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Research article

Uncertainty distributions of solutions to nabla Caputo uncertain difference equations and application to a logistic model

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Abstract: The nabla fractional-order uncertain difference equation with Caputo-type was analyzed in this article. To begin, the existence and uniqueness theorem of solutions for nabla Caputo uncertain difference equations with almost surely bounded uncertain variables was presented. Furthermore, the uncertainty distributions of the solutions for the proposed equations were obtained by establishing a connection between the solutions of equations and their α-paths based on new comparison theorems. Finally, an application of the uncertain difference equations in a logistic population model involving Allee effect was provided and examples were performed to demonstrate the validity of the theoretical results presented.

Keywords: uncertainty distribution; α-paths; nabla difference; fractional-order difference equation; logistic model
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1. Introduction

Fractional-order calculus is a refinement and extension of integer-order derivative and integral to arbitrary order. Fractional-order differential equations (FDEs) have been a powerful tool for describing many real-world systems with the memory and hereditary properties. In recent years, with the popularization of computer technology, discrete fractional-order calculus, a generalization of integer-order difference and sum to arbitrary order, is receiving increasing attention. FDEs have been a very promising tool to characterize the discrete systems with the memory and hereditary properties, for instance, biology [1], economics [2], population dynamics [3], and heat transfer [4]. Delta fractional-order difference and sum are defined in [5]. However, delta fractional-order sum will change the domain of a function performed, as do also delta fractional-order difference. Fortunately, the nabla fractional-order difference proposed by Atici and Eloe overcomes this deficiency [6] and has gained considerable interest [7, 8]. Compared with the Riemann-Liouville FDEs, the initial conditions of the

Caputo ones will be more easily obtained for modeling the practical problems. Consequently, Caputo type nabla fractional-order difference is considered in this paper.

In the real world, systems would be inevitably affected by uncertainty. The uncertainty theory proposed by Liu in 2007 [9] is an effective tool to deal with the uncertainty influenced by subjective and imprecise factors. Systems affected by the uncertainty above are called uncertain systems, driven by uncertain differential or difference equations [10]. To describe the uncertain systems with memory effects, the fractional-order uncertain differential equation was first introduced by Zhu in 2015 [11]. Immediately, fractional-order uncertain difference equations (FUDEs) with Riemann-Liouville type have been proposed in [12]. It may be a potential tool to model discrete systems with uncertain factors and memory effects. Currently, some properties of solutions to FUDEs have been mentioned in [13–17]. Existence and uniqueness and the uncertainty distribution of the solution for the FUDEs with fractional-order delta difference are developed in [13, 14, 16, 17]. Existence and uniqueness of the solution for the FUDEs with Riemann-Liouville type fractional-order nabla difference and fractional-order between 0 and 1 are provided in [15]. Therefore, this article is dedicated to existence and the uncertainty distribution of the solution for nabla Caputo FUDEs (CFUDEs) with any order.

It is worth noting that FDEs have been widely utilized in the fields of physics, engineering, and control. Unfortunately, few people have applied uncertainty theory to discrete systems in real-world applications. Recently, uncertainty theory is successfully employed in optimal control [18, 19], logistic model [20–22], and infectious disease simulation [23, 24]. A logistic population model incorporating the Allee effect based on the uncertainty theory in the continuous case was proposed in [21]. In fact, as early as 1948, Hutchinson highlighted that the events from nature are not continuous [25] and discrete maps may better describe the evolution of populations. Moreover, the stability results of a general discrete population model with Allee effect are investigated and analyzed, and a discrete logistic equation involving Allee effect in the Caputo-Fabrizio sense is defined in [26] and [27], respectively. As a consequence, this paper presents a novel effort to expand the uncertain logistic population model from its continuous form to the discrete case.

Inspired by the references mentioned, this article mainly studies the uncertainty distribution of solutions to CFUDEs by the obtained comparison theorems and presents an application of the proposed equations in a class of logistic growth models. The specific contributions of this paper are concluded in the following:

i) Existence and uniqueness of a class of CFUDEs with almost surely bounded uncertain variables and any fractional-order is proposed by the Banach fixed theorem, which is different from the equations investigated in [12, 13, 15, 16].

ii) Uncertainty distributions of solutions to CFUDEs are given based on new comparison theorems of Caputo fractional-order difference equations. The Lipschitz constant of coefficient functions of CFUDEs here is not bounded by the fractional-order, which does not agree with the results of FUDEs with Riemann-Liouville type in [14].

iii) A novel uncertain logistic population model with Allee effect is presented as a discrete extension of the logistic population model in [21].

The organization of this article is as below: In Section 2, we will review the relevant knowledge of the uncertainty theory and nabla fractional-order difference first. The existence and uniqueness theorem of solutions for CFUDEs with almost surely bounded uncertain variables will be presented and proven in Section 3. Uncertainty distributions of solutions to CFUDEs will be obtained by establishing a new comparison principle in Section 4. Finally, in Section 5, a discrete uncertain population model involving Allee effect will be introduced, along with an example.

2. Preliminaries

The fundamental facts about uncertainty theory and nabla fractional-order difference could be referenced to [9] and [28], respectively.

In this article, Γ is a nonempty set named universal set and $\mathfrak{L}$ is a σ-algebra over Γ. A set function $\mathcal{M}$ defined on σ-algebra $\mathfrak{L}$ stands for uncertain measure. The triple $(\Gamma, \mathfrak{L}, \mathcal{M})$ is called an uncertainty space. E denotes a set whose uncertain measure is 0. More details would be found in [9].

Definition 2.1. *[9] An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathfrak{L}, \mathcal{M})$ to a set of real numbers $\mathbb{R}$, i.e., for any Borel set $\mathbf{B}$ of real numbers, the set*

$$\{\xi \in \mathbf{B}\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in \mathbf{B}\}$$

is an event called the element in $\mathfrak{L}$.

Let ξ be an uncertain variable. Then, its uncertainty distribution Φ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leqslant x\}, \forall x \in \mathbb{R}.$$

For example, consider ξ is a linear uncertain variable $\mathcal{L}(a_1, a_2)$ with an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & x \leqslant a_1, \\ (x - a_1)/(a_2 - a_1), & a_1 < x < a_2, \\ 1, & x \geqslant a_2, \end{cases}$$

where a_1 and a_2 are real numbers with $a_1 < a_2$. See Figure 1.

