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Research article

Some insights into rank conditions of vector subspaces

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Abstract: We introduce the general notion of a rank on a vector space, which includes both tensor rank and conventional matrix rank, but incorporates other examples as well. Extending this concept, we investigate vector spaces consisting of vectors with a lower bound on their rank. Our main result shows that bases for such spaces of maximum dimension can be chosen to consist exclusively of vectors of minimal rank. This generalization extends the results of [15, 36], with potential applications in different areas.

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1. Introduction

Research on linear matrix spaces of matrices with bounded rank has a rich history and continues to be an active area of investigation, driven by both theoretical developments and practical applications across various disciplines. For any positive integer $k \leqslant \min\{m, n\}$, three interesting types of subspaces exist within the space of $m \times n$ real matrices. These include subspaces where the rank of each matrix is bounded above by k, subspaces where the rank of each non-zero matrix is at least k, and subspaces consisting of matrices with a fixed rank k and a zero matrix, referred to as k-spaces of matrices. The study of matrices with a bounded rank dates back to the early days of linear algebra, with expanding applications across disciplines.

One of the initial breakthroughs in this field is the determination of the maximum possible dimension of a subspace consisting of real $n \times n$ full-rank matrices (i.e., a subspace in which every non-zero matrix is invertible), known as the Radon-Hurwitz number [20, 37]. This corresponds to the problem of vector fields on spheres, as noted by J. F. Adams [1]. E. G. Rees [38] showed that for each k, where $0 < k < m \leqslant n$, there exist linear subspaces of real $n \times m$ matrices with dimensions

$(m - k)(n - k)$, in which every non-zero matrix has a rank of at least k. K. Y. Lam and P. Yiu [26] determined the largest possible dimensions of linear spaces consisting of real $n \times n$ matrices with fixed ranks of $n - 1$ or $n - 2$, using topological K-theory. For further exploration, we recommend papers [3, 13, 22, 27, 31, 40] for various results in this field.

Of particular interest is the paper by Handel [15], investigating subspaces of the tensor product of two finite-dimensional vector spaces $V \otimes W$ that do not contain rank-1 tensors. This corresponds to the existence of non-singular bilinear maps. Determining the minimum possible dimension of the vector space U for which there exists a vector space homomorphism $f : V \otimes W \to U$ such that:

$$f(v \otimes w) = 0 \implies v = 0 \text{ or } w = 0$$

is equivalent to finding a subspace of maximum dimension M that does not contain rank-1 matrices. When the underlying field is the field of real numbers, the existence of non-singular bilinear maps has important applications in topology. Handel's paper presents two important results. First, he showed that maximal subspaces meeting the specified condition possess bases consisting only of elements with ranks two and three. This result was crucial in establishing his second main result: the existence of subspaces with maximum dimension that satisfy the aforementioned condition and have a basis consisting exclusively of rank-2 tensors. Expanding on Handel's work in [36] (see also [35]), the author generalized these results to include cases where the subspaces do not contain non-zero matrices with a rank less than the specified number. The author showed the existence of a subspace of maximum dimension consisting entirely of matrices having rank k or higher (for any given $k \geqslant 2$), together with the zero matrix, which has a basis consisting only of rank k matrices. Corresponding results related to the spaces of symmetric and skew-symmetric matrices were also discussed.

The paper is structured as follows: In Section 2, we revisit the notions of tensor rank and CP decomposition, due to their fundamental importance. Section 3 introduces a general rank function applicable to all vector spaces and provides some simple examples. As far as we know, this rank function does not appear elsewhere in the existing literature. Section 4 investigates subspaces with rank conditions and generalizes results from [36]. The proofs closely resemble those in the aforementioned paper, as we distilled from the usual tensor rank what is necessary to make our proofs go through. Section 5 presents several more advanced examples. Finally, in Section 6, we point out that our rank function is not related to rank functions arising in Strassen's works on asymptotic spectra.

2. Tensor rank and CP decomposition

This section introduces basic concepts related to tensor rank and CP decomposition. Let $\mathbb{F}$ denote a field, and $V_1, V_2, \ldots, V_n$ be $\mathbb{F}$-vector spaces. A non-zero tensor $\mathbf{t} \in V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is rank-one if it can be written as $v_1 \otimes v_2 \otimes \cdots \otimes v_n$, where $v_j \in V_j$ for $j = 1, \ldots, n$.

