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Research article

A new fixed point approach for solutions of a p-Laplacian fractional q-difference boundary value problem with an integral boundary condition

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Abstract: We explored a class of quantum calculus boundary value problems that include fractional q-difference integrals. Sufficient and necessary conditions for demonstrating the existence and uniqueness of positive solutions were stated using fixed point theorems in partially ordered spaces. Moreover, the existence of a positive solution for a boundary value problem with a Riemann-Liouville fractional derivative and an integral boundary condition was examined by utilizing a novel fixed point theorem that included a α-η-Geraghty contraction. Several examples were provided to demonstrate the efficacy of the outcomes.

Keywords: fractional derivatives; quantom calculus; differential equations; boundary value problems; positive solution
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1. Introduction

As fractional calculus has grown and developed, numerous investigations have been undertaken on the existence and uniqueness of the solutions for initial and boundary value problems, including fractional equations [1–4]. Recently, with the generalization of fractional operators on arbitrary spaces, the study of fractional initial and boundary value problems (BVPs) has attracted the attention of many scientists [5, 6]. One of these generalizations led to the definition of the fractional derivative on the time scale interval space $\mathbb{T} = \{q^k : k \in \mathbb{N}, 0 < q < 1\}$ and the study on the existence and uniqueness of solutions for initial and boundary value problems incorporating fractional differential equations were conducted on this space [7–10]. However, there are few papers about the investigation of q-difference BVPs within the p-Laplacian operator [11–13].

In [14], Miao and co-authors used some fixed point theorems on partially or- dered sets and p-Laplacian operators to study the positive solutions of q-difference BVP

$$\begin{cases} D_q^{\upsilon}\phi_p(D_q^{\mu}\zeta(\tau)) + h(\tau,\zeta(\tau)) = 0, \quad 0 < \tau < 1, 2 < \mu, \nu < 3, \\ \zeta(0) = D_q\zeta(0) = 0, \quad D_q\zeta(1) = \gamma D_q\zeta(\xi), \end{cases}$$

where $0 < \upsilon < 1, 2 < \mu < 3,\ 0 < \gamma\xi^{\mu-2} < 1, D_q^{\mu}$ is the Riemann-Liouville derivative and $\phi_p(\zeta) = |\zeta|^{p-2}\zeta$ is p-Laplacian operator, $p > 1$.

In [15], Mardanov and co-authors used some known fixed point theorems to study the q-difference BVP of the form

$$\begin{cases} {}^cD^{\beta}_{q,0^+}\phi_p({}^cD^{\alpha}_{q,0^+}\zeta)(\tau) = h(\tau,\zeta(\tau)), \quad 0 \le \tau \le 1, \\ \zeta(0) = \eta\zeta(1), \quad {}^cD^{\alpha}_{q,0^+}\zeta(0) = \gamma^c D^{\alpha}_{q,0^+}\zeta(1), \end{cases}$$

where $\phi_p(\zeta) = |\zeta|^{p-2}\zeta$ is p-Laplacian operator, $p > 1, \phi_p^{-1}(\zeta) = \phi_s(\zeta)$, where $p^{-1} + s^{-1} = 1$ and $0 < \alpha, \beta \le 1$.

In a previous work by Aktuğlu and Özarslan [16], they investigated a Caputo-type q-fractional boundary value problem involving the p-Laplacian operator. This problem can be expressed as:

$$D_q(\varphi_p({}^CD_q^{\alpha}x(t))) = f(t, x(t)), \quad 0 < t < 1 \tag{1.1}$$

subject to the boundary conditions:

$$\begin{aligned} &D_q^k x(0) = 0, \quad k = 2, 3, \dots, n-1, \\ &x(0) = a_0x(1), x(0) = a_0x(1), \quad D_qx(0) = a_1D_qx(1), \end{aligned}$$

where a_0, $a_1 \neq 0$, $\alpha > 1$ and $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$. Aktuğlu and Özarslan employed the Banach contraction mapping principle to establish the existence and uniqueness of a solution for this boundary value problem under specific conditions.

Yan and Hou in [17] applied the Avery-Peterson fixed point to obtain some existing results for the q-difference BVP

$$\begin{cases} {}^cD^{\beta}_{q,0^+}(\phi_p(D_q\zeta(\tau))) + h(\tau)f(\tau,\zeta(\tau),D_q\zeta(\tau)) = 0, \quad \tau \in (0,1), \\ \zeta(0) - aD_q\zeta(0) = \int_0^1 g_1(\varsigma)\zeta(\varsigma)d_q\varsigma, \\ \zeta(1) + \eta D_q\zeta(1) = \int_0^1 g_2(\varsigma)\zeta(\varsigma)d_q\varsigma, \end{cases}$$

where $a, \eta \ge 0,\ f \in C([0,1] \times [0,+\infty) \times \mathbb{R}, [0,+\infty)), h \in C([0,1] \times [0,+\infty))$ and g_i is nonnegative, integrable and $\int_0^1 g_i(\varsigma)d_q\varsigma \in [0,1), i = 1, 2$.

In 2020, Ragoub and co-authors in [18], investigated the q-fractional boundary value problem with the p-Laplace operator

$$\begin{aligned} &aD_q^{\beta}(\phi_p(aD_q^{\alpha}u(t)) + Q(t)\phi_p(u(t)) = 0, \quad t \in (a,b), \\ &u(a) = 0, \quad u(b) = Au(\xi), \\ &aD_q^{\alpha}u(a) = 0, \quad aD_q^{\alpha}u(b) = BaD_q^{\alpha}u(\delta), \end{aligned}$$

where $aD_q^{\alpha}, aD_q^{\beta}$ are the fractional q-derivative of the riemann-Liouville type with $1 < \alpha, \beta < 2$, $0 \leq A, B \leq 1, 0 < \xi, \delta < 1, \phi_p(s) = |s|^{p-2}s, p > 1, \phi_p^{-1} = \phi_r, \frac{1}{p} + \frac{1}{r} = 1$, and $Q : [a,b] \to \mathbb{R}$ is a continuous function on $[a,b]$.