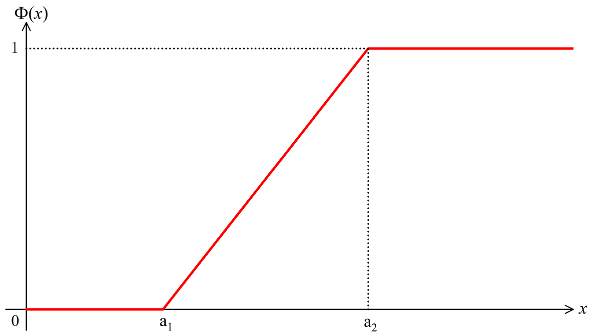

Figure 1. Uncertainty distribution $\mathcal{L}(a_1, a_2)$ of linear uncertain variable.

The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if

$$\mathcal{M}\{\bigcap_{i=1}^{m}(\xi_i \in \mathbf{B_i})\} = \bigwedge_{i=1}^{m} \mathcal{M}\{\xi_i \in \mathbf{B_i}\}$$

for any Borel sets $\mathbf{B_1}, \mathbf{B_2}, \cdots, \mathbf{B_m}$ of real numbers. If the independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ have the same uncertainty distribution, then they are called to be independent identically distribution (i.i.d.) uncertain variables.

Definition 2.2. *[9] For each $\gamma \in \Gamma \setminus E$, an uncertain variable ξ is almost surely bounded if and only if there is a positive real number r such that*

$$\mathcal{M}\{|\xi(\gamma)| \leqslant r\} = 1.$$

For linear uncertain variable $\mathcal{L}(a_1, a_2)$, denote $W = |a_1| \bigvee |a_2|$. It follows that $|\xi(\gamma)| \leqslant W$ for any $\gamma \in \Gamma \setminus \{(\xi < a_1) \bigcup (\xi > a_2)\}$. Thereby, ξ is almost surely bounded.

Definition 2.3. *[28] Let y be defined on $\mathbb{N}_a = \{a, a+1, \cdots\}$. Then the nabla left fractional sum of order $\nu > 0$ is defined by*

$$\nabla_a^{-\nu} y(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} [t - \rho(s)]^{\overline{\nu-1}} y(s), \quad t \in \mathbb{N}_{a+1},$$

where the backward jump operator $\rho(s) = s - 1$, the rising function $t^{\overline{\nu}} = \frac{\Gamma(t+\nu)}{\Gamma(t)}$, and $\Gamma(t)$ is the gamma function.

Definition 2.4. *[28] For any function f defined on $\mathbb{N}_a$, $n-1 < \nu < n$, $n \in \mathbb{N}_1$, then*

$$\nabla_a^{\nu} f(t) = \nabla^n \nabla_a^{-(n-\nu)} f(t)$$

is called the Riemann-Liouville-like fractional-order difference.

The Caputo-like fractional-order difference is given by

$${}^{C}\nabla_a^{\nu} f(t) = \nabla_a^{-(n-\nu)} \nabla^n f(t).$$

Definition 2.5. *[12] Assume that $\nu > 0$, $a \in \mathbb{R}$ and there exists an uncertain sequence ζ_t indexed by $t \in \mathbb{N}_a$. The nabla fractional-order sum of order ν for ζ_t is given by*

$$\nabla_a^{-\nu} \zeta_t = \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\nu-1}} \zeta_s, \forall t \in \mathbb{N}_a,$$

where ζ_t is an uncertain variable for a fixed t.

The nabla fractional-order difference of uncertain sequence ζ_t with Caputo-type is defined by

$${}^{C}\nabla_a^{\nu} \zeta_t = \nabla_a^{-(n-\nu)} \nabla^n \zeta_t, \forall t \in \mathbb{N}_a,$$

where $n-1 < \nu < n$, and $n \in \mathbb{N}_1$.

Lemma 2.1. *[28] For $p > 0$, $q > -1$, $t \in \mathbb{N}_a$, one has*

$$\nabla_a^{-p} (t-a)^{\overline{q}} = \frac{\Gamma(q+1)}{\Gamma(q+p+1)} (t-a)^{\overline{p+q}}.$$

Lemma 2.2. *[28] Assume that function f is defined on $\mathbb{N}_a$ and $\nu > 0$, and p is a positive integer. We have*

$$(\nabla_a^{-\nu} \nabla^p f)(t) = (\nabla^p \nabla_a^{-\nu} f)(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\overline{\nu-p+k}}}{\Gamma(\nu+k-p+1)} \nabla^k f(a).$$

Lemma 2.3. *[28] Suppose that function h is defined on $\mathbb{N}_a$ and k is a nonnegative integer. The following equality holds:*

$$(\nabla_a^{-\nu C}\nabla_a^{\nu}h)(t) = h(t) - \sum_{k=0}^{n-1}\frac{(t-a)^{\overline{k}}}{k!}\nabla^k h(a),$$

where $\nu \in (n-1, n]$ and n is a positive integer. In particular, if $0 < \nu \leqslant 1$, then

$$(\nabla_a^{-\nu C}\nabla_a^{\nu}h)(t) = h(t) - h(a), t \in \mathbb{N}_a.$$

3. Existence and uniqueness theorem

In this section, the following CFUDE will be analyzed:

$$({}^C\nabla_{a+n-1}^{\nu}y)(t) = P(t, y(t)) + Q(t, y(t))\zeta_t, t \in \mathbb{N}_{a+n} \quad (3.1)$$

with the initial conditions (i.c.s)

$$\nabla^i y(t) \mid_{t=a+n-1} = y_i, i = 0, 1, \cdots, n-1, \quad (3.2)$$

where ${}^C\nabla_{a+n-1}^{\nu}$ denotes the Caputo-like nabla difference with $n-1 < \nu \leqslant n$, $a \in \mathbb{R}$, n is a positive integer, $t \in J = \mathbb{N}_{a+n} \bigcap [a, \tau]$, P, Q are two functions defined on $J \times \mathbb{R}$, and $\zeta_{a+n}, \zeta_{a+n+1}, \cdots, \zeta_t$ are mutually independent and almost surely bounded uncertain variables with the bound M.

Following the approach of Theorem 2 in [12], the existence and uniqueness of the solution of the CFUDE (3.1) subject to i.c.s (3.2) will be demonstrated by the Banach fixed point theorem.