The rank of the zero tensor is defined to be zero. Clearly, each tensor admits a decomposition into a sum of k rank-1 tensors with a corresponding non-negative integer k.

Definition 1. *The rank of a tensor* $\mathbf{t} \in V_1 \otimes V_2 \otimes \cdots \otimes V_n$*, denoted by* $\mathrm{rank}(\mathbf{t})$*, is the minimal number r of rank-*1 *tensors needed to express* $\mathbf{t}$ *as a sum:*

$$\mathbf{t} = \sum_{i=1}^{r} v_1^i \otimes v_2^i \otimes \cdots \otimes v_n^i, \quad v_j^i \in V_j, \; j = 1, \ldots, n. \tag{2.1}$$

The CP decomposition (2.1), introduced by Hitchcock [18, 19], represents **t** as a sum of rank-one tensors, and is also known as CANDECOMP/PARAFAC [6, 16].

For an n-th order tensor **t** of size $m_1 \times m_2 \times \cdots \times m_n$, the CP decomposition (2.1) can be equivalently expressed as

$$\mathbf{t} = [\![U_1, U_2, \ldots, U_n]\!], \tag{2.2}$$

where $U_j \in \mathbb{R}^{m_j \times r}$.

Difficulties in tensor rank research include the lack of efficient algorithms for determining the rank of tensors when $n > 2$ [17] and variations in rank based on the underlying field [25].

Uniqueness in tensor decomposition, as shown by Kruskal's criteria, demonstrates distinct properties of tensor rank compared to matrices [24, 39, 41]. It may seem unexpected that higher-order tensors show an advantage in this regard. While the decomposition of matrices into a sum of rank-1 matrices lacks uniqueness, higher-order tensors generally possess a unique CP decomposition (up to permutations and scalings) under relatively mild conditions. To establish uniqueness criteria, Kruskal introduced the concept of a *k-rank*, indicating the largest number k such that every k vector in the vector collection $\{v_1, \ldots, v_n\}$ is linearly independent, or equivalently, the largest number k such that $\dim(\operatorname{span}\{v_{i_1}, \ldots, v_{i_k}\}) = k$ for any subset of k vectors. If

$$k_j = k\text{-}rank\left(\{v_1^j, v_2^j, \ldots, v_r^j\}\right), \ \ j = 1, \ldots, n, \ \text{where } n \geqslant 3,$$

then, according to Kruskal's theorem, a sufficient condition for the uniqueness of the CP decomposition (2.1) is given by:

$$\sum_{j=1}^{n} k_j \geqslant 2r + n - 1. \tag{2.3}$$

For recent advances in the generalization of Kruskal's theorem, we recommend consulting [10, 14, 28].

Various concepts related to tensor ranks, such as symmetric rank, border rank, and Schmidt rank, among others, are actively researched [7, 8, 11, 33]. In addition, new concepts have recently been introduced, which will be discussed in more detail in Section 5.

For detailed discussions of tensor rank properties, we recommend the paper by T. Kolda and B. Bader [23], and for applications in quantum information theory, Bruzda et al. [4].

While higher-order tensors represent a natural extension of matrices, their rank properties exhibit distinct characteristics. In this paper, we demonstrate that certain properties related to matrix spaces, originally established in the paper by Z. Petrović [36], can be extended to higher-order tensor spaces.

At the end of this section, let us consider the following example:

Example 1. *Consider the positive integer $k \geqslant 2$ and the real vector spaces V_1, V_2, and V_3 of dimensions m, n, and p, respectively. Assume $m = sk$, where s is a positive integer, and $m < n < p$. Choose any set of linearly independent vectors (e.g., standard basis vectors) $\{u_1, \ldots, u_m\} \subset V_1$, $\{v_1, \ldots, v_n\} \subset V_2$, and $\{w_1, \ldots, w_p\} \subset V_3$. Define the tensors:*

$$\mathbf{t}_l = \sum_{j=(l-1)k+1}^{lk} u_j \otimes v_j \otimes w_j, \ l = 1, \ldots, s. \tag{2.4}$$

The set $\{u_j \otimes v_j \otimes w_j : 1 \leqslant j \leqslant m\}$ contains linearly independent tensors due to the linear independence of the respective vectors in V_1, V_2, and V_3. Furthermore, $\mathbf{t}_l$ is a rank-k tensor. Since $k \geqslant 2$, Kruskal's

condition (2.3) ensures unique CP decompositions for $\mathbf{t}_l$. *It follows that*

$$\text{span}\{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_s\}$$

represents a tensor subspace where every non-zero tensor has a rank of at least k.

