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Research article

Noble-Abel gas diffusion at convex corners of the two-dimensional compressible magnetohydrodynamic system

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Abstract: In this paper, we study the expansion of Noble-Abel gas into a vacuum around the convex corner of the two-dimensional compressible magnetohydrodynamic system. We reduce this problem to the interaction of a centered simple wave emanating from the convex corner with a backward planar simple wave. Mathematically, this is a Goursat problem. By using the method of characteristic decomposition and construction of invariant regions, combining C^0 and C^1 estimation as well as hyperbolicity estimation, we obtain the existence of a global classical solution by extending the local classical solution.

Keywords: gas diffusion; convex corner; compressible magnetohydrodynamic system; Noble-Able gas; goursat problem
Mathematics Subject Classification: 35A01, 76N25

1. Introduction

A large number of fluids exist in nature. Neglecting viscosity and heat transfer, some fluids in nature can be viewed as ideal fluids [1–4]. The study of ideal fluid flow around a convex corner can be traced back to 1948, as proposed by Courant and Friedrichs [5]. They discovered the existence of a solution for this problem. Subsequently, Sheng and You [6] studied the problem of diffusion of the pseudo-steady supersonic flow of the polytropic gas into a vacuum at a convex corner and obtained the existence of a solution. Sheng and Yao [7] studied the problem of two-dimensional pseudo-steady isentropic irrotational supersonic flow of the polytropic gas around a convex corner. They obtained the structure of non-completely centered simple wave solutions. In 2023, Chen, Shen, and Yin [8] studied the problem of supersonic diffusion of a non-ideal gas around a convex corner into the vacuum. They proved the existence of classical solutions in the region of interaction between planar and centered simple waves. Li and Sheng [9] studied the expansion of the Van der Waals gas into the vacuum around a convex corner. The problem is studied in terms of the interaction of a completely centered simple wave with a backward planar simple wave and an incompletely centered simple wave with a backward planar simple wave. They constructively obtained the global existence of solutions to the gas expansion problem. For other studies on the convex corner winding problem, we refer to [10–25].

Since magnetic fields may affect fluids during flow, it is natural to consider compressible magnetohydrodynamic equations. There have been some results in recent years about the flow of an ideal magnetic fluid around a convex corner. In 2020, Chen, Yin, and You [26] investigated the two-dimensional pseudo-steady compressible magnetohydrodynamic equations for the polytropic gas expansion into the vacuum at the convex corner. They obtained a classical solution for interacting with centered and planar simple waves.

The two-dimensional isentropic compressible magnetohydrodynamic equations can be written as

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + (\rho u^2 + p + \frac{\mu k_0^2}{2}\rho^2)_x + (\rho uv)_y = 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p + \frac{\mu k_0^2}{2}\rho^2)_y = 0, \end{cases} \tag{1.1}$$

where $\vec{q} = (u, v)$ denotes the velocity, μ is the magnetic permeability, ρ is the density, k_0 is the positive constant, t denotes time, and p is the gas pressure.

The Noble-Abel gas is

$$p(\rho) = \frac{A\rho^\gamma}{(1 - a\rho)^\gamma}, \quad 0 < a\rho < 1, \quad A > 0, \quad 1 < \gamma < 3, \tag{1.2}$$

where $a > 0$ is constant and represents the compressibility limit of a gas molecule, and γ represents the adiabatic index.

It is assumed that the supersonic incoming flow in the second quadrant travels along the horizontal solid wall to the origin with a constant velocity $\vec{q}_0 = (u_0, 0)$, and the rest of the region is a vacuum (see Figure 1). The supersonic flow around a convex corner problem has the following initial value conditions:

$$(u, v, \rho)(x, y, t) = \begin{cases} (u_0, 0, \rho_0), & t = 0, x < 0, y \geq 0, \\ vacuum, & t = 0, x > 0, y \geq x \tan\theta, \end{cases} \tag{1.3}$$

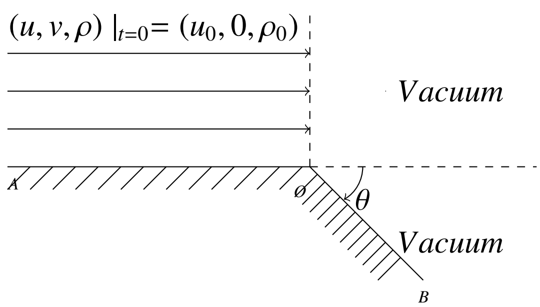

Figure 1. Supersonic flow around convex corners to vacuum.

And the boundary data

$$\begin{cases} (\rho v)(x,0,t) = 0, & x<0, y=0, t\geq 0, \\ (\rho v)(x,y,t) = (\rho u)(x,y,t)\tan\theta, & y<0, x = y\cot\theta, t\geq 0, \end{cases} \tag{1.4}$$

where ρ_0 denotes the incoming flow density, is a constant, and θ is the solid wall inclination.

For the sake of the discussion that follows, we propose the following notation,

$$\mu^2(\rho) = \frac{(\gamma-1+2a\rho)c^2 + (1-a\rho)b^2}{(\gamma+1)c^2 + 3(1-a\rho)b^2}, \tag{1.5}$$

$$\left[\mu^2(\rho)\right]' = \frac{2a\rho(\gamma+1)c^4 + \left[2(\gamma-2)^2 + 14a\rho(\gamma-1) + a\rho(18a\rho-8)\right]b^2c^2}{\rho\left[(\gamma+1)c^2 + (1-a\rho)3b^2\right]}, \tag{1.6}$$

$$m(\rho) = \frac{(3-\gamma-4a\rho)c^2 + (1-a\rho)b^2}{(\gamma+1)c^2 + 3(1-a\rho)b^2}, \tag{1.7}$$

$$m'(\rho) = \frac{-a\rho\left[\gamma+\gamma^2+4(1-a\rho)\right]c^4 + \left[27a\rho(1-\gamma) + a\rho(17-36a\rho) - 4(\gamma-2)^2\right](1-a\rho)b^2c^2}{\left((\gamma+1)c^2 + (1-a\rho)b^2\right)^2(1-a\rho)}, \tag{1.8}$$

$$\Pi(\hat{\rho}) = -\frac{\hat{w}^2}{\gamma(\gamma-1)}\left[\gamma(\gamma+1) - 2(\gamma+1)a\hat{\rho} + 2a^2\hat{\rho}^2\right], \tag{1.9}$$

$$\begin{aligned} \Pi'(\hat{\rho}) = &-\frac{A\gamma\hat{\rho}^{\gamma-2}(\gamma-1+2a\hat{\rho}) + \mu\kappa_0^2(1-a\hat{\rho})^{\gamma+2}}{\gamma(\gamma-1)(1-a\hat{\rho})^{\gamma+2}}\left[\gamma(\gamma+1) - 2(\gamma+1)a\hat{\rho} + 2a^2\hat{\rho}^2\right] \\ &-\frac{\hat{w}^2}{\gamma(\gamma-1)}\left[-2(\gamma+1)a + 4a^2\rho\right], \end{aligned} \tag{1.10}$$

$$\alpha_0 = \arcsin\frac{w_0}{u_0}, \quad \alpha_v = \int_{\rho_0}^{0}\frac{\hat{w}\sqrt{\hat{q}^2-\hat{w}^2}}{\hat{q}^2\hat{\rho}}d\hat{\rho}, \quad \bar{\delta}(\rho) = \arctan\sqrt{m(\rho)},$$

$$\hat{q}^2 = \hat{u}^2 + \hat{v}^2, \quad c^2 = \frac{A\gamma\rho^{\gamma-1}}{(1-a\rho)^{\gamma+1}}, \quad b^2 = \mu k_0^2\rho, \quad c_0^2 = \frac{A\gamma\rho_0^{\gamma-1}}{(1-a\rho_0)^{\gamma+1}}, \quad b_0^2 = \mu k_0^2\rho_0, \tag{1.11}$$

$$\bar{\delta}_* = \bar{\delta}(0) = \begin{cases} \arctan\sqrt{\frac{3-\gamma}{\gamma+1}}\,(1<\gamma<2) \\ \frac{\pi}{6}\,(2\leq\gamma<3) \end{cases}, \quad w = \sqrt{c^2+b^2}, \quad w_0 = \sqrt{c_0^2+b_0^2},$$

where $u_0 > w_0$ denotes the incoming horizontal velocity as a constant and w_0 denotes the incoming magnetoacoustic velocity as a constant. α_0 is the maximum characteristic inclination of the C_+ characteristic curve of the centered simple wave at point O, and α_v is the minimum characteristic inclination of the C_+ characteristic curve of the centered simple wave at point O, which is determined by (3.26). The variable $(u,v,\rho)(\xi,\eta)$ is controlled by (ξ,η), and in order to solve for the main part of the properties of the centered simple wave, we have changed the variable to a variable controlled by a single parameter α, $(\hat{u},\hat{v},\hat{\rho})(\alpha)$, and $\hat{w}$, $\hat{u}$, and $\hat{v}$ are determined by (3.25). b as the Alfven velocity, c as the speed of sound, and $w = \sqrt{c^2+b^2}$ as the speed of magneto-sound. b_0, c_0, and $w_0 = \sqrt{c_0^2+b_0^2}$ denote the Alfven velocity of the incoming flow at t = 0, the speed of sound, and the speed of magnetoacoustic sound, respectively. All other symbols are for ease of expression and calculation and have no real physical meaning.

For the convenience of the following proof, we assume that there exists a constant ρ_1, such that the following condition holds for any $\rho < \rho_1$ when $0 < a\rho_1 < 1$

$$\mu^2(\rho) > 0, \quad \left[\mu^2(\rho)\right]' > 0, \quad m(\rho) > 0, \quad m'(\rho) < 0, \quad \Pi(\rho) < 0, \quad \Pi'(\rho) < 0. \tag{1.12}$$

In this paper, we study the Riemannian problem of a supersonic magnetic fluid with Noble-Abel gas diffusing into the vacuum around a convex corner, which is essentially the interaction of a centered simple wave with a planar simple wave and which can be solved by reducing it to a Goursat problem. The solution's hyperbolicity and a priori C^1 estimates are established using characteristic decompositions and invariant regions. In addition, pentagonal invariant regions are constructed to obtain global solutions. In addition, the generality of this gas, the sub-invariant region, is constructed, and the solved hyperbolicity is obtained based on the continuity of the sub-invariant region. Finally, the global existence of the solution to the gas expansion problem is constructively obtained. The main results of this paper are as follows:

Theorem 1.1. *If* $u_0 > w_0$, $\theta \leq \alpha_v$, $\rho_0 < \rho_1, \max\left(\alpha_0 - 4\bar{\delta}(\rho_0) + 4\bar{\delta}_* - \frac{\pi}{2}, \alpha_0 - \alpha_v - 2\bar{\delta}(\rho_0)\right) < 0$, *then the problem of a magnetoacoustic velocity flow with Noble–Abel gas expanding into vacuum at the convex corner* (1.1)–(1.4) *has a global classical solution.*

The structure of the paper is as follows: In Section 2, the characteristic forms and characteristic decompositions related to the characteristic direction, the pseudo-flow direction, and the Mach angle are given. In Section 3, we provide the expressions for the centered simple and planar simple waves. In Section 4, the primary study is the interaction problem of centered simple and planar simple waves. The C^0, C^1, and hyperbolicity estimates of the solutions are obtained. Then, we obtain the existence of global classical solutions. In Section 5, we discuss the shortcomings of Theorem 1.1.