Theorem 3.1. *Let functions $P(t, y)$ and $Q(t, y)$ satisfy the Lipschitz condition*

$$|P(t, y_1) - P(t, y_2)| + |Q(t, y_1) - Q(t, y_2)| \leqslant L|y_1 - y_2|$$

Additionally, the Lipschitz constant L meets the inequality below:

$$L < \frac{\Gamma(\nu+1)\Gamma(\tau+1-a-n)}{(1+M)\Gamma(\tau+1-a-n+\nu)}.$$

Then, the CFUDE (3.1) *subject to i.c.s* (3.2) *for $t \in \mathbb{N}_{a+n} \bigcap [a, \tau]$ has a unique solution almost surely.*

Proof. We define a nonempty set and a norm as follows:

$$l_a^k = \{x | x = \{x(t)\}_{a+n}^k, k \in \mathbb{N}_1\}, \| x \| = \max_{t \in \mathbb{N}_{a+n} \bigcap [a,\tau]} |x(t)|,$$

where $\{x(t)\}_{a+n}^k$ is a finite real sequence having k terms. It is obvious that $(l_a^k, \| \cdot \|)$ is a Banach space. For any X_t, $Y_t \in l_a^k$, the definition of the operator U is given as follows:

$$UY_t = \sum_{i=0}^{n-1}\frac{(t-a-n+1)^{\overline{i}}}{i!}y_i + \frac{1}{\Gamma(\nu)}\sum_{s=a+n}^{t}[t-\rho(s)]^{\overline{\nu-1}}[P(s, y(s)) + Q(s, y(s))\zeta_s].$$

As ζ_t is an almost surely bounded uncertain variable with the bound M for each $t \in \mathbb{N}_{a+n} \bigcap [a, \tau]$, there is a positive real number M such that $|\zeta(\gamma)| \leqslant M$ for any $\gamma \in \Gamma \setminus E$.

By Lemma 2.1, for any given $\gamma \in \Gamma \setminus E$, we get

$$\begin{aligned}
\| UX_t(\gamma) - UY_t(\gamma) \| &= \max_{t \in \mathbb{N}_{a+n} \bigcap [a,\tau]} |UX_t(\gamma) - UY_t(\gamma)| \\
&\leqslant \frac{1}{\Gamma(\nu)} \max_{t \in \mathbb{N}_{a+n} \bigcap [a,\tau]} \sum_{s=a+n}^{t} [t-\rho(s)]^{\overline{\nu-1}} \\
&\quad \times [|P(s, X_s(\gamma)) - P(s, Y_s(\gamma))| + |Q(s, X_s(\gamma)) - Q(s, Y_s(\gamma))\zeta_s|] \\
&\leqslant \frac{1}{\Gamma(\nu)} \max_{t \in \mathbb{N}_{a+n} \bigcap [a,\tau]} \sum_{s=a+n}^{t} [t-\rho(s)]^{\overline{\nu-1}} \\
&\quad \times [|P(s, X_s(\gamma)) - P(s, Y_s(\gamma))| + M|Q(s, X_s(\gamma)) - Q(s, Y_s(\gamma))|] \\
&\leqslant \frac{L(1+M)}{\Gamma(\nu)} \max_{t \in \mathbb{N}_{a+n} \bigcap [a,\tau]} \sum_{s=a+n}^{t} [t-\rho(s)]^{\overline{\nu-1}} |X_s(\gamma) - Y_s(\gamma)| \\
&\leqslant L(1+M) \| X_t(\gamma) - Y_t(\gamma) \| \max_{t \in \mathbb{N}_{a+n} \bigcap [a,\tau]} (\nabla_{a+n-1}^{-\nu} (t-(a+n-1))^{\overline{0}}) \\
&= L(1+M) \| X_t(\gamma) - Y_t(\gamma) \| \max_{t \in \mathbb{N}_{a+n} \bigcap [a,\tau]} \frac{(t-(a+n-1))^{\overline{\nu}}}{\Gamma(\nu+1)} \\
&= \frac{L(1+M)(\tau+1-a-n)^{\overline{\nu}}}{\Gamma(\nu+1)} \| X_t(\gamma) - Y_t(\gamma) \| \\
&= \frac{L(1+M)\Gamma(\tau+1-a-n+\nu)}{\Gamma(\nu+1)\Gamma(\tau+1-a-n)} \| X_t(\gamma) - Y_t(\gamma) \| .
\end{aligned}$$

According to $L < \frac{\Gamma(\nu+1)\Gamma(\tau+1-a-n)}{(1+M)\Gamma(\tau+1-a-n+\nu)}$, the operator U is a contraction mapping in l_a^k. Consequently, a unique fixed point $X_t(\gamma)$ for CFUDE (3.1) of U in l_a^k can be obtained by the Banach fixed point theorem. Furthermore, let $X_t(\gamma) = \lim_{m\to\infty} X_t^m(\gamma)$, where $X_t^m(\gamma) = U(X_t^{m-1}(\gamma))$ with $X_t^0(\gamma) = \sum_{i=0}^{n-1} \frac{(t-a-n+1)^{\overline{i}}}{i!} y_i$. Obviously, $X_t^0(\gamma)$ is an uncertain variable.

Note that functions P and Q are Lipschitz continuous. So, the operator U is measurable for any given $t \in \mathbb{N}_{a+n} \bigcap [a, \tau]$. Since a real-valued measurable function of uncertain variables is an uncertain variable, $X_t^1, X_t^2, \cdots, X_t^n, \cdots$ are uncertain variables. Therefore, $X_t(\gamma) = \lim_{m\to\infty} X_t^m(\gamma)$ is an uncertain variable by Theorem 3 in [29].

Consequently, the CFUDE (3.1) subject to i.c.s (3.2) has the unique solution X_t for $t \in \mathbb{N}_{a+n} \bigcap [a, \tau]$ almost surely. □

Example 3.1. Consider the CFUDE as follows:

$$\begin{cases} {}^C\nabla_2^{1.5} y(t) = \frac{\sin y(t)}{10+t^2} + 0.1\zeta_t, \quad t \in \mathbb{N}_3 \bigcap [1,5], \\ y(2) = \nabla y(2) = 0, \end{cases} \tag{3.3}$$

where $\zeta_3, \zeta_4, \zeta_5$ are mutually independent uncertain variables with the same linear uncertainty distribution $\mathcal{L}(0, 1)$.