Inspired by the aforementioned research, in this research, we examine q-difference BVP

$$\begin{cases} D_q^{\nu}\varphi_r\left(D_q^{\mu}\zeta(\tau)\right) = \Lambda(\tau, \zeta(\tau)), \quad \tau \in [0,1], \\ \zeta(0) = D_q\zeta(0) = D_q\zeta(1) = D_q^{\mu}\zeta(0) = D_q^{\mu+1}\zeta(0) = 0, \\ D_q^{\mu+1}\zeta(1) = \lambda[\mu-1]_q \int_0^1 g(\theta)\zeta(\theta)d_q\theta, \end{cases} \tag{1.2}$$

where $2 < \nu, \mu < 3$, D_q^{μ} is the Riemann-Liouville fractional derivative and $\phi_r(\zeta) = |\zeta|^{r-2}\zeta$ is p-Laplacian operator, $r > 1$, $\phi_r^{-1}(\zeta) = \phi_s(\zeta)$, $\frac{1}{s} + \frac{1}{r} = 1$, $g : [0,1] \to [0,\infty)$ be a function such that $\tau^{\mu-1}g(\tau) \in L^1[0,1]$ and λ is a constant such that $\lambda \int_0^1 \tau^{\mu-1}g(\tau)d_q\tau < 1$. Unlike earlier studied equations, Eq (1.2) is formed using the p-Laplace operator and a high order fractional derivative, making Green's function analysis challenging. Using a new fixed point theorem that incorporates an $\mathfrak{a} - \eta$-Geraghty contraction, the existence of a positive solution for Eq (1.2) is investigated.

This is how the remainder of the paper is structured. In Section 2, we present a few baseline data and relevant resources. In Section 3, the Green function of the problem and its required attributes are calculated. In Section 4, the existence and uniqueness of positive solutions are demonstrated using a new fixed point theorem. Further, we study the existence of the solution to the fractional q-difference boundary value issue of the Riemann-Liouville derivative by using a $\mathfrak{a} - \eta$-Geraghty contraction. Finally, a few examples are provided to demonstrate the usefulness of our findings. We concluded the paper by drawing a brief conclusion.

2. Preliminaries

Prior to delving into the primary notion of this work, a few fundamental ideas must be introduced, and several helpful instruments used in this work should be mentioned (see [19, 20]).

Let $q \in (0,1)$, we denote

$$[\nu]_q = \frac{1-q^{\nu}}{1-q} = q^{\nu-1} + \cdots + 1, \qquad \nu \in \mathbb{R}. \tag{2.1}$$

The q-analog of the power function $(\nu - \omega)^{\kappa}$ with $\kappa \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is

$$(\nu-\omega)^0 = 1, \quad (\nu-\omega)^{(\kappa)} = \prod_{i=0}^{\kappa-1}(\nu - \omega q^i), \qquad \kappa \in \mathbb{N}, \quad \nu, \omega \in \mathbb{R}. \tag{2.2}$$

Furthermore, let $\gamma \in \mathbb{R}$, then

$$(\nu-\omega)^{(\gamma)} = \nu^{\gamma}\prod_{i=0}^{\infty}\frac{\nu-\omega q^i}{\nu-\omega q^{i+\gamma}} \qquad \nu \neq 0. \tag{2.3}$$

If $\omega = 0$, then $\nu^{(\gamma)} = \nu^{\gamma}$ and we denote $0^{(\gamma)} = 0$ for $\gamma \geq 0$.

The q-gamma function is given by

$$\Gamma_q(\mu) = \frac{(1-q)^{(\mu-1)}}{(1-q)^{\mu-1}}, \qquad \mu \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}. \tag{2.4}$$

Similar to the gamma function, the following relationship is also established for the q-gamma function.

$$\Gamma_q(\mu+1) = [\mu]_q \Gamma_q(\mu), \quad \Gamma_q(\mu) = [\mu-1]_q.$$

The q-integral on $[0, b]$ is given by

$$(\Im_q f)(\tau) = \int_0^b f(\tau) d_q\tau = \tau(1-q) \sum_{n=0}^{\infty} f(\tau q^n) q^n. \tag{2.5}$$

Let $a \in [0, b]$ and f be given on $[0, b]$, then its q-integral from a to b stated as follows:

$$\int_a^b f(\tau) d_q\tau = \int_0^b f(\tau) d_q\tau - \int_0^a f(\tau) d_q\tau. \tag{2.6}$$

Definition 2.1. *Let $\upsilon \geq 0$ and f be a real function on $[a, b]$. The RL q-integral of order υ is given by*

$$\begin{aligned} &(\Im_q^0 \tau) = f(\tau), \\ &(\Im_q^\upsilon f)(\tau) = \frac{1}{\Gamma_q(\upsilon)} \int_a^\tau (\tau - q\varsigma)^{(\upsilon-1)} f(\varsigma) d_q\varsigma, \qquad \upsilon > 0, \quad \tau \in [a, b]. \end{aligned} \tag{2.7}$$

Definition 2.2. *The fractional q-derivative of the RL- type of order $\upsilon \geq 0$ for the function f is defined by*

$$({}^0\mathfrak{D}_q^\upsilon f)(\tau) = f(\tau)$$

and

$$({}^0\mathfrak{D}_q^\upsilon f)(\tau) = (\mathfrak{D}_q^{[\upsilon]} \Im_q^{[\upsilon]-\upsilon} f)(\tau),$$

where $[\upsilon]$ is the smallest integer greater than or equal to υ.

Definition 2.3. *Let $\upsilon \geq 0$ and the Caputo q-derivatives of f be given by*

$$({}^c\mathfrak{D}_q^\upsilon f)(\tau) = (\Im_q^{[\upsilon]-\upsilon} \mathfrak{D}_q^{[\upsilon]} f)(\tau), \tag{2.8}$$

where $[\upsilon]$ is the smallest integer greater than or equal to υ.