2. Characteristic analysis of compressible magnetohydrodynamic equations

2.1. The characteristic equations for α, β and w

By the self-similar transformation $(\xi, \eta) = (x/t, y/t)$, the system of Eq (1.1) can be written as

$$\begin{cases} (\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\ (\rho U^2 + p + \frac{\mu k_0^2}{2}\rho^2)_\xi + (\rho UV)_\eta + 3\rho U = 0, \\ (\rho UV)_\xi + (\rho V^2 + p + \frac{\mu k_0^2}{2}\rho^2)_\eta + 3\rho V = 0, \end{cases} \tag{2.1}$$

where $(U, V) = (u-\xi, v-\eta)$ is the pseudo-flow velocity. The initial edge value condition (1.3) and (1.4) are transformed into a boundary conditions in the (ξ, η) plane (refer to Figure 2).

$$(u, v, \rho)(\xi, \eta) = \begin{cases} (u_0, 0, \rho_0), & \xi < 0, \eta \geq 0, \xi^2 + \eta^2 \to \infty, \\ vacuum, & \xi > 0, \eta \geq \xi tan\theta, \xi^2 + \eta^2 \to \infty, \end{cases} \tag{2.2}$$

and

$$\begin{cases} (\rho v)(\xi,\eta) = 0, & \xi < 0, \eta = 0, \\ (\rho v)(\xi,\eta) = (\rho u)(\xi,\eta)\tan\theta, & \eta < 0, \xi = \eta\cot\theta, \end{cases} \tag{2.3}$$

where u_0 and ρ_0 are two different constants for the horizontal velocity and density of the incoming flow, respectively, and θ is the solid wall inclination.

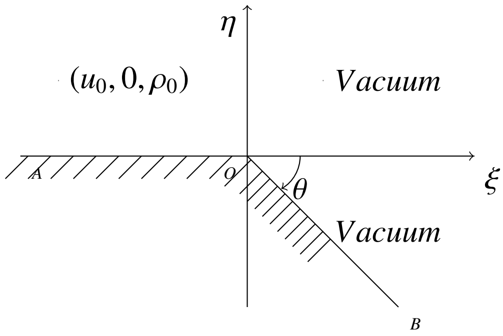

Figure 2. Initial frontier conditions in self-similar plane.

For smooth solutions, the system of Eq (2.1) can be written in the following form:

$$\begin{cases} (\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\ UU_\xi + VU_\eta + U + \frac{1}{\rho}(p + \frac{\mu k_0^2}{2}\rho^2)_\xi = 0, \\ UV_\xi + VV_\eta + V + \frac{1}{\rho}(p + \frac{\mu k_0^2}{2}\rho^2)_\eta = 0. \end{cases} \tag{2.4}$$

In the case of the irrotational condition $u_\eta = v_\xi$, there exist potential functions φ such that $\varphi_\xi = U$ and $\varphi_\eta = V$. By the last two equations of (2.4), the pseudo-Bernoulli's law is obtained as follows:

$$\frac{1}{2}(U^2 + V^2) + \frac{(\gamma - a\rho)}{(\gamma - 1)} \cdot \frac{A\rho^{\gamma-1}}{(1 - a\rho)^\gamma} + \mu k_0^2\rho + \varphi = constant. \tag{2.5}$$

We define $b = \sqrt{\mu k_0^2 \rho}$ as the Alfven velocity, $c = \sqrt{\frac{A\gamma\rho^{\gamma-1}}{(1-a\rho)^{\gamma+1}}}$ as the speed of sound, and $w = \sqrt{c^2 + b^2}$ as the speed of magneto-sound. Through the irrotational condition $u_\eta = v_\xi$, the system of Eq (2.4) can be transformed into

$$\begin{cases} (w^2 - U^2)u_\xi - UV(u_\xi + v_\eta) + (w^2 - V^2)v_\eta = 0, \\ v_\xi - u_\eta = 0. \end{cases} \tag{2.6}$$

The matrix form of system (2.6) is as follows:

$$\begin{pmatrix} w^2 - U^2 & -UV \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_\xi + \begin{pmatrix} -UV & w^2 - V^2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_\eta = 0. \tag{2.7}$$

The eigenvalues Λ of system (2.7) are given by

$$(w^2 - U^2)\Lambda^2 + 2UV\Lambda + (w^2 - V^2) = 0. \tag{2.8}$$

That is

$$(V - \Lambda U)^2 - w^2(1 + \Lambda^2) = 0, \tag{2.9}$$

which yields

$$\Lambda_{\pm} = \frac{UV \pm \sqrt{w^2 \left(U^2 + V^2 - w^2\right)}}{U^2 - w^2}. \tag{2.10}$$

(2.10) indicates that when $U^2 + V^2 > w^2$, the system (2.7) is hyperbolic, with two families of wave characteristics defined by

$$\frac{d\eta}{d\xi} = \Lambda_{\pm}. \tag{2.11}$$

Taking into account Eq (2.9), it is evident that w is essentially a projection of the pseudo-flow velocity in the normal direction of the $C_{\pm}$ feature line. The eigenvectors on the left side that correspond to the eigenvalues $\Lambda_{\pm}$ are

$$l_{\pm} = \left(1, \mp \sqrt{w^2 \left(U^2 + V^2 - w^2\right)}\right). \tag{2.12}$$

By left-multiplying (2.12) by (2.7), the system of Eq (2.7) can be transformed into the following characteristic form:

$$\begin{cases} \partial_+ u + \Lambda_- \partial_+ v = 0, \\ \partial_- u + \Lambda_+ \partial_- v = 0, \end{cases} \tag{2.13}$$

where $\partial_{\pm} = \partial_{\xi} + \Lambda_{\pm} \partial_{\eta}$.

We define the characteristic inclinations α, β of Λ_+ and Λ_- as follows:

$$\tan\alpha = \Lambda_+, \quad \tan\beta = \Lambda_-. \tag{2.14}$$

Joining (2.14) and (2.10) yields

$$\tan\alpha - \tan\beta = \frac{2\sqrt{w^2 \left(U^2 + V^2 - w^2\right)}}{U^2 - w^2}, \quad \tan\alpha \tan\beta = \frac{V^2 - w^2}{U^2 - w^2}. \tag{2.15}$$

That is

$$U^4 \sin^2(\alpha - \beta) - 2U^2 w^2 \left(\cos^2\alpha + \cos^2\beta\right) + w^4 \sin^2(\alpha + \beta) = 0. \tag{2.16}$$

The pseudo-streamline inclination is $\sigma = \frac{\alpha+\beta}{2}$ and the Mach angle is $\delta = \frac{\alpha-\beta}{2}$ as shown in Figure 3, we have

$$u = \xi + w\frac{\cos\sigma}{\sin\delta}, \quad v = \eta + w\frac{\sin\sigma}{\sin\delta}, \tag{2.17}$$

$$\delta = \arcsin\left(\frac{w}{\sqrt{U^2 + V^2}}\right). \tag{2.18}$$

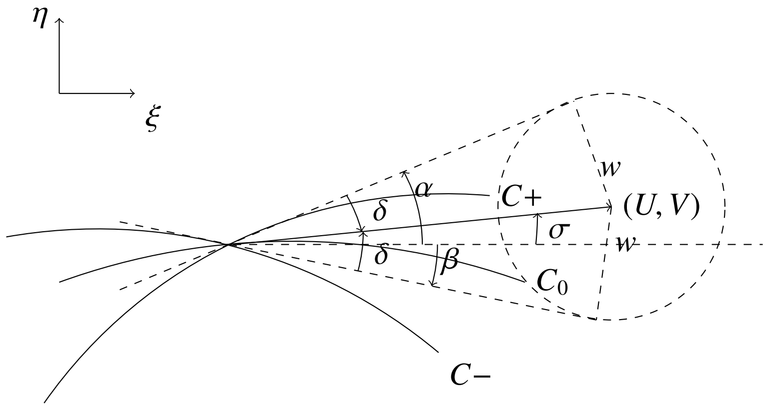

Figure 3. The relations between characteristics and local magneto-scoustic speed.

By the first equation of (2.17), we have

$$u_\xi = 1 + \frac{cc_\xi + bb_\xi}{\sqrt{c^2+b^2}}\frac{\cos\sigma}{\sin\delta} - \frac{\sqrt{c^2+b^2}}{2}\frac{\alpha_\xi\cos\beta - \beta_\xi\cos\alpha}{\sin^2\delta}, \tag{2.19}$$

$$u_\eta = \frac{cc_\eta + bb_\eta}{\sqrt{c^2+b^2}}\frac{\cos\sigma}{\sin\delta} - \frac{\sqrt{c^2+b^2}}{2}\frac{\alpha_\eta\cos\beta - \beta_\eta\cos\alpha}{\sin^2\delta}. \tag{2.20}$$

Thus

$$\bar{\partial}_\pm u = \cos(\sigma \pm \delta) + \frac{\cos\sigma}{\sin\delta}\bar{\partial}_\pm w - \frac{w\cos\beta\bar{\partial}_\pm\alpha - w\cos\alpha\bar{\partial}_\pm\beta}{2\sin^2\delta}, \tag{2.21}$$

where

$$\bar{\partial}_+ = \cos\alpha\partial_\xi + \sin\alpha\partial_\eta, \quad \bar{\partial}_- = \cos\beta\partial_\xi + \sin\beta\partial_\eta. \tag{2.22}$$

From the second equation (2.17), we can conclude that

$$v_\xi = \frac{cc_\xi + bb_\xi}{\sqrt{c^2+b^2}}\frac{\sin\sigma}{\sin\delta} - \frac{\sqrt{c^2+b^2}}{2}\frac{\alpha_\xi\sin\beta - \beta_\xi\sin\alpha}{\sin^2\delta}, \tag{2.23}$$

$$v_\eta = 1 + \frac{cc_\eta + bb_\eta}{\sqrt{c^2+b^2}}\frac{\sin\sigma}{\sin\delta} - \frac{\sqrt{c^2+b^2}}{2}\frac{\alpha_\eta\sin\beta - \beta_\eta\sin\alpha}{\sin^2\delta}. \tag{2.24}$$

Subsequently

$$\bar{\partial}_\pm v = \sin(\sigma \pm \delta) + \frac{\sin\sigma}{\sin\delta}\bar{\partial}_\pm w - \frac{w\sin\beta\bar{\partial}_\pm\alpha - w\sin\alpha\bar{\partial}_\pm\beta}{2\sin^2\delta}, \tag{2.25}$$

$$\bar{\partial}_\pm w = \frac{c\bar{\partial}_\pm c + b\bar{\partial}_\pm b}{\sqrt{c^2+b^2}} = \frac{A\gamma\rho^{\gamma-2}\left[(\gamma-1)+2a\rho\right] + \mu\kappa_0^2(1-a\rho)^{\gamma+2}}{2w(1-a\rho)^{\gamma+2}}\bar{\partial}_\pm\rho. \tag{2.26}$$

According to (2.5), we have

$$U\bar{\partial}_\pm u + V\bar{\partial}_\pm v + \tilde{p}'(\rho)\,\tilde{\partial}_\pm\rho = 0, \tag{2.27}$$

where

$$\tilde{p}'(\rho) = \frac{A\gamma\rho^{\gamma-2}}{(1-a\rho)^{\gamma+1}} + \mu\kappa_0^2. \tag{2.28}$$

(2.13) and (2.27) lead to

$$\bar{\partial}_+ u = \frac{\Lambda_- \tilde{p}'(\rho)}{V - \Lambda_- U} \bar{\partial}_+ \rho, \tag{2.29}$$

$$\bar{\partial}_+ v = \frac{\tilde{p}'(\rho)}{U\Lambda_- - V} \bar{\partial}_+ \rho, \tag{2.30}$$

$$\bar{\partial}_- u = \frac{\Lambda_+ \tilde{p}'(\rho)}{V - \Lambda_+ U} \bar{\partial}_- \rho, \tag{2.31}$$

$$\bar{\partial}_- v = \frac{\tilde{p}'(\rho)}{U\Lambda_+ - V} \bar{\partial}_- \rho. \tag{2.32}$$

Substituting (2.14), (2.17), (2.26), and (2.28) into (2.29)–(2.32) reaches

$$\bar{\partial}_+ u = \frac{2\left(c^2 + b^2\right)\sin\beta}{M(\rho)\, c^2 + b^2} \bar{\partial}_+ w, \quad \bar{\partial}_- u = -\frac{2\left(c^2 + b^2\right)\sin\alpha}{M(\rho)\, c^2 + b^2} \bar{\partial}_- w, \tag{2.33}$$

$$\bar{\partial}_+ v = -\frac{2\left(c^2 + b^2\right)\cos\beta}{M(\rho)\, c^2 + b^2} \bar{\partial}_+ w, \quad \bar{\partial}_- v = \frac{2\left(c^2 + b^2\right)\cos\alpha}{M(\rho)\, c^2 + b^2} \bar{\partial}_- w, \tag{2.34}$$

where

$$M(\rho) = \frac{\gamma - 1 + 2a\rho}{1 - a\rho}. \tag{2.35}$$

Associating (2.21), (2.25), (2.33), and (2.34), we obtain

$$w\bar{\partial}_+ \alpha = -\frac{\sin 2\delta}{2\mu^2(\rho)} \Omega(\delta, \rho)\, \bar{\partial}_+ w, \tag{2.36}$$

$$w\bar{\partial}_+ \beta = -\frac{\tan\delta}{\mu^2(\rho)} \bar{\partial}_+ w - 2\sin^2\delta, \tag{2.37}$$

$$w\bar{\partial}_- \alpha = \frac{\tan\delta}{\mu^2(\rho)} \bar{\partial}_- w + 2\sin^2\delta, \tag{2.38}$$

$$w\bar{\partial}_- \beta = \frac{\sin 2\delta}{2\mu^2(\rho)} \Omega(\delta, \rho)\, \bar{\partial}_- w, \tag{2.39}$$

where

$$\Omega(\delta, \rho) = m(\rho) - \tan^2\delta. \tag{2.40}$$

2.2. Characteristic decomposition of the variable w

In this section, we compute the characteristic decomposition of w using the commutator relations and the characteristic equations of α, β, w, and u.