Considering $\nu = 1.5, n = 2, \tau = 5, a = 1, M = 1$, it is easy to validate that

$$|P(t, y_1) - P(t, y_2)| + |Q(t, y_1) - Q(t, y_2)| \leqslant 0.1|y_1 - y_2|$$

and

$$\frac{\Gamma(1.5+1)\Gamma(5+1-1-2)}{(1+1)\Gamma(5+1-1-2+1.5)} \approx 0.11 > 0.1.$$

By Theorem 3.1, the CFUDE (3.3) has a unique solution almost surely.

4. Uncertainty distribution of the solutions to CFUDEs

Uncertainty distribution of solutions to CFUDEs will be presented in this section. The concept of α-path for the proposed CFUDEs will be first introduced, which provides a way to analyze the behavior of the solutions. Novel comparison theorems for FDEs will be provided, and a link between solutions of CFUDEs and their α-paths will be established based on the proposed comparison theorems.

Definition 4.1. *The CFUDE* (3.1) *with i.c.s* (3.2) *is called to have an α-path X_t^α if it is the solution of the corresponding FDE*

$${}^C\nabla^\nu_{a+n} X_t^\alpha = P(t, X_t^\alpha) + |Q(t, X_t^\alpha)|\Upsilon^{-1}(\alpha),$$

with the same i.c.s (3.2)*, where $\alpha \in (0,1)$ and $\Upsilon^{-1}(\alpha)$ is the inverse uncertainty distribution of ζ_t, i.e., for each $t \in \mathbb{N}_{a+n} \bigcap [a,\tau]$, ζ_t has the same regular uncertainty distribution, the inverse function with $\Upsilon^{-1}(\alpha)$.*

In Example 3.1, the associated FDE of CFUDE (3.3)

$$\begin{cases} {}^C\nabla^{1.5}_2 y(t) = \frac{siny(t)}{10+t^2} + 0.1\Upsilon^{-1}(\alpha), \quad t \in \mathbb{N}_3 \bigcap [1,5], \\ y(2) = \nabla y(2) = 0, \end{cases}$$

has a solution

$$y(t) = \frac{1}{\Gamma(1.5)} \sum_{s=3}^{t} (t-\rho(s))^{\overline{0.5}} [\frac{siny(s)}{10+s^2} + 0.1\alpha].$$

Consequently, the CFUDE (3.3) has an α-path

$$y_t^\alpha = \frac{1}{\Gamma(1.5)} \sum_{s=3}^{t} (t-\rho(s))^{\overline{0.5}} [\frac{siny(s)}{10+s^2} + 0.1\alpha].$$

In order to obtain a correlation between solutions of CFUDEs and their α-paths, the comparison theorems of the FDEs in the Caputo form will be innovated.

Inspired by [14], we present a comparison theorem for FDEs with Caputo difference when $0 < \nu < 1$.

Theorem 4.1. *For $\nu \in (0,1)$, $z(t,u)$ and $h(t,u)$ are two real-valued functions defined on $\mathbb{R}^+ \times \mathbb{R}$. Let $h(t,u)$ be Lipschitz continuous with respect to u. If $u_1(t)$ and $u_2(t)$ are, respectively, unique solutions of the following FDEs:*

$$\begin{cases} ({}^C\nabla^\nu_a u)(t) = z(t,u(t)), \ t \in \mathbb{N}_{a+1}, \\ u(a) = u_0, \end{cases}$$

and

$$\begin{cases} ({}^C\nabla^\nu_a u)(t) = h(t,u(t)), \ t \in \mathbb{N}_{a+1}, \\ u(a) = u_0. \end{cases}$$

(i) When $z(t,u) \leqslant h(t,u)$, $u_1(t) \leqslant u_2(t)$ for $t \in \mathbb{N}_a$;
(ii) When $z(t,u) > h(t,u)$, $u_1(t) > u_2(t)$ for $t \in \mathbb{N}_{a+1}$.

Proof. (i) Assume that the condition $u_1(t) \leqslant u_2(t)$ is not valid. There exists at least one $t_0 \in \mathbb{N}_a$ such that $u_1(t_0) > u_2(t_0)$ holds. Let $u(t) = u_1(t) - u_2(t)$ and $t_1 = \min\{t \in \mathbb{N}_{a+1} : u_1(t) > u_2(t)\}$. It is obvious that

$$u(t_1) > 0, \tag{4.1}$$

$$u(t) \leqslant 0, t \in \mathbb{N}_a \cap [0, t_1 - 1]. \tag{4.2}$$

From the definition of the Caputo difference operator, we can conclude that

$$\begin{aligned}
({}^C\nabla_a^\nu u)(t_1) &= \frac{1}{\Gamma(1-\nu)} \sum_{s=a+1}^{t_1} (t_1 - \rho(s))^{\overline{-\nu}} \nabla u(s) \\
&= \frac{1}{\Gamma(1-\nu)} \sum_{s=a+1}^{t_1} (t_1 - s + 1)^{\overline{-\nu}} (u(s) - u(s-1)) \\
&= \frac{1}{\Gamma(1-\nu)} (t_1 - a)^{\overline{-\nu}} [u(a+1) - u(a)] \\
&\quad + \frac{1}{\Gamma(1-\nu)} (t_1 - a - 1)^{\overline{-\nu}} [u(a+2) - u(a+1)] + \cdots \\
&\quad + \frac{1}{\Gamma(1-\nu)} (2)^{\overline{-\nu}} [u(t_1 - 1) - u(t_1 - 2)] + \frac{1}{\Gamma(1-\nu)} (1)^{\overline{-\nu}} [u(t_1) - u(t_1 - 1)] \\
&= u(t_1) + (-\nu) u(t_1 - 1) + \frac{-\nu(1-\nu)}{2} u(t_1 - 2) \\
&\quad + \frac{-\nu(-\nu+1)(-\nu+2)}{6} u(t_1 - 3) + \cdots + \frac{-\nu\Gamma(t_1 - a - 1 - \nu)}{\Gamma(1-\nu)\Gamma(t_1 - a)} u(a+1).
\end{aligned} \tag{4.3}$$