Remark. Note that the tensors $\mathbf{t}_l$ are actually diagonal tensors with frontal slices $E_{ij} = [\delta_{ij}]$, where δ_{ij} is the Kronecker delta, $i, j = (l-1)k+1, \ldots, lk$. Visualizing them as third-order tensors, they resemble cuboids with frontal slices E_{ij}.

3. Rank function on a vector space

Let us first revisit an example from [35]. Consider $m \times n$ matrices (assuming $m \leqslant n$) of the following form:

$$\begin{bmatrix} 0 & \cdots & 0 & x_1 & x_2 & \cdots & & \cdots & & x_{n-k+1} \\ 0 & \cdots & x_1 & x_2 & \cdots & & \cdots & x_{n-k+1} & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ x_1 & x_2 & \cdots & \cdots & \cdots & x_{n-k+1} & & \cdots & & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & & \cdots & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & & \cdots & & 0 \end{bmatrix}$$

Except when $x_1 = \cdots = x_{n-k+1} = 0$, these matrices all have rank k. Thus, the lower bound on the maximum dimension of subspaces generated by $m \times n$ matrices with fixed rank k is at least $\max\{m, n\} - k + 1$. Determining the maximum possible dimensions of such spaces involves various techniques from algebraic topology and algebraic geometry. However, these methods do not readily extend to higher-order tensors.

Motivated by the results from [36], we propose a rank function that can be applied to any vector space.

Definition 2. *Let V be an arbitrary vector space over the field* $\mathbb{F}$, *and let* $\mathbb{N}$ *denote the set of non-negative integers. A numerical function* $\rho \colon V \to \mathbb{N}$ *is called a* rank *function if it satisfies the following conditions:*

(1) $\rho(v) = 0$ *if and only if* $v = 0$.
(2) $\rho(\alpha v) = \rho(v)$ *for all* $\alpha \in \mathbb{F} \setminus \{0\}$.
(3) $\rho(v + w) \leqslant \rho(v) + \rho(w)$ *for all* $v, w \in V$.
(4) *If* $\rho(v) = k$, *there exists* $v_1, \ldots, v_k \in V$ *such that* $v = v_1 + \cdots + v_k$ *and* $\rho(v_i) = 1$, $i = 1, \ldots, k$.

Proposition 1. *Let V be an arbitrary vector space and* ρ *a rank function on V. Then* $\rho(v)$ *is the smallest number k such that* $v = v_1 + \cdots + v_k$, *where* $\rho(v_i) = 1$ *for all i. If we have two rank functions* ρ_1 *and* ρ_2 *satisfying these conditions, and if for all* $v \in V$, $\rho_1(v) = 1$ *if and only if* $\rho_2(v) = 1$, *then* $\rho_1(v) = \rho_2(v)$ *for all* $v \in V$.

Proof. The first statement follows immediately from conditions (3) and (4). Suppose $\rho(v) = k$. By condition (4), v can be expressed as a sum of k vectors of rank 1. If v could also be expressed as

$v = w_1 + \cdots + w_l$ with $l < k$ and $\rho(w_i) = 1$ for all i, then by condition (3), we would have $\rho(v) \leq l < k$. This contradicts $\rho(v) = k$, so v must indeed be expressible as a sum of k rank 1 vectors.

Therefore, the rank function is completely determined once we know which vectors have rank 1, and the second statement follows. □

Such rank functions do indeed exist. We present a few examples here; additional examples will be presented in Section 5. The simplest yet crucial one for our purposes is the following:

Example 2. *If V is a space consisting entirely of tensors of a certain order, the usual tensor rank serves as a rank function.*

We can also construct rank functions on a general vector space in a straightforward manner.

