein polynomials, was mentioned. In [7], a steering law called range polynomial guidance is presented for the pulse angle control problem by shaping the viewing light angle with relative range by applying a polynomial shaping method.

Polynomials are used as a cryptographic tool in important issues such as protecting information and the secure transfer of images. An example is [8], where classical Bernstein polynomials were used. In that study, which aimed to ensure secure transmission of the image with the public key cryptographic technique with the help of the Bernstein polynomial, the original image was compressed using the Bernstein image. The compressed image was decrypted by the receiver using a cryptographic algorithm, and the decrypted image was decompressed to recover the main image.

Polynomials are used in many sub-branches of mathematics. One of these is approximation theory. This theory, which is of great importance for applied fields, has recently been an area where researchers have worked extensively. Bernstein operators, Szasz operators, Schurer operators, etc., are among the important operators used in approximation research. In addition, studies [9–11] can be used as examples of different operators defined.

The beginning of the operators in approximation theory is based on the Bernstein polynomial. Bernstein introduced a polynomial sequence named after him in [12]. Bernstein polynomials have many uses such as statistics, numerical analysis, and mechanical components. After the introduction of quantum and post-quantum ideas into many fields, the application of q-calculus and (p,q) has made important contributions to the development of approximation theory, applied mathematics, and engineering. With the development of q-analysis, many operators have been defined. For example, q generalization of a Bernstein-type operator is given in [13]. Although q-Bernstein polynomials and classical Bernstein polynomials are similar in terms of some properties, their convergence properties are different from each other. The convergence properties of q-Bernstein polynomials vary depending on whether q is in the range $(0,1)$ or not. Some of the studies carried out today involve the (p,q) versions of the operators as important study subjects, as in [14].

Mursaleen et al., in [15], gave the structure of (p,q)-Bernstein operators. For (p,q)-Bernstein operators, whether the operator is convergent or not depends on the range in which p and q are defined. Many different generalizations have been made about this operator. For example, a modification of the two-dimensional Bernstein-Stancu operator is studied in [16]. Karahan and Izgi gave a modification of this operator and Cevik examined this modification on a certain interval in [17,18], respectively.

On the other hand, Schurer defined a new operator in [19]. Cai and Zhou examined the important features of the Kantorovich-type Bernstein-Stancu-Schurer operator in [20]. Then, q- and (p,q)-Bernstein analogues of these operators were introduced in [21,22]. Many different modifications of the Schurer operator have been defined in the sense of q and then (p,q), e.g., [23,24]. The appear of the concept of statistical convergence has enabled important studies to be carried out in many sub-branches of functional analysis. [25] After the transfer of statistical convergence theory to approximation theory, it was investigated whether existing or newly defined operators in the literature would approximate statistical convergence [26]. Important properties on the variable range of one- and two-variable operator generalization were also examined in [27].

After summarizing the literature on approximation theory, before moving on to the requirements for the use of polynomials in cryptology, we will discuss image transmission, which is one of the data types for which different encryption algorithms are used. The most important problem in the use of wireless transmission of images, especially in the fields of the defense industry and the transfer of personal information to those concerned, is security. The data transmitted by wireless communication can be copied by espionage and unwanted malicious people and can reveal negative situations. In the Russia-Ukraine war, where UAVs were widely used, it was seen that the aforementioned situations were experienced and the side that gained superiority on the frontline in this way changed instantly. Because UAV systems communicate wirelessly with the ground control station and transmit real-time images, the above-mentioned situations occur if the security of the UAV platforms is not developed to resist electronic attacks. For this reason, war technologies are sometimes prepared before the war, and sometimes, as in this war, they are developed and made more effective in line with the needs during the war. Different encryption algorithms are used to prevent security problems during the transmission of images, and as technology develops, efforts to prevent the decrypt ability of ciphers cover the development of new methods or the strengthening of existing ciphers.

For this purpose, efficient image encryption algorithms that consume minimum power and work fast, along with expressions with mathematical difficulty, should be utilized. Since the 2000s, many mathematical concepts such as matrices, modular arithmetic, and polynomials have been used in encryption algorithms as the security of encryption methods is mostly based on the solution of mathematical problems. Without the use of quantum computers, there was no algorithm that worked in polynomial time to solve problems such as factorization and discrete logarithms. After quantum concepts are utilized, many problems that cannot be solved in polynomial time are transformed into ones that can be solved in polynomial time.

When determining an image encryption method, security, format suitability, time, and suitability for compression should be taken into account. For this reason, we consider the compression of the original image with the help of a sequence of polynomials using the polynomial image we define. Taking advantage of the difficulties of post-quantum numbers, the defined operator is based on the method described in [8], which uses the classical Bernstein polynomial for image transmission.

The contributions of this study to the literature can be presented as follows:

- The study contains important approximation properties of the (p,q)-Bernstein-Schurer operator on a variable bounded interval.
- Since the defined basis functions provide De Casteljau steps, a mathematical argument is presented for model validation and simulation studies that can be performed using Bezier curves such as designing air vehicle airfoil curves and route planning of unmanned aerial vehicles with the help of mathematical modeling.
- This work includes a cryptological algorithm based on (p,q)-Bernstein-Schurer functions that we define for the security of images that need to be protected during transmission between wireless communication devices, which is of great importance for local control, smart object design.
- The image transfer algorithm given in [8] using the classical Bernstein algorithm is modified with the innovations brought by the operator we defined with the aim of increasing the security so that the password cannot be decrypted by third parties thanks to the difficult calculations of (p, q)-numbers.
- As the defined operator has statistical approximation properties, it is a good reference for researchers who study different types of approximation.

In this study, considering the rapid developments in technology, medicine, and engineering, it is our aim to define and make available a new operator approaching functions, which can be used in situations where it is not possible to work with some functions in these areas. This study was created in order to fill the relevant gap in the literature as an intersection of (p,q)-operators and mobilization of the working interval. In our work, we give a new modification of (p,q)-Bernstein-Schurer-type operators inspired by [16–19,22–24,26]. We examinate the convergence properties and give moments of this operator. Also, important examples are given to support the theoretical information visually and numerically. Then, it is shown that the defined operator is suitable for statistical convergence by showing that it satisfies the Korovkin-type theorem. In Section 4, the (p,q)-Bernstein-Schurer basis functions that provide some properties used in De Casteljau algorithms are introduced and various equations are presented. In Section 5, matrix implementations of the defined basis functions are given, taking into account the applications of matrices in decision-making and cryptology. In the 6th part of the study, which includes cryptological data, an algorithm is given to encrypt the image for secure transmission and decrypt it by the receiver with the help of the (p,q)-operator, which we defined based on the method applied in [8] for the classical Bernstein operator.

2. Materials and methods

Now, we remind the reader of some definitions of (*p*,*q*)-analysis, which is worked in [28–30]. For all $p,q>0$, $[s]_{pq}$ are defined by

$$[s]_{pq}=\begin{cases}\frac{p^s-q^s}{p-q}, & \text{if } p\neq q\neq 1;\\ \frac{1-q^s}{1-q}, & \text{if } p=1;\\ s, & \text{if } p=q=1.\end{cases} \tag{1}$$

$[s]_{pq}!$ and $\begin{bmatrix} s \\ v \end{bmatrix}_{pq}$ are defined as follows:

$$[s]_{pq}!=\begin{cases}[s]_{pq}[s-1]_{pq}\cdots[1]_{pq} & ,if\ s\geq 1\\ 1 & ,if\ s=0\end{cases} \tag{2}$$

$$\begin{bmatrix} s \\ v \end{bmatrix}_{pq}=\frac{[s]_{pq}!}{[s-v]_{pq}![v]_{pq}!}. \tag{3}$$

For $m>0$, $(z-t)_{pq}^m$ is defined as

$$(z-t)_{pq}^m=\prod_{r=0}^{m-1}(p^rz-q^rt)=(z-t)(pz-qt)\ \times(p^2z-q^2t)\cdots(p^{m-1}z-q^{m-1}t). \tag{4}$$

For $m\leq 0$, $\prod_{r=0}^{m-1}(p^rz-q^rt)=0$. In this study, for $n=0$, $0^n:=1$ is taken. This section contains definitions and lemmas as a preparation for the main section. We introduce our new operators and the relevant equations are given by applying the operator to the test functions. Here, we define the next operator on $C\left[0,\frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$.