If $f(\tau) = \tau^{\gamma-1}$ for $\gamma \notin \mathbb{N}$, then

$${}^c\mathfrak{D}_q^\upsilon f(\tau) = \frac{\Gamma_q(\gamma)}{\Gamma_q(\gamma-\upsilon)} \tau^{\gamma-\upsilon-1}. \tag{2.9}$$

Lemma 2.4. *[19] Let $\upsilon, \gamma \geq 0$ and $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$. Then*

$$\mathfrak{D}_q^\upsilon (I_q^\upsilon f)(\tau) = f(\tau), \quad \Im_q^\upsilon \Im_q^\gamma f(\tau) = \Im_q^{\upsilon+\gamma} f(\tau).$$

Lemma 2.5. *[19] Let $f : [a, b] \to \mathbb{R}$ be differentiable and p be a positive integer. Then*

$$\Im_q^\upsilon \mathfrak{D}_q^p f(t) = \mathfrak{D}_q^p \Im_q^\upsilon f(t) - \sum_{k=0}^{p-1} \frac{t^{\upsilon-p+k}}{\Gamma_q(\upsilon-p+k+1)} (\mathfrak{D}_q^k f)(0), \qquad t \in [a, b].$$

Definition 2.6. *We set Ψ be the set of ψ that satisfy the following condition*

(1) $\psi : [0,\infty) \to [0,\infty)$ *is a continuous and nondecreasing;*
(2) $\psi(\tau) > 0$ *for all* $\tau \in (0,\infty)$*;*
(3) $\psi(0) = 0$*;*
(4) $\lim_{\tau\to\infty} \psi(\tau) = \infty$.

Theorem 2.7. *[21] Suppose $(\mathfrak{E}, d)$ be a partially ordered (with respect the order $\leq$) complete metric space such that satisfy the following conditions*

i) If $\{\zeta_n\}$ is a nondecreasing convergent in $\mathfrak{E}$ ($\lim_{n\to\infty} \zeta_n = \zeta$) then $\zeta_n < \zeta,\ n \in \mathbb{N}$;
ii) Let $\psi \in \Psi$, $F : \mathfrak{E} \to \mathfrak{E}$ be a nondecreasing with

$$d(F\zeta, F\omega) \leq d(\zeta,\omega) - \psi(d(\zeta,\omega)), \quad \zeta \geq \omega,$$

and $\zeta_0 \leq F(\zeta_0)$.

Then F has a fixed point.

Theorem 2.8. *[22] By considering the following extra condition,*

iii) For each pair ζ and ω in $\mathfrak{E}$, there exists a member like ϖ in $\mathfrak{E}$ such that it is comparable to ζ and ω.

to the assumptions of the previous theorem, we reach the uniqueness of the fixed point.

3. Green function

This section is dedicated to constructing the Green's function of equations and demonstrating some of its features.

Lemma 3.1. *[23] Assume $\varrho : [0,1] \to [0,\infty)$ be a continuous, then fractional q-difference BVP*

$$\begin{array}{c} D_q^{\nu} y(\tau) = \varrho(\tau) \quad 0 \leq \tau \leq 1, \quad 2 < \nu < 3, \\ y(0) = D_q y(0) = D_q y(1) = 0, \end{array} \tag{3.1}$$

is equivalent to:

$$y(\tau) = -\int_0^1 H_\nu(\tau, q\varsigma)\varrho(\varsigma) d_q\varsigma, \tag{3.2}$$

where

$$H_\nu(\tau, q\varsigma) = \begin{cases} \frac{1}{\Gamma_q(\nu)} \left(\tau^{\nu-1}(1-q\varsigma)^{(\nu-2)} - (\tau - q\varsigma)^{(\nu-1)} \right) & q\varsigma \leq \tau, \\ \frac{1}{\Gamma_q(\nu)} \tau^{\nu-1}(1-q\varsigma)^{(\nu-2)} & q\varsigma \geq \tau. \end{cases} \tag{3.3}$$

Lemma 3.2. *Let $h \in C([0,1]), 2 < \mu \leq 3$, then fractional q-difference BVP*

$$\begin{cases} D_q^{\mu}\zeta(\tau) + h(\tau) = 0, \tau \in [0,1], \\ \zeta(0) = D_q\zeta(0) = 0, D_q\zeta(1) = \lambda[\mu-1]_q \int_0^1 g(\theta)\zeta(\theta) d_q\theta, \end{cases} \tag{3.4}$$

is equivalent to:

$$\zeta(\tau) = \int_0^1 G(\tau, q\varsigma)h(\varsigma)d_q\varsigma, \tag{3.5}$$

where

$$G(\tau, \varsigma) = H_\nu(\tau, q\varsigma) + K(\tau, q\varsigma), \tag{3.6}$$

with

$$K(\tau, q\varsigma) = \frac{\lambda\tau^{\mu-1}}{\Gamma_q(\mu)(1 - \lambda\int_0^1 \theta^{\mu-1}g(\theta)d_q\theta)} \int_0^1 g(\theta)H_\mu(\theta, q\varsigma)d\theta. \tag{3.7}$$

Proof. By integrating of order μ from Eq (3.4) one can get

$$\zeta(\tau) = -\frac{1}{\Gamma_q(\mu)} \int_0^\tau (\tau - q\varsigma)^{(\mu-1)}h(\varsigma)d_q\varsigma + c_1\tau^{\mu-1} + c_2\tau^{\mu-2} + c_3\tau^{\mu-3}. \tag{3.8}$$

Utilizing the conditions $\zeta(0) = 0$, we get $c_3 = 0$. On the other hand by differentiating from relation (3.8) we have

$$(D_q\zeta)(\tau) = -\frac{1}{\Gamma_q(\mu)} \int_0^\tau [\mu - 1]_q(\tau - q\varsigma)^{(\mu-2)}h(\varsigma)d_q\varsigma + [\mu - 1]_q c_1\tau^{\mu-2} + [\mu - 2]_q c_2\tau^{\mu-3},$$

by applying boundary condition $D_q\zeta(0) = 0$, we have $c_2 = 0$, and from the last boundary condition we get

$$\begin{aligned} D_q\zeta(1) &= -\frac{1}{\Gamma_q(\mu)} \int_0^1 [\mu - 1]_q(1 - q\varsigma)^{(\mu-2)}h(\varsigma)d_q\varsigma + [\mu - 1]_q c_1 \\ &= \lambda[\mu - 1]_q \int_0^1 g(\theta)\zeta(\theta)d_q\theta. \end{aligned}$$

So

$$c_1 = \frac{1}{\Gamma_q(\mu)} \int_0^1 [\mu - 1]_q(1 - q\varsigma)^{(\mu-2)}h(\varsigma)d_q\varsigma + \lambda[\mu - 1]_q \int_0^1 g(\theta)\zeta(\theta)d_q\theta. \tag{3.9}$$