On the other hand, from (4.2), we have

$$\begin{aligned}
({}^C\nabla_a^\nu u)(t_1) &= ({}^C\nabla_a^\nu u_1)(t_1) - ({}^C\nabla_a^\nu u_2)(t_1) \\
&= z(t_1, u_1(t_1)) - h(t_1, u_2(t_1)) \\
&\leqslant h(t_1, u_1(t_1)) - h(t_1, u_2(t_1)) \\
&\leqslant -\mathbf{L}_h (u_1(t_1) - u_2(t_1)) = -\mathbf{L}_h (u(t_1)).
\end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), it can be obtained that

$$\begin{aligned}
(1 + \mathbf{L}_h) u(t_1) &\leqslant \nu u(t_1 - 1) + \frac{\nu(1-\nu)}{2} u(t_1 - 2) \\
&\quad + \frac{\nu(-\nu+1)(-\nu+2)}{6} u(t_1 - 3) + \cdots + \frac{\nu\Gamma(t_1 - a - 1 - \nu)}{\Gamma(1-\nu)\Gamma(t_1 - a)} u(a+1).
\end{aligned} \tag{4.5}$$

For each $0 < \nu < 1$, the coefficient to the right side of the inequality (4.5) is always nonnegative. In fact, using $z(t,u) \leqslant h(t,u)$, we can see that

$$u(a+2) = u_1(a+2) - u_2(a+2)$$

$$= [u_0 + \nu z(a+1, u(a+1)) + z(a+2, u(a+2))]$$
$$- [u_0 + \nu h(a+1, u(a+1)) + h(a+2, u(a+2))]$$
$$= \nu(z(a+1, u(a+1)) - h(a+1, u(a+1)))$$
$$+ z(a+2, u(a+2)) - h(a+2, u(a+2)) \leqslant 0.$$

This verifies that $t_1 > a+2$. It is evident that $\frac{\Gamma(t_1-a-1-\nu)}{\Gamma(1-\nu)} = \frac{\Gamma((t_1-a-2)+(1-\nu))}{\Gamma(1-\nu)} = (1-\nu)(2-\nu)\cdots(t_1-a-3) \geqslant 0$, when $t_1 > a+2$. Therefore, $\frac{\nu\Gamma(t_1-a-1-\nu)}{\Gamma(1-\nu)\Gamma(t_1-a)} \geqslant 0$ is valid for $0 < \nu < 1$.

Since Lipschitz constant $\mathbf{L}_h > 0$ and the right side of the inequality (4.5) is nonpositive, we can see $u(t_1) \leqslant 0$, which is in conflict with (4.1).

(ii) The proof can be similar to the proof of (i). □

In the sequel, Theorem 4.1 will be extended to the case of the fractional-order $\nu \in (n-1, n)$.

Theorem 4.2. *For $\nu \in (n-1, n)$, $z(t,u)$ and $h(t,u)$ are two real-valued functions defined on $\mathbb{R}^+ \times \mathbb{R}$. Let $h(t,u)$ be Lipschitz continuous with respect to u. If $u_1(t)$ and $u_2(t)$ are, respectively, unique solutions for the following FDEs:*

$$\begin{cases} ({}^C\nabla^{\nu}_{a+n-1}u)(t) = z(t, u(t)), \quad t \in \mathbb{N}_{a+n}, \\ \nabla^i u(t)\mid_{t=a+n-1} = u_i, i = 0, 1, \cdots, n-1, \end{cases} \tag{4.6}$$

and

$$\begin{cases} ({}^C\nabla^{\nu}_{a+n-1}u)(t) = h(t, u(t)), \quad t \in \mathbb{N}_{a+n}, \\ \nabla^i u(t)\mid_{t=a+n-1} = u_i, i = 0, 1, \cdots, n-1. \end{cases} \tag{4.7}$$

(i) When $z(t,u) \leqslant h(t,u)$, $u_1(t) \leqslant u_2(t)$ for $t \in \mathbb{N}_{a+n-1}$;

(ii) When $z(t,u) > h(t,u)$, $u_1(t) > u_2(t)$ for $t \in \mathbb{N}_{a+n}$.

Proof. (i) Given that ${}^C\nabla^{\nu}_{a+n-1}u(t) = {}^C\nabla^{\mu}_{b}\nabla^{n-1}u(t)$, where $b = a+n-1$ and $\mu = 1-n+\nu \in (0,1)$, then the FDEs (4.6) and (4.7) can be transformed into the case of the fractional-order $\nu \in (0,1)$.

Denote $y(t) = \nabla^{n-1}u(t)$. By Lemma 2.2, we can derive that

$$u(t) = \sum_{i=0}^{n-2} \frac{t^{\bar{i}}}{\Gamma(i+1)} u_i + \frac{1}{\Gamma(n-1)} \sum_{s=0}^{t} (t-\rho(s))^{\overline{n-2}} y(s). \tag{4.8}$$

Hence, the FDEs (4.6) and (4.7) can be written as

$${}^C\nabla^{\mu}_{b}y(t) = z(t, \sum_{i=0}^{n-2} \frac{t^{\bar{i}}}{\Gamma(i+1)} u_i + \frac{1}{\Gamma(n-1)} \sum_{s=0}^{t} (t-\rho(s))^{\overline{n-2}} y(s)) \tag{4.9}$$

and

$${}^C\nabla^{\mu}_{b}y(t) = h(t, \sum_{i=0}^{n-2} \frac{t^{\bar{i}}}{\Gamma(i+1)} u_i + \frac{1}{\Gamma(n-1)} \sum_{s=0}^{t} (t-\rho(s))^{\overline{n-2}} y(s)) \tag{4.10}$$

with the same initial condition $y_0 = y(b) = u_{n-1}$.

From assumption $z(t,u) \leqslant h(t,u)$, we can obtain

$$\begin{aligned} & z(t, \sum_{i=0}^{n-2} \frac{t^{\bar{i}}}{\Gamma(i+1)} u_i + \frac{1}{\Gamma(n-1)} \sum_{s=0}^{t} (t-\rho(s))^{\overline{n-2}} y(s)) \\ \leqslant\; & h(t, \sum_{i=0}^{n-2} \frac{t^{\bar{i}}}{\Gamma(i+1)} u_i + \frac{1}{\Gamma(n-1)} \sum_{s=0}^{t} (t-\rho(s))^{\overline{n-2}} y(s)). \end{aligned}$$

Let y_1 and y_2 be the unique solutions of the FDEs (4.9) and (4.10), respectively. Then, it is easy to validate that $y_1(t) \leqslant y_2(t)$ by Theorem 4.1.

Thus, from Eq (4.8), we obtain $u_1(t) \leqslant u_2(t)$.