Definition 1. Let $p>q,p,q\in(0,1]$ and $0\leq x\leq\frac{[s+1]_{pq}}{[s+2]_{pq}}$. For every $f\in C\left[0,\frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$, the new type (*p*,*q*)-Bernstein-Schurer operators are defined by

$$\tilde{L}_s^{(p,q)}(f;x):=\frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}}\sum_{k=0}^{s+\eta}M_s(x)\times f\left(\frac{p^{s+\eta-k}[k]_{pq}[1+s]_{pq}}{[s]_{pq}[2+s]_{pq}}\right). \tag{5}$$

Here,

$$M_s(x):=\begin{bmatrix} s+\eta \\ k \end{bmatrix}_{pq}\left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta}\left(p^{\frac{k(k-1)}{2}}x^k\right)\prod_{t=0}^{s+\eta-k-1}\left(p^t\frac{[s+1]_{pq}}{[s+2]_{pq}}-q^tx\right). \tag{6}$$

Lemma 1. Let $p>q,f\in C\left[0,\frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$, and $p,q\in(0,1]$. $\tilde{L}_s^{(p,q)}(f;x)$ is positive and linear.

Lemma 2. For $1\geq p>q>0$, $x\in\left[0,\frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$,

$$\text{i.}\ \tilde{L}_s^{(p,q)}(1;x)=1, \tag{7}$$

$$\text{ii.}\, \tilde{L}_s^{(p,q)}(y;x) = x\frac{[s+\eta]_{pq}}{[s]_{pq}}, \tag{8}$$

$$\text{iii.}\ \tilde{L}_s^{(p,q)}(y^2;x) = \frac{[s+\eta]_{pq}[s+\eta-1]_{pq}}{[s]_{pq}^2} qx^2 + \frac{[s+\eta]_{pq}[s+1]_{pq}}{[s+2]_{pq}[s]_{pq}^2} p^{s+\eta-1}x. \tag{9}$$

Proof. i. Actually, by the definition of $\tilde{L}_s^{(p,q)}(f;x)$,

$$\tilde{L}_s^{(p,q)}(1;x) = \frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}}\sum_{k=0}^{s+\eta} M_s(x)$$

$$= \frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}}\sum_{k=0}^{s+\eta}\left(\left(\frac{[s+2]_{p,q}}{[s+1]_{p,q}}\right)^{s+\eta}\begin{bmatrix} s+\eta \\ k\end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}}x^k \prod_{t=0}^{s+\eta-k-1}\left(p^t\frac{[s+1]_{p,q}}{[s+2]_{p,q}} - q^t x\right)\right)$$

So we get $\tilde{L}_s^{(p,q)}(1;x) = 1$.

ii. From the definition of the operators, we have

$$\tilde{L}_s^{(p,q)}(y;x) = \frac{1}{p^{\frac{(\eta+s)(s-1+\eta)}{2}}} \times \sum_{k=0}^{s+\eta}\left(\left(\frac{[s+1]_{pq}}{[s+2]_{pq}[s]_{pq}}[k]_{pq}p^{s+\eta-k}\right)M_s(x)\right)$$

$$= \left(\frac{p^{s+\eta}[s+\eta]_{pq}}{p^{\frac{(\eta-1+s)}{2}(\eta+s)}[s]_{pq}}\left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta-1}\right)\sum_{k=0}^{s+\eta}\left(p^{-k}\begin{bmatrix} s+\eta-1 \\ k\end{bmatrix}_{pq}[k]_{pq}x^k p^{\frac{(k-1)k}{2}}\right.$$

$$\left.\times \prod_{t=0}^{-k+s+\eta-1}\left(p^t\frac{[1+s]_{pq}}{[2+s]_{pq}} - q^t x\right)\right)$$

$$= x\frac{[s+\eta]_{pq}}{[s]_{pq}}.$$

iii. $$\tilde{L}_s^{(p,q)}(y^2;x) = \frac{p^{2(s+\eta)}[s+\eta]_{pq}}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}[s]_{pq}^2}\left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta-2}$$

$$\times \sum_{k=0}^{s+\eta-1}\left(\begin{bmatrix} s+\eta-1 \\ k\end{bmatrix}_{pq}[1+k]_{pq}p^{\frac{(k+1)(k-4)}{2}}x^{k+1}\right.$$

$$\left.\times \prod_{t=0}^{-k+\eta+s-2}\left(p^t\frac{[1+s]_{pq}}{[2+s]_{pq}} - q^t x\right)\right).$$

From $[1+u]_{pq} = p^u + [u]_{pq}q$, we have

$$
\begin{aligned}
\tilde{L}_s^{(p,q)}(y^2;x) &= \frac{xp^{2(\eta+s-1)}[s+\eta]_{pq}}{[s]_{pq}^2} p^{\frac{(\eta-1+s)(s+\eta-2)}{2}} \frac{[1+s]_{pq}}{[2+s]_{pq}} \\
&\quad + \frac{p^{2(s+\eta)}q[s+\eta]_{pq}[s+\eta-1]_{pq}}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}[s]_{pq}^2} \left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta-2} \sum_{k=0}^{s+\eta-2} \left(\begin{bmatrix} s+\eta-2 \\ k \end{bmatrix}_{pq} \right. \\
&\quad \left. \times p^{\frac{(k+2)}{2}(k-3)} x^{k+2} \left(\frac{[s+1]_{pq}}{[s+2]_{pq}} - x\right)^{s+\eta-k-3} \right) \\
&= \frac{[\eta+s]_{pq}}{[s]_{pq}} qx^2 \frac{[s+\eta-1]_{pq}}{[s]_{pq}} + \frac{[s+1]_{pq}[s+\eta]_{pq}}{[s]_{pq}^2[s+2]_{pq}} p^{s+\eta-1}x.
\end{aligned}
$$

Now, from these equations, we can give important theorems about the operators. We will show that the operator we defined in this section satisfies the Korovkin theorem. The applicability of the operator is shown by giving the graphs of the approach.