Example 3. *Let V be an arbitrary vector space over a field $\mathbb{F}$. Take any basis of V, and define $\rho(v) = k$ if and only if v is a linear combination of k elements of this basis with non-zero coefficients.*

This example illustrates that there exist infinitely many rank functions on a given vector space. It is straightforward to check that a function defined in this manner satisfies all the conditions specified in Definition 2. Note that this rank function does not coincide with the usual rank if the elements of V are tensors.

Example 4. *For a field $\mathbb{F}$ and $V = \mathbb{F}[X_1, \ldots, X_n]$, the previous example provides us with a rank function ρ, where $\rho(p)$ counts the monomials appearing in the polynomial $p \in V$.*

It depends on the structure of elements in a particular vector space which other rank functions one may introduce. For example, in the space of skew-symmetric matrices of a given order over fields of characteristic not equal to 2, the most natural rank function is half of the usual rank (since the usual rank of skew-symmetric matrices is an even number).

It is noteworthy that rank functions exhibit a rather important property derived from conditions (3) and (4):

Proposition 2. *If $\rho(v) = l > 1$ and $1 \leqslant k < l$, then there exist vectors w and w' such that:*

$$\rho(w) = k, \quad \rho(w') = l - k, \quad and \quad v = w + w'.$$

Proof. This can be easily proved using Definition 2. We have $v = v_1 + \cdots + v_l$ for some vectors v_i such that $\rho(v_i) = 1$ for all i. If we take $w = v_1 + \cdots + v_k$ and $w' = v_{k+1} + \cdots + v_k$ (in that order), we observe that according to the third condition:

$$v = w + w', \quad \rho(w) \leqslant k, \quad and \quad \rho(w') \leqslant l - k.$$

Moreover, if $\rho(w) < k$ or $\rho(w') < l - k$, condition (3) would imply that $\rho(v) < l$. Thus, the proposition is established. □

Let us rephrase the fundamental notion of an unshrinkable basis by Handel [15, Definition 2.1].

Definition 3. *Let V be a vector space with a rank function ρ, W be a subspace of V, and $e = [v_1, \ldots, v_r]$ be a basis of W. Define:*

$$\rho(e) := \sum_i \rho(v_i).$$

Then e is called unshrinkable if $\rho(e) = \min\{\rho(e') : e'$ is a basis of $W\}$. If e is an unshrinkable basis satisfying $\rho(v_1) \leqslant \cdots \leqslant \rho(v_r)$, it is called a sorted unshrinkable basis. We define $\rho(W)$ to be $\min\{\rho(v) : v \in W, v \neq 0\}$.

These bases always exist and they have useful properties (cf. [15, Lemmas 2.3 and 2.4]), one of which is crucial for this paper, expressed through the rank function ρ:

Lemma 1. *For an unshrinkable basis $e = [v_1, \ldots, v_m]$ of W, we have:*

$$\rho\left(\sum_i \alpha_i v_i\right) \geqslant \max\{\rho(v_i) \mid \alpha_i \neq 0\}.$$

Proof. Suppose $\rho(\sum_i \alpha_i v_i) < \rho(v_j)$ for some j where $\alpha_j \neq 0$. If we replace v_j with $\sum_i \alpha_i v_i$ in our basis e, while keeping the other vectors unchanged, we obtain another basis e'. Since $\alpha_j \neq 0$, v_j is a linear combination of vectors from e', ensuring e' is indeed a basis. However, this new basis e' has

$$\rho(e') < \rho(e),$$

which contradicts the fact that e is unshrinkable. $\square$

4. Subspaces with rank conditions

We are now ready to state the main results.

Theorem 1. *Let $k \geqslant 2$ be an integer and W a subspace of a finite-dimensional vector space V equipped with the rank function ρ. Suppose W is maximal among those subspaces W' with $\rho(W') = k$. Let $[v_1, \ldots, v_s]$ be an unshrinkable basis of W. Then, for all $i = 1, \ldots, s$, we have:*

$$k \leqslant \rho(v_i) \leqslant 3k - 3.$$

Proof. Consider a subspace U of W spanned by all v_i's such that $k \leqslant \rho(v_i) \leqslant 3k - 3$.