Lemma 2.1. *The commutator relations (Li, Zhang, and Zheng [27]).*

$$\bar{\partial}_- \bar{\partial}_+ - \bar{\partial}_+ \bar{\partial}_- = \frac{1}{\sin(2\delta)} \left\{ \left(\cos(2\delta)\, \bar{\partial}_+ \beta - \bar{\partial}_- \alpha \right) \bar{\partial}_- - \left(\bar{\partial}_+ \beta - \cos(2\delta)\, \bar{\partial}_- \alpha \right) \bar{\partial}_+ \right\}. \tag{2.41}$$

Lemma 2.2.

$$w\bar{\partial}_+\bar{\partial}_-w=\bar{\partial}_-w\left\{\sin 2\delta+\frac{\bar{\partial}_-w}{2\mu^2(\rho)\cos^2\delta}+\left(1+\frac{\Omega(\delta,\rho)\,cos2\delta}{2\mu^2(\rho)}+N(\rho)\right)\bar{\partial}_+w\right\},\tag{2.42}$$

$$w\bar{\partial}_-\bar{\partial}_+w=\bar{\partial}_+w\left\{\sin 2\delta+\frac{\bar{\partial}_+w}{2\mu^2(\rho)\cos^2\delta}+\left(1+\frac{\Omega(\delta,\rho)\,cos2\delta}{2\mu^2(\rho)}+N(\rho)\right)\bar{\partial}_-w\right\},\tag{2.43}$$

where

$$N(\rho)=\frac{a(\gamma+1)\rho c^2\left(c^2+b^2\right)+c^2b^2(\gamma-2+3a\rho)^2}{\left((\gamma-1+2a\rho)c^2+(1-a\rho)b^2\right)^2}.\tag{2.44}$$

Proof. From Lemma 2.1

$$\bar{\partial}_+\bar{\partial}_-u-\bar{\partial}_-\bar{\partial}_+u=\frac{-1}{\sin(2\delta)}\left\{\left(\cos(2\delta)\bar{\partial}_+\beta-\bar{\partial}_-\alpha\right)\bar{\partial}_-u-\left(\bar{\partial}_+\beta-\cos(2\delta)\bar{\partial}_-\alpha\right)\bar{\partial}_+u\right\}.\tag{2.45}$$

Substituting (2.33) into (2.45), we have

$$\begin{aligned}&\bar{\partial}_+\left(-\frac{2\left(c^2+b^2\right)\sin\alpha}{M(\rho)c^2+b^2}\bar{\partial}_-w\right)-\bar{\partial}_-\left(\frac{2\left(c^2+b^2\right)\sin\beta}{M(\rho)c^2+b^2}\bar{\partial}_+w\right)\\&=\frac{-1}{\sin 2\delta}\left\{\left(\cos(2\delta)\bar{\partial}_+\beta-\bar{\partial}_-\alpha\right)\left(-\frac{2\left(c^2+b^2\right)\sin\alpha}{M(\rho)c^2+b^2}\right)\bar{\partial}_-w-\left(\bar{\partial}_+\beta-\cos(2\delta)\bar{\partial}_-\alpha\right)\frac{2\left(c^2+b^2\right)\sin\beta}{M(\rho)c^2+b^2}\bar{\partial}_+w\right\}.\end{aligned}\tag{2.46}$$

Simplifying and combining gives

$$\begin{aligned}&\sin\alpha\bar{\partial}_+\bar{\partial}_-w+\sin\beta\bar{\partial}_-\bar{\partial}_+w\\&=\left\{\frac{-1}{\sin 2\delta}\left(\cos(2\delta)\bar{\partial}_+\beta-\bar{\partial}_-\alpha\right)\sin\alpha+2N(\rho)\sin\alpha\cdot w\bar{\partial}_+w-\frac{\cos\alpha}{w}\cdot\frac{2\Theta\cos^2\delta-\cos 2\delta}{\Theta\sin 2\delta}\bar{\partial}_+w\right\}\bar{\partial}_-w\\&+\left\{\frac{\sin\beta}{\sin 2\delta}\left(\cos(2\delta)\bar{\partial}_-\alpha-\bar{\partial}_+\beta\right)+2N(\rho)\sin\beta\cdot w\bar{\partial}_-w+\frac{\cos\beta}{w}\cdot\frac{2\Theta\cos^2\delta-\cos 2\delta}{\Theta\sin 2\delta}\bar{\partial}_-w\right\}\bar{\partial}_+w,\end{aligned}\tag{2.47}$$

where

$$\Theta(\delta,\rho)=\frac{M(\rho)c^2+b^2}{2\left[2w^2\sin^2\delta+M(\rho)c^2+b^2\right]}.\tag{2.48}$$

Inserting (2.37) and (2.38) into (2.47) and using the commutator relation (2.41), we obtain

$$\begin{aligned}w(\sin\alpha+\sin\beta)\bar{\partial}_+\bar{\partial}_-w&=\frac{-w}{\sin 2\delta}\left(\cos(2\delta)\bar{\partial}_+\beta-\bar{\partial}_-\alpha\right)(\sin\alpha+\sin\beta)\bar{\partial}_-w\\&+2N(\rho)(\sin\alpha+\sin\beta)\bar{\partial}_+w\bar{\partial}_-w+(\cos\alpha-\cos\beta)\frac{\sin 2\delta}{2\mu^2(\rho)}\Omega(\delta,\rho)\bar{\partial}_+w\bar{\partial}_-w.\end{aligned}\tag{2.49}$$

Thus, we have

$$w\bar{\partial}_+\bar{\partial}_-w=\left\{\sin 2\delta-\frac{2\Theta\cos^2\delta-1}{\Theta\sin^2(2\delta)}\bar{\partial}_-w+\left(\frac{2\Theta\left(4\sin^2\delta-1\right)\cos^2(2\delta)}{\Theta\sin^2(2\delta)}+2N(\rho)\right)\bar{\partial}_+w\right\}\bar{\partial}_-w.\tag{2.50}$$

Using a simple calculation, we have

$$\frac{2\Theta(\delta,\rho)\left(4\sin^2\delta-1\right)\cos^2\delta+\cos^2(2\delta)}{\Theta(\delta,\rho)\sin^2(2\delta)}=1+\frac{\Omega(\delta,\rho)\cos 2\delta}{2\mu^2(\rho)}, \tag{2.51}$$

and

$$\frac{2\Theta(\delta,\rho)\cos^2\delta-1}{\Theta(\delta,\rho)\sin^2(2\delta)}=\frac{-1}{2\mu^2(\rho)\cos^2\delta}. \tag{2.52}$$

Therefore, (2.42) is established. The proof of (2.43) is similar, so we will not prove it in detail here. □

3. Centered simple waves and planar simple waves

In this section, we focus on constructing centered and planar simple waves.

3.1. Centered simple wave

In order to construct the centered simple wave solution for isentropic irrotational pseudo-steady flow at convex corners, we first discuss the nature of the general centered simple wave principal part of the equation.

According to the definition [28] of a centered simple wave, its solution $(u,v,\rho)(\xi,\eta)$ can be determined by the direct characteristic line $\eta=\xi\tan\alpha$ and the function $(u,v,\rho)(\xi,\eta)=(\overline{u},\overline{v},\overline{\rho})(r,\alpha)$ defined on the rectangular region $\widetilde{\Lambda}(t)=\{(\xi,\eta)\mid 0\le r\le\zeta,\tilde{\alpha}_2\le\alpha\le\tilde{\alpha}_1\}$, see Figure 4. It is easy to see that $\xi=r\cos\alpha$ and $\eta=r\sin\alpha$, there are

$$\begin{aligned}
&\frac{\partial r}{\partial\xi}=\cos\alpha,\quad \frac{\partial r}{\partial\eta}=\sin\alpha,\quad \frac{\partial\alpha}{\partial\xi}=-\frac{1}{r}\sin\alpha,\quad \frac{\partial\alpha}{\partial\eta}=\frac{1}{r}\cos\alpha,\\
&(u,v,\rho)(\xi,\eta)=(\overline{u},\overline{v},\overline{\rho})(\xi,\alpha),\\
&\frac{\partial}{\partial\xi}=\frac{\partial}{\partial\xi}+\tan\alpha\frac{\partial}{\partial\eta},\quad \frac{\partial}{\partial\alpha}=\frac{\xi}{\cos^2\alpha}\frac{\partial}{\partial\eta}.
\end{aligned} \tag{3.1}$$

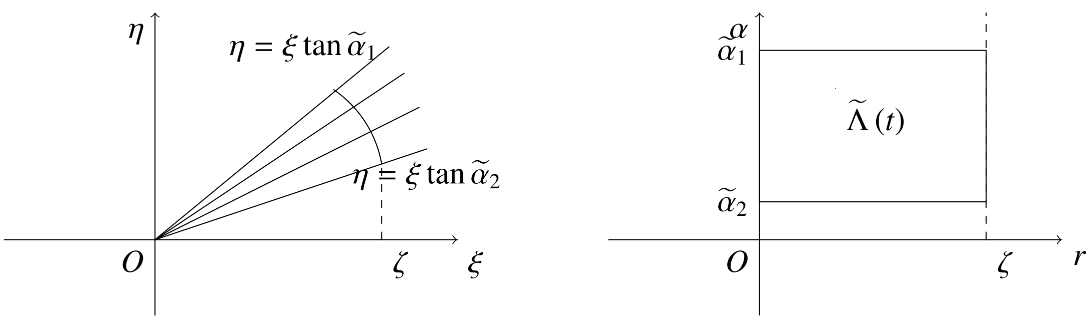

Figure 4. The centered simple wave flow region in the (ξ,η) and (r,α) planes.