Remark 1. Let $1 \geq p_s \geq q_s > 0$ and

$$\lim_{s\to\infty} p_s = \lim_{s\to\infty} q_s = 1. \tag{10}$$

Then, the following equalities hold:

$$\lim_{s\to\infty} \frac{p_s^s}{[s]_{p_s q_s}} = 0, \ \lim_{s\to\infty} \frac{q_s[s+\eta-1]_{p_s q_s}}{[s]_{p_s q_s}} = 1. \tag{11}$$

Theorem 1. Let $\lim_{s\to\infty} p_s = \lim_{s\to\infty} q_s = 1$ with $1 \geq p_s \geq q_s > 0$. In this case, $\tilde{L}_s^{(p_s,q_s)}(f,x)$ uniformly converges to f on $\breve{A} \coloneqq \left[0, \frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$, i.e., for every $f \in C(\breve{A})$,

$$\lim_{s\to\infty} \left\| \tilde{L}_s^{(p,q)}(f;x) - f(x) \right\|_{C(\breve{A})} = 0. \tag{12}$$

Proof. From the Korovkin theorem, it is enough to prove the next properties

$$\lim_{s\to\infty} \left\| \tilde{L}_s^{(p,q)}(y^\mu;x) - x^\mu \right\|_{C(\breve{A})} = 0.$$

Since

$$\lim_{s\to\infty} \left\| \tilde{L}_s^{(p,q)}(1;x) - 1 \right\|_{C(\breve{A})} = 0, \tag{13}$$

we can see that for $\mu = 1,2$

$$\lim_{s\to\infty} \left\| \tilde{L}_s^{(p,q)}(y^\mu;x) - x^\mu \right\|_{C(\breve{A})} = 0. \tag{14}$$

For $0 < q_s < p_s \leq 1$, using

$$\lim_{s\to\infty} \frac{[s+\eta]_{p_s q_s}}{[s]_{p_s q_s}} = \lim_{s\to\infty} \frac{[-1+s+\eta]_{p_s q_s}}{[s]_{p_s q_s}} q_s = 1, \tag{15}$$

we get

$$\lim_{s\to\infty} \left\| \tilde{L}_s^{(p_s,q_s)}(y;x) - x \right\|_{C(\breve{A})} \leq \lim_{s\to\infty} \left\| \left(-1 + \frac{[s+\eta]_{p_s q_s}}{[s]_{p_s q_s}}\right) \frac{[s+1]_{p_s q_s}}{[s+2]_{p_s q_s}} \right\|_{C(\breve{A})} = 0, \tag{16}$$

$$\lim_{s\to\infty} \left\| \tilde{L}_s^{(p_s,q_s)}(y^2;x) - x^2 \right\|_{C(\breve{A})} \leq \lim_{s\to\infty} \left\| \left(\frac{[s+\eta]_{p_s q_s} [s+\eta-1]_{p_s q_s}}{[s]_{p_s q_s}^2} q_s \right) \right\|_{C(\breve{A})}$$

$$+ \lim_{s\to\infty} \left\| \left(\frac{[s+1]_{p_s q_s}}{[s+2]_{p_s q_s}} \right)^2 \right\|_{C(\breve{A})} + \lim_{s\to\infty} \left\| \left(\frac{[s+\eta]_{p_s q_s}}{[s]_{p_s q_s}^2} p^{s+\eta-1} - 1 \right) \left(\frac{[s+1]_{p_s q_s}}{[s+2]_{p_s q_s}} \right)^2 \right\|_{C(\breve{A})} = 0.$$

Example 1. Let $f(x) = 2x^3 \left(\left(e^{\sin(x+1)}\right)^2 - \left(e^{\cos(2x^2)}\right) \right)$. We are given the graphs of the approximation of the function (blue) using different p and q values (see Figure 1).

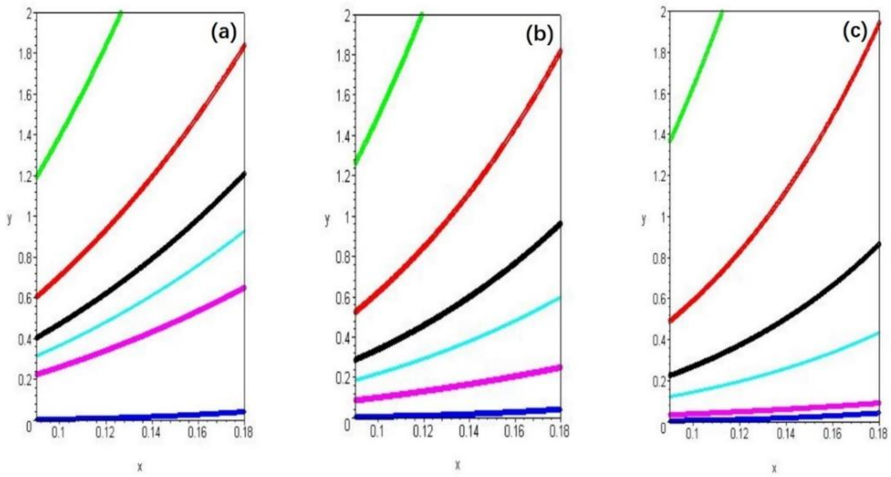


Figure 1. Approximation by operators for (p, q = 0.99, 0.70), (p, q = 0.99, 0.85), and (p, q = 0.99, 0.97), respectively.

The algorithm prepared with the Maple program, which gives the operator's approach to some n values, is presented below.

```
restart;
with(plots):
> f:=x->4*(x^3)*((exp(sin(x+1))^2)-exp(cos(2*x^2)))/2:
> q:=0.7: p:=0.99:
> m:=17: eta:=3:
> int1:=((p)^k-(q)^k)/(p-q):
> int4:=((p)^n-(q)^n)/(p-q):
```

```
> int2:=(((p)^(n+1))-((q)^(n+1)))/(p-q):
> int3:=(((p)^(n+2))-((q)^(n+2)))/(p-q):
> M:=Product((p^t)*(int2/int3)-((q^t)*x),t=0..(n+eta-k-1)):
> for n from 1 to m do
> L[n](f,x):=(1/(p^((n+eta)*((n+eta-1)/2))))*sum(f((((int2)
*(int1))*p^(n+eta-k))/((int3)*(int4))) *evalf(simplify(sum(((p^(n+eta-i+1))
>-(q^(n+eta-i+1)))/(p-q), i=0..k)))*(p^((k*(k-1))/2))
*(x^k)*M*((int3/int2)^(n+eta)),k=0..n+eta);
>end do:
M1:=plot(f(x),x=0.09..0.18,y=0..2,color=blue, style=point,
symbol=diamond, numpoints=380,symbolsize=8):
> M2:=plot(L[2](f,x),x=0.09..0.18,y=0..2,color=green, style=point,
symbol=circle, numpoints=380,symbolsize=8):
> M3:=plot(L[3](f,x),x=0.09..0.18,y=0..2,color=red,style=point,
symbol=circle, numpoints=380,symbolsize=8):
> M4:=plot(L[4](f,x),x=0.09..0.18,y=0..2,color=black, style=point,
symbol=circle, numpoints=380,symbolsize=8):
> M5:=plot(L[5](f,x),x=0.09..0.18,y=0..2,color=cyan, style=point,
symbol=circle, numpoints=380,symbolsize=7):
> M6:=plot(L[8](f,x),x=0.09..0.18,y=0..2, color=magenta, style=point,
symbol=circle, numpoints=380,symbolsize=8):
> display([M1,M2,M3,M4,M5,M6]);
```

The second example, which gives the operator's approach to the g function for some different values of p and q, is as follows:

Example 2. Let $g(x)=\frac{x^2-1}{5cos(x^3+1)+4sin(x^3+1)+90}$. We are given the graphs of the approximation of the function (blue) using different p and q values (see Figure 2).

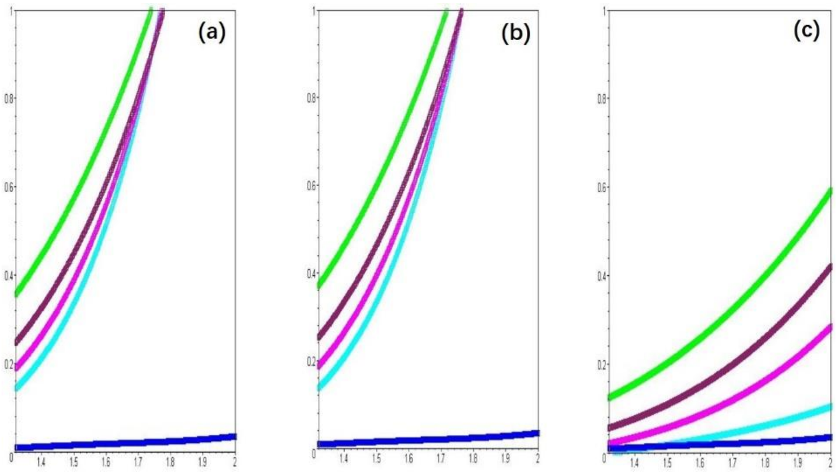


Figure 2. Approximation by operators for (p, q = 0.95, 0.75), (p, q = 0.89, 0.82), and (p, q = 0.79, 0.76), respectively.