We claim that $W = U$. Assume, to the contrary, that $W \neq U$ and let $x \in W \setminus U$. By Lemma 1, we have $\rho(x) > 3k - 3$. To establish $W = U$, we construct an epimorphism $f : V \to V/U$ with kernel W. This proves $\dim W = \dim U$, hence $W = U$.

Let $x \in V$. We want to show that there exist $y \in W$ and $z \in V$ such that $\rho(z) < k$ and $x = y + z$.

If $x \in W$, then we can take $y = x$ and $z = 0$.

Let $x \in V \setminus W$. Since W is maximal among those subspaces W' with $\rho(W') = k$, the subspace spanned by W and x must contain a vector z_1 with $\rho(z_1) < k$. Thus, there exist z_1 and y_1 such that

$$z_1 = \alpha x + y_1,$$

where $\rho(z_1) < k$, $y_1 \in W$, and $\alpha \in \mathbb{F}$. Since $z_1 \notin W$, $\alpha \neq 0$, and we conclude that x can be expressed as

$$x = y + z, \quad y \in W, \quad \rho(z) < k.$$

If $x = y' + z'$ is another such expression for x, then

$$z - z' = y' - y \in W,$$

and

$$\rho(z - z') \leqslant \rho(z) + \rho(z') \leqslant 2(k - 1).$$

Therefore, $z - z' \in U$. Since the previous expression for x is also unique $(\bmod\, U)$ when $x \in W$, we conclude that the following function is well defined:

$$f : V \to V/U$$

$$f(x) = z + U, \text{ if } x = y + z, \; y \in W, \; \rho(z) < k.$$

Let us now show that f is a linear map. Consider $x = y + z$ as a previous representation of x. For any $c \in \mathbb{F}$, we have $cx = cy + cz$, which gives the corresponding representation of cx. Hence, $f(cx) = cz + U = c(z + U)$.

Next, suppose $x = y + z$, $x' = y' + z'$, and $x + x' = y'' + z''$ for appropriate y, z, y', z', y'', z''. We then have:

$$f(x) + f(x') = z + z' + U; \quad f(x + x') = z'' + U.$$

We have $z'' - z - z' = y + y' - y'' \in U$ and $\rho(z'' - z - z') \leqslant 3(k - 1)$. Therefore, $z'' - z - z' \in U$ and we conclude that $f(x + x') = f(x) + f(x')$. The kernel of f consists of those x where the corresponding z belongs to U. However, if $x = y + z$ where $y \in W$ and $z \in U$, then $z \in W$. Therefore, $\mathrm{Ker}\,(f) = W$.

To show that f is onto, note that every vector in V is a sum of vectors where the rank function takes the value 1. It suffices to show that for any z with $\rho(z) = 1$, there exists an x such that $f(x) = z + U$. But,

$$f(z) = f(0 + z) = z + U.$$

Thus, f is onto, completing the proof. $\qquad\square$

Theorem 2. *Let $k \geqslant 2$ be an integer, and let q be the largest integer for which a finite-dimensional vector space V with rank function ρ admits a vector subspace W' of dimension q such that $\rho(W') = k$. Then, there exists such a subspace W which admits a basis entirely composed of elements w_i such that $\rho(w_i) = k$ for all i.*

Proof. Among those subspaces W of maximum dimension q, which contain no non-zero elements of rank less than k, let us choose one such that $\rho(e)$ is minimal, where e is a sorted unshrinkable basis of W. It suffices to prove $\rho(e) = kq$.

To prove $\rho(e) = kq$, assume otherwise, $\rho(e) > kq$. Let $e = [v_1, \ldots, v_q]$. Since $\rho(e) > kq$, it follows that $\rho(v_q) = l > k$. We know there exist w and w' such that $v_q = w + w'$, $\rho(w) = k$, and $\rho(w') = l - k$.

We prove that $e' = \{v_1, \ldots, v_{q-1}, w\}$ is linearly independent, and the subspace W' spanned by e' satisfies $\rho(W') = k$.