From the systems (2.13) and (2.5), it is clear that the main parts $(u,v,\rho)(\xi,\eta)$ of the C_+ type centered simple wave satisfy

$$\begin{cases} \frac{\partial \overline{u}}{\partial \xi} + \tan\beta \frac{\partial \overline{v}}{\partial \xi} = 0, \\ -\xi\left(1+\tan^2\alpha\right)\frac{\partial \overline{v}}{\partial \xi} + \frac{\partial \overline{u}}{\partial \alpha} + \tan\alpha \frac{\partial \overline{v}}{\partial \alpha} = 0, \\ \frac{1}{2}\left[(\overline{u}-\xi)^2 + (\overline{v}-\xi\tan\alpha)^2\right] + \frac{(\gamma-a\overline{\rho})}{(\gamma-1)}\cdot\frac{A\overline{\rho}^{\gamma-1}}{(1-a\overline{\rho})^{\gamma}} + \overline{b}^2 + \overline{\varphi} = constant. \end{cases} \tag{3.2}$$

For the potential function φ, we have

$$\xi\frac{\partial\overline{\varphi}}{\partial\xi} = \frac{\tan\alpha + \cot\sigma}{1+\tan^2\alpha}\frac{\partial\overline{\varphi}}{\partial\alpha}. \tag{3.3}$$

Let $\xi \to 0$, then (3.2) and (3.3) become

$$\frac{d\hat{u}}{d\alpha} + \tan\alpha\frac{d\hat{v}(\alpha)}{d\alpha} = 0, \quad \hat{\varphi}(\alpha) = constant, \tag{3.4}$$

$$\frac{1}{2}\left(\hat{u}^2(\alpha) + \hat{v}^2(\alpha)\right) + \frac{(\gamma-a\hat{\rho})}{(\gamma-1)}\cdot\frac{A\hat{\rho}^{\gamma-1}}{(1-a\hat{\rho})^{\gamma}} + \hat{b}^2(\alpha) = constant, \tag{3.5}$$

$$\tan\alpha = \frac{\hat{u}(\alpha)\hat{v}(\alpha) + \hat{w}(\alpha)\sqrt{\hat{u}^2(\alpha)+\hat{v}^2(\alpha)-\hat{w}^2(\alpha)}}{\hat{u}^2(\alpha)-\hat{w}^2(\alpha)}, \tag{3.6}$$

where $(\overline{u}(0,\alpha), \overline{v}(0,\alpha), \overline{\varphi}(0,\alpha), \overline{\rho}(0,\alpha)) = (\hat{u}(\alpha), \hat{v}(\alpha), \hat{\varphi}(\alpha), \hat{\rho}(\alpha))$.

Lemma 3.1. *If the principal part* $(\overline{u}, \overline{v}, \overline{\rho})(r,\alpha) = (\hat{u}, \hat{v}, \hat{\rho})(\alpha)$ *satisfies*

$$\frac{d\hat{u}}{d\alpha} + \tan\alpha\frac{d\hat{v}(\alpha)}{d\alpha} = 0, \tag{3.7}$$

$$\frac{1}{2}\left(\hat{u}^2(\alpha) + \hat{v}^2(\alpha)\right) + \frac{(\gamma-a\hat{\rho})}{(\gamma-1)}\cdot\frac{A\hat{\rho}^{\gamma-1}}{(1-a\hat{\rho})^{\gamma}} + \hat{b}^2(\alpha) = constant, \tag{3.8}$$

$$\tan\alpha = \frac{\hat{u}(\alpha)\hat{v}(\alpha) + \hat{w}(\alpha)\sqrt{\hat{u}^2(\alpha)+\hat{v}^2(\alpha)-\hat{w}^2(\alpha)}}{\hat{u}^2(\alpha)-\hat{w}^2(\alpha)}, \tag{3.9}$$

then $(u,v,\rho)(\xi,\eta) = (\hat{u},\hat{v},\hat{\rho})(\alpha), \eta = \xi\tan\alpha$ *is the* C_+ *centered simple wave solution of Eq (2.13) at the origin.*

Proof. This proof is divided into the following three steps:

Step 1. First: prove that for any given α, the line $\eta = \xi\tan\alpha$ is the C_+ characteristic curve. This only requires proving that $\eta = \xi\tan\alpha$ is circular with the speed of sound $(\hat{u}(\alpha)-\xi)^2 + (\hat{v}(\alpha)-\eta)^2 = \hat{w}(\alpha)^2$ tangent. In fact, substituting $\eta = \xi\tan\alpha$ into the velocity circle expression of sound is obtained

$$\left(\hat{u}^2+\hat{v}^2-\hat{w}^2\right) - 2\left[\hat{u}+\hat{v}\tan\alpha\right]\xi + \left(1+\tan^2\alpha\right)\xi^2 = 0. \tag{3.10}$$

Substituting (3.8) into (3.9) yields the discriminant of the (3.10) root

$$\Delta = -4\left[\left(\hat{u}^2-\hat{v}^2\right)\tan^2\alpha - 2\hat{u}\hat{v}\tan\alpha + \left(\hat{v}^2-\hat{w}^2\right)\right] = 0. \tag{3.11}$$

So, for any given line, $\eta = \xi \tan\alpha$ is a C_+ straight eigenline.

Step 2. Prove that $(u, v, \rho)(\xi, \eta)$ satisfies Eq (2.13) in the theorem. According to the definition of (u, v, ρ), it can be known that along C_+ straight eigenline is constant, that is, $\partial_+ u = \partial_+ v = \partial_+ \rho = 0$ and

$$\partial_+ u + \lambda_- \partial_+ v = 0. \tag{3.12}$$

Under condition (3.4), it is obtained that

$$\partial_- u + \lambda_+ \partial_- v = \left(\frac{d\hat{u}}{d\alpha} + \tan\alpha \frac{d\hat{v}(\alpha)}{d\alpha}\right)\partial_- \alpha = 0. \tag{3.13}$$

Step 3. Prove that $(u, v, \rho)(\xi, \eta)$ satisfies the proposed Bernoulli law in the theorem. This can be obtained from Eqs (3.4) and (3.5).

$$\begin{aligned}
&\partial_+ \left(\frac{1}{2}\left(U^2 + V^2\right) + \frac{(\gamma - A\rho)}{\gamma - 1} \cdot \frac{A\rho^{\gamma-1}}{(1 - a\rho)^{\gamma}} + \mu\kappa_0^2 \rho + \varphi\right) \\
&= (u - \xi)\partial_+ u + (v - \eta)\partial_+ v + \frac{A\gamma\rho^{\gamma-2}}{(1 - a\rho)^{\gamma+1}}\partial_+ \rho + \mu\kappa_0^2 \partial_+ \rho \\
&= 0,
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
&\partial_- \left(\frac{1}{2}\left(U^2 + V^2\right) + \frac{(\gamma - A\rho)}{\gamma - 1} \cdot \frac{A\rho^{\gamma-1}}{(1 - a\rho)^{\gamma}} + \mu\kappa_0^2 \rho + \varphi\right) \\
&= (u - \xi)\partial_- u + (v - \eta)\partial_- v + \frac{A\gamma\rho^{\gamma-2}}{(1 - a\rho)^{\gamma+1}}\partial_- \rho + \mu\kappa_0^2 \partial_- \rho \\
&= \left((\hat{u} - \xi)\frac{d\hat{u}}{d\alpha} + (\hat{v} - \eta)\frac{d\hat{v}}{d\alpha} + \frac{A\gamma\hat{\rho}^{\gamma-2}}{(1 - a\hat{\rho})^{\gamma+1}}\frac{d\hat{\rho}}{d\alpha} + \mu\kappa_0^2 \frac{d\hat{\rho}}{d\alpha}\right)\partial_- \alpha \\
&= \left(\hat{u}\frac{d\hat{u}}{d\alpha} + \hat{v}\frac{d\hat{v}}{d\alpha} + \frac{A\gamma\hat{\rho}^{\gamma-2}}{(1 - a\hat{\rho})^{\gamma+1}}\frac{d\hat{\rho}}{d\alpha} + \mu\kappa_0^2 \frac{d\hat{\rho}}{d\alpha}\right)\partial_- \alpha - \xi\left(\frac{d\hat{u}}{d\alpha} + \tan\alpha\frac{d\hat{v}}{d\alpha}\right)\partial_- \alpha \\
&= 0,
\end{aligned} \tag{3.15}$$

(3.14) and (3.15) show that Bernoulli's law holds. □

Next, we obtain the expression of the centered simple wave solution. Since $(u, v, \rho)(\xi, \eta) = (\hat{u}, \hat{v}, \hat{\rho})(\alpha)$ is the principal part of the centered simple wave for type C_+ and α is the characteristic angle of the C_+ characteristic line $\eta = \xi \tan\alpha$. We decompose the pseudo-flow velocity $(U, V) = (\hat{u}(\alpha) - \xi, \hat{v}(\alpha) - \eta)$ along the $(\sin\alpha, -\cos\alpha)$ and $(\cos\alpha, \sin\alpha)$ directions, respectively. I obtain the velocity components w and g as follows:

$$\overline{w}(\xi, \alpha) = (\hat{u}(\alpha) - \xi)\sin\alpha - (\hat{v}(\alpha) - \xi\tan\alpha)\cos\alpha, \quad \overline{g}(\xi, \alpha) = (\hat{u}(\alpha) - \xi)\cos\alpha + (\hat{v}(\alpha) - \xi\tan\alpha)\sin\alpha. \tag{3.16}$$

When $\xi \to 0$, we get

$$\begin{aligned}
&\hat{w}(\alpha) = \hat{u}(\alpha)\sin\alpha - \hat{v}(\alpha)\cos\alpha, \quad \hat{g}(\alpha) = \hat{u}(\alpha)\cos\alpha + \hat{v}(\alpha)\sin\alpha, \\
&\hat{u}(\alpha) = \hat{g}(\alpha)\cos\alpha + \hat{w}(\alpha)\sin\alpha, \quad \hat{v}(\alpha) = \hat{g}(\alpha)\sin\alpha - \hat{w}(\alpha)\cos\alpha.
\end{aligned} \tag{3.17}$$

It is easy to see that $\hat{g}(\alpha)^2 + \hat{w}(\alpha)^2 = \hat{u}(\alpha)^2 + \hat{v}(\alpha)^2$, derivation of α in each of the last two equations of (3.17) yields

$$\frac{d\hat{u}}{d\alpha} = \frac{d\hat{g}}{d\alpha}\cos\alpha - \hat{g}\sin\alpha + \frac{d\hat{w}}{d\alpha}\sin\alpha + \hat{w}\cos\alpha, \quad \frac{d\hat{v}}{d\alpha} = \frac{d\hat{g}}{d\alpha}\sin\alpha + g\,\hat{\cos}\,\alpha - \frac{d\hat{w}}{d\alpha}\cos\alpha + \hat{w}\sin\alpha. \quad (3.18)$$

Substituting Eq (3.18) into Eq (3.4), we have

$$\hat{g}_\alpha = -\hat{w}. \quad (3.19)$$

The derivation of α in Bernoulli's law (3.5) and then the union (3.19) yield

$$\hat{g} = \frac{(\gamma+1)\hat{c}^2 + 3(1-a\hat{\rho})\hat{b}^2}{(\gamma-1+2a\hat{\rho})\hat{c}^2 + (1-a\hat{\rho})\hat{b}^2}\hat{w}_\alpha = \frac{\hat{w}_\alpha}{\mu^2(\hat{\rho})}. \quad (3.20)$$

Then the derivation of α in (3.20) yields

$$\frac{(\gamma+1)\hat{c}^2 + 3(1-a\hat{\rho})\hat{b}^2}{(\gamma-1+2a\hat{\rho})\hat{c}^2 + (1-a\hat{\rho})\hat{b}^2}\hat{w}_{\alpha\alpha} - \frac{\left[(\gamma+1)a\hat{\rho}\hat{w}^2 + (2-\gamma-3a\hat{\rho})^2\hat{b}^2\right](1-a\hat{\rho})\hat{c}^2}{\left[(\gamma-1+2a\hat{\rho})\hat{c}^2 + (1-a\hat{\rho})\hat{b}^2\right]^3}4\hat{w}(\hat{w}_\alpha)^2 + \hat{w} = 0. \quad (3.21)$$

Using the initial value condition $(\hat{u}, \hat{v}, \hat{\rho})(\alpha_0) = (u_0, 0, \rho_0)$ to solve Eq (3.21), we have

$$\hat{w} = \int_{\alpha_0}^{\alpha} \mu^2(\hat{\rho})\sqrt{\Pi(\hat{\rho}) + \Pi_0}\,d\alpha + w_0, \quad \hat{g} = \frac{\hat{w}_\alpha}{\mu^2(\hat{\rho})} = \sqrt{\Pi(\hat{\rho}) + \Pi_0}, \quad (3.22)$$

in which

$$\Pi(\hat{\rho}) = -\frac{\hat{w}^2}{\gamma(\gamma-1)}\left[\gamma(\gamma+1) - 2(\gamma+1)a\hat{\rho} + 2a^2\hat{\rho}^2\right], \quad (3.23)$$

and

$$\Pi_0 = u_0^2 - w_0^2 + \frac{w_0^2}{\gamma(\gamma-1)}\left[\gamma(\gamma+1) - 2(\gamma+1)a\rho_0 + 2a^2\rho_0^2\right]. \quad (3.24)$$

According to (3.17), it is not hard to obtain

$$\begin{aligned} \hat{u} &= \sqrt{\Pi(\hat{\rho}) + \Pi_0}\cos\alpha + \hat{w}\sin\alpha, \\ \hat{v} &= \sqrt{\Pi(\hat{\rho}) + \Pi_0}\sin\alpha - \hat{w}\cos\alpha, \\ \hat{w} &= \int_{\alpha_0}^{\alpha} \mu^2(\hat{\rho})\sqrt{\Pi(\hat{\rho}) + \Pi_0}\,d\alpha + w_0. \end{aligned} \quad (3.25)$$

Lemma 3.2. *Suppose $\rho_0 < \rho_1$ and $\theta \le \alpha_v$, then R_C is a complete simple wave that connects the constant state with the vacuum (see Figure 5), where α_0 and α_v satisfy*

$$\alpha_0 = \arcsin\frac{w_0}{u_0}, \quad \alpha_v = \int_{\rho_0}^{0} \frac{\hat{w}\sqrt{\hat{q}^2 - \hat{w}^2}}{\hat{q}^2\hat{\rho}}d\hat{\rho}, \quad (3.26)$$

and $\hat{q}^2 = \hat{u}^2 + \hat{v}^2$.

