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Research article

A generalized viscosity forward-backward splitting scheme with inertial terms for solving monotone inclusion problems and its applications

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Abstract: Our aim was to establish a novel generalized viscosity forward-backward splitting scheme that incorporates inertial terms for addressing monotone inclusion problems within a Hilbert space. By incorporating appropriate control conditions, we achieved strong convergence. The significance of this theorem lies in its applicability to resolve convex minimization problems. To demonstrate the efficacy of our proposed algorithm, we conducted a comparative analysis of its convergence behavior against other algorithms. Finally, we showcased the performance of our proposed method through numerical experiments aimed at addressing image restoration problems.

Keywords: inertial method; forward-backward algorithm; monotone inclusion problem
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1. Introduction

Let $\mathcal{H}$ represent a real Hilbert space featuring an inner product $\langle\cdot,\cdot\rangle$ and its corresponding norm denoted as $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$. Our aim is to address the monotone inclusion problem, which seeks to find $x \in \mathcal{H}$ such that

$$0 \in Ax + Bx, \tag{1.1}$$

where $A : \mathcal{H} \to \mathcal{H}$ is a single-valued mapping, and $B : \mathcal{H} \to 2^{\mathcal{H}}$ denotes a multi-valued mapping. The set of all solutions to problem (1.1) is denoted as $(A+B)^{-1}(0)$. Many intriguing problems can be framed within the framework of the monotone inclusion problem (1.1), such as convex minimization problems, variational inequalities, equilibrium problems, image processing challenges, and more. One of the most renowned algorithms used to approximate the solution of problem (1.1) is the forward-backward algorithm (FB), as highlighted in references such as [1–3]. This algorithm (FB) was initially introduced by Brezis and Lions [4] and is defined by a sequence $(x_k)_{k\geq 1}$ according to the recurrence relation:

$$x_{k+1} = J_\lambda^B(x_k - \lambda A x_k), \quad \text{for all } k \geq 1, \tag{1.2}$$

where $J_\lambda^B = (Id + \lambda B)^{-1}$ represents the resolvent of the operator B, $\lambda > 0$, and Id denotes the identity mapping. This method is used in the context of solving monotone inclusion problems and has been widely studied and applied in various fields such as optimization and signal processing. After that, Moudafi [5] later introduced the viscosity approximation method to address issues of strong convergence. This method combines the forward-backward splitting algorithm with contraction mappings. The viscosity approach ensures strong convergence of the iterative sequences, which is particularly useful in various applications like fixed-point problems.

The method involves both the proximal point algorithm and the gradient method, as evidenced in references such as [6–12].

On the other hand, the concept of the heavy ball method (or inertial method), introduced by Polyak in 1964 [13], was an early example of incorporating inertia into optimization algorithms. The discretized form of this method has inspired various optimization algorithms that incorporate momentum to accelerate convergence.

In 2001, Alvarez and Attouch [14] introduced a new algorithm based on the inertial method outlined in [13]. This method is expressed as follows:

$$\begin{cases} w_k = x_k + \theta_k(x_k - x_{k-1}), \\ x_{k+1} = (Id + \lambda_k B)^{-1} w_k, \quad \text{for all } k \geq 1. \end{cases} \tag{1.3}$$

They established that the sequence $(x_k)_{k\geq 1}$ generated by algorithm (1.3) converges weakly to a zero point of the operator B under the conditions $(\theta_k)_{k\geq 1} \subseteq [0, 1]$ and $(\lambda_k)_{k\geq 1}$ being non-decreasing, with the constraint

$$\sum_{k=1}^{\infty} \theta_k \|x_k - x_{k-1}\|^2 < \infty. \tag{1.4}$$

See [15, 16] for other types of conditions on θ_k, which no longer rely on the iterates.

Moudafi and Oliny [17] proposed an iterative method that incorporates the concept of the inertial method to address problem (1.1). They also proved weak convergence of the iterates under the following conditions:

(i) The condition (1.4) holds.
(ii) $\lambda_k < 2/L$ with L the Lipschitz constant of A.

Their algorithm is defined by

$$\begin{cases} w_k = x_k + \theta_k(x_k - x_{k-1}), \\ x_{k+1} = (Id + \lambda_k B)^{-1}(w_k - \lambda_k A w_k), \quad \text{for all } k \geq 1, \end{cases} \tag{1.5}$$

where $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{H} \to 2^{\mathcal{H}}$.

We explore several methods proposed in recent studies for tackling monotone inclusion problems, each with its unique approach and contributions. Cholamjiak et al. [18] extended the inertial forward-backward splitting method to Banach spaces; this study focuses on applications in compressed sensing, showcasing the method's efficacy in real-world scenarios. After that, Shehu et al. [19] introduced inertial terms into iterative methods for nonexpansive mappings; we introduce the advantages of inertial techniques in improving convergence rates and handling compressed sensing problems. In 2020, Artsawang and Ungchittrakool [20] aimed at solving monotone inclusion and image restoration problems; this method develops an inertial Mann-type algorithm for nonexpansive mappings, showcasing its applicability in diverse domains. The approach utilizes the Mann-type algorithm, as demonstrated by references such as [21–24]. Alternatively, image restoration problems typically require the restoration of a high-quality image from a degraded or noisy version. Monotone inclusion methods provide powerful frameworks for addressing such problems, as can be seen in [25–31].