Lemma 3. The following formulas are obtained for the moments.

$$i. \tilde{L}_s^{(p,q)}((y-x)^0;x) = 1, \tag{17}$$

$$ii. \tilde{L}_s^{(p,q)}((y-x)^1;x) = x\frac{[s+\eta]_{pq}}{[s]_{pq}} - x, \tag{18}$$

$$iii.\ \tilde{L}_s^{(p,q)}((y-x)^2;x) = \left(\frac{q[s+\eta]_{pq}[s+\eta-1]_{pq}}{[s]_{pq}^2}\right)x^2 + \left(1-\frac{2[s+\eta]_{pq}}{[s]_{pq}}\right)x^2$$

$$+\frac{[s+1]_{pq}}{[s+2]_{pq}}\frac{[s+\eta]_{pq}}{[s]_{pq}^2}p^{s+\eta-1}x. \tag{19}$$

Proof. From Lemma 2,

i. $\tilde{L}_s^{(p,q)}((y-x)^0;x) = 1.$

If $(y-x)$ is substituted in the operator, we obtain the following

ii. $\tilde{L}_s^{(p,q)}((y-x)^1;x) = \left(\frac{[s+\eta]_{pq}}{[s]_{pq}} - 1\right)x.$

iii. From equations valid for $\tilde{L}_s^{(p,q)}(y^2;x)$ and $\tilde{L}_s^{(p,q)}(y;x)$,

$$\tilde{L}_s^{(p,q)}((y-x)^2;x) = x^2 + \tilde{L}_s^{(p,q)}(y^2;x) - 2x\tilde{L}_s^{(p,q)}(y;x)$$

$$= \left(\frac{q[s+\eta]_{pq}[s+\eta-1]_{pq}}{[s]_{pq}^2} - \frac{2[s+\eta]_{pq}}{[s]_{pq}} + 1\right)x^2 + \frac{[s+\eta]_{pq}[s+1]_{pq}}{[s]_{pq}^2[s+2]_{pq}}p^{s+\eta-1}x.$$

3. Results

Let $x_1, x_2 \in \left[0, \frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$. The modulus of continuity is given by

$$w(f,\delta) = \sup_{|x_1-x_2| \le \delta} |f(x_1) - f(x_2)|. \tag{20}$$

For $\sigma \in (0,1]$, $M > 0$ and $x_1, x_2 \in \left[0, \frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$, then $f \in Lip_M(\sigma)$ if

$$|f(x_1) - f(x_2)| \le M|t-x|^\sigma. \tag{21}$$

From the definitions in the literature (e.g., [22]), we can give the following theorems.

Theorem 2. Let $f \in C\left[0, \frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$, and in this case

$$\left|\tilde{L}_s^{(p,q)}(f;x) - f(x)\right| \le \left(\frac{[s+1]_{pq}}{[s+2]_{pq}} + 1\right)\omega(f;\delta_s), \tag{22}$$

where $\delta_s = \left(\frac{[s+1]_{pq}}{[s+2]_{pq}}\right)\left(\frac{[s+\eta]_{pq}[s+\eta-1]_{pq}}{[s]_{pq}^2}q + \frac{[s+\eta]_{pq}}{[s]_{pq}^2}p^{s+\eta-1} - \frac{2[s+\eta]_{pq}}{[s]_{pq}} + 1\right)^{\frac{1}{2}}$.

Proof. From the definition of $w(f,\delta)$, the following inequality,

$$|f(y) - f(x)| \leq \left(1 + \frac{|y-x|}{\delta_s}\right)\omega(f;\delta_s), \tag{23}$$

can be written. Applying the operator to this inequality and using linearity properties, we get

$$\left|\tilde{L}_s^{(p,q)}(f;x) - f(x)\right| \leq \left(1 + \frac{1}{\delta_s}\tilde{L}_s^{(p,q)}(|y-x|;x)\right)\omega(f;\delta_s).$$

From the Cauchy-Schwarz inequality, we have

$$\left|\tilde{L}_s^{(p,q)}(f;x) - f(x)\right| \leq \left(1 + \frac{1}{\delta_s}\sqrt{\tilde{L}_s^{(p,q)}((y-x)^2;x)}\right)\omega(f;\delta_s). \tag{24}$$

If $\tilde{L}_s^{(p,q)}((y-x)^2;x)$ from Lemma 2 is substituted in the last inequality with the selection of the (q_s) satisfying the conditions in Remark 1,

$$\left\|\tilde{L}_s^{(p,q)}(f;x) - f(x)\right\|_{C\left[0,\frac{[s+1]_{pq}}{[s+2]_{pq}}\right]} \leq \left(1 + \frac{1}{\delta_s}\sqrt{\tilde{L}_s^{(p,q)}((y-x)^2;x)}\right)\omega(f;\delta_s)$$

$$\leq \left(\frac{1}{\delta_s}\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}\delta_s\right) + 1\right) \times \omega(f;\delta_s)$$

is obtained. In this last inequality, if we choose

$$\delta_s = \left(\frac{[s+1]_{pq}}{[s+2]_{pq}}\right)\left(\frac{[s+\eta]_{p,q}[s+\eta-1]_{pq}}{[s]_{pq}^2}q + \frac{[s+\eta]_{pq}}{[s]_{pq}^2}p^{s+\eta-1} - \frac{2[s+\eta]_{pq}}{[s]_{pq}} + 1\right)^{\frac{1}{2}},$$

then

$$\left|\tilde{L}_s^{(p,q)}(f;x) - f(x)\right| \leq \left(1 + \frac{[s+1]_{pq}}{[s+2]_{pq}}\right)\omega(f;\delta_s)$$

is found.

The numerical calculations of error bounded to the functions are given in Table 1 below.

Table 1. The error bound of function f and g.

s	Error bound for f	s	Error bound for g
10	0.5713948672	10	0.2023919644
10^2	0.00687324736	10^2	0.03644928772
10^3	0.01388441546	10^3	0.03862481462
10^4	0.01388446384	10^4	0.03862482420
10^5	0.01388503849	10^5	0.03862321282
10^6	0.01394821027	10^6	0.03861813728
10^7	0.01443107879	10^7	0.03856046586
10^8	0.02076262260	10^8	0.03698847304
10^9	0.06986735306	10^9	0.03130302222

Example 3. *The error bound of the functions for* $x \in \left[0, \frac{[s+1]_{pq}}{[s+2]_{pq}}\right], f(x) = \frac{1}{3} exp(x^3+1),\ q = 0.97, p = 0.99, \eta = 3,$ *and* $g(x) = sin\left(\frac{1}{3}\right) + (x^2+1), q = 0.94, p = 0.98, \eta = 5.$

Theorem 3. Let $f \in Lip_M(\sigma), 0 < q_s < p_s \leq 1.$ In this case, we have the following inequality,

$$\left|\tilde{L}_s^{(p_s,q_s)}(f;x) - f(x)\right| \leq M(\delta_s)^{\frac{\sigma}{2}}. \tag{25}$$

Proof. The proof can be done easily using Lemma 3.