Proof. Deriving and substituting $\hat{u} = \hat{q}\cos\sigma$ and $\hat{v} = \hat{q}\sin\sigma$ into the first equation of (3.4), and also deriving Eq (3.5), we have

$$\frac{d\sigma}{d\hat{q}} = -\frac{\cos\delta}{\hat{q}\sin\delta} = -\frac{\sqrt{\hat{q}^2 - \hat{w}^2}}{\hat{q}\hat{w}}, \quad \frac{d\hat{q}}{d\hat{\rho}} = -\frac{\hat{w}^2}{\hat{q}\hat{\rho}}. \tag{3.27}$$

Therefore, we have

$$\sigma = \int_{\rho_0}^{\hat{\rho}} \frac{\hat{w}\sqrt{\hat{q}^2 - \hat{w}^2}}{\hat{q}^2\hat{\rho}} d\hat{\rho}, \tag{3.28}$$

it is easy to see that $\alpha_v = \lim_{\rho\to 0}\sigma$. By deflating α_v one obtains

$$\alpha_v \in \left(-\frac{1}{u_0}\left(\frac{\gamma - a\rho_0}{\gamma - 1}\cdot\frac{A\rho_0^{\gamma-1}}{(1-a\rho_0)^{\gamma}} + \mu\kappa_0^2\rho_0\right), 0\right). \tag{3.29}$$

So α_v is bounded. □

3.2. Planar simple wave

By (1.1) and its initial value condition

$$(u, v, \rho)(x, y, 0) = \begin{cases} (u_0, 0, \rho_0), & n_1x + n_2y < 0, \\ Vacuum, & n_1x + n_2y > 0, \end{cases} \tag{3.30}$$

where $n_1^2 + n_2^2 = 1$.

By coordinate transformations $\hat{x} = n_1x + n_2y$, $\hat{y} = -n_2x + n_1y$, let $\hat{u} = n_1u + n_2v$, $\hat{v} = -n_2u + n_1v$. Solving a 1-D Riemann problem, we obtain

$$(\hat{u}, \hat{v}, \hat{\rho})(\hat{x}, \hat{y}, t) = \begin{cases} Vacuum, & \hat{\xi} > \hat{\xi}_1, \\ (u_r, 0, \rho_r), & \hat{\xi}_2 \le \hat{\xi} \le \hat{\xi}_1, \\ (u_0, 0, \rho_0), & \hat{\xi} < \hat{\xi}_2, \end{cases} \tag{3.31}$$

where $\hat{\xi} = \frac{\hat{x}}{t}$, $\hat{\xi}_1 = \lim_{\rho_r\to 0} u_r(\rho_r)$, $\hat{\xi}_2 = u_0 - w(\rho_0)$. The functions $u_r\left(\hat{\xi}\right)$ and $\rho_r\left(\hat{\xi}\right)$ are implicitly determined by

$$u_r = u_0 + \int_{\rho_0}^{\rho_r} \frac{\sqrt{c^2 + b^2}}{\rho} d\rho\,(\rho_r < \rho_0), \quad \hat{\xi} = u_r - w(\rho_r). \tag{3.32}$$

Therefore, the solutions of (1.1) and (3.30) are obtained as follows:

$$(u, v, \rho)(x, y, t) = \begin{cases} Vacuum, & \hat{\xi} > \hat{\xi}_1, \\ (n_1u_r, n_2u_r, \rho_r), & \hat{\xi}_2 \le \hat{\xi} \le \hat{\xi}_1, \\ (u_0, 0, \rho_0), & \hat{\xi} < \hat{\xi}_2. \end{cases} \tag{3.33}$$

4. Interaction of centered and planar rarefaction waves

Suppose that the centered simple wave Rc and the planar simple wave Rp start to interact at point $I\left(u_0 - w_0, w_0\sqrt{\frac{u_0-w_0}{u_0+w_0}}\right)$ to form an interaction region Ω. Ω is enclosed by the C_- penetrating characteristic curve $\widetilde{IH}$ of the centered simple wave Rc, the C_+ penetrating characteristic curve $\widetilde{IK}$ of the backward planar simple wave Rp, and the interface $\widetilde{HK}$ between the gas and the vacuum region, where $\widetilde{IH}$ is determined by the centered simple wave Rc and point I, while $\widetilde{IK}$ is determined by the planar simple wave Rp and point I; see Figure 5.

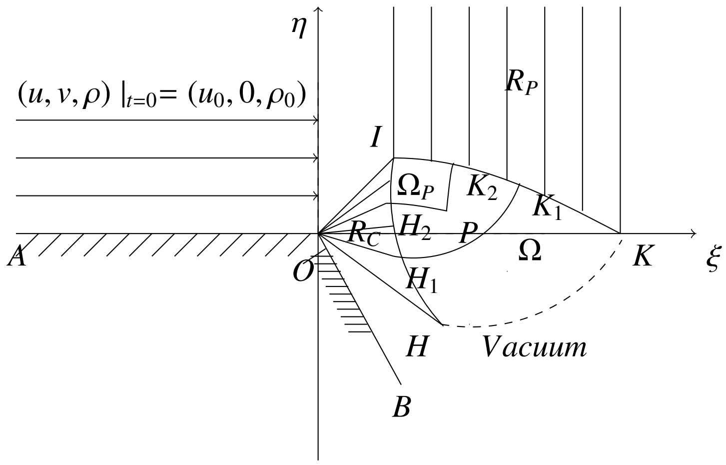

Figure 5. Interaction region.

According to [29], a simple calculation yields the following parametric expression for $\widetilde{IK}$ with respect to the density ρ

$$\begin{cases} \xi = u_0 - w + \int_{\rho_0}^{\rho} \frac{\sqrt{\frac{A\gamma\tau^{\gamma-1}}{(1-a\tau)^{\gamma+1}} + \mu\kappa_0^2\tau}}{\tau} d\tau, \\ \eta = \left\{ \frac{\rho w}{\rho_0 w_0} \left[w_0^2 \frac{u_0-w_0}{u_0+w_0} - \rho_0 w_0 \int_{\rho_0}^{\rho} \frac{\frac{(\gamma+1)A\gamma\tau^{\gamma-2}}{(1-a\tau)^{\gamma+2}} + 3\mu\kappa_0^2}{2\tau\sqrt{\frac{A\gamma\tau^{\gamma-1}}{(1-a\tau)^{\gamma+1}} + \mu\kappa_0^2\tau}} d\tau \right] \right\}^{\frac{1}{2}}, \end{cases} \tag{4.1}$$

where

$$w_0 = \sqrt{\frac{A\gamma\rho_0^{\gamma-1}}{(1-a\rho_0)^{\gamma+1}} + \mu\kappa_0^2\rho_0}. \tag{4.2}$$

From (4.1), when $\rho = 0$, the coordinates of the point K can be obtained as

$$K(\xi, \eta) = \left(u_0 - \int_0^{\rho_0} \frac{w}{\rho} d\rho, 0\right). \tag{4.3}$$

The solution of the system of Eqs (2.1)–(2.3) outside the interaction region a consists of the constant state $(u_0, 0, w_0)$, the vacuum state, the centered simple wave Rc, and the backward planar simple wave Rp; see Figure 5. Solving the solution of the system of Eq (2.1) in the interaction region Ω boils down to a Goursat problem, i.e., the system of Eq (2.1) as well as the boundary conditions

$$(u,v,\rho)(\xi,\eta)=\begin{cases}(u_+,v_+,\rho_+)(\xi,\eta), & on \quad \widetilde{IK},\\ (u_-,v_-,\rho_-)(\xi,\eta), & on \quad \widetilde{IH},\end{cases} \tag{4.4}$$

where $(u_\pm,v_\pm,\rho_\pm)(\xi,\eta)$ is determined by the simple wave Rp (Rc).

Lemma 4.1. *For any $\rho\in(0,\rho_0)$, Ω_{w_ρ} denotes the region enclosed by the C_+ characteristic curve $\widetilde{IK_1}$, the C_- characteristic curve $\widetilde{IH_1}$, and the isomagnetoacoustic velocity line $w(\xi,\eta)=w_\rho$. Here K_1 and H_1 are the intersections of the isomagnetoacoustic velocity line $w(\xi,\eta)=w_\rho$ with $\widetilde{IK}$ and $\widetilde{IH}$, respectively. When ρ is sufficiently close to ρ_0, then there exists a unique C^1 solution to the Goursat problem (2.1) and (4.4) on Ω_{w_ρ}.*

Proof. According to the theory of the existence of local classical solutions of hyperbolic equations [28], there is a local C^1 solution to the Goursat problems (2.4) and (4.4). Thus, there is a sufficiently small normal number such that ε Goursat problems (2.4) and (4.4) have a C^1 solution on Σ^ε, where Σ^ε is the closed region bounded by $\widetilde{IK}$, $\widetilde{IH}$, and the straight line

$$\xi=u_0-w_0+\varepsilon. \tag{4.5}$$

Let

$$l=\sup_{\xi=u_0-w_0+\varepsilon} w(\xi,\eta). \tag{4.6}$$

In the planar simple wave Rp, we have $\beta=-\frac{\pi}{2}$. Combining with Eq (2.36), on the C_+ characteristic curve $\widetilde{IK}$, we obtain

$$\bar\partial_+ w=-\left[\frac{\frac{\gamma-1+2a\rho}{1-a\rho}c^2+b^2}{\frac{\gamma+1}{1-a\rho}c^2+3b^2}\right]\sin(2\delta)<0. \tag{4.7}$$

Similarly, in the centered simple wave Rc, from Eq (2.37), on the C_- characteristic curve $\widetilde{IH}$, we have

$$\bar\partial_- w=\frac{\mu^2(\rho)}{\tan\alpha}\left[w\bar\partial_-\alpha-2\sin^2\delta\right]<0. \tag{4.8}$$

Combining with (4.7), (4.8), (2.42), and (2.43), we have $\bar\partial_\pm w<0$ in the region Σ^ε, evidently, $l<w_0$. Therefore, there exists a unique C^1 solution to the Goursat problem on Ω_{w_ρ}. □

Lemma 4.2. *On the C_+ characteristic curve $\widetilde{IK}$, there is*

$$\bar\delta(\rho_0)<\delta\le\frac{\alpha_0}{2}+\frac{\pi}{4},\quad \beta=-\frac{\pi}{2}. \tag{4.9}$$

Meanwhile, on the C_- characteristic curve $\widetilde{IH}$, there is

$$\alpha_v\le\alpha\le\alpha_0,\quad \min\left(\alpha_v-2\bar\delta_*,-\frac{\pi}{2}\right)\le\beta<\alpha_0-2\bar\delta(\rho_0), \tag{4.10}$$

where

$$\bar\delta(\rho)=\arctan\sqrt{m(\rho)},\quad \bar\delta_*=\begin{cases}\arctan\sqrt{\frac{3-\gamma}{\gamma+1}}\,(1<\gamma<2)\\ \frac{\pi}{6}\,(2\le\gamma<3)\end{cases},\quad \bar\delta(\rho(E))=\delta(E). \tag{4.11}$$

Proof. In view of (2.36), we have

$$w\bar{\delta}_+\delta = -\frac{\sin(2\delta)}{4\mu^2(\rho)}\left[m(\rho) - \tan^2\delta\right]\bar{\partial}_+ w. \tag{4.12}$$

Due to

$$m(\rho) > 0, \quad m'(\rho) < 0. \tag{4.13}$$

We have

$$\bar{\delta}'(\rho) < 0, \quad \lim_{\rho\to 0}\bar{\delta} = \bar{\delta}_* < \frac{\pi}{4}. \tag{4.14}$$

Moreover

$$2\delta(\rho_0) = \alpha_0 + \frac{\pi}{2} > 2\bar{\delta}(\rho_0). \tag{4.15}$$

Due to $\bar{\partial}_+ m(\rho) > 0$, we have the following two possible cases, see Figure 6.