Kitkuan et al. [32] recently introduced a viscosity approximation algorithm using the inertial forward-backward approach to solve problem (1.1). Their algorithm is formulated as follows:

$$\begin{cases} w_k = x_k + \theta_k(x_k - x_{k-1}), \\ x_{k+1} = \gamma_k(f(x_k)) + (1-\gamma_k)J^B_{\lambda_k}(w_k - \lambda_k A w_k), \quad \text{for all } k \geq 1, \end{cases} \tag{1.6}$$

where $A : \mathcal{H} \to \mathcal{H}$ represents a μ-inverse strongly monotone operator with $\mu > 0$, $B : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator, and $f : \mathcal{H} \to \mathcal{H}$ is a contraction with constant constraint $c \in (0, 1)$. They also proved the strong convergence of their proposed method under certain appropriate conditions imposed on the parameters.

In 2020, Kitkuan et al. [33] introduced a novel method that combines the Halpern-type method and the forward-backward splitting method to solve the monotone inclusion problem (1.1). Their method is described as:

$$\begin{cases} u, x_1 \in \mathcal{H}, \\ z_k = \alpha_k x_k + (1-\alpha_k)J^B_{\lambda_k}(x_k - \lambda_k A x_k), \\ y_k = \beta_k x_k + (1-\beta_k)J^B_{\lambda_k}(z_k - \lambda_k A z_k), \\ x_{k+1} = \gamma_k u + (1-\gamma_k)y_k, \quad \text{for all } k \geq 1, \end{cases} \tag{1.7}$$

where $J^B_{\lambda_k} = (Id + \lambda_k B)^{-1}$ denotes the resolvent of B, and $\alpha_k, \beta_k, \gamma_k \in (0, 1)$. Strong convergence results are obtained under certain appropriate conditions.

By drawing inspiration from the inertial viscosity forward-backward splitting algorithms pioneered by Kitkuan et al. [32, 33], we propose the following algorithm:

$$(\textbf{Algorithm 1}) \quad \begin{cases} w_k = x_k + \theta_k(x_k - x_{k-1}), \\ z_k = \alpha_k w_k + (1-\alpha_k)J^B_{\lambda_k}(w_k - \lambda_k A w_k), \\ y_k = \beta_k w_k + (1-\beta_k)J^B_{\lambda_k}(z_k - \lambda_k A z_k), \\ x_{k+1} = \gamma_k f(x_n) + (1-\gamma_k)y_k, \quad \text{for all } k \geq 1, \end{cases} \tag{1.8}$$

where $(\theta_k)_{k\geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$ and $(\alpha_k)_{k\geq 1}, (\beta_k)_{k\geq 1}$ and $(\gamma_k)_{k\geq 1}$ are sequences in $[0, 1]$.

Remark 1.1. If $\alpha_k = 1$ in Algorithm 1, we have the inertial viscosity forward-backward splitting algorithm (1.6).

If $\theta_k = 0$ and setting $f(x_k) = u$ in Algorithm 1, we have generalized Halpern-type forward-backward splitting method (1.7).

The structure of this work is as follows. In Section 2, we revisit and compile essential definitions and properties crucial to this study. In Section 3, we prove convergence results for our proposed method addressing problem (1.1). After that, Section 4 assesses the proposed method's performance through numerical experiments. Finally, we conclude this work by offering some closing remarks in Section 5.

2. Preliminaries

An operator $T : \mathcal{H} \to \mathcal{H}$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$. We denote the set of all fixed points of the operator T as $\mathbf{Fix}(T) := \{x \in \mathcal{H} : Tx = x\}$.

Let C be a nonempty closed convex subset of $\mathcal{H}$. The metric projection of $\mathcal{H}$ onto C, denoted as $\mathbf{proj}_C : \mathcal{H} \to C$, is defined by $\mathbf{proj}_C(x) = \arg\min_{c\in C} \|x - c\|$ for all $x \in \mathcal{H}$. It is known that

$$\langle x - \mathbf{proj}_C(x), y - \mathbf{proj}_C(x)\rangle \leq 0$$

for all $x \in \mathcal{H}$ and $y \in C$.

A mapping $K : \mathcal{H} \to \mathcal{H}$ is called monotone if for all $x, y \in \mathcal{H}$, $\langle Kx - Ky, x - y\rangle \geq 0$ and it is said to be β-inverse strongly monotone with parameter $\beta > 0$ if, there exists a constant $\beta > 0$ such that

$$\langle Kx - Ky, x - y\rangle \geq \beta\|Kx - Ky\|^2$$

for all $x, y \in \mathcal{H}$.