If we replace $\zeta(\tau)$ from the relation (3.8) into the relation (3.9) we have

$$\begin{aligned} c_1 &= \frac{1}{\Gamma_q(\mu)} \int_0^1 (1 - q\varsigma)^{(\mu-2)}h(\varsigma)d_q\varsigma \\ &\quad +\lambda \int_0^1 g(\theta)\left(-\frac{1}{\Gamma_q(\mu)} \int_0^\theta (\theta - q\varsigma)^{(\mu-1)}h(\varsigma)d_q\varsigma + c_1\theta^{\mu-1}\right)d_q\theta \\ &= \frac{1}{\Gamma_q(\mu)} \int_0^1 (1 - q\varsigma)^{(\mu-2)}h(\varsigma)d_q\varsigma + c_1\lambda \int_0^1 \theta^{\mu-1}g(\theta)d_q\theta \\ &\quad -\frac{\lambda}{\Gamma_q(\mu)} \int_0^1 g(\theta) \int_0^\theta (\theta - q\varsigma)^{(\mu-1)}h(\varsigma)d_q\varsigma d_q\theta. \end{aligned}$$

Hence

$$c_1\left(1-\lambda\int_0^1\theta^{\mu-1}g(\theta)d_q\theta\right) = \frac{1}{\Gamma_q(\mu)}\int_0^1(1-q\varsigma)^{(\mu-2)}h(\varsigma)d_q\varsigma - \frac{\lambda}{\Gamma_q(\mu)}\int_0^1 g(\theta)\int_0^\theta(\theta-q\varsigma)^{(\mu-1)}h(\varsigma)d_q\varsigma d_q\theta.$$

So

$$\begin{aligned}c_1 &= \frac{1}{\Gamma_q(\mu)\left(1-\lambda\int_0^1\theta^{\mu-1}g(\theta)d_q\theta\right)}\int_0^1(1-q\varsigma)^{(\mu-2)}h(\varsigma)d_q\varsigma\\ &-\frac{\lambda}{\Gamma_q(\mu)\left(1-\lambda\int_0^1\theta^{\mu-1}g(\theta)d_q\theta\right)}\int_0^1 g(\theta)\int_0^\theta(\theta-q\varsigma)^{(\mu-1)}h(\varsigma)d_q\varsigma d_q\theta.\end{aligned}$$

Consequently,

$$\begin{aligned}\zeta(\tau) &= -\frac{1}{\Gamma_q(\mu)}\int_0^\tau(\tau-q\varsigma)^{(\mu-1)}h(\varsigma)d_q\varsigma\\ &+\frac{\tau^{\mu-1}}{\Gamma_q(\mu)\left(1-\lambda\int_0^1\theta^{\mu-1}g(\theta)d_q\theta\right)}\int_0^1(1-q\varsigma)^{(\mu-2)}h(\varsigma)d_q\varsigma\\ &-\frac{\lambda\tau^{\mu-1}}{\Gamma_q(\mu)\left(1-\lambda\int_0^1\theta^{\mu-1}g(\theta)d_q\theta\right)}\int_0^1 g(\theta)\int_0^\theta(\theta-\varsigma)^{(\mu-1)}h(\varsigma)d_q\varsigma d_q\theta\\ &= -\frac{1}{\Gamma_q(\mu)}\int_0^\tau(\tau-q\varsigma)^{(\mu-1)}h(\varsigma)d_q\varsigma+\frac{\tau^{\mu-1}}{\Gamma_q(\mu)}\int_0^1(1-q\varsigma)^{(\mu-2)}h(\varsigma)d_q\varsigma\\ &+\frac{\lambda\tau^{\mu-1}}{\Gamma_q(\mu)\left(1-\lambda\int_0^1\theta^{\mu-1}g(\theta)d_q\theta\right)}\int_0^1\theta^{\mu-1}g(\theta)d_q\theta\int_0^1(1-q\varsigma)^{(\mu-2)}h(\varsigma)d_q\varsigma\\ &-\frac{\lambda\tau^{\mu-1}}{\Gamma_q(\mu)\left(1-\lambda\int_0^1\theta^{\mu-1}g(\theta)d_q\theta\right)}\int_0^1 g(\theta)\int_0^\theta(\theta-q\varsigma)h(\varsigma)d_q\varsigma d_q\theta\\ &= \int_0^1 H_\mu(\tau,q\varsigma)h(\varsigma)d_q\varsigma+\\ &+\frac{\lambda\tau^{\mu-1}}{\Gamma_q(\mu)\left(1-\lambda\int_0^1\theta^{\mu-1}g(\theta)d_q\theta\right)}\int_0^1 g(\theta)\int_0^1 H_\mu(\theta,q\varsigma)h(\varsigma)d_q\varsigma d_q\theta\\ &= \int_0^1 H_\mu(\tau,q\varsigma)h(\varsigma)d_q\varsigma\\ &+\int_0^1\frac{\lambda\tau^{\mu-1}}{\Gamma_q(\mu)\left(1-\lambda\int_0^1\theta^{\mu-1}g(\theta)d_q\theta\right)}\int_0^1 g(\theta)H_\mu(\theta,q\varsigma)h(\varsigma)d_q\theta d_q\varsigma\\ &= \int_0^1 H_\mu(\tau,q\varsigma)h(\varsigma)d_q\varsigma+\int_0^1 K(\tau,q\varsigma)h(\varsigma)d_q\varsigma=\int_0^1 G(\tau,q\varsigma)h(\varsigma)d_q\varsigma.\end{aligned}$$

□

Lemma 3.3. *Fractional BVP* (1.2) *is equivalent to:*

$$\zeta(\tau) = \int_0^1 G(\tau, q\varsigma)\varphi_s\left(\int_0^{\varsigma} H_\nu(\varsigma, q\xi)\Lambda(\xi, \zeta(\xi))d_q\xi\right)d_q\varsigma, \tag{3.10}$$

where G and H were given by (3.3) *and* (3.6) *respectively.*

Proof. Let $\varrho(\tau) = \varphi_s(\zeta(\tau))$, $h(\tau) = -\Lambda(\tau, \zeta(\tau))$. Then from Lemma 3.1 we get

$$\varrho(\tau) = \varphi_s\left(\int_0^1 H_\nu(\tau, q\varsigma)\Lambda(\varsigma, \zeta(\varsigma))d_q\varsigma\right).$$