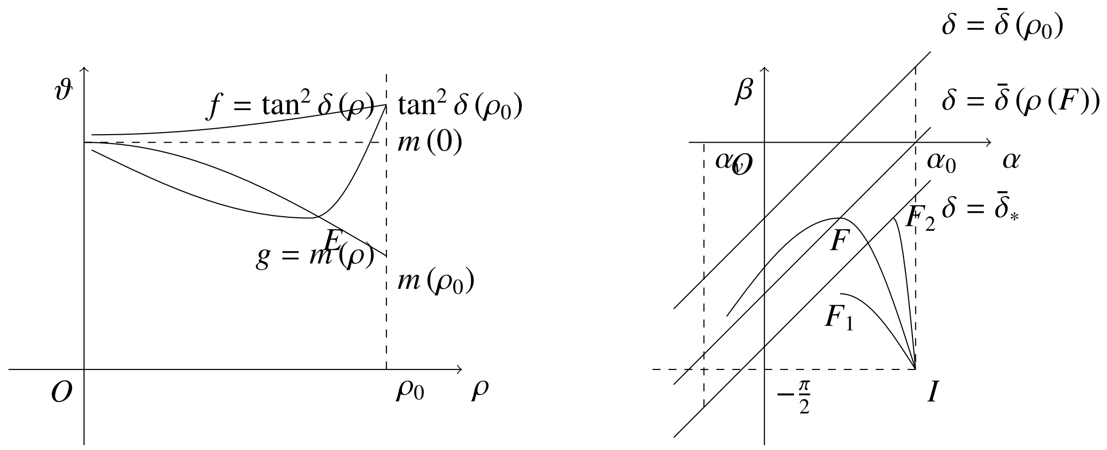

Figure 6. Boundary data estimates.

Case 1. If the functions f and g have no intersection within $(0, \rho_0)$, then we have

$$\bar{\delta}_* \le \delta \le \frac{\alpha_0}{2} + \frac{\pi}{4}, \quad 2\bar{\delta}_* - \frac{\pi}{2} \le \alpha \le \alpha_0, \quad \beta = -\frac{\pi}{2}, \quad \delta > \bar{\delta}(\rho) \quad on \quad \widetilde{IK}. \tag{4.16}$$

Case 2. If the functions f and g intersect at point E in the range $(0, \rho_0)$, then we have $m(\rho(E)) = \tan^2\delta(E)$. Clearly, $\tan^2\delta > m(\rho)$ holds on IE. Then, we get $\bar{\partial}_+\delta\,|_{\widetilde{IE}} < 0$ and

$$\bar{\delta}(\rho(E)) \le \delta \le \frac{\alpha_0}{2} + \frac{\pi}{4}, \quad 2\bar{\delta}(\rho(E)) - \frac{\pi}{2} < \alpha < \alpha_0, \quad \beta = -\frac{\pi}{2} \quad on \quad \widetilde{IE}. \tag{4.17}$$

Furthermore, there are $\tan^2\delta < m(\rho)$ holds on EK; otherwise, there exists a point E_1 on EK such that $\bar{\partial}_+\delta(E_1) = 0$, and $m'(\rho) < 0$. According to (4.12), $\bar{\partial}_+\delta(E_1) < 0$ holds, which contradicts the hypothesis. It is not difficult to obtain

$$\bar{\delta}(\rho(E)) < \delta < \bar{\delta}_*, \quad 2\bar{\delta}(\rho(E)) - \frac{\pi}{2} < \alpha < 2\bar{\delta}_* - \frac{\pi}{2}, \quad \beta = -\frac{\pi}{2} \quad on \quad \widetilde{EK}. \tag{4.18}$$

Based on $\bar{\delta}_* > \bar{\delta}(\rho_0)$ and $\bar{\delta}(\rho(E)) > \bar{\delta}(\rho_0)$, it is not difficult to obtain (4.9).

Similarly, by (2.39), on the C_- cross characteristic curve $\widetilde{IH}$, we obtain

$$w\bar{\partial}_-\beta = \frac{\sin(2\delta)}{2\mu^2(\rho)}\left[m(\rho) - \tan^2\delta\right]\bar{\partial}_- w, \tag{4.19}$$

we discuss this in the following two cases:

Case 1. If $\bar{\delta}(\rho) \le \delta$ as $\alpha_v \le \alpha \le \alpha_0$. According to Eq (4.19), we have

$$-\frac{\pi}{2} \le \beta < \alpha_0 - 2\bar{\delta}_*, \quad \bar{\delta}(\rho) \le \delta \quad on \quad \widetilde{IH}. \tag{4.20}$$

Case 2. If not, there must exist a point F between $\delta = \bar{\delta}(\rho_0)$ and $\delta = \bar{\delta}_*$ such that $\delta(F) = \bar{\delta}(\rho(F))$. Clearly, $\delta > \bar{\delta}$ holds on IF. Then, we obtain $\bar{\partial}_-\beta\,|_{IF} > 0$ and

$$-\frac{\pi}{2} \le \beta \le \beta(F) \quad on \quad \widetilde{IF}. \tag{4.21}$$

Additionally, there is $\delta < \bar{\delta}$ holds on $\widetilde{FH}$; otherwise, there exists a point F_1 on $\widetilde{FH}$ such that $\bar{\partial}_-\beta(F_1) = 0$, and $m'(\rho) < 0$. According to (4.19), $\bar{\partial}_-\beta(F_1) < 0$ holds, which contradicts the hypothesis. It is not difficult to get

$$\alpha_v - 2\bar{\delta}_* \le \beta \le \beta(F) \quad on \quad \widetilde{FH}. \tag{4.22}$$

Based on $\alpha_0 - 2\bar{\delta}_* < \alpha_0 - 2\bar{\delta}(\rho_0)$ and $\beta(F) < \alpha_0 - 2\bar{\delta}(\rho_0)$, it is not difficult to obtain (4.10). Then, this lemma can be obtained. □

Lemma 4.3. *If there exists a C^1 solution to the Goursat problem (2.1) and (4.4) in $\Omega_{\tilde{\rho}}$, and* $\max\left(\alpha_0 - 4\bar{\delta}(\rho_0) + 4\bar{\delta}_* - \frac{\pi}{2}, \alpha_0 - \alpha_v - 2\bar{\delta}(\rho_0)\right) < 0$*, then the characteristic inclination (α,β) within $\Omega_{\tilde{\rho}}$ satisfies*

$$\underline{\alpha} < \alpha < \overline{\alpha}, \quad \underline{\beta} < \beta < \overline{\beta}, \quad 2\underline{\delta} < \alpha - \beta < \pi, \tag{4.23}$$

where

$$\begin{aligned} &\overline{\alpha} = 2\bar{\delta}_* + \alpha_0 - 2\bar{\delta}(\rho_0), \quad \underline{\alpha} = \min\left(2\bar{\delta}(\rho_0) - \frac{\pi}{2}, \alpha_v\right), \quad \overline{\beta} = \alpha_0 - 2\bar{\delta}(\rho_0), \\ &\underline{\beta} = \min\left(\alpha_v - 2\bar{\delta}_*, -\frac{\pi}{2} + 2\bar{\delta}(\rho_0) - 2\bar{\delta}_*\right), \quad \overline{\alpha} - \overline{\beta} = 2\bar{\delta}_*, \quad \underline{\alpha} - \underline{\beta} = 2\bar{\delta}_*, \end{aligned} \tag{4.24}$$

for every $\tilde{\rho} \in (0, \rho_0)$ (Figure 7).

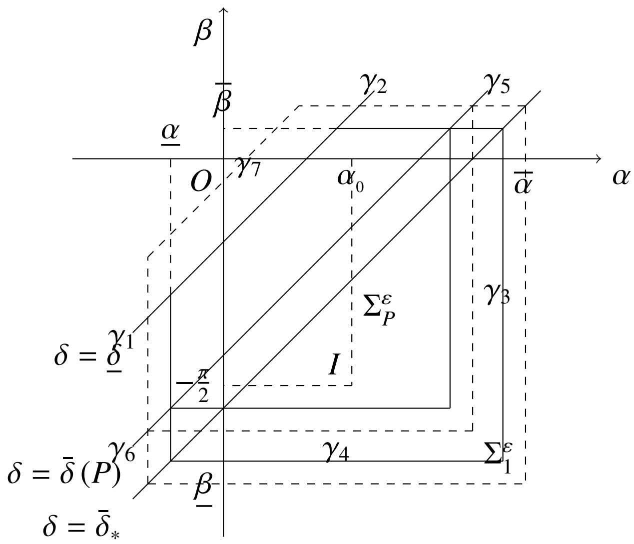

Figure 7. Invariant region.

Proof. Assume

$$\Sigma_1^\varepsilon = \left\{(\alpha,\beta) \mid \underline{\alpha} - \varepsilon < \alpha < \overline{\alpha} + \varepsilon, \underline{\beta} - \varepsilon < \beta < \overline{\beta} + \varepsilon, 2\underline{\delta} < \alpha - \beta\right\}, \tag{4.25}$$

where ε is any positive number close to zero and $\underline{\delta}$ is a sufficiently small constant satisfied by $0 < \underline{\delta} < \frac{\alpha_v}{2} - \frac{\alpha_0}{2} + \bar{\delta}(\rho_0)$, such that

$$1 - \frac{\left(1 - \cos^2\delta \cdot \Omega\right) M_1}{\mu^2(\rho)\sin(2\delta)} > 0 \quad as \quad \delta \le \underline{\delta}. \tag{4.26}$$

It is easy to see that when $\max\left(\alpha_0 - 4\bar{\delta}(\rho_0) + 4\bar{\delta}_* - \frac{\pi}{2}, \alpha_0 - \alpha_v - 2\bar{\delta}(\rho_0)\right) < 0$, we have

$$\begin{aligned}(\overline{\alpha} + \varepsilon) - \left(\underline{\beta} - \varepsilon\right) &= 2\bar{\delta}_* + \alpha_0 - 2\bar{\delta}(\rho_0) + \varepsilon - \min\left(\alpha_v - 2\bar{\delta}_* - \varepsilon, -\frac{\pi}{2} + 2\bar{\delta}(\rho_0) - 2\bar{\delta}_* - \varepsilon\right)\\ &= \max\left(\alpha_0 + 4\bar{\delta}_* - 2\bar{\delta}(\rho_0) + 2\varepsilon, \alpha_0 + 4\bar{\delta}_* - 4\bar{\delta}(\rho_0) + \frac{\pi}{2} + 2\varepsilon\right) < \pi.\end{aligned} \tag{4.27}$$

Therefore, we have $0 < 2\underline{\delta} < \alpha - \beta < \pi$, which shows that Σ_1^ε satisfies the hyperbolicity condition. Choose any point P in $\Omega_{\tilde{\rho}}$, we have a C_+ characteristic curve $\widetilde{PH_2}$ and a C_- characteristic curve $\widetilde{PK_2}$, where $H_2 \in \widetilde{IH}$ and $K_2 \in \widetilde{IK}$. Ω_P is a closed region delimited by the corresponding characteristic curves $\widetilde{PK_2}$, $\widetilde{PH_2}$, $\widetilde{IH_2}$ and $\widetilde{IK_2}$.