Let $L : \mathcal{H} \to 2^{\mathcal{H}}$ be a set-values operator. We denote by $\mathbf{gra}(L) := \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in Lx\}$ its graph of L. The operator L is called monotone if,

$$\langle u - v, x - y\rangle \geq 0$$

for all $(x, u), (y, v) \in \mathbf{gra}(L)$. It is classified as maximal monotone if there exists no proper monotone extension of its graph.

The *resolvent* of L and parameter $\lambda \geq 0$, $J^L_\lambda : \mathcal{H} \to 2^{\mathcal{H}}$ defined by $J^L_\lambda := (Id + \lambda L)^{-1}$, where Id is the identity operator from $\mathcal{H}$ to $\mathcal{H}$. If L is maximally monotone, J^L_λ is a single-valued operator.

Next, we present several results in real Hilbert spaces that will prove to be valuable in our convergence analysis.

Lemma 2.1. [34] *Let $\mathcal{H}$ be a real Hilbert space. Then, the following conditions are satisfied:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in \mathcal{H}$.
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in \mathcal{H}$.
(iii) $\|\tau x + (1 - \tau)y\|^2 = \tau\|x\|^2 + (1 - \tau)\|y\|^2 - \tau(1 - \tau)\|x - y\|^2$ *for all* $\tau \in [0, 1]$ *and* $x, y \in \mathcal{H}$.

In order to show the convergence results, we also require the following tools.

Lemma 2.2. [35, Lemma 2.5] *Let $(S_k)_{k\geq1}$ be a sequence of nonnegative real numbers satisfying the following inequalities*

$$S_{k+1} \leq (1 - \rho_k)S_k + \rho_k\sigma_k \ \ \forall k \geq 1 \text{ and } S_{k+1} \leq S_k - \eta_k + \pi_k \ \ \forall k \geq 1,$$

where $(\rho_k)_{k\geq1}$ forms a sequence within $(0, 1)$, $(\eta_k)_{k\geq1}$ constitutes a sequence of nonnegative real numbers, and both $(\sigma_k)_{k\geq1}$ and $(\pi_k)_{k\geq1}$ are real sequences, satisfying the conditions:

(i) $\sum_{k\geq 1} \rho_k = \infty$.
(ii) $\lim_{k\to\infty} \pi_k = 0$.
(iii) $\lim_{i\to\infty} \eta_{k_i} = 0$ *implies* $\limsup_{i\to\infty} \sigma_{k_i} \leq 0$ *for any subsequence* $(\eta_{k_i})_{i\geq 1}$ *of* $(\eta_k)_{k\geq 1}$.

Then the sequence $(S_k)_{k\geq 1}$ *converges to* 0.

For convenience, the following notation will be used

$$\Gamma_\lambda^{A,B} := (Id + \lambda B)^{-1}(Id - \lambda A),\ \lambda \geq 0.$$

Lemma 2.3. [36] *Let A be an μ-inverse strongly monotone operator from a real Hilbert space $\mathcal{H}$ into itself and $B : \mathcal{H} \to 2^{\mathcal{H}}$ a maximal monotone operator. Then, the following inequalities hold.*

$$\begin{aligned}\|\Gamma_\lambda^{A,B}x - \Gamma_\lambda^{A,B}y\|^2 \leq \|x-y\|^2 - \lambda(2\mu-\lambda)\|Ax-Ay\|^2 \\ - \|(Id - J_\lambda^B)(Id-\lambda A)x - (Id - J_\lambda^B)(Id-\lambda A)y\|^2\end{aligned} \tag{2.1}$$

for all $x, y \in B_\lambda := \{z \in \mathcal{H} : \|z\| \leq \lambda\}$.

Lemma 2.4. [36] *Let A be an μ-inverse strongly monotone operator from a real Hilbert space $\mathcal{H}$ into itself and $B : \mathcal{H} \to 2^{\mathcal{H}}$ a maximal monotone operator. Then, the following conditions hold.*

(i) *For $\lambda > 0$,* $\mathbf{Fix}(\Gamma_\lambda^{A,B}) = (A+B)^{-1}(0)$.
(ii) *For $0 < \delta \leq \lambda$ and $x \in \mathcal{H}$,* $\|x - \Gamma_\delta^{A,B}x\| \leq 2\|x - \Gamma_\lambda^{A,B}x\|$.

Theorem 2.5. [37] *Let $\mathcal{H}$ be a real Hilbert space with a nonempty closed convex subset C, consider a nonexpansive mapping $T : C \to C$ with $\mathbf{Fix}(T) \neq \emptyset$. For every $u \in C$ and any $t \in (0,1)$, the unique fixed point x_t within C derived from the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly towards a fixed point of T as t tends to zero.*

3. Main results

In this section, we delve into the intricate details of the convergence analysis for our main results.