(ii) The proof is similar to the proof of (i). □

Remark 4.1. *In comparison with the results presented in [14] for the Riemann-Liouville type, the comparison theorem obtained in this paper is established based on the properties of the Caputo backward difference operator. Moreover, the range of the Lipschitz constant in Theorem 4.2 is not restricted by the fractional-order.*

Remark 4.2. *Comparing with the results presented in [17] for the Liouville-Caputo type, the order is extended to be a real number between adjacent positive integers and eliminates the monotonicity conditions. These modifications create a foundation for proposing the uncertainty distributions of the solutions to CFUDEs.*

Based on the existence and uniqueness theorem and comparison theorems, the uncertainty distributions of solutions of CFUDEs with bounded and symmetrical uncertain variables can be further analyzed. The definition of the symmetrical uncertain variable is first revisited.

Definition 4.2. *[9] Assuming that a regular uncertainty distribution $\Upsilon(x)$ exists for an uncertain variable ξ, we call ξ symmetrical if*

$$\Upsilon(x) + \Upsilon(-x) = 1.$$

For example, the linear uncertain variable $\mathcal{L}(-a,a)$ is symmetrical, which can be obtained from Figure 1.

Furthermore, an important lemma that reveals the relationship between the solutions of CFUDEs and their α-paths can be presented.

Lemma 4.1. *Suppose that CFUDE* (3.1) *with i.c.s* (3.2) *has a solution X_t and an α-path X_t^{α}. If $P + |Q|\Upsilon^{-1}(\alpha)$ is Lipschitz continuous with respect to y, and the uncertain variables $\zeta_{a+n}, \zeta_{a+n+1}, \cdots, \zeta_t$ in the CFUDE* (3.1) *are i.i.d. almost surely bounded and symmetrical for any $t \in \mathbb{N}_{a+n} \bigcap [a,\tau]$, then we obtain*

$$\mathcal{M}\{X_t \leqslant X_t^{\alpha}\} = \alpha, \quad \mathcal{M}\{X_t > X_t^{\alpha}\} = 1 - \alpha.$$

Proof. The proof can refer to Theorem 4.1 in [14].

To begin, write

$$F^+ = \{t \in \mathbb{N}_{a+n} \cap [a,\tau] : Q(t,x) \geqslant 0\},$$

$$F^- = \{t \in \mathbb{N}_{a+n} \cap [a, \tau] : Q(t, x) < 0\};$$
$$W^+ = \{\gamma | \zeta_t(\gamma) \leqslant \zeta^{-1}(\alpha), t \in F^+\},$$
$$W^- = \{\gamma | \zeta_t(\gamma) \geqslant \zeta^{-1}(1 - \alpha), t \in F^-\}.$$

It is clear that $F^+ \cap F^{-1} = \emptyset$ and $F^+ \cup F^{-1} = \mathbb{N}_{a+n} \cap [a, \tau]$. Since $\zeta_{a+n}, \zeta_{a+n+1}, \cdots, \zeta_t$ in the CFUDE (3.1) are i.i.d. and symmetrical for $t \in \mathbb{N}_{a+n} \cap [a, \tau]$, we have

$$\mathcal{M}\{W^+ \cap W^-\} = \mathcal{M}\{W^+\} \wedge \mathcal{M}\{W^-\} = \alpha$$

and

$$\Phi^{-1}(1 - \alpha) + \Phi^{-1}(\alpha) = 0.$$

Hence, the inequality $Q(t, x)\zeta_t(\gamma) \leqslant |Q(t, x)|\Phi^{-1}(\alpha)$ holds for each $\gamma \in W^+ \cap W^-$ and $t \in \mathbb{N}_{a+n} \cap [a, \tau]$. Note that $X_t(\gamma)$ and X_t^α are, respectively, the unique solution and α-path of CFUDE (3.1) with *i.c.s* (3.2) for each $\gamma \in W^+ \cap W^-$, and it can be deduced that $X_t(\gamma) \leqslant X_t^\alpha$ by Theorem 4.2. Thus, we obtain $W^+ \cap W^- \subseteq \{X_t \leqslant X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\}$. Moreover, based on the monotonicity theorem in [9], we can derive the following inequality:

$$\mathcal{M}\{\{X_t \leqslant X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\} \geqslant \mathcal{M}\{W^+ \cap W^-\} = \alpha.$$

On the other hand, denote $\Omega^+ = \{\gamma | \xi_t(\gamma) > \Phi^{-1}(\alpha), t \in F^+\}, \Omega^- = \{\gamma | \xi_t(\gamma) < \Phi^{-1}(1 - \alpha), t \in F^-\}$. Similarly, we obtain $\Omega^+ \bigcap \Omega^- \subseteq \{X_t > X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \bigcap [a, \tau]\}$, which leads to

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\} \geqslant \mathcal{M}\{\Omega^+ \cap \Omega^-\} = 1 - \alpha.$$

Since $\mathcal{M}\{X_t > X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\} + \mathcal{M}\{X_t \leqslant X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\} \leqslant 1$, we obtain

$$\mathcal{M}\{X_t \leqslant X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\} = \alpha,$$
$$\mathcal{M}\{X_t > X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\} = 1 - \alpha.$$

□

Theorem 4.3. *Suppose that CFUDE* (3.1) *with i.c.s* (3.2) *has a solution X_t and an α-path X_t^α. If $P + |Q|\Upsilon^{-1}(\alpha)$ is Lipschitz continuous with respect to y, and the uncertain variables ζ_{a+n}, ζ_{a+n+1}, $\cdots$, ζ_t in the CFUDE* (3.1) *are i.i.d. almost surely bounded and symmetrical, then the inverse uncertainty distribution of X_t could be formulated by*

$$\Upsilon_t^{-1}(\alpha) = X_t^\alpha.$$

Proof. The proof of the theorem refers to the Theorem 5.1 in [14].
Note that

$$\{X_t \leqslant X_t^\alpha, \forall t\} \supset \{X_t \leqslant X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\},$$

and

$$\{X_t > X_t^\alpha, \forall t\} \supset \{X_t > X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\}.$$

According to the monotonicity of uncertain measure in [9], we have

$$\mathcal{M}\{X_t \leqslant X_t^\alpha, \forall t\} \geqslant \mathcal{M}\{X_t \leqslant X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\} = \alpha, \tag{4.11}$$

and

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} \geqslant \mathcal{M}\{X_t > X_t^\alpha, \forall t \in \mathbb{N}_{a+n} \cap [a, \tau]\} = 1 - \alpha. \tag{4.12}$$

$\mathcal{M}\{X_t \leqslant X_t^\alpha\}$ and $\mathcal{M}\{X_t > X_t^\alpha\}$ are opposite events with each other for each $t \in \mathbb{N}_{a+n} \cap [a, \tau]$, which means $\mathcal{M}\{X_t \leqslant X_t^\alpha\} + \mathcal{M}\{X_t > X_t^\alpha\} = 1$. It follows from Eqs (4.11) and (4.12) that $\mathcal{M}\{X_t \leqslant X_t^\alpha\} = \alpha$. Thus, $\Upsilon_t^{-1}(\alpha) = X_t^\alpha$ is the inverse uncertainty distribution of X_t. □

Example 4.1. In this example, we deal with the following CFUDE:

$${}^{C}\nabla_{-1}^{0.25} y(t) = 0.025 y^2(t) + \zeta_t, \quad y_0 = y(0) = 0.5, \quad t \in \mathbb{N}_0 \bigcap [0, 2], \tag{4.13}$$

where $\zeta_0, \zeta_1, \zeta_2$ are independent with the same uncertainty distribution $\mathcal{L}(-3, 3)$.