Now from Lemma 3.2 we have

$$\zeta(\tau) = \int_0^1 G(\tau, q\varsigma)\varphi_s\left(\int_0^{\varsigma} H_\nu(\varsigma, q\xi)\Lambda(\xi, \zeta(\xi))d_q\xi\right)d_q\varsigma.$$

□

Lemma 3.4. *[14,23] Let $\alpha := \mu$ or $\alpha = \nu$, also let $H(\tau, q\varsigma) := H_\nu(\tau, q\varsigma)$ or $H(\tau, q\varsigma) := H_\mu(\tau, q\varsigma)$, then H satisfies the following conditions*

(1) H is continuous and $H(\tau, q\varsigma) \geq 0$ for all $\tau, \varsigma \in [0, 1]$;
(2) H is a strictly increasing function concerning the first variable;
(3) $\tau^{\alpha-1}H(1, q\varsigma) \leq H(\tau, q\varsigma) \leq H(1, q\varsigma)$ for all $\tau, \varsigma \in [0, 1]$.

Lemma 3.5. *Let G be the function that defined by* (3.6)*, then G satisfies the following conditions.*

(1) G is a continuous function and $G(\tau, q\varsigma) \geq 0$ for all $\tau, \varsigma \in [0, 1]$;
(2) G is a strictly increasing function concerning the first variable;
(3) $\tau^{\mu-1}G(1, q\varsigma) \leq G(\tau, q\varsigma) \leq G(1, q\varsigma)$ for all $\tau, \varsigma \in [0, 1]$.

The proof is straightforwardly attained from Lemma 3.4.

4. Main results

Here, we will use all of the items reported in the preceding sections to develop the primary findings of this research.

Let $\mathcal{P}$ be defined as

$$\mathcal{P} = \{\zeta \in C([0, 1]) : \zeta(\tau) \geq 0\}.$$

It is easy to check that $\mathcal{P}$ is a cone and since it is a closed set of $C([0, 1])$, hence it is a complete metric space that is equipped with the meter

$$d(\zeta, \omega) = \sup_{\tau\in[0,1]} |\zeta(\tau) - \omega(\tau)|,$$

also for convenience, we set

$$\Delta = \phi_s\left(\int_0^{\varsigma} H_\mu(\varsigma, q\xi)d_q\xi\right)\sup_{0\leq\tau\leq 1}\int_0^1 G(\tau, q\varsigma)d_q\varsigma.$$

Now consider the $F : \mathcal{P} \to \mathcal{P}$ defined by

$$F\zeta(\tau) = \int_0^1 G(\tau, q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi, q\xi)\Lambda(\xi, \zeta(\xi))d_q\xi\right)d_q\varsigma. \tag{4.1}$$

Since BVP (1.2) is equivalent with (3.10), so the solutions of BVP (1.2) are the fixed points of the operator (4.1). To prove the existence of fixed points of (3.10), we apply Theorems 2.7 and 2.8.

We make use of the following conditions:

(A1) Let $g : [0,1] \to [0,\infty)$ such that $\tau^{\mu-1}g(\tau) \in L^1[0,1]$ and λ is a constant such that $\lambda\int_0^1 \tau^{\mu-1}g(\tau)d_q\tau < 1$.

(A2) $\Lambda \in C([0,1]\times[0,\infty),[0,\infty))$ and it is non-decreasing respect to the second variable.

(A3) There exists $0 < (\eta+1)\Delta < 1$ such that for all $0 \le \omega \le \zeta < \infty$ we have

$$\phi_r(\ln(\omega+2)) \le \Lambda(\tau,\omega) \le \Lambda(\tau,\zeta) \le \phi_r(\ln(\zeta+2)(\zeta-\omega+1)^\eta).$$

Theorem 4.1. *Assume that (A1), (A2) and (A3) hold, then BVP* (1.2) *has a unique positive solution.*

Proof. Given Lemma 3.5 and (A1), it is concluded that $F(\mathcal{P}) \subset \mathcal{P}$. Now we check all conditions of Theorems 2.7 and 2.8 for the operator (3.10). Let $\zeta, \omega \in \mathcal{P}$ and $\zeta \ge \omega$, by (A1) we get

$$\begin{aligned}
F\zeta(\tau) &= \int_0^1 G(\tau, q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi, q\xi)\Lambda(\xi, \zeta(\xi))d_q\xi\right)d_q\varsigma\\
&\ge \int_0^1 G(\tau, q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi, q\xi)\Lambda(\xi, \omega(\xi))d_q\xi\right)d_q\varsigma\\
&= F\omega(\tau).
\end{aligned}$$

That is the operator F is nondecreasing.

Now let $\zeta \ge \omega$, in view of (A2) we obtain

$$\begin{aligned}
& d(F\zeta, F\omega)\\
\le & \sup_{0\le\tau\le1} |(F\zeta(\tau) - F\omega(\tau))\\
= & \sup_{0\le\tau\le1}\left[\int_0^1 G(\tau,q\varsigma)\phi_s\left(\int_0^\varsigma H_\nu(\varsigma,q\xi)\Lambda(\xi,\zeta(\xi))d_q\xi\right)d_q\varsigma\right.\\
& \left. - \int_0^1 G(\tau,q\varsigma)\phi_s\left(\int_0^\varsigma H_\nu(\varsigma,q\xi)\Lambda(\xi,\omega(\xi))d_q\xi\right)d_q\varsigma\right]\\
\le & (\ln(\zeta+2)(\zeta-\omega+1)^\eta - \ln(\omega+2)) \sup_{0\le\tau\le1}\int_0^1 G(\tau,q\varsigma)\phi_s\left(\int_0^\varsigma H_\nu(\varsigma,q\xi)d_q\xi\right)d_q\varsigma\\
\le & \ln\frac{(\zeta+2)(\zeta-\omega+1)^\eta}{\omega+2}\sup_{0\le\tau\le1}\int_0^1 G(\tau,q\varsigma)\phi_s\left(\int_0^\varsigma H_\nu(\varsigma,q\xi)d_q\xi\right)d_q\varsigma\\
\le & (\eta+1)\ln(\zeta-\omega+1)\sup_{0\le\tau\le1}\int_0^1 G(\tau,q\varsigma)\phi_s\left(\int_0^\varsigma H_\nu(\varsigma,q\xi)d_q\xi\right)d_q\varsigma\\
\le & (\ln(\zeta+2)(\zeta-\omega+1) - \ln(\omega+2))\phi_s\left(\int_0^\varsigma H_\nu(\varsigma,q\xi)d_q\xi\right)\sup_{0\le\tau\le1}\int_0^1 G(\tau,q\varsigma)d_q\varsigma.
\end{aligned}$$