Theorem 3.1. *Let $A : \mathcal{H} \to \mathcal{H}$ be an μ-inverse strongly monotone operator on a real Hilbert space $\mathcal{H}$ with $\mu > 0$, and $B : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone operator such that $(A+B)^{-1}(0) \neq \emptyset$. Let $f : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with constant $c \in (0,1)$. Let $(x_k)_{k\geq 1}$ be generated by* **Algorithm 1**. *Assume that the following conditions hold:*

(i) $\lim_{k\to\infty} \gamma_k = 0$ *and* $\sum_{k\geq 1} \gamma_k = +\infty$.
(ii) $\lim_{k\to\infty} \frac{\theta_k}{\gamma_k}\|x_k - x_{k-1}\| = 0$.
(iii) $0 < \liminf_{k\to+\infty} \lambda_k \leq \limsup_{k\to+\infty} \lambda_k < 2\mu$.
(iv) $\liminf_{k\to+\infty}(1-\alpha_k)(1-\beta_k) > 0$.

Then, the sequence $(x_k)_{k\geq 1}$ converges strongly to $\overline{x} := \mathbf{proj}_{(A+B)^{-1}(0)}(f(\overline{x}))$.

Proof. Let $\Gamma_k = J_{\lambda_k}^B(Id - \lambda_k A)$. By Lemma, we have for each $k \in \mathbb{N}$ Γ_k is nonexpansive mapping. By Lemma 2.4, we obtain that $(A+B)^{-1}(0) = \mathbf{Fix}(\Gamma_k)$.

We expect that $(x_k)_{k\geq 1}$ is bounded. Since f is contraction mapping and $\mathbf{proj}_{(A+B)^{-1}(0)}(\cdot)$ is nonexpansive, we have $\mathbf{proj}_{(A+B)^{-1}(0)}(f(\cdot))$ is contraction mapping. Then, there exists the unique fixed point $\overline{x} \in (A+B)^{-1}(0)$ such that $\overline{x} = \mathbf{proj}_{(A+B)^{-1}(0)}(f(\overline{x}))$. Thus $\overline{x} \in \mathbf{Fix}(\Gamma_k)$. It follows that

$$\begin{aligned}\|z_k - \overline{x}\| &= \|\alpha_k w_k + (1-\alpha_k)\Gamma_k w_k - \overline{x}\| \\ &\leq \alpha_k\|w_k - \overline{x}\| + (1-\alpha_k)\|\Gamma_k w_k - \overline{x}\| \\ &\leq \|w_k - \overline{x}\|, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\|y_k - \overline{x}\| &= \|\beta_k w_k + (1-\beta_k)\Gamma_k z_k - \overline{x}\| \\ &\leq \beta_k\|w_k - \overline{x}\| + (1-\beta_k)\|\Gamma_k z_k - \overline{x}\| \\ &\leq \beta_k\|w_k - \overline{x}\| + (1-\beta_k)\|z_k - \overline{x}\|. \end{aligned} \tag{3.2}$$

On the other hand, we consider

$$\begin{aligned}\|w_k - \overline{x}\| &= \|x_k + \theta_k(x_k - x_{k-1} - \overline{x})\| \\ &\leq \|x_k - \overline{x}\| + \theta_k\|x_k - x_{k-1}\|. \end{aligned} \tag{3.3}$$

Combining (3.1)–(3.3), we obtain that

$$\begin{aligned}\|x_{k+1} - \overline{x}\| &= \|\gamma_k f(x_k) + (1-\gamma_k)y_k - \overline{x}\| \\ &\leq \gamma_k\|f(x_k) - \overline{x}\| + (1-\gamma_k)\|y_k - \overline{x}\| \\ &\leq \gamma_k\|f(x_k) - f(\overline{x})\| + \gamma_k\|f(\overline{x}) - \overline{x}\| + (1-\gamma_k)\|w_k - \overline{x}\| \\ &\leq \gamma_k c\|x_k - \overline{x}\| + \gamma_k\|f(\overline{x}) - \overline{x}\| + (1-\gamma_k)\|x_k - \overline{x}\| \\ &\quad + (1-\gamma_k)\theta_k\|x_k - x_{k-1}\| \\ &\leq (1-\gamma_k(1-c))\|x_k - \overline{x}\| + \gamma_k\|f(\overline{x}) - \overline{x}\| \\ &\quad + (1-\gamma_k)\theta_k\|x_k - x_{k-1}\| \\ &\leq (1-\gamma_k(1-c))\|x_k - \overline{x}\| + \gamma_k\|f(\overline{x}) - \overline{x}\| \\ &\quad + (1-\gamma_k(1-c))\theta_k\|x_k - x_{k-1}\|. \end{aligned} \tag{3.4}$$

Since $\lim_{k\to\infty} \frac{\theta_k}{\gamma_k}\|x_k - x_{k-1}\| = 0$, there exists $M > 0$ such that

$$\frac{(1-\gamma_k(1-c))\theta_k}{\gamma_k}\|x_k - x_{k-1}\| \leq M \text{ for all } k \in \mathbb{N}.$$

From (3.4), we can obtain that

$$\|x_{k+1} - \overline{x}\| \leq (1-\gamma_k(1-c))\|x_k - \overline{x}\| + \gamma_k(1-c)\left(\frac{\|f(\overline{x}) - \overline{x}\| + M}{1-c}\right).$$