Now, let us examine whether this operator, which we define on a variable-bounded interval and test its suitability for the classical convergence method, is a suitable operator for statistical convergence.

In this section, we employed the following methodology and algorithm, which must be followed to illustrate the approximation of a linear positive operator, as can be seen from the literature (see, e.g., [31]):

1) Assume that $C(\breve{A})$ is the space of all continuous functions in the interval $\left[0, \frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$, and first get all functions from $C(\breve{A})$.

2) Determine the properties that the sequences (p_s) and (q_s) must satisfy in order to achieve convergence.

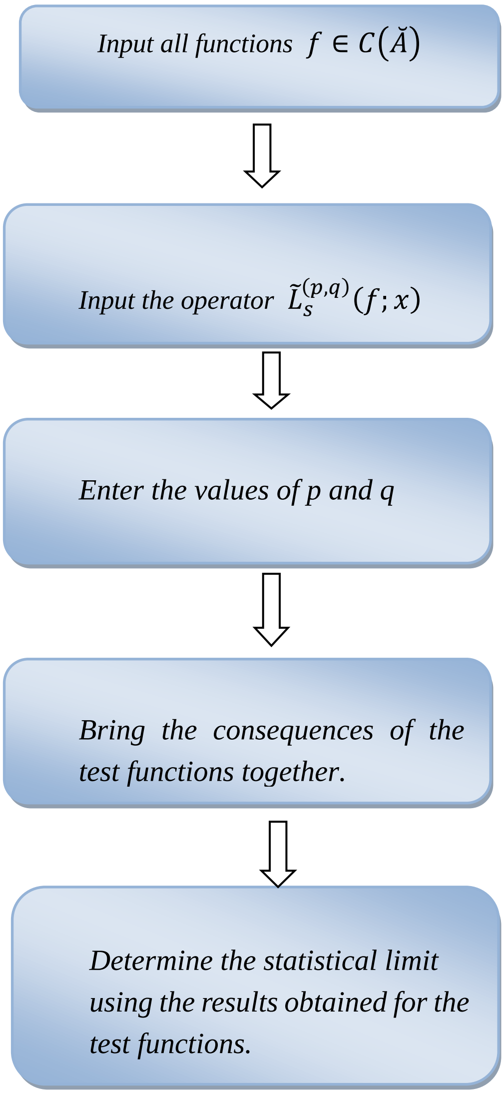
3) Implement the new type (p,q)-Bernstein-Schurer operators defined by (5) on continuous functions.

4) Compute ancillary results for test functions $(1, t, t^2)$ using the Korovkin theorem, which shows the existence of a positive linear operator.

5) Determine the statistical limits to find convergence in a statistical sense through (5).

The next algorithm gives us a method to select the function from $C(\breve{A})$ depending on our needs.

Algorithm.



We see by the following theorem that, Korovkin type theorem is also valid in the case of statistical convergence.

Theorem 4. *Let* $|f(y) - f(x)| \leq w(f, |y - x|)$ and for $0 < q_s < p_s < 1$, $\lim_{s\to\infty} p_s = \lim_{s\to\infty} q_s = 1$. Also for $c, d \in (0,1]$, $\lim_{s\to\infty} {p_s}^s = c$, $\lim_{s\to\infty} {q_s}^s = d$, and $\lim_{s\to\infty} [s]_{p_s q_s} = \infty$. In this case, for all f on $C(\breve{A})$,

$$st - \lim_{s\to\infty} \left\| \tilde{L}_s^{(p_s,q_s)}(f; x) - f(x) \right\|_{C(\breve{A})} = 0. \tag{26}$$

Proof. Using $\tilde{L}_s^{(p,q)}(1; x) = 1$, we get

$$st - \lim_{s\to\infty} \left\| \tilde{L}_s^{(p_s,q_s)}(1; x) - 1 \right\|_{C(\breve{A})} = 0.$$

Similarly, using $\tilde{L}_s^{(p,q)}(y;x) = x\frac{[s+\eta]_{pq}}{[s]_{pq}}$, we have

$$\left\|\tilde{L}_s^{(p_s,q_s)}(y;x) - x\right\|_{C(\breve{A})} = \left\|-x + [s+\eta]_{pq}\frac{x}{[s]_{pq}}\right\|.$$

For a given $\varepsilon > 0$, we define the next sets:

$$Y_1 = \{s: \|\tilde{L}_s(t;x) - x\| \geq \varepsilon\}, Y_2 = \left\{s: \frac{-[s+\eta]_{pq}}{[s]_{pq}} + 1 \geq \varepsilon\right\}.$$

Then, if we use $\delta\left\{k \leq s: \frac{-[s+\eta]_{pq}}{[s]_{pq}} + 1 \geq \varepsilon\right\}$, we get

$$st - \lim_{s\to\infty}\left\|\tilde{L}_s^{(p_s,q_s)}(y;x) - x\right\|_{C(\breve{A})} = 0.$$

Now, we show

$$st - \lim_{s\to\infty}\left\|\tilde{L}_s^{(p_s,q_s)}(y^2;x) - x^2\right\|_{C(\breve{A})} = 0.$$

$$\left\|\tilde{L}_s^{(p_s,q_s)}(y^2;x) - x^2\right\|_{C(\breve{A})} = \left(\frac{[s+\eta]_{pq}}{[s]_{pq}}\frac{[-1+s+\eta]_{pq}}{[s]_{pq}}q - 1\right)x^2$$

$$+\frac{[s+1]_{pq}}{[2+s]_{pq}}p^{a+\eta-1}x\frac{[s+\eta]_{pq}}{[s]^2{}_{pq}}.$$

Here, we take

$$B_s = -1 + \frac{[-1+s+\eta]_{pq}}{[s]_{pq}}q\frac{[s+\eta]_{pq}}{[s]_{pq}}, u_s = \frac{[s+\eta]_{pq}}{[s]^2{}_{pq}} + \frac{[1+s]_{pq}}{[s+2]_{pq}}p^{s+\eta-1},$$

And then, $st - \lim_{s\to\infty} B_s = st - \lim_{s\to\infty} u_s = 0$.

For given $\varepsilon > 0$, we define the next sets:

$$M_1 = \{s: \|\tilde{L}_s(y^2;x) - x^2\| \geq \varepsilon\}, M_2 = \left\{s: B_s \geq \frac{\varepsilon}{2}\right\}, M_3 = \left\{s: u_s \geq \frac{\varepsilon}{2}\right\}.$$

Using $M_1 \subseteq M_2 \cup M_3$, we get

$$\delta\{k \leq s: \|\tilde{L}_s(y^2;x) - x^2\| \geq \varepsilon\} \leq \delta\left\{k \leq s: B_s \geq \frac{\varepsilon}{2}\right\} + \delta\left\{k \leq s: u_s \geq \frac{\varepsilon}{2}\right\}.$$

Then we have

$$st - \lim_{s\to\infty}\left\|\tilde{L}_s^{(p,q)}(y^2;x) - x^2\right\|_{C(\breve{A})} = 0.$$

So, for all $f \in C\left[0, \frac{[1+s]_{pq}}{[s+2]_{pq}}\right]$,

$$st - \lim_{s\to\infty} \left\| \tilde{L}_s^{(p,q)}(f;x) - f(x) \right\|_{C(\breve{A})} = 0.$$