Provide that $(\alpha,\beta)(\xi,\eta) \in \Sigma_1^\varepsilon$ for $(\xi,\eta) \in \Omega_P \setminus \{P\}$, then we have

$$(\alpha,\beta)(\xi,\eta) \in \Sigma_P^\varepsilon, \tag{4.28}$$

for all $(\xi,\eta) \in \Omega_P$, where

$$\begin{aligned}&\Sigma_P^\varepsilon = \left\{(\alpha,\beta) \mid \underline{\alpha} - \varepsilon < \alpha < \overline{\alpha}_P + \varepsilon, \underline{\beta}_P - \varepsilon < \beta < \overline{\beta} + \varepsilon, 2\underline{\delta} < \alpha - \beta < \overline{\alpha}_P - \underline{\beta}_P\right\},\\ &\overline{\alpha}_P - \overline{\beta} = 2\bar{\delta}_P, \quad \underline{\alpha} - \underline{\beta}_P = 2\bar{\delta}_P.\end{aligned} \tag{4.29}$$

This lemma can be obtained. From Lemma 4.2, we obtain

$$(\alpha,\beta)(\xi,\eta) \in \Sigma_P^\varepsilon, \quad as \quad (\xi,\eta) \in \widetilde{IH}_2 \cup \widetilde{IK}_2. \tag{4.30}$$

Let

$$\gamma_1 = \left\{\alpha = \underline{\alpha} - \varepsilon, \underline{\beta}_P - \varepsilon < \beta < \overline{\beta} + \varepsilon, \alpha - \beta > 2\underline{\delta}\right\},$$
$$\gamma_2 = \left\{\underline{\alpha} - \varepsilon < \alpha < \overline{\alpha}_P + \varepsilon, \beta = \overline{\beta} + \varepsilon, \alpha - \beta > 2\underline{\delta}\right\},$$
$$\gamma_3 = \left\{\alpha = \overline{\alpha}_P + \varepsilon, \underline{\beta}_P - \varepsilon \le \beta < \overline{\beta} + \varepsilon\right\},$$
$$\gamma_4 = \left\{\underline{\alpha} - \varepsilon < \alpha < \overline{\alpha}_P + \varepsilon, \beta = \underline{\beta}_P - \varepsilon\right\},$$
$$\gamma_5 = \left\{\alpha = \overline{\alpha}_P + \varepsilon, \beta = \overline{\beta} + \varepsilon\right\},$$
$$\gamma_6 = \left\{\alpha = \underline{\alpha} - \varepsilon, \beta = \underline{\beta}_P - \varepsilon\right\},$$
$$\gamma_7 = \left\{\underline{\alpha} - \varepsilon \le \alpha \le \overline{\alpha}_P + \varepsilon, \underline{\beta}_P - \varepsilon \le \beta \le \overline{\beta} + \varepsilon, \alpha - \beta = 2\underline{\delta}\right\}.$$

Assuming that (4.28) does not hold, there exists a point Q in the region Ω_P for which $(\alpha,\beta)(Q) \in \bigcup_{i=1}^{6}\gamma_i$ and $(\alpha,\beta)(\xi,\eta) \in \Sigma_P^\varepsilon$ hold for all $(\xi,\eta) \in \Omega_Q \setminus \{Q\}$, where Ω_Q is the closed region bounded by the characteristic curves $\widetilde{QK}_3$, $\widetilde{QH}_3$, $\widetilde{IH}_3$, and $\widetilde{IK}_3$. The C_- characteristic curve through point Q intersects $\widetilde{IK}$ at point K_3, and the C_+ characteristic curve through point Q intersects $\widetilde{IH}$ at point H_3.

If $(\alpha,\beta)(Q) \in \gamma_1$, on the basis of the assumed condition that $(\alpha,\beta)(\xi,\eta) \in \Sigma_1^\varepsilon$ as $(\xi,\eta) \in \Omega_P \setminus \{P\}$, we have Q=P, according to (2.36), we obtain

$$w\bar{\partial}_+\alpha(Q) = -\frac{\sin 2\delta}{2\mu^2(\rho)}\left[\tan^2\bar{\delta}_P - \tan^2\delta(\rho(Q))\right]\bar{\partial}_+ w > 0. \tag{4.31}$$

But based on our assumptions, we have

$$\alpha(\xi,\eta) > \alpha(Q) = \underline{\alpha} - \varepsilon, \quad \forall(\xi,\eta) \in \widetilde{H_3Q} \setminus \{Q\}. \tag{4.32}$$

Thus, we obtain $\bar{\partial}_+\alpha(Q) \le 0$, and that contradicts my assumptions.

If $(\alpha,\beta)(Q) \in \gamma_2$, we can export contradictions similarly.

If $(\alpha,\beta)(Q) \in \gamma_3$, it is not hard to obtain $\bar{\delta}(\rho(Q)) \le \bar{\delta}(\rho(P)) < \delta(\rho(Q))$. Then, we obtain

$$w\bar{\partial}_+\alpha(Q) = -\frac{\sin 2\delta}{2\mu^2(\rho)}\left[\tan^2\bar{\delta}_Q - \tan^2\delta(\rho(Q))\right]\bar{\partial}_+ w < 0. \tag{4.33}$$

But based on our assumptions, we have

$$\alpha(\xi,\eta) < \alpha(Q) = \overline{\alpha}_P + \varepsilon, \quad \forall(\xi,\eta) \in \widetilde{H_3Q} \setminus \{Q\}. \tag{4.34}$$

Similarly, we obtain $\bar{\partial}_+\alpha(Q) \ge 0$. And that contradicts my assumptions.

If $(\alpha,\beta)(Q) \in \gamma_4$, we can export contradictions similarly.

If $(\alpha,\beta)(Q) \in \gamma_5$, we define $\tilde{\alpha}$ on $\widetilde{H_3Q}$ as follows:

$$\begin{cases} w\bar{\partial}_+\tilde{\alpha} = -\frac{\sin 2\delta}{2\mu^2(\rho)}\left[\tan^2\bar{\delta}_P - \tan^2\left(\frac{\tilde{\alpha}-\overline{\alpha}_P-\varepsilon}{2}\right)\right]\bar{\partial}_+ w, \\ \tilde{\alpha}(H_3) = \alpha(H_3). \end{cases} \tag{4.35}$$

Let $z = \tan\frac{\tilde{\alpha}-\overline{\alpha}_P-\varepsilon}{2}$; (4.35) becomes

$$\begin{cases} w\bar{\partial}_+ z = \frac{-1}{4\mu^2(\rho)}\sin(2\delta)\left(1+z^2\right)\left(\tan^2\bar{\delta}_P - z^2\right), \\ z(H_3) = \tan\frac{\alpha(H_3)-\overline{\alpha}_P-\varepsilon}{2} < \tan\bar{\delta}_P. \end{cases} \tag{4.36}$$

A simple calculation based on the method of Lemma 4.2 in [6] yields

$$\tilde{\alpha}(Q) < 2\bar{\delta}_P + \overline{\alpha}_P + \varepsilon. \tag{4.37}$$

Combining (2.36) and (4.35), we have

$$\begin{cases} w\bar{\partial}_+(\tilde{\alpha}-\alpha) = -\frac{\sin 2\delta}{2\mu^2(\rho)}\left[\tan^2\bar{\delta}_P - \tan^2\bar{\delta}\right]\bar{\partial}_+ w + \frac{\sin 2\delta}{2\mu^2(\rho)}\left[\tan^2\left(\frac{\tilde{\alpha}-\overline{\alpha}_P-\varepsilon}{2}\right) - \tan^2\left(\frac{\alpha-\beta}{2}\right)\right]\bar{\partial}_+ w, \\ (\tilde{\alpha}-\alpha)(H_3) = 0. \end{cases} \tag{4.38}$$

Substituting $\tilde{\alpha}(H_3) = \alpha(H_3)$ into the first equation of (4.38), we obtain $\bar{\partial}_+(\tilde{\alpha}-\alpha)(H_3) > 0$. We assert that $(\tilde{\alpha}-\alpha)\left(\overline{H}\right) \geq 0$, $\overline{H} \in \widetilde{H_3Q}$. If not, there must be a point $H_* \in \widetilde{H_3Q} \setminus \{H_3, Q\}$ such that $(\tilde{\alpha}-\alpha)(H_*) = 0$ and $(\tilde{\alpha}-\alpha)(\xi,\eta) > 0$, $\forall(\xi,\eta) \in \widetilde{H_3H_*} \setminus \{H_3, H_*\}$. Thus, $\bar{\partial}_+(\tilde{\alpha}-\alpha)(H_*) \leq 0$. On the other hand, according to the hypothesis, there are $(\alpha,\beta)(H_*) \in \Sigma_P^\varepsilon$ holds, and we obtain $\bar{\partial}_+(\tilde{\alpha}-\alpha)(H_*) > 0$ according to equation (4.38), which contradicts $\bar{\partial}_+(\tilde{\alpha}-\alpha)(H_*) \leq 0$. Therefore, we have $\alpha(Q) \leq \tilde{\alpha}(Q) < 2\bar{\delta}_P + \overline{\alpha}_P + \varepsilon$, and that contradicts our assumptions.

If $(\alpha,\beta)(Q) \in \gamma_6$, we can export contradictions similarly.

If $(\alpha,\beta)(Q) \in \gamma_7$, we have $\bar{\partial}_-\delta(Q) \leq 0$. According to (2.38), (2.39), and (4.26), it is obtained that

$$\begin{aligned} w\bar{\partial}_-\delta_Q &= \frac{1}{2}\left[\frac{\tan\delta}{\mu^2(\rho)}\bar{\partial}_- w + 2\sin^2\delta - \frac{\sin(2\delta)}{2\mu^2(\rho)}\Omega(\delta,\rho)\bar{\partial}_- w\right] \\ &= \frac{\tan\delta}{2\mu^2(\rho)}\left(1-\cos^2\delta\cdot\Omega\right)\bar{\partial}_- w + \sin^2\delta \\ &> \left(1 - \frac{\left(1-\cos^2\delta\cdot\Omega\right)M_1}{\mu^2(\rho)\sin(2\delta)}\right)\sin^2\delta > 0, \end{aligned} \tag{4.39}$$

and that contradicts our assumptions. Based on the above discussion, when ε tends to zero, it is easy to get this lemma. □

Lemma 4.4. *If there exists a C^1 solution to the Goursat problem (2.1) and (4.4) in $\Omega_{\widetilde{\rho}}$, then there exists a constant $M_0 > 0$ independent of $\widetilde{\rho}$ such that*

$$\|(u, v, \rho, \varphi)\|_{C^0(\Omega_{\tilde{\rho}})} < M_0, \tag{4.40}$$

where $\tilde{\rho} \in (0, \rho_0)$.