It follows that

$$\|x_{k+1} - \overline{x}\| \leq \max\left\{\|x_k - \overline{x}\|, \frac{\|f(\overline{x}) - \overline{x}\| + M}{1-c}\right\}$$

$$\vdots$$
$$\leq \max\left\{\|x_1 - \overline{x}\|, \frac{\|f(\overline{x}) - \overline{x}\| + M}{1 - c}\right\}. \tag{3.5}$$

Therefore, $(x_k)_{k\geq 1}$ is bounded. So $(w_k)_{k\geq 1}$, $(z_k)_{k\geq 1}$ and $(y_k)_{k\geq 1}$ also bounded. Using the condition (2.1) in Lemma 2.1 and the definition of $(z_k)_{k\geq 1}$ and $(y_k)_{k\geq 1}$, we get that

$$\begin{aligned} \|z_k - \overline{x}\|^2 &= \|\alpha_k w_k + (1-\alpha_k)\Gamma_k w_k - \overline{x}\|^2 \\ &\leq \alpha_k\|w_k - \overline{x}\|^2 + (1-\alpha_k)\|\Gamma_k w_k - \overline{x}\|^2, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \|y_k - \overline{x}\|^2 &= \|\beta_k w_k + (1-\beta_k)\Gamma_k z_k - \overline{x}\|^2 \\ &\leq \beta_k\|w_k - \overline{x}\|^2 + (1-\beta_k)\|\Gamma_k z_k - \overline{x}\|^2. \end{aligned} \tag{3.7}$$

Now, consider terms $\|\Gamma_k w_k - \overline{x}\|^2$ and $\|\Gamma_k z_k - \overline{x}\|^2$ using Lemma 2.3, we have

$$\begin{aligned} \|\Gamma_k w_k - \overline{x}\|^2 &= \|\Gamma_k w_k - \Gamma_k\overline{x}\|^2 \\ &\leq \|w_k - \overline{x}\|^2 - \lambda_k(2\mu - \lambda_k)\|Aw_k - A\overline{x}\|^2 \\ &\quad - \|w_k - \lambda_k Aw_k - \Gamma_k w_k + \lambda_k A\overline{x}\|^2, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \|\Gamma_k z_k - \overline{x}\|^2 &= \|\Gamma_k z_k - \Gamma_k\overline{x}\|^2 \\ &\leq \|z_k - \overline{x}\|^2 - \lambda_k(2\mu - \lambda_k)\|Az_k - A\overline{x}\|^2 \\ &\quad - \|z_k - \lambda_k Az_k - \Gamma_k z_k + \lambda_k A\overline{x}\|^2. \end{aligned} \tag{3.9}$$

Substituting (3.8) into (3.6), we have

$$\begin{aligned} \|z_k - \overline{x}\|^2 \leq \|w_k - \overline{x}\|^2 &- (1-\alpha_k)\lambda_k(2\mu - \lambda_k)\|Aw_k - A\overline{x}\|^2 \\ &- (1-\alpha_k)\|w_k - \lambda_k Aw_k - \Gamma_k w_k + \lambda_k A\overline{x}\|^2. \end{aligned} \tag{3.10}$$

Substituting (3.9) into (3.7), we have

$$\begin{aligned} \|y_k - \overline{x}\|^2 \leq \beta_k\|w_k - \overline{x}\|^2 &+ (1-\beta_k)\|z_k - \overline{x}\|^2 \\ &- (1-\beta_k)\lambda_k(2\mu - \lambda_k)\|Az_k - A\overline{x}\|^2 \\ &- (1-\beta_k)\|z_k - \lambda_k Az_k - \Gamma_k z_k + \lambda_k A\overline{x}\|^2. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we can imply that

$$\begin{aligned} \|y_k - \overline{x}\|^2 \leq \|w_k - \overline{x}\|^2 &+ (1-\beta_k)(1-\alpha_k)\lambda_k(2\mu - \lambda_k)\|Aw_k - A\overline{x}\|^2 \\ &- (1-\beta_k)(1-\alpha_k)\|w_k - \lambda_k Aw_k - \Gamma_k w_k + \lambda_k A\overline{x}\|^2 \\ &- (1-\beta_k)\lambda_k(2\mu - \lambda_k)\|Az_k - A\overline{x}\|^2 \\ &- (1-\beta_k)\|z_k - \lambda_k Az_k - \Gamma_k z_k + \lambda_k A\overline{x}\|^2. \end{aligned} \tag{3.12}$$