Suppose for some linear combination z of elements in e':

$$z = \sum_{i=1}^{q-1} \alpha_i v_i + \beta w, \quad \rho(z) \leqslant k - 1.$$

This will establish the properties we want to prove. If $\beta = 0$, then $z = 0$ since $\rho(W) = k$. Consequently, all α_i's are zero. Thus, $\beta \neq 0$, and we define an element t by:

$$t = \sum_{i=1}^{q-1} \alpha_i v_i + \beta v_q.$$

Since $\beta \neq 0$, by Lemma 1, $\rho(t) \geqslant l = \rho(v_q)$. On the other hand:

$$t = z + \beta(v_q - w).$$

Then,

$$\rho(t) \leqslant \rho(z) + \rho(v_q - w) \leqslant k - 1 + l - k = l - 1.$$

Let W' be the subspace spanned by e'. Then $\dim W' = \dim W$, and $\rho(e') < \rho(e)$. Therefore W' is a subspace of maximum dimension containing no non-zero elements of rank less than k, and it has an unshrinkable basis e' such that $\rho(e') < \rho(e)$. Since we may transform this basis into a sorted one, this contradicts our choice of W.

Thus, we have a contradiction, which completes the proof. □

For a finite-dimensional vector space V, define

$$L(V, \rho, k) := \max\{\dim W : W \text{ is a subspace of } V \text{ such that } (\forall v \in W \setminus \{0\}) \, \rho(v) \geq k\}.$$

We immediately obtain the following corollary of Theorem 2.

Corollary 1. *For a given finite-dimensional vector space V over a field $\mathbb{F}$, rank function ρ, and integer $k \geqslant 2$, we have that $L(V, \rho, k)$ is the maximal number s such that there exists $x_1, \ldots, x_s \in V$, with $\rho(x_1) = \cdots = \rho(x_s) = k$, for which $\rho(\alpha_1 x_1 + \cdots + \alpha_s x_s) \geqslant k$ for every non-trivial linear combination ($\alpha_i \in \mathbb{F} \setminus \{0\}$ for at least one index i).* □

Therefore, if one wants to find $L(V, \rho, k)$ it is enough to consider elements of rank k whose non-trivial linear combinations are of rank at least k.

5. Examples

In this section, we present additional examples of rank functions in vector spaces. It is easy to check that they all satisfy the conditions in Definition 2.

Example 5. Symmetric tensor and symmetric tensor rank.

Let $\mathbb{F}$ be a field and V a finite-dimensional vector space over $\mathbb{F}$. An element $\mathcal{A} \in \overbrace{V \otimes V \otimes \cdots \otimes V}^{n}$ is symmetric if $\sigma(\mathcal{A}) = \mathcal{A}$ for all $\sigma \in \mathbb{S}_n$, where σ acts on $\mathcal{A}$ in the usual way. The symmetric tensor rank of $\mathcal{A}$ is defined as the minimal r such that

$$\mathcal{A} = \sum_{i=1}^{r} v_i \otimes v_i \otimes \cdots \otimes v_i,$$

for some $v_i \in V \setminus \{0\}$. For more details on this rank, we refer the reader to the paper by Comon [7].

Example 6. Slice and partition rank.
The notion of slice rank was introduced by Tao in his blog [42] and partition rank was introduced by Naslund in [32]. They are of crucial importance in additive combinatorics.

Definition 4. *Let $X_1, \ldots, X_n$ be finite sets and $\mathbb{F}$ a field. We say that a function $h \colon X_1 \times \cdots \times X_n \to \mathbb{F}$ has slice rank 1, denoted by* srank, *if*

$$h(x_1, \ldots, x_n) = f(x_i)g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n),$$

for some $i \in \{1, \ldots, n\}$ and functions f and g. Then the slice rank is defined as

$$\mathrm{srank}(h) := \min\{r : h = h_1 + \cdots + h_r, \ \mathrm{srank}(h_i) = 1\}.$$

Definition 5. *Let $X_1, \ldots, X_n$ be finite sets and $\mathbb{F}$ a field. We say that a function $h : X_1 \times \cdots \times X_n \to \mathbb{F}$ has partition rank 1, denoted by* prank, *if*

$$h(x_1, \ldots, x_n) = f_1(x_{s_{11}}, \ldots, x_{s_{1i_1}}) \cdots f_k(x_{s_{k1}}, \ldots, x_{s_{ki_k}}),$$

for some k and functions $f_i : S_i \to \mathbb{F}$, where $\{1, \ldots, n\} = S_1 \sqcup \cdots \sqcup S_k$ and $S_r = \{s_{r1}, \ldots, s_{ri_r}\}$. Then the partition rank is defined as

$$\mathrm{prank}(h) := \min\{r : h = h_1 + \cdots + h_r, \ \mathrm{prank}(h_i) = 1\}.$$

If we denote the usual tensor rank by trank, *then we have the inequalities*

$$\mathrm{prank}(h) \leqslant \mathrm{srank}(h) \leqslant \mathrm{trank}(h)$$

for every h.