The function $g(\zeta) := \ln(\zeta + 1)$ is a nondecreasing function, so by (A2), we have

$$\begin{aligned} d(F\zeta, F\omega) \leq& (\eta + 1)\ln(\|\zeta - \omega\| + 1)\phi_s\left(\int_0^{\varsigma} H_\nu(\varsigma, q\xi)d_q\xi\right)\sup_{0\leq\tau\leq 1}\int_0^1 G(\tau, q\varsigma)d_q\varsigma \\ =& (\eta + 1)\ln(\|\zeta - \omega\| + 1)\Delta \\ \leq& \|\zeta - \omega\| - (\|\zeta - \omega\| - \ln(\|\zeta - \omega\| + 1)). \end{aligned}$$

Now if we set $\psi(\zeta) := \zeta - \ln(\zeta + 1)$, then $\psi \in \Psi$. Thus for all $\zeta \geq \omega$ we obtain

$$d(F\zeta, F\omega) \leq d(\zeta, \omega) - \psi(d(\zeta, \omega)).$$

Since $G(\tau, q\varsigma) \geq 0$, $H_\nu(\tau, q\varsigma) \geq 0$ and $\Lambda \geq 0$, so

$$(F0)(\tau) = \int_0^1 G(\tau, q\varsigma)\phi_s\left(\int_0^{\varsigma} H(\varsigma, q\xi)\Lambda(\xi, 0)d_q\xi\right)d_q\varsigma \geq 0,$$

hence by Theorem 2.7, BVP (1.2) has at least one positive solution. On the other hand since $(\mathcal{P}, \leq)$ satisfies condition (iii) of Theorem 2.8, hence, the BVP (1.2) has a unique positive solution. □

Let Θ contains all $\theta : \mathbb{R}^+ \to [0, 1)$ which satisfy the condition: $\theta(t_n) \to 1$ implies $t_n \to 0$.

Definition 4.2. *[24] Let (X, d) is MS and $\mathfrak{a}, \vartheta : X \times X \to \mathbb{R}^+$ two functions. $g : X \to X$ is said to be an $\mathfrak{a}$-ϑ-Geraghty contraction if there exists $\theta \in \Theta$ such that for $\nu, \omega \in X$,*

$$\mathfrak{a}(\nu, \omega) \geq \vartheta(\nu, \omega) \Rightarrow d(g\nu, g\omega) \leq \theta(d(\nu, \omega))d(\nu, \omega).$$

Definition 4.3. *[24] Let $g : X \to X$ and $\mathfrak{a}, \vartheta : X \times X \to \mathbb{R}^+$ be given. Then g is called $\mathfrak{a}$-admissible with respect to ϑ, if for $\nu, \omega \in X$,*

$$\mathfrak{a}(\nu, \omega) \geq \vartheta(\nu, \omega) \Rightarrow \mathfrak{a}(g\nu, g\omega) \geq \vartheta(g\nu, g\omega).$$

Theorem 4.4. *[24] Let (X, d) be a complete metric space and $\varphi : X \to X$ be a $\mathfrak{a} - \theta$-Geraghty contraction such that*
(i) φ is $\mathfrak{a}$-admissible respect to ϑ;
(ii) $\exists\, w_0 \in X$ with $\mathfrak{a}(w_0, \varphi w_0) \geq \vartheta(w_0, \varphi w_0)$;
(iii) φ is continuous.
Then φ has a fixed point.

Theorem 4.5. *Let (X, d) be a complete metric space and $\varphi : X \to X$ be a $\mathfrak{a} - \theta$-Geraghty contraction such that*
(i) φ is $\mathfrak{a}$-admissible respect to ϑ;
(ii) $\exists\, w_0 \in X$ with $\mathfrak{a}(w_0, \varphi w_0) \geq \vartheta(w_0, \varphi w_0)$;
(iii) $\{w_n\} \subseteq X$, $w_n \to u$ in X and $\mathfrak{a}(w_n, w_{n+1}) \geq \vartheta(w_n, w_{n+1})$ then $\mathfrak{a}(w_n, w) \geq \vartheta(w_n, w)$.
Then φ has a fixed point.

The proof can be concluded by following the same arguments as in the proof of Theorem 2.4 in [25].

Theorem 4.6. *Suppose that (A1) hold and there exist $\rho : \mathbb{R}^2 \to \mathbb{R}$ and $\theta \in \Theta$ with the following property:*

(i) For all $\zeta, \omega \in C([0,1])$ *and* $\tau \in [0,1]$ *we have*

$$|\Lambda(\tau,\zeta) - \Lambda(\tau,\omega)| \leq |\phi_r(\frac{1}{\Delta}\omega\theta(\zeta-\omega)) - \phi_r(\frac{1}{\Delta}\zeta\theta(\zeta-\omega))|;$$

(ii) For $\zeta, \omega \in C([0,1])$, $\rho(\zeta(\tau),\omega(\tau)) \geq 0$ *and there exists* $\zeta_0 \in C([0,1])$ *with*

$$\rho\left(\zeta_0(\tau), \int_0^1 G(\tau,q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi,q\xi)\Lambda(\xi,\zeta_0(\xi))d_q\xi\right)d_q\varsigma\right) \geq 0;$$

(iii) If $\rho(\zeta(\xi),\omega(\xi)) \geq 0$*, then*

$$\rho\left(\int_0^1 G(\tau,q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi,q\xi)\Lambda(\xi,\zeta(\xi))d_q\xi\right)d_q\varsigma,\right.$$
$$\left.\int_0^1 G(\tau,q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi,q\xi)\Lambda(\xi,\omega(\xi))d_q\xi\right)d_q\varsigma\right) \geq 0;$$

(iv) if $\{\zeta_n\} \subseteq C([0,1])$, $\zeta_n \to \zeta$ *in* $C([0,1])$*, and* $\rho(\zeta_n,\zeta_{n+1}) \geq 0$*, then* $\rho(\zeta_n,\zeta) \geq 0$.