Proof. According to Lemma 4.3 and Eqs (2.5) and (2.17), we have

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \frac{w}{\sin\delta}\begin{pmatrix} \cos\sigma \\ \sin\sigma \end{pmatrix}, \\ \frac{1}{2}(U^2+V^2) + \frac{(\gamma-a\rho)}{(\gamma-1)}\cdot\frac{A\rho^{\gamma-1}}{(1-a\rho)^\gamma} + \mu\kappa_0^2\rho + \varphi = constant, \end{cases} \tag{4.41}$$

in the domain $\Omega_{\widetilde{\rho}}$, then the lemma is easy to proved. □

Lemma 4.5. *If there exists a C^1 solution to the Goursat problem (2.1) and (4.4) in $\Omega_{\widetilde{\rho}}$, then we have*

$$\left(\bar{\partial}_+ w, \bar{\partial}_- w\right) \in (-M_1, 0) \times (-M_1, 0), \tag{4.42}$$

in the domain $\Omega_{\widetilde{\rho}}$, where

$$M_1 = \max\left\{2\mu^2(\rho_0), \frac{\sqrt{\Pi_0}}{\sqrt{u_0^2 - w_0^2}}\mu^2(\rho_0)\right\}. \tag{4.43}$$

Proof. From Eq (4.7) and $\mu^2(\rho) > 0$, we obtain

$$-\mu^2(\rho_0) \leq -\mu^2(\rho) = -\left[\frac{\frac{\gamma-1+2a\rho}{1-a\rho}c^2 + b^2}{\frac{\gamma+1}{1-a\rho}c^2 + 3b^2}\right] < \bar{\partial}_+ w \mid_{\widetilde{IK}} < 0. \tag{4.44}$$

Moreover, according to (3.22), (1.12), and (3.1), they lead to

$$\hat{w}_\alpha = \mu^2(\rho)\sqrt{\Pi(\rho) + \Pi_0} > 0, \tag{4.45}$$

$$\begin{aligned}\bar{\partial}_- w \mid_{\widetilde{IH}} &= \hat{w}_\alpha\left(\cos\beta \cdot \partial_\xi \alpha + \sin\beta \cdot \partial_\eta \alpha\right) = -\frac{\hat{w}_\alpha}{g}\sin(2\delta) \\ &= -\mu^2(\rho)\sqrt{\Pi(\rho) + \Pi_0} \cdot \frac{\sin(2\delta)}{g} > -\frac{\sqrt{\Pi_0}\cdot\mu^2(\rho_0)}{\sqrt{u_0^2 - w_0^2}}.\end{aligned} \tag{4.46}$$

Based on the characteristic decompositions (2.42) and (2.43), we find by calculation that

$$\begin{aligned}1 + \frac{\Omega(\xi,\eta)\cos(2\delta)}{2\mu^2(\rho)} &= 1 + \frac{\left(m(\rho) - \tan^2\delta\right)\cos(2\delta)}{2\mu^2(\rho)} = 1 + \frac{\left[\left(1 - 2\mu^2(\rho)\right)\cos^2\delta - \sin^2\delta\right]\cos(2\delta)}{2\mu^2(\rho)\cos^2\delta} \\ &= \frac{\mu^2(\rho)\left(1 - 2\cos^2(2\delta)\right) + \cos^2(2\delta)}{2\mu^2(\rho)\cos^2\delta} = \frac{\mu^2(\rho) + m(\rho)\cos^2(2\delta)}{2\mu^2(\rho)\cos^2\delta} > 0.\end{aligned} \tag{4.47}$$

Then, we obtain

$$1 + \frac{\Omega(\xi,\eta)\cos(2\delta)}{2\mu^2(\rho)} + \frac{\left(c^2 + b^2\right)c^2 a(\gamma+1)\rho + c^2 b^2(\gamma - 2 + 3a\rho)^2}{\left[(\gamma - 1 + 2a\rho)c^2 + (1 - a\rho)b^2\right]^2} > 0. \tag{4.48}$$

Next, using the converse, we prove the consistent boundedness of $\bar{\partial}_\pm w$. Suppose that the conclusion of Lemma 4.5 does not hold, and that there exists a point P_0 inside $\Omega_{\widetilde{\rho}}$ such that $\bar{\partial}_- w(P_0) = -M_1$, $\bar{\partial}_+ w(P_0) \in [-M_1, 0)$ and $\left(\bar{\partial}_+ w, \bar{\partial}_- w\right) \in (-M_1, 0) \times (-M_1, 0)$ in $\Omega_{P_0} \setminus \{P_0\}$, where Ω_{P_0} is the region bounded by the C_+ characteristic curve $\widetilde{P_0H_0}$ over the point P_0 and the C_- characteristic curve $\widetilde{P_0K_0}$ over the point P_0 and $\widetilde{IK}$, $\widetilde{IH}$. According to (2.42), we have

$$\begin{aligned}&w\bar{\partial}_+\bar{\partial}_- w(P_0) \\ &= \bar{\partial}_- w\left\{\sin 2\delta + \frac{\bar{\partial}_- w}{2\mu^2(\rho)\cos^2\delta} + \left(1 + \frac{\Omega(\delta,\rho)\, cos2\delta}{2\mu^2(\rho)} + \frac{\left(c^2 + b^2\right)c^2 a(\gamma+1)\rho + c^2 b^2(\gamma - 2 + 3a\rho)^2}{\left[(\gamma - 1 + 2a\rho)c^2 + (1 - a\rho)b^2\right]^2}\right)\bar{\partial}_+ w\right\} \\ &> \bar{\partial}_- w\left\{\sin 2\delta + \frac{\bar{\partial}_- w}{2\mu^2(\rho)\cos^2\delta}\right\} \geq \bar{\partial}_- w\left\{\sin 2\delta - \frac{1}{\cos^2\delta}\right\} > 0.\end{aligned} \tag{4.49}$$

However, this would contradict the hypothesis, so $\bar{\partial}_- w(P_0) \in (-M_1, 0)$. Similarly, $\bar{\partial}_+ w(P_0) \in (-M_1, 0)$ can be proved, and applying the continuity method yields $\left(\bar{\partial}_+ w, \bar{\partial}_- w\right) \in (-M_1, 0) \times (-M_1, 0)$ in the domain $\Omega_{\widetilde{\rho}}$. □

Lemma 4.6. *If there exists a C^1 solution to Goursat problem (2.1) and (4.4) in $\Omega_{\widetilde{\rho}}$, then there exists a constant $M_2 > 0$ independent of $\widetilde{\rho}$ such that*

$$\|Du, Dv, D\rho\|_{C^0(\Omega_{\widetilde{\rho}})} < M_2, \tag{4.50}$$

where $\tilde{\rho} \in (0, \rho_0)$.

Proof. The derivation of both sides of $w^2 = c^2 + b^2$ in the direction of the $C_\pm$ characteristic curve can be obtained

$$\bar{\partial}_\pm \rho = \frac{2(1-a\rho)\sqrt{A\gamma\rho^{\gamma-1}(1-a\rho)^{\gamma+1} + \mu\kappa_0^2\rho(1-a\rho)^{2\gamma+2}}}{A\gamma\rho^{\gamma-2}(\gamma-1+2a\rho) + \mu\kappa_0^2(1-a\rho)^{\gamma+2}} \bar{\partial}_\pm w. \tag{4.51}$$

Observe

$$\partial_\xi = -\frac{\sin\beta\bar{\partial}_+ - \sin\alpha\bar{\partial}_-}{\sin(2\delta)}, \quad \partial_\eta = \frac{\cos\beta\bar{\partial}_+ - \cos\alpha\bar{\partial}_-}{\sin(2\delta)}. \tag{4.52}$$

From Lemma 4.5, (2.33), and (2.34), it is known that there exists a normal number M_2 independent of $\widetilde{\rho}$. □

Lemma 4.7. *There is a global classical solution for the boundary problem (2.1) and (4.4) in the region Ω, where the Ω region is bounded by the C_+ characteristic curve $\widetilde{IK}$, the C_- characteristic curve $\widetilde{IH}$, and the vacuum boundary $\widetilde{HK}$.*

Proof. Suppose that the Goursat problem has a C^1 solution on the area $\Omega_{\widetilde{\rho}}$, where $\widetilde{\rho} \in (0, \rho_0)$. Similarly to the proof of Theorem 4.12 in [6], we have that the isomagnetic sound velocity line $\rho(\xi, \eta) = \widetilde{\rho}$ is Lipschitz continuous. The isomagnetic sound velocity line $\rho(\xi, \eta) = \widetilde{\rho}$ is solvable to length. Let $P_0, P_1, P_2, \ldots, P_n$ be $\rho(\xi, \eta) = \widetilde{\rho}$ on sequentially different n + 1 points; P_0 is located in $\widetilde{IH}$, and P_n is located in $\widetilde{IK}$. The C_+ eigencurve through point P_i intersects the C_- characteristic curve through the point P_{i-1} at point I_i, where $i = 0, 1, \ldots, n-1$. The contour $\rho(\xi, \eta) = \widetilde{\rho}$ is a non-characteristic curve, for any $i = 0, 1, \ldots, n-1$, both $P_i \neq I_i$ and $P_{i-1} \neq I_i$. For any $i = 0, 1, \ldots, n-1$, the system of Eq (2.1) is characterized by the characteristic curves $\widetilde{P_iI_i}$ and $\widetilde{P_{i-1}I_i}$. The Goursat problem is in the quadrilateral region $\Omega_{\widetilde{\rho}}$ bounded by $\widetilde{P_iI_i}$, $\widetilde{P_{i-1}I_i}$, $\widetilde{P_{i-1}T_i}$, and $\widetilde{P_iT_i}$. There is a global smooth classical solution on $\Omega_{\widetilde{\rho}}$, where $\widetilde{P_iT_i}$ is the C_- characteristic curve of the point P_i, and $\widetilde{P_{i-1}T_i}$ is the C_+ characteristic curve of the point P_{i-1}.

Let $B = T_0$ and $A = T_{n+1}$. For each $i = 0, 1, \ldots, n$, there is a $0 < \widetilde{\rho}_i < \widetilde{\rho}$ such that there is a C^1 solution on the Goursat problem $\rho(\xi, \eta) = \widetilde{\rho}_i$, $\widetilde{P_iT_i}$ and $\widetilde{P_iT_{i+1}}$ bounded by the characteristic curves $\widetilde{P_iT_i}$ and $\widetilde{P_iT_{i+1}}$. Let $\widetilde{\rho}_\varepsilon = \max\left\{\widetilde{\rho}_0, \widetilde{\rho}_1, \ldots, \widetilde{\rho}_n, \rho(T_1), \rho(T_2), \ldots, \rho(T_n)\right\}$. We obviously have a relationship $\widetilde{\rho}_\varepsilon < \widetilde{\rho}$ established. Then, we obtain the solution to the Goursat problem (2.1) and (4.4) in the region $\Omega_{\widetilde{\rho}}$. By repeating the above process, we can solve the problem at the regional Ω. So we obtain this lemma. □

Thus, we have completed all the discussions and obtained Theorem 1.1. The ideal magnetohydrodynamic system described in Theorem 1.1 can be used to model phenomena in astrophysics, laboratory plasmas, solar physics, etc. It represents an ideal plasma flow interacting with a magnetic field and is described by partial differential equations, including conservation of mass, momentum, total energy, and magnetic field. The conclusion of Theorem 1.1 shows that we only need to control the density ρ_0 and velocity u_0 of the incoming flow and the solid–wall inclination θ so that they satisfy the conditions. We can guarantee that the problem of expanding a magnetic fluid with a Noble–Abel gas into the vacuum at a convex corner has a global classical solution.

5. Conclusions and discussions

We study the Riemann problem of a supersonic magnetic fluid with Noble-Abel gas diffusing into the vacuum around a convex corner, which is solved by reducing it to a Goursat problem. The solution's hyperbolicity and a priori C^0,C^1 estimates are established using characteristic decompositions and invariant regions. In addition, pentagonal invariant areas are constructed to obtain a global solution. In addition, sub-invariant regions are built, and the hyperbolicity of the solution is obtained based on the continuity of the sub-invariant regions. Finally, the global existence of the solution to the gas expansion problem is constructively received.

This paper deals with this problem under the special assumption of the solid wall angle θ. For more general cases, the problem requires further attempts and discussions, for example, when the incoming flow velocity is not supersonic but sonic or subsonic. The authors will continue to study these issues later. In addition, we study the complete expansion problem in this work. How about incomplete expansion problems? These problems are interesting. Follow-up studies are needed.

Use of AI tools declaration

The author declares she have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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