From (3.12), we obtain

$$
\begin{aligned}
\|x_{k+1}-\overline{x}\|^2 &= \langle \gamma_k f(x_n) + (1-\gamma_k)y_k - \overline{x}, x_{k+1}-\overline{x}\rangle \\
&= \langle \gamma_k(f(x_k)-\overline{x}), x_{k+1}-\overline{x}\rangle + \langle (1-\gamma_k)(y_k-\overline{x}), x_{k+1}-\overline{x}\rangle \\
&= \gamma_k\langle f(x_k)-f(\overline{x}), x_{k+1}-\overline{x}\rangle + \gamma_k\langle f(\overline{x})-\overline{x}, x_{k+1}-\overline{x}\rangle \\
&\quad + (1-\gamma_k)\langle y_k-\overline{x}, x_{k+1}-\overline{x}\rangle \\
&\le \gamma_k\|f(x_k)-f(\overline{x})\|\|x_{k+1}-\overline{x}\| + \gamma_k\langle f(\overline{x})-\overline{x}, x_{k+1}-\overline{x}\rangle \\
&\quad + (1-\gamma_k)\|y_k-\overline{x}\|\|x_{k+1}-\overline{x}\| \\
&\le \frac{\gamma_k}{2}\left(\|f(x_k)-f(\overline{x})\|^2 + \|x_{k+1}-\overline{x}\|^2\right) + \gamma_k\langle f(\overline{x})-\overline{x}, x_{k+1}-\overline{x}\rangle \\
&\quad + \frac{(1-\gamma_k)}{2}\left(\|y_k-\overline{x}\|^2 + \|x_{k+1}-\overline{x}\|^2\right) \\
&\le \frac{\gamma_k c^2}{2}\|x_k-\overline{x}\|^2 + \frac{\gamma_k}{2}\|x_{k+1}-\overline{x}\|^2 + \gamma_k\langle f(\overline{x})-\overline{x}, x_{k+1}-\overline{x}\rangle \\
&\quad + \frac{(1-\gamma_k)}{2}\|w_k-\overline{x}\|^2 \\
&\quad - \frac{(1-\gamma_k)(1-\beta_k)(1-\alpha_k)}{2}\lambda_k(2\mu-\lambda_k)\|Aw_k-A\overline{x}\|^2 \\
&\quad - \frac{(1-\gamma_k)(1-\beta_k)(1-\alpha_k)}{2}\|w_k-\lambda_k Aw_k-\Gamma_k w_k+\lambda_k A\overline{x}\|^2 \\
&\quad - \frac{(1-\gamma_k)(1-\beta_k)\lambda_k(2\mu-\lambda_k)}{2}\|Az_k-A\overline{x}\|^2 \\
&\quad - \frac{(1-\gamma_k)(1-\beta_k)}{2}\|z_k-\lambda_k Az_k-\Gamma_k z_k+\lambda_k A\overline{x}\|^2 \\
&\quad + \frac{(1-\gamma_k)}{2}\|x_{k+1}-\overline{x}\|^2 \\
&\le \frac{\gamma_k c^2}{2}\|x_k-\overline{x}\|^2 + \frac{1}{2}\|x_{k+1}-\overline{x}\|^2 + \gamma_k\langle f(\overline{x})-\overline{x}, x_{k+1}-\overline{x}\rangle \\
&\quad + \frac{(1-\gamma_k)}{2}\left(\|x_k-\overline{x}\|^2 + 2\theta_k\langle x_k-x_{k-1}, w_k-\overline{x}\rangle\right) \\
&\quad - \frac{(1-\gamma_k)(1-\beta_k)(1-\alpha_k)}{2}\lambda_k(2\mu-\lambda_k)\|Aw_k-A\overline{x}\|^2 \\
&\quad - \frac{(1-\gamma_k)(1-\beta_k)(1-\alpha_k)}{2}\|w_k-\lambda_k Aw_k-\Gamma_k w_k+\lambda_k A\overline{x}\|^2 \\
&\quad - \frac{(1-\gamma_k)(1-\beta_k)\lambda_k(2\mu-\lambda_k)}{2}\|Az_k-A\overline{x}\|^2 \\
&\quad - \frac{(1-\gamma_k)(1-\beta_k)}{2}\|z_k-\lambda_k Az_k-\Gamma_k z_k+\lambda_k A\overline{x}\|^2 \\
&\le \frac{(1-\gamma_k(1-c^2))}{2}\|x_k-\overline{x}\|^2 + \frac{1}{2}\|x_{k+1}-\overline{x}\|^2 + \gamma_k\langle f(\overline{x})-\overline{x}, x_{k+1}-\overline{x}\rangle \\
&\quad + (1-\gamma_k)\theta_k\langle x_k-x_{k-1}, w_k-\overline{x}\rangle \\
&\quad - \frac{(1-\gamma_k)(1-\beta_k)(1-\alpha_k)}{2}\lambda_k(2\mu-\lambda_k)\|Aw_k-A\overline{x}\|^2
\end{aligned}
$$

$$- \frac{(1-\gamma_k)(1-\beta_k)(1-\alpha_k)}{2}\|w_k - \lambda_k A w_k - \Gamma_k w_k + \lambda_k A\overline{x}\|^2$$
$$- \frac{(1-\gamma_k)(1-\beta_k)\lambda_k(2\mu-\lambda_k)}{2}\|Az_k - A\overline{x}\|^2$$
$$- \frac{(1-\gamma_k)(1-\beta_k)}{2}\|z_k - \lambda_k A z_k - \Gamma_k z_k + \lambda_k A\overline{x}\|^2. \tag{3.13}$$