As a result, Eq (4.13) has a unique solution almost surely, which can be easily confirmed through Theorem 3.1.

Consider

$$P(t, y) + |Q(t, y)|\Upsilon^{-1}(\alpha) = 0.025 y^2(t) + 6\alpha - 3.$$

It is easy to see that the above formula is Lipschitz continuous with respect to $y \in [-3, 3]$ for $0 < \alpha < 1$. According to Lemma 2.3, the inverse distribution of the solution of Eq (4.13) could be determined by solving the following FDE:

$$y(t) = y_0 + \frac{1}{\Gamma(\frac{1}{4})} \sum_{s=0}^{t} (t - \rho(s))^{\overline{-\frac{3}{4}}} (0.025 y^2(s) + 6\alpha - 3), t \in \mathbb{N}_0 \bigcap [0, 2].$$

Therefore, the uncertainty distribution $\Phi_t(x)$ of the solution of Eq (4.13) is as shown in Figure 2 with $y_0 = 0.5$. In fact, the solution of Eq (4.13) is an uncertain sequence, which is an uncertain variable for each given t. The point $(-1.87, 0.09)$ in Figure 2 means that there is 9% possibility that the solution of Eq (4.13) takes the value as -1.87 at $t = 0$. It can also been seen from Figure 2 that $\Phi_t(x)$ is a monotonic increasing function about x for each t.

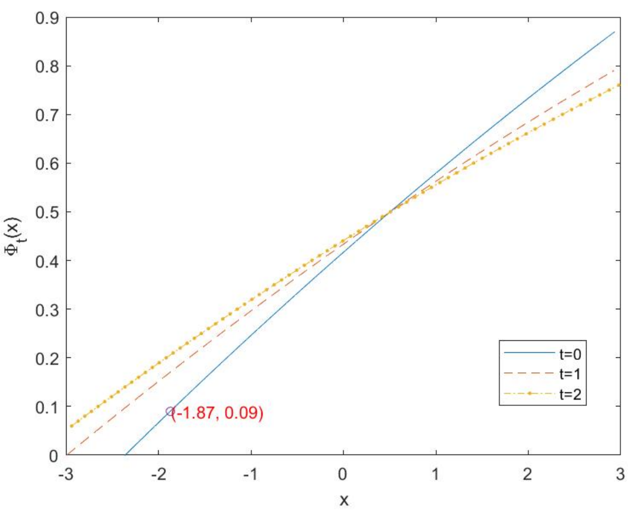

Figure 2. Uncertainty distribution of the solution of Eq (4.13).

Remark 4.3. *The uncertainty distribution of the solution for the proposed CFUDE shows that at a time t the solution takes a value with a possibility.*

Corollary 4.1. *Suppose that CFUDE* (3.1) *with i.c.s* (3.2) *has a solution X_t and an α-path X_t^{α}. If $P + |Q|\Upsilon^{-1}(\alpha)$ is Lipschitz continuous with respect to y, the uncertain variables $\zeta_{a+n}, \zeta_{a+n+1}, \cdots, \zeta_t$ in the CFUED* (3.1) *are i.i.d. almost surely bounded and symmetrical and the expected valued $E[X_t]$ exists, then*

$$E[X_t] = \int_0^1 X_t^{\alpha} d\alpha.$$

Proof. It can be directly inferred from Theorem 4.3. □

5. Discrete uncertain logistic model

An uncertain model with Allee effect has been proposed in [21] as the equation below:

$$\frac{dN_t}{dt} = AN_t(1 - \frac{N_t}{K})(1 - \frac{U}{N_t}) + \sigma N_t(1 - \frac{N_t}{K})(1 - \frac{U}{N_t})\frac{dC_t}{dt},$$

where N_t denotes the size or density of population at time t, A stands for the maximum average rate of change, U stands for the minimum density, K is the environment capacity, σ is considered as the intensity of the noise and the noise term $\frac{dC_t}{dt}$ is taken as an uncertain variable.

Hutchinson has pointed out that the events from the natural world are not successive in [25] and discrete maps can better describe the evolution of populations. Furthermore, the fractional difference equation and the corresponding fractional delayed logistic maps have been introduced in [30]. Moreover, a discrete logistic equation with Allee effect in Caputo-Fabrizio sense from a nonlinear fractional differential equation has been defined in [27]. These prompt us to generalize the uncertain logistic population model in [21] from a continuous framework to a discrete framework.

Inspired by the aforementioned research, this article focuses on the following form of uncertain logistic population model

$$^{C}\nabla_a^{\nu} N_t = AN_t(1 - \frac{N_t}{K})(1 - \frac{U}{N_t}) + \sigma N_t(1 - \frac{N_t}{K})(1 - \frac{U}{N_t})\zeta_t, \quad t \in \mathbb{N}_{a+1} \bigcap [a, \tau], \tag{5.1}$$

where the noise term ζ_t for each $t \in \mathbb{N}_{a+1} \bigcap [a, \tau]$ is considered as a linear uncertain variable.

This model comprehensively considers Allee effect and uncertain factors, making it more meaningful and reasonable in practical applications and theoretical research. Furthermore, in ecology, population dynamics are better described using a discrete-time model, rather than a continuous case, particularly when the population has nonoverlapping generations.