4. New type (*p*,*q*)-Bernstein-Schurer functions

Definition 2. For $p > q$, $p,\ q \in (0,1]$, and $0 \le u \le \frac{[s+1]_{pq}}{[s+2]_{pq}}$, the new type (*p*,*q*)-Bernstein-Schurer functions are defined by

$$\tilde{L}_{s+\eta,k}^{(p,q)}(u) := \frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}} M_s(u), \tag{27}$$

where

$$M_s(u) := \begin{bmatrix} s+\eta \\ k \end{bmatrix}_{pq} \left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta} \left(p^{\frac{k(k-1)}{2}} u^k\right) \prod_{t=0}^{s+\eta-k-1} \left(p^t \frac{[s+1]_{pq}}{[s+2]_{pq}} - q^t u\right).$$

Theorem 5. The new type (*p*,*q*)-Bernstein-Schurer functions have the following properties on $\left[0, \frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$:

i. Non-Negativity: For $k = 0,1,\dots,s+\eta$ and $u \in \left[0, \frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$, $\tilde{L}_{s+\eta,k}^{(p,q)}(u) \ge 0$.

ii. Partition of Unity: For all $u \in \left[0, \frac{[s+1]_{pq}}{[s+2]_{pq}}\right]$, $\sum_{k=0}^{s+\eta} \tilde{L}_{s+\eta,k}^{(p,q)}(u) = 1$.

iii. End-Point Property: $\tilde{L}_{s+\eta,k}^{(p,q)}(0) = 0$ and $\tilde{L}_{s+\eta,k}^{(p,q)}\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}\right) = 0$.

Proof. The property i. is easily obtained from the definition.

ii. $\sum_{k=0}^{s+\eta} \tilde{L}_{s+\eta,k}^{(p,q)}(u) = \sum_{k=0}^{s+\eta} \frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}} M_s(u)$

$$= \frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}} \sum_{k=0}^{s+\eta} \begin{bmatrix} s+\eta \\ k \end{bmatrix}_{pq} \left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta} \left(p^{\frac{k(k-1)}{2}} u^k\right) \prod_{t=0}^{s+\eta-k-1} \left(p^t \frac{[s+1]_{pq}}{[s+2]_{pq}} - q^t u\right) = 1.$$

iii. $\tilde{L}_{s+\eta,k}^{(p,q)}(0) = \frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}} M_s(0)$

$$= \frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}} \begin{bmatrix} s+\eta \\ k \end{bmatrix}_{pq} \left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta} \left(p^{\frac{k(k-1)}{2}} 0^k\right) \prod_{t=0}^{s+\eta-k-1} \left(p^t \frac{[s+1]_{pq}}{[s+2]_{pq}} - q^t 0\right)$$

$$= \frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}} \begin{bmatrix} s+\eta \\ k \end{bmatrix}_{pq} \left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta} \left(p^{\frac{k(k-1)}{2}} 0^k\right) \prod_{t=0}^{s+\eta-k-1} \left(p^t \frac{[s+1]_{pq}}{[s+2]_{pq}}\right).$$

So, we get $\tilde{L}_{s+\eta,k}^{(p,q)}(0)=0$.

On the other hand,

$$\tilde{L}_{s+\eta,k}^{(p,q)}\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}\right)=\frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}}\begin{bmatrix}s+\eta\\k\end{bmatrix}_{pq}\left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta}\left(p^{\frac{k(k-1)}{2}}\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}\right)^{k}\right)$$

$$\times\prod_{t=0}^{s+\eta-k-1}\left(p^t\frac{[s+1]_{pq}}{[s+2]_{pq}}-q^t\frac{[s+1]_{pq}}{[s+2]_{pq}}\right)$$

$$=\frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}}\begin{bmatrix}s+\eta\\k\end{bmatrix}_{pq}\left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta}\left(p^{\frac{k(k-1)}{2}}\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}\right)^{k}\right)$$

$$\times\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}\right)^{s+\eta-k-1}\prod_{t=0}^{s+\eta-k-1}(p^t-q^t).$$

Here, if $t=0$, $\tilde{L}_{s+\eta,k}^{(p,q)}\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}\right)=0$, and if $t\neq 0$ and $p=q$, then $\tilde{L}_{s+\eta,k}^{(p,q)}\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}\right)=0$.

Lemma 4. Some (p,q)-Bernstein-Schurer polynomials according to their degrees are calculated as follows:

$$\tilde{L}_{0,0}^{(p,q)}(u)=\tilde{L}_{0,1}^{(p,q)}(u)=\tilde{L}_{0,2}^{(p,q)}(u)=\cdots=0,$$

$$\tilde{L}_{1,2}^{(p,q)}(u)=\tilde{L}_{1,3}^{(p,q)}(u)=\tilde{L}_{1,4}^{(p,q)}(u)=\cdots=0,$$

$$\tilde{L}_{1,1}^{(p,q)}(u)=\tilde{L}_{2,2}^{(p,q)}(u)=\tilde{L}_{3,3}^{(p,q)}(u)=\cdots=0,$$

$$\tilde{L}_{1,0}^{(p,q)}(u)=\left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}-u\right),$$

$$\tilde{L}_{2,0}^{(p,q)}(u)=\frac{1}{p}\left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^2\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}-u\right)\left(p\frac{[s+1]_{pq}}{[s+2]_{pq}}-qu\right).$$

5. Some special matrix applications

In this section, considering that matrices also have applications in areas such as decision-making processes and cryptology, some important matrices have been constructed using (p, q)-Bernstein basis functions and made available to researchers. Let $p>q$, $p,\ q\in(0,1]$, and $0\leq u\leq\frac{[s+1]_{pq}}{[s+2]_{pq}}$. Various applications can be made regarding the (p,q) version of some special matrices using the (p, q)-operators and polynomials we defined in this section.

For $a_s^{p,q} = \left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta}$, we write two (p,q)-Rhaly-type matrices $\mathfrak{R}_{a_s^{p,q}}$, $\widetilde{\mathfrak{R}}_{a_s^{p,q}}$:

$$\mathfrak{R}_{a_s^{p,q}} = \begin{pmatrix} a_1^{p,q} & 0 & 0 & \cdots \\ a_2^{p,q} & a_2^{p,q} & 0 & \cdots \\ a_3^{p,q} & a_3^{p,q} & a_3^{p,q} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \left(\frac{[3]_{pq}}{[2]_{pq}}\right)^{1+\eta} & 0 & 0 & \ldots \\ \left(\frac{[4]_{pq}}{[3]_{pq}}\right)^{2+\eta} & \left(\frac{[4]_{pq}}{[3]_{pq}}\right)^{2+\eta} & 0 & \ddots \\ \left(\frac{[5]_{pq}}{[4]_{pq}}\right)^{3+\eta} & \left(\frac{[5]_{pq}}{[4]_{pq}}\right)^{3+\eta} & \left(\frac{[5]_{pq}}{[4]_{pq}}\right)^{3+\eta} & \ddots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

and

$$\widetilde{\mathfrak{R}}_{a_s^{p,q}} = \begin{pmatrix} a_1^{p,q} & 0 & 0 & \ldots \\ \frac{a_2^{p,q}}{[2]_{pq}} & \frac{a_1^{p,q}}{[2]_{pq}} & 0 & \cdots \\ \frac{a_3^{p,q}}{[3]_{pq}} & \frac{a_2^{p,q}}{[3]_{pq}} & \frac{a_1^{p,q}}{[3]_{pq}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{[3]_{pq}}{[2]_{pq}}\right)^{1+\eta} & 0 & 0 & \ldots \\ \frac{1}{[2]_{pq}}\left(\frac{[4]_{pq}}{[3]_{pq}}\right)^{2+\eta} & \frac{1}{[2]_{pq}}\left(\frac{[4]_{pq}}{[3]_{pq}}\right)^{1+\eta} & 0 & \ddots \\ \frac{1}{[3]_{pq}}\left(\frac{[5]_{pq}}{[4]_{pq}}\right)^{3+\eta} & \frac{1}{[3]_{pq}}\left(\frac{[4]_{pq}}{[3]_{pq}}\right)^{2+\eta} & \frac{1}{[3]_{pq}}\left(\frac{[3]_{pq}}{[2]_{pq}}\right)^{1+\eta} & \ddots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