Then Problem (3.1) *has at least one solution.*

Proof. From Lemma 3.3, $\zeta \in C([0,1])$ is a solution of (3.1) if and only if is a solution of

$$F\zeta(\tau) = \int_0^1 G(\tau,q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi,q\xi)\Lambda(\xi,\zeta(\xi))d_q\xi\right)d_q\varsigma.$$

Thus we find the fixed point of $F : C([0,1]) \to C([0,1])$ given by

$$F\zeta(\tau) = \int_0^1 G(\tau,q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi,q\xi)\Lambda(\xi,\zeta(\xi))d_q\xi\right)d_q\varsigma.$$

Let $\zeta, \omega \in C([0,1])$ with $\rho(\zeta(\tau),\omega(\tau)) \geq 0$. By (i), we obtain

$$\begin{aligned}
&\Big|d(F\zeta,F\omega)| = \sup_{0\leq\tau\leq1} |(F\zeta(\tau) - F\omega(\tau))\Big| \\
&= \sup_{0\leq\tau\leq1}\Big|\int_0^1 G(\tau,q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi,q\xi)\Lambda(\xi,\zeta(\xi))d_q\xi\right)d_q\varsigma \\
&\quad - \int_0^1 G(\tau,q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi,q\xi)\Lambda(\xi,\omega(\xi))d_q\xi\right)d_q\varsigma\Big| \\
&\leq |\Lambda(\xi,\omega(\xi)) - \Lambda(\xi,\zeta(\xi))| \sup_{0\leq\tau\leq1}\int_0^1 G(\tau,q\varsigma)\phi_s\left(\int_0^\varsigma H_\nu(\varsigma,q\xi)d_q\xi\right)d_q\varsigma \\
&\leq |\frac{1}{\Delta}\omega\theta(\zeta-\omega)) - \frac{1}{\Delta}\zeta\theta(\zeta-\omega))| \sup_{0\leq\tau\leq1}\int_0^1 G(\tau,q\varsigma)\phi_s\left(\int_0^\varsigma H_\nu(\varsigma,q\xi)d_q\xi\right)d_q\varsigma \\
&\leq \frac{1}{\Delta}(\zeta-\omega)\theta(\zeta-\omega)\Delta = (\zeta-\omega)\theta(\zeta-\omega) \\
&\leq \|\zeta-\omega\|_\infty\theta(\|\zeta-\omega\|_\infty) = d(\zeta,\omega)\theta(d(\zeta,\omega)).
\end{aligned}$$

Let $\mathfrak{a} : C([0,1]) \times C([0,1]) \to \mathbb{R}^+$ be stated as:

$$\mathfrak{a}(\zeta,\omega) = \begin{cases} \vartheta(\zeta(\tau),\omega(\tau)) & \rho(\zeta(\tau),\omega(\tau)) \geq 0, \quad \tau \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$\mathfrak{a}(\zeta,\omega) d(F\zeta, F\omega) \leq \mathfrak{a}(\zeta,\omega)\theta(d(\zeta,\omega)).$$

Then F is an $\mathfrak{a}$-θ-contractive. From (iii) and the definition of $\mathfrak{a}$ we get

$$\begin{aligned} \mathfrak{a}(\zeta,\omega) \geq \vartheta(\zeta(\tau),\omega(\tau)) &\Rightarrow \rho(\zeta(\tau),\omega(\tau)) \geq 0 \\ &\Rightarrow \rho(F(\zeta),F(\omega)) \geq 0 \\ &\Rightarrow \mathfrak{a}(F(\zeta),F(\omega)) \geq \vartheta(F(\zeta(\tau)),F(\omega(\tau))), \end{aligned}$$

for $\zeta,\omega \in C([0,1])$. Thus, F is $\mathfrak{a}$-admissible. By (ii) $\exists\, \zeta_0 \in C([0,1])$ with $\mathfrak{a}(\zeta_0, F\zeta_0) \geq \vartheta(\zeta_0(\tau), F(\zeta_0(\tau)))$. From (iv) and Theorem 4.5, there is $\zeta^* \in C([0,1])$ with $\zeta^* = F\zeta^*$. Hence ζ^* is a solution of the problem. □

5. Examples

The following are two supportive examples that adhere to all theoretical presumptions.

Example 5.1. *Consider the q-difference BVP*

$$\begin{cases} D_q^{\nu}\varphi_r\left(D_q^{\mu}\zeta(\tau)\right) = \Lambda(\tau,\zeta(\tau)), \quad \tau \in [0,1], \\ \zeta(0) = D_q\zeta(0) = D_q\zeta(1) = D_q^{\mu}\zeta(0) = D_q^{\mu+1}\zeta(0) = 0, \\ D_q^{\mu+1}\zeta(1) = \lambda[\mu-1]\int_0^1 g(\theta)\zeta(\theta)d_q\theta, \end{cases} \tag{5.1}$$

where $\mu = \nu = \frac{5}{2}$, $q = \lambda = \frac{1}{2}$, $g(\tau) = \frac{1}{10}\tau$, $r = \frac{7}{3}$ *and* $\Lambda(\tau,\zeta(\tau)) = (\frac{1}{100}\sin^2\tau + \frac{1}{2})\ln(2+\zeta(\tau))$. *It is easy to see that* Λ *is a continuous function and for* $\tau \in [0,1]$ *we have* $\Lambda(\tau,\zeta) \neq 0$. *Also* Λ *is a nondecreasing with respect to the second variable. Now since in this problem* $\mu = \nu$, *we let* $H := H_\mu = H_\nu$ *and due to the fact that*

$$\int_0^{\varsigma} H(\varsigma,q\xi)d_q\xi \leq \int_0^1 H(\varsigma,q\xi)d_q\xi \leq \int_0^1 \frac{(1-q\xi)^{(\nu-2)}}{\Gamma_q(\nu)}d_q\xi \leq \int_0^1 \frac{1}{\Gamma_{0.5}(2.5)}d_{0.5}\xi \approx 0.8397.$$

By using Mathematica software, one can calculate the following quantities:

$$K(\tau,q\varsigma) \leq 0.4316, \qquad \int_0^1 G(\tau,q\varsigma)d_q\varsigma \leq 0.43615,$$

$$\Delta = \phi_s\left(\int_0^{\varsigma} H_\mu(\varsigma,q\xi)d_q\xi\right)\sup_{0\leq\tau\leq1}\int_0^1 G(\tau,q\varsigma)d_q\varsigma \leq 0.37945.$$