Example 7. Waring rank.

Let k be an algebraically closed field of characteristic zero, and consider $k[X_1, \ldots, X_n]$, the graded polynomial ring in n indeterminates over k. Let F be a non-zero homogeneous d-form. The Waring rank of this form is defined as the smallest r such that

$$F = \sum_{i=1}^{r} L_i^d,$$

where L_i are linear forms. This rank satisfies our conditions, with forms of rank 1 being powers of linear forms. Condition (2) is satisfied since the field k is algebraically closed.

The Waring rank is related to the classical Waring problem in number theory and also to Hilbert's 17th problem. For more about this, we refer the reader to [5, 12, 21].

Example 8. Schmidt rank.

This concept arises in quantum mechanics and is significant in quantum information theory. Simply put, for two finite-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ over $\mathbb{C}$, any vector $w \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can be expressed as

$$w = \lambda_1 u_1 \otimes v_1 + \cdots + \lambda_r u_r \otimes v_r,$$

where $u_1, \ldots, u_r \in \mathcal{H}_1$ and $v_1, \ldots, v_r \in \mathcal{H}_2$ are orthonormal vectors, and λ_i are non-zero complex numbers. The Schmidt rank of w is defined to be the number r, denoted $\mathrm{Sch}(w) = r$.

In [9], the authors discuss the problem of finding $L(\mathbb{C}^d \otimes C^d, \mathrm{Sch}, k)$. See also [43], where the notion of a Schmidt number for density matrices, related to Schmidt rank, is introduced.

Example 9. Stabilizer rank.

This concept is another important notion from quantum information theory. For a quantum state ψ, the stabilizer rank is the minimal r such that, in Dirac ket notation:

$$|\psi\rangle = \sum_{j=1}^{r} c_j |\varphi_j\rangle,$$

where $c_j \in \mathbb{C}$ and $|\varphi_j\rangle$ are stabilizer states. For more details on stabilizer rank, refer to [29, 30, 34].

Condition (4) in Definition 2 is rather restrictive. Let us discuss briefly the border rank of a tensor, denoted here by brank(**t**). Simply put, the border rank of a tensor **t** is defined as the smallest r such that **t** lies in the closure of the subset of tensors of rank r in the appropriate projective space (for more details and precise definitions, see, e.g., [2] and [11]).

Since tensors of rank 1 form a closed subset, we have that for any tensor **t**, trank(**t**) = 1 if and only if brank(**t**) = 1. Therefore, if brank were to satisfy condition (4), then from Proposition 1, it would follow that brank(**t**) = trank(**t**) for all tensors **t**. However, since this is not the case, we conclude that brank is not a rank function in our sense.

6. Discussion

As pointed out in Section 2, the usual rank properties of matrices do not carry over to higher-order tensors. However, Theorems 1 and 2 reveal that certain aspects of rank properties may be extended to tensors.

In establishing the main results of this paper, the introduction of the rank function ρ was essential. This function was purposefully chosen to capture the fundamental properties of the tensor rank. However, as demonstrated by several examples presented, there exist interesting rank functions that satisfy our conditions. By customizing rank functions for specific problem domains, valuable insights can potentially be obtained.

Finally, it should be noted that our rank function is distinct from various rank functions arising in connection with Strassen's work on asymptotic spectra (for a comprehensive presentation of these results, from the modern viewpoint which include (Strassen) preorder on semirings, see [44]). Our rank function is simpler – it takes values in the set of non-negative integers, can be defined on all vector spaces, and does not require any deeper structural assumptions.

Author contributions

Petrović: Conceptualization, Investigation, Methodology, Validation, Writing – review & editing; Pucanović: Conceptualization, Investigation, Funding acqusition, Project administration, Supervision, Writing – original draft; Pešović: Conceptualization, Validation; Kovačević: Conceptualization. Project administration.

All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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