The determinant of the 3×3 type $\mathfrak{R}_{a_s^{p,q}}$ and $\widetilde{\mathfrak{R}}_{a_s^{p,q}}$ matrices is as follows.

$$\left|\mathfrak{R}_{a_s^{p,q}}{}_{3\times 3}\right| = \left(\frac{[5]^3{}_{pq}}{[4]_{pq}!}\right)\left(\frac{[5]_{pq}}{[2]_{pq}}\right)^{\eta},$$

$$\left|\widetilde{\mathfrak{R}}_{a_s^{p,q}}{}_{3\times 3}\right| = \frac{(p^2+q^2)^{1+\eta}}{[3]_{pq}!}\left(\frac{[3]_{pq}}{[2]_{pq}}\right)^{1+\eta}.$$

Using the (p,q)-Bernstein-Schurer polynomials, we can build another terraced-type matrix as follows.

$$\tilde{L}_{a_s^{p,q}} = \begin{pmatrix} \tilde{L}_{1,1}^{(p,q)}(u) & 0 & 0 & \cdots \\ \tilde{L}_{2,1}^{(p,q)}(u) & \tilde{L}_{2,2}^{(p,q)}(u) & 0 & \cdots \\ \tilde{L}_{3,1}^{(p,q)}(u) & \tilde{L}_{3,2}^{(p,q)}(u) & \tilde{L}_{3,3}^{(p,q)}(u) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \tilde{L}_{2,1}^{(p,q)}(u) & 0 & 0 & \cdots \\ \tilde{L}_{3,1}^{(p,q)}(u) & \tilde{L}_{3,2}^{(p,q)}(u) & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Here are a few values of $\tilde{L}_{i,j}^{(p,q)}(u)$ in the matrix:

$$\tilde{L}_{2,1}^{(p,q)}(u) = \frac{1}{p}\begin{bmatrix}3\\1\end{bmatrix}_{pq}\left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^2\left(\frac{[s+1]_{pq}}{[s+2]_{pq}}u - q^t u^2\right),$$

$$\tilde{L}_{3,1}^{(p,q)}(u) = \frac{1}{p^3}\begin{bmatrix}3\\1\end{bmatrix}_{pq}\left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^3 u^3\left(\frac{[s+1]_{pq}}{[s+2]_{pq}} - u\right)\left(p\frac{[s+1]_{pq}}{[s+2]_{pq}} - qu\right),$$

$$\tilde{L}_{3,2}^{(p,q)}(u) = \frac{1}{p^2}\begin{bmatrix}3\\2\end{bmatrix}_{pq}\left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^3 u^2\left(\frac{[s+1]_{pq}}{[s+2]_{pq}} - u\right).$$

The determinant of the 3×3 type $\tilde{L}_{a_s^{p,q}}$ matrix is $\left|L_{a_s^{p,q}}{}_{3\times 3}\right| = 0.$

Another useful matrix that can be constructed using the above polynomials is as follows.

$$\tilde{\tilde{L}}_{a_s^{p,q}} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \tilde{L}_{1,1}^{(p,q)}(u) & 1 & 0 & \cdots \\ \tilde{L}_{2,1}^{(p,q)}(u) & \tilde{L}_{2,2}^{(p,q)}(u) & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \tilde{L}_{2,1}^{(p,q)}(u) & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Also, the determinant of the 3×3 type $\tilde{\tilde{L}}_{a_s^{p,q}}$ matrix is 0. That is $\left|\tilde{\tilde{L}}_{a_s^{p,q}}{}_{3\times 3}\right| = 0.$

6. Secure image transmission and the (*p*,*q*)-Bernstein-Schurer polynomial

In this section, we bring together the Rivest Shamir Adleman (RSA) method, from the literature, which we prefer because it is secure depending on the key state being used, and an approach that uses polynomials to increase password security. We present a modification of the method in [8] that uses polynomials to compress the encrypted image and decrypt it at the receiver to restore the image. According to this method, the compressed image is encrypted at the sender's end and decrypted by the receiver using the proposed encryption algorithm. The decrypted image is decompressed to recover the master image. The difficulty with the RSA method is that it requires the plaintext to be represented as an integer [8]. To overcome these difficulties and take advantage of the characteristics of the method, in our study, we propose a (p,q)-polynomial method that encrypts the plaintext based on the polynomial points we define.

The components of the encryption and decryption process are the algorithm used and the keys. A key is a secret numerical data shared in advance between the sender and the receiver, unknown to others, and of a certain length for each cryptosystem. Encryption algorithms are called secret-key cryptosystems when the keys of the sender and receiver are the same, and public-key cryptosystems when the keys are different. In the second method, there are two keys, one public and one private, with a mathematical relation between them. The first one used for encryption is publicly available. In this study, the Rivest Shamir Adleman (RSA) method, which is one of the public key algorithms, is used because of the lower risk of key disclosure and interception. Although it takes longer to be decrypted by the recipient, RSA is still the preferred method because the method uses less memory for encrypted data and third parties cannot easily crack the password.

There are many methods to transfer information with minimal loss, and the basic idea is to make different changes and complicate the bits of the pixels. In digital systems, information can be expressed in the form of a sequence of bits. When converting a decimal number to a binary number, the number is divided by 2 until the quotient is less than 2, the remainder is taken from each division, and the remainder is written sequentially from left to right, starting from the last division. When converting a binary number to a decimal number, the numbers consisting of 0 and 1 are multiplied by the power of 2, starting from zero and increasing by one, to be used in order from right to left. The results are summed and the decimal number is found. In the binary notation of a number, the leftmost digit, which is represented by the largest exponential value in base 2, is the most significant value for that number, called the most significant bit (MSB). On the other hand, the rightmost digit of the number, which is expressed with the smallest exponential value in base 2, is the least significant value (LSB) for that number. Therefore, this indicates that changing the MSB digit of a number in binary notation is the largest change that can be made to that number, and changing the LSB digit is the smallest change [32].

In this section, an approach that uses (p,q)-polynomials to increase password security aims to minimize losses and obtain a more detailed image than the classical Bernstein operator. In addition, a more secure encryption is achieved by making the operator dependent on p and q variables, similar to the increase in security obtained by using multivariate polynomials.

First of all, we should start by reminding the reader of the notation of the (p,q)-Bernstein-Schurer basic functions we defined.

$$\tilde{L}_{s+\eta,k}^{(p,q)}(u) = \frac{1}{p^{\frac{(s+\eta)(s+\eta-1)}{2}}} M_s(u), \text{ where}$$

$$M_s(u) = \begin{bmatrix} s+\eta \\ k \end{bmatrix}_{pq} \left(\frac{[s+2]_{pq}}{[s+1]_{pq}}\right)^{s+\eta} \left(p^{\frac{k(k-1)}{2}} u^k\right) \prod_{t=0}^{s+\eta-k-1} \left(p^t \frac{[s+1]_{pq}}{[s+2]_{pq}} - q^t u\right).$$

(x,y) coordinates are calculated according to the polynomial degree and the image is obtained for (p,q)-Bernstein-Schurer coordinate elements. This method uses the (p,q)-Bernstein-Schurer image to squeeze the input image and then the squeezed image is encrypted using a prepared algorithm. At the time of compression, the input is partitioned with the help of the (p,q)-Bernstein-Schurer image. This algorithm is an adaptation of the algorithm presented in [8] for the classical Bernstein operator to the operator we defined.