Remark 5.1. *The uncertainty distribution of solutions would be used to analyze the statistical characteristics of population dynamics with uncertainty described by uncertain FDEs. For example, the expected value of the solution can reflect the trend of the average population growth rate, while variance can reflect the intensity of the impact of uncertain factors on the population growth process and population volatility.*

Example 5.1. In 1999, the number of red-crowned cranes in the Xiaosanjiang Plain of China was 115. If we substitute $A = 0.1, K = 119, U = 8, \sigma = 0.1$, and N_0=115 as the initial condition into the uncertain model considered above, the Eq (5.1) can be transformed into the following form:

$$\begin{cases} {}^{C}\nabla_{-1}^{0.5} N_t = 0.1N_t(1 - \frac{N_t}{119})(1 - \frac{8}{N_t}) + 0.1N_t(1 - \frac{N_t}{119})(1 - \frac{8}{N_t})\zeta_t, \\ N_0 = 115, \end{cases} \tag{5.2}$$

where $t \in \mathbb{N}_1 \bigcap [0, 3]$ and $\zeta_1, \zeta_2, \zeta_3$ are mutually independent uncertain variables with the same uncertainty distribution $\mathcal{L}(-1, 1)$.

Therefore, the system has a unique solution almost surely, a fact that can be confirmed by Theorem 3.1.

Consider

$$P(t, y) + |Q(t, y)|\Upsilon^{-1}(\alpha) = 0.1(-\frac{y^2}{119} + \frac{127}{119}y - 8) + 0.1| - \frac{y^2}{119} + \frac{127}{119}y - 8|(2\alpha - 1).$$

It is easy to see that the above formula is Lipschitz continuous with respect to $y \in [114, 118]$.

According to Lemma 2.3, the inverse distribution of the solution for the system can be concluded by solving the following FDE:

$$\begin{aligned} N_t = N_0 + \frac{1}{\Gamma(\frac{1}{2})} \sum_{s=1}^{t} (t - \rho(s))^{\overline{-\frac{1}{2}}} [(0.1N_s)(1 - \frac{N_s}{119})(1 - \frac{8}{N_s}) \\ + 0.1|N_s(1 - \frac{N_s}{119})(1 - \frac{8}{N_s})|(2\alpha - 1)], t \in \mathbb{N}_1 \bigcap [0, 3]. \end{aligned}$$

Therefore, the uncertainty distribution $\Psi_t(x)$ of the solution for the system is shown in Figure 3 with $N_0 = 115$.

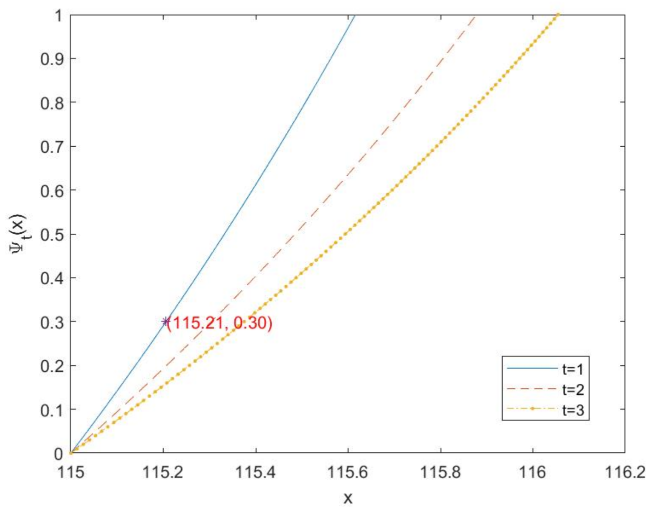

Figure 3. Uncertainty distribution of the solution N_t of Eq (5.2).

Moreover, as illustrated in Figure 3, we can see that when $t = 1$, the value of α is 0.3 with N_1=115.2062. This indicates that the belief degree regarding the count of red-crowned cranes in the

the Xiaosanjiang Plain of China was 115.2062 in 2000 ,which is 30%. The α-path of the solution N_t of Eq (5.2), which shows the possibility of the evolution of red-crowned cranes in the Xiaosanjiang Plain of China, is revealed in Figure 4. For example, there is a 90% probability that evolution of red-crowned cranes can be characterized by the discrete points labeled by *. From Figure 3, it is shown that $\Psi_t(x)$ is a monotonic increasing function about x for each fixed t. It can be also known that α-path N_t^α is also a monotonic increasing function about t for the same α from Figure 4, which shows that the quantity of red-crowned cranes increases over time in the short term for the same possibility.

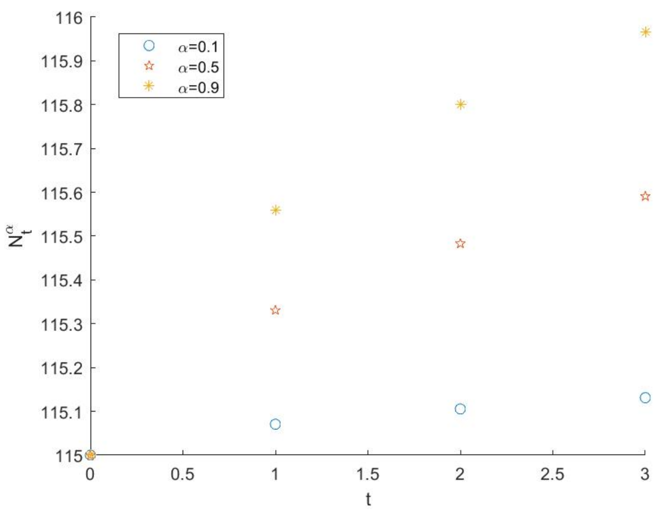

Figure 4. α-path of the solution N_t of Eq (5.2).

Additionally, Table 1 displays the expected values obtained according to Corollary 4.1.

Table 1. The expected value of N_t.

t	1	2	3
$E[N_t]$	115.3189	115.4698	115.5715

The table displays the expected values of solution for the proposed logistic equation, which were 115.3189, 115.4698, and 115.5715, respectively, showing the average values of population density and size in Xiaosanjiang Plain, China in 2000, 2001, and 2002.

6. Conclusions

The existence and uniqueness theorem and uncertainty distribution of solutions for the proposed Caputo fractional-order equations are provided in this article. Based on the results achieved, a new discrete logistic model involving Allee effect is introduced and analyzed. However, the stability of the proposed fractional-order equations is not involved, which is a crucial aspect. Therefore, our future work is primarily devoted to the Mittag-Leffler stability and finite-time stability of the solutions for CFUDEs.

Author contributions

Qinyun Lu, Ya Li, Hai Zhang and Hongmei Zhang: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. All authors contributed equally to the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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