Then (3.13) reducing to the following:

$$\begin{aligned}\|x_{k+1}-\overline{x}\|^2 \le\ & (1-\gamma_k(1-c^2))\|x_k-\overline{x}\|^2 + 2\gamma_k\langle f(\overline{x})-\overline{x}, x_{k+1}-\overline{x}\rangle \\ & + 2(1-\gamma_k)\theta_k\langle x_k - x_{k-1}, w_k - \overline{x}\rangle \\ & - (1-\gamma_k)(1-\beta_k)(1-\alpha_k)\lambda_k(2\mu-\lambda_k)\|Aw_k - A\overline{x}\|^2 \\ & - (1-\gamma_k)(1-\beta_k)(1-\alpha_k)\|w_k - \lambda_k A w_k - \Gamma_k w_k + \lambda_k A\overline{x}\|^2 \\ & - (1-\gamma_k)(1-\beta_k)\lambda_k(2\mu-\lambda_k)\|Az_k - A\overline{x}\|^2 \\ & - (1-\gamma_k)(1-\beta_k)\|z_k - \lambda_k A z_k - \Gamma_k z_k + \lambda_k A\overline{x}\|^2. \end{aligned} \tag{3.14}$$

For each $k \in \mathbb{N}$, we set
$S_k = \|x_{k+1} - \overline{x}\|^2$,
$\rho_k = \gamma_k(1-c^2)$, $\pi_k = \rho_k\sigma_k$,
$\sigma_k = \frac{2}{(1-c^2)}\langle f(\overline{x})-\overline{x}, x_{k+1}-\overline{x}\rangle + \frac{2(1-\gamma_k)\theta_k}{\gamma_k(1-c^2)}\langle x_k - x_{k-1}, w_k - \overline{x}\rangle$ and

$$\begin{aligned}\eta_k =\ & (1-\gamma_k)(1-\beta_k)(1-\alpha_k)\lambda_k(2\mu-\lambda_k)\|Aw_k - A\overline{x}\|^2 \\ & + (1-\gamma_k)(1-\beta_k)(1-\alpha_k)\|w_k - \lambda_k A w_k - \Gamma_k w_k + \lambda_k A\overline{x}\|^2 \\ & + (1-\gamma_k)(1-\beta_k)\lambda_k(2\mu-\lambda_k)\|Az_k - A\overline{x}\|^2 \\ & + (1-\gamma_k)(1-\beta_k)\|z_k - \lambda_k A z_k - \Gamma_k z_k + \lambda_k A\overline{x}\|^2.\end{aligned}$$

As a result, inequality (3.14) reduces to the following:

$$S_{k+1} \le (1-\rho_k)S_k + \rho_k\sigma_k \text{ and } S_{k+1} \le S_k - \eta_k + \pi_k.$$

By the condition (i), we get that $\sum_{k\ge1}\rho_k = \infty$ and $\lim_{k\to\infty}\pi_k = 0$. In order to complete proof, by applying Lemma 2.2, it is sufficient to show that $\lim_{k\to\infty}\eta_{k_i} = 0$ implies $\limsup_{i\to\infty}\sigma_{k_i} \le 0$ for any subsequence $(\eta_{k_i})_{i\ge1}$ of $(\eta_k)_{k\ge1}$.

Let $(\eta_{k_i})_{i\ge1}$ be a subsequence of $(\eta_k)_{k\ge1}$ such that $\lim_{i\to\infty}\eta_{k_i} = 0$. Therefore, by the assumptions of Lemma 2.2, we can conclude that

$$\lim_{i\to\infty}\|Aw_{k_i} - A\overline{x}\| = 0;$$
$$\lim_{i\to\infty}\|Az_{k_i} - A\overline{x}\| = 0;$$
$$\lim_{i\to\infty}\|w_{k_i} - \lambda_{k_i}Aw_{k_i} - \Gamma_{k_i}w_{k_i} + \lambda_{k_i}A\overline{x}\| = 0;$$
$$\lim_{i\to\infty}\|z_{k_i} - \lambda_{k_i}Az_{k_i} - \Gamma_{k_i}z_{k_i} + \lambda_{k_i}A\overline{x}\| = 0.$$

This implies that

$$\lim_{i\to\infty}\|\Gamma_{k_i}w_{k_i} - w_{k_i}\| = 0; \tag{3.15}$$

$$\lim_{i\to\infty} \|\Gamma_{k_i} z_{k_i} - z_{k_i}\| = 0. \tag{3.16}$$

From (3.1), we have

$$\|w_{k_i} - x_{k_i}\| = \theta_{k_i}\|x_{k_i} - x_{k_{i-1}}\| \to 0 \ (i \to \infty). \tag{3.17}$$

On the other hand, we get

$$\begin{aligned}\|\Gamma_{k_i} z_{k_i} - w_{k_i}\| &\le \|\Gamma_{k_i} z_{k_i} - z_{k_i}\| + \|z_{k_i} - w_{k_i}\| \\ &= \|\Gamma_{k_i} z_{k_i} - z_{k_i}\| + (1-\alpha_{k_i})\|\Gamma_{k_i} w_{k_i} - w_{k_i}\|.\end{aligned} \tag{3.18}$$