Stage 1: Determine the numbers p and q. Partition the input with the help of the (p,q)-Bernstein-Schurer image.
Stage 2: Select key pairs $(\text{Ꝃ}_u, \text{ꝃ}_r)$, where Ꝃ_u is the public key and ꝃ_r is the private key.
Stage 3: Select the hidden reference value Ꝃ on the curve based on the (p,q)-Bernstein-Schurer polynomial.
Stage 4: The original text points are considered as pixel values of the squeezed image.

Stage 5: Let $\text{ꝑ} \coloneqq (s+\eta)^2 + (s+\eta) - 4I$. Determine $(\text{Ꝁ}/\text{Ꝃ}_u)\big(modGF(\text{ꝑ})\big) = \text{æ}$ (perform) achieved by the point division formula.

Using the pixel value Ꝁ of the squeezed image, $\text{æ} = \text{Ꝁ} + (s+\eta)(\text{Ꝃ}_u - \text{Ꝁ})(modGF(\text{ꝑ}))$.
Stage 6: For hidden reference value Ꝃ, perform $(\text{æ}/\text{Ꝃ})(modGF(\text{ꝑ})) = \mathfrak{S}$ to produce the encrypted image where $\mathfrak{S} = \text{æ} + (s+\eta)(\text{Ꝃ} - \text{æ})(modGF(\text{ꝑ}))$.
Stage 7: The encrypted image is sent to the receiving end.
Stage 8 (decryption): The receiver gets the encrypted image and starts to perform decryption.
Stage 9: Perform point multiplication on $\mathfrak{S}$ and Ꝃ $(modGF(\text{ꝑ})) = \tilde{\text{æ}}$ using the hidden reference value Ꝃ to be shared between the two sides using any secure key exchange algorithm, Ꝃ $\tilde{\text{æ}} = \big((s+\eta)\text{Ꝃ} - \mathfrak{S}\big)/(s+\eta-1)\,(modGF(\text{ꝑ}))$.

Stage 9: Decryption performs $(\tilde{\text{æ}}/\text{ꝃ}_r)(modGF(\text{ꝑ})) = \text{Ꝁ}$, where $\text{Ꝁ} = \tilde{\text{б}} + 1/\Big((s+\eta)\big(\text{ꝃ}_r - \tilde{\text{б}}\big)\Big)\,(modGF(\text{ꝑ}))$. If the receiver uses a suitable private key, this is the pixel value of the compressed image. Here, the relationship between Ꝃ_u and ꝃ_r is $\text{ꝃ}_r = (s+\eta)\Big[\Big(\big(\text{б} - (s+\eta)\text{Ꝃ}_u\big)/\big(1-(s+\eta)\big) + \big(\text{б}\big(1-(s+\eta)\big)/(s+\eta)\big)\Big)\Big]\,(modGF(\text{ꝑ}))$. When æ, $\tilde{\text{æ}}$ are substituted in the polynomial, the resulting value is б, $\tilde{\text{б}}$, respectively.
Stage 10: The decrypted image is decompressed to get the parent image.

7. Discussion

Cryptology plays a major role in the dynamic development process regarding information security. The most important part of cryptography is the data encryption method. Many different methods have been developed and are being developed for this purpose. In this paper, we exploit the difficulties of (p,q)-numbers and polynomials to reduce the decrypt ability of encryption in order to prepare useful material for the data encryption algorithm.

For this purpose, important approximation results of the (p,q)-operator defined here for the first time were given. We defined a modification of the (p,q)-Bernstein-Schurer operators and we saw that the choice of p and q was important in order to reveal the approximation. We believe that we have brought a useful operator to the literature where the interval is important, since we can achieve the goals related to the approach situation that we have set out while defining the operator. The advantage of our study compared to many other articles in this field is that it contains numerical calculations and visual elements obtained with a mathematical drawing program.

In the study given in [33], the Kantorovich family of operators was used to process the image with computed tomography in research aimed at facilitating the adaptation of the vessel to heart contraction-relaxation movements by increasing the lumen of the vessel, allowing the image to be reconstructed and improved. Considering that the two-variable generalizations of the operator we have defined will detail the image and enable better imaging, this new operator will be important for future studies in the health field. As a note for future researchers, based on the result obtained above that the operator gives better approximation graphs when p and q are chosen close to each other, it is predicted that similar results will be obtained in the two-dimensional version of the operator.

In [34], Bezier curves, a concept related to Bernstein polynomials, were used to propose a solution to a real-world problem. In this study, important results were presented regarding a collision-free arrival at the desired target in multiple drone use. With a similar idea, the results can be analyzed with the help of control points and polynomials of appropriate degree by taking into account the results of Section 4 in our study to prevent collisions in the process of maximum hover performance with minimum energy and reaching the desired target in the use of multiple drones.

Therefore, our results can be used for potential applications in both medical (diagnostic imaging) and strategic security (UAV route determination) areas, where we have presented similar application examples.

Moreover, we defined some special matrices with the help of (p,q)-polynomials obtained from the operator. For classical forms of matrices, see, for example, [35] and [36]. These matrices can be transferred to different application areas with many algebraic operations in matrix theory.

In our study, we modified the method in [8], which was developed for classical Bernstein polynomials, by using both (p,q)-integers and the Schurer version of polynomials to increase password security. With this method, a useful encryption method was presented by reducing the processing time and computational complexity. The security of the algorithm has been increased through the information shared between the parties with the help of any secure key exchange algorithm.

8. Conclusions

We have presented a modification of Bernstein-Schurer operators and then calculated the moments of those operators. We also gave convergence properties of this generalization. We conclude that the approximation properties provided for different modifications of this operator are provided for our operator, which we defined to include the right endpoint of the interval. Researchers who want to work in this field can produce new operators by combining a different operator with this operator or by including a polynomial, thus contributing to the development of approximation theory. In addition, researchers in this field can examine different types of convergence for the operators, since we have shown that the statistical convergence conditions are met, e.g., [37].

As it is known, matrix theory is an important field of mathematics where active studies continue and contains important tools for many scientific fields. In this sense, in order to offer a useful and dynamic topic to researchers who would be inspired by our work, we included some small information from matrix theory, which contains the keys to new important topics. By using the special matrices included in the study, many different special matrices can be created with the help of (p,q)-polynomials obtained from the operator we defined, and transition can be made to the field where these matrices are applied or needed.

In the last part of our study, we adapted the method in [8] for classical Bernstein polynomials to the Bernstein-Schurer operator by using the properties of (p,q)-integers and polynomials in order to increase information security. With this method, the algorithm was made more secure with the help of a secure key exchange algorithm, and the decrypt ability of the cipher by third parties is reduced due to the difficulties of the sum formula and polynomials. Therefore, based on this study, researchers can develop different encryption algorithms by using this operator or different operators with useful features.

This study can be combined with studies such as [38–41], which use different methods and include new image encryption algorithms to provide the basic features of a good encryption structure, and important studies containing comprehensive cryptological information can be produced.

As another application of our study, future studies can be prepared on the polynomials related to the operator we defined, or special cases of them can be used in order to obtain the approximate numerical solution of partial differential equations or kinetic equations, which are important tools of joint studies in applied mathematics and physics, e.g., [42,43].

Author contributions

Nazmiye Gonul Bilgin: Conceptualization, Investigation, Methodology, Validation, Writing-review and editing, Formal analysis, Software; Yusuf Kaya: Conceptualization, Methodology, Formal analysis, Writing-review and editing, Supervision; Melis Eren: Methodology, Writing-original draft preparation, Visualization, Validation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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