Moreover, we get

$$
\begin{aligned}
\Lambda(\tau,\zeta)-\Lambda(\tau,\omega)=&\left(\frac{1}{100}\sin^2\tau+\frac{1}{2}\right))\ln(2+\zeta)-\left(\frac{1}{100}\sin^2\tau+\frac{1}{2}\right)\ln(2+\omega)\\
\leq&\left(\frac{1}{100}\sin^2\tau+\frac{1}{2}\right)\ln\left(\frac{2+\zeta}{2+\omega}\right)\\
=&\left(\frac{1}{100}\sin^2\tau+\frac{1}{2}\right)\ln\left(\frac{2+\omega+\zeta-\omega}{2+\omega}\right)\\
=&\left(\frac{1}{100}\sin^2\tau+\frac{1}{2}\right)\ln\left(1+\frac{\zeta-\omega}{2+\omega}\right)\\
\leq&\left(\frac{1}{100}\sin^2\tau+\frac{1}{2}\right)\ln(1+(\zeta-\omega))\\
\leq&0.51\ln(1+\zeta-\omega).
\end{aligned}
$$

So $\eta=0.51$ and $(\eta+1)\Delta=1.51\times0.3794=0.5728<1$. Consequently all conditions of the Theorem 4.1 hold and the q-difference BVP has a unique positive solution like $\zeta(\tau)$ that satisfies

$$
\zeta(\tau)=\int_0^1 G(\tau,q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi,q\xi)\Lambda(\xi,\zeta(\xi))d_q\xi\right)d_q\varsigma.
$$

Example 5.2. *Let $\theta(\tau)=(\cos(\tau))^{\frac{3}{4}}$, $\rho(y,z)=yz$, $\zeta_n(\tau)=\dfrac{\tau}{n^2+1}$. Consider $\Lambda:\mathcal{I}\times C(\mathcal{I})\to[0,\infty]$ and the BVP*

$$
\begin{cases}
D_q^\nu\varphi_r\left(D_q^\mu\zeta(\tau)\right)=\Lambda(\tau,\zeta(\tau)), \quad \tau\in[0,1],\\
\zeta(0)=D_q\zeta(0)=D_q\zeta(1)=D_q^\mu\zeta(0)=D_q^{\mu+1}\zeta(0)=0,\\
D_q^{\mu+1}\zeta(1)=\lambda[\mu-1]\int_0^1 g(\theta)\zeta(\theta)d_q\theta.
\end{cases}
\tag{5.2}
$$

One can easily see that $\Lambda(\tau,\zeta(\tau))=\frac{1}{4}\sin^2 2(\zeta(\tau))$, $(\tau,v(\tau))\in\mathcal{I}\times[1,\infty)$. $\theta(t_n)\to1$ implies $t_n\to0$, hence $\theta\in\Theta$.

Furthermore, we get

$$
\begin{aligned}
&\frac{1}{4}|\sin^2(2\zeta(\tau))-\sin^2(2\omega(\tau))|=\frac{1}{4}|\sin(2\zeta(\tau))-\sin(2\omega(\tau))(\sin(2\zeta(\tau))+\sin(2\omega(\tau))|\\
=&|\sin(\zeta(\tau)-\omega(\tau))||\cos(\zeta(\tau)+\omega(\tau))||\sin(\zeta(\tau)+\omega(\tau))||\cos(\zeta(\tau)-\omega(\tau))|\\
\leq&|\zeta(\tau)-\omega(\tau)||\cos(\zeta(\tau)-\omega(\tau))|\\
\leq&(\frac{1}{./38}\zeta(\tau))^{\frac{4}{3}}\cos(\zeta(\tau)-\omega(\tau)))-(\frac{1}{./38}\omega(\tau)\cos(\zeta(\tau)-\omega(\tau)))^{\frac{4}{3}}|\\
=&|\phi_r(\frac{1}{\Delta}\omega\theta(\zeta-\omega))-\phi_r(\frac{1}{\Delta}\zeta\theta(\zeta-\omega))|,
\end{aligned}
$$

when $\tau\in\mathcal{I}$ and $\zeta(\tau),\omega(\tau)\in[1,\infty)$ with $\rho(\zeta(\tau),\omega(\tau))\geq0$. So the condition (i) from Theorem 4.6 hold.

If $\zeta_0(\tau)=\tau$, then

$$
\rho(\zeta_0(\tau)),\int_0^1 G(\tau,q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi,q\xi)\Lambda(\xi,\zeta_0(\xi))d_q\xi\right)d_q\varsigma\geq0,
$$

for $\tau\in\mathcal{I}$. Further, $\rho(\zeta(\tau),\omega(\tau))=\zeta(\tau)\omega(\tau)\geq0$ implies that

$$
\rho\left(\int_0^1 G(\tau,q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi,q\xi)\Lambda(\xi,\zeta(\xi))d_q\xi\right)d_q\varsigma,\right.
$$

$$\int_0^1 G(\tau, q\varsigma)\varphi_s\left(\int_0^\varsigma H_\nu(\xi, q\xi)\Lambda(\xi, \omega(\xi))d_q\xi\right)d_q\varsigma\right) \geq 0.$$

It is obvious that condition (iv) in Theorem 4.6 hold. Hence, the all conditions of Theorem 4.6 are satisfied. Thus, Eq (1.2) *has at least one solution.*

6. Conclusions

There are few papers in the literature about p-Laplacian q-fractional boundary value problems, thus we investigated a class of p-Laplacian q-fractional boundary value problems with an integral boundary condition. The Green function of the problem was computed and some properties of the Green function were determined. By using a new fixed point theorem that involves a α-η-Geraghty contraction, the existence of positive solutions was proved. Two examples are provided to support the theoretical findings.

The technique applied in this paper is different and may be used effectively to verify the existence of solutions to many sorts of equations. Moreover, one can use this technique to verify the existence of positive solutions for some boundary value problems including a system of q-fractional differential equations in the future.

Author contributions

H. Afshari: Conceptualization, Investigation; A. Ahmadkhanlu: Conceptualization, Formal analysis, Writing –review; J. Azabut: Writing–original draft, Editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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