From (3.15) and (3.16), we obtain that

$$\lim_{i\to\infty} \|\Gamma_{k_i} z_{k_i} - w_{k_i}\| = 0. \tag{3.19}$$

Given that $\liminf_{k\to+\infty} \lambda_k > 0$, we can find a positive real number $\lambda > 0$ such that $\lambda_k \ge \lambda$ for all $k \in \mathbb{N}$. Specifically, this implies that $\lambda_{k_i} \ge \lambda$ for all $i \in \mathbb{N}$. By the condition (2.4) of Lemma 2.4, one has

$$\|\Gamma_\lambda^{A,B} w_{k_i} - w_{k_i}\| \le 2\|\Gamma_{k_i} w_{k_i} - w_{k_i}\|. \tag{3.20}$$

From (3.20), we can imply that

$$\lim_{i\to\infty} \|\Gamma_\lambda^{A,B} w_{k_i} - w_{k_i}\| = 0. \tag{3.21}$$

Let

$$z_t = tf(\overline{x}) + (1-t)\Gamma_\lambda^{A,B} z_t, \ t \in (0,1). \tag{3.22}$$

By utilizing Theorem 2.5, z_t exhibits strong convergence towards the unique fixed point $\overline{x} = \mathbf{proj}_{(A+B)^{-1}(0)}(f(\overline{x}))$ as $t \to 0$. Consequently, we can conclude that

$$\begin{aligned}\|z_t - w_{k_i}\|^2 &= \|t(f(\overline{x}) - w_{k_i}) + (1-t)(\Gamma_\lambda^{A,B} z_t - w_{k_i})\|^2 \\ &\le (1-t)^2\|\Gamma_\lambda^{A,B} z_t - w_{k_i}\|^2 + 2t\langle f(\overline{x}) - z_t, z_t - w_{k_i}\rangle \\ &\quad + 2t\langle z_t - w_{k_i}, z_t - w_{k_i}\rangle \\ &\le (1-t)^2(\|\Gamma_\lambda^{A,B} z_t - \Gamma_\lambda^{A,B} w_{k_i}\| + \|\Gamma_\lambda^{A,B} w_{k_i} - w_{k_i}\|)^2 \\ &\quad + 2t\langle f(\overline{x}) - z_t, z_t - w_{k_i}\rangle + 2t\|z_t - w_{k_i}\|^2 \\ &\le (1-t)^2(\|z_t - w_{k_i}\| + \|\Gamma_\lambda^{A,B} w_{k_i} - w_{k_i}\|)^2 \\ &\quad + 2t\langle f(\overline{x}) - z_t, z_t - w_{k_i}\rangle + 2t\|z_t - w_{k_i}\|^2.\end{aligned} \tag{3.23}$$

The inequality (3.23) reduces the following:

$$\begin{aligned}&\langle z_t - f(\overline{x}), z_t - w_{k_i}\rangle \\ &\quad \le \frac{(1-t)^2}{2t}(\|z_t - w_{k_i}\| + \|\Gamma_\lambda^{A,B} w_{k_i} - w_{k_i}\|)^2 + \frac{(2t-1)}{2t}\|z_t - w_{k_i}\|^2.\end{aligned} \tag{3.24}$$

Combining (3.19) and (3.24), we get that

$$\limsup_{i\to+\infty}\langle z_t - f(\overline{x}), z_t - w_{k_i}\rangle \le \frac{1}{2t}[(1-t)^2 + (2t-1)]M_0^2, \tag{3.25}$$

where $M_0 = \sup_{i\in\mathbb{N},t\in(0,1)}\|z_t - w_{k_i}\|$. We take $k \to +\infty$ in (3.25), we obtain that

$$\limsup_{i\to\infty}\langle \overline{x} - f(\overline{x}), \overline{x} - w_{k_i}\rangle \le 0. \tag{3.26}$$

Let us consider,

$$\begin{aligned}\langle z - f(\overline{x}), z - x_{k_i}\rangle &= \langle z - f(\overline{x}), z - w_{k_i}\rangle + \theta_{k_i}\langle z - f(\overline{x}), x_{k_i} - x_{k_{i-1}}\rangle \\ &\le \langle z - f(\overline{x}), z - w_{k_i}\rangle + \theta_{k_i}\|z - f(\overline{x})\|\|x_{k_i} - x_{k_{i-1}}\|. \end{aligned}\tag{3.27}$$

From (3.27), one has

$$\limsup_{i\to\infty}\langle \overline{x} - f(\overline{x}), \overline{x} - x_{k_i}\rangle \le 0. \tag{3.28}$$

Next, we claim that $\lim_{i\to+\infty}\|x_{k_i+1} - x_{k_i}\| = 0$. By Algorithm 1, we have the following estimates:

$$\begin{aligned}\|x_{k_{i+1}} - x_{k_i}\| &\le \gamma_{k_i}\|f(\overline{x}) - x_{k_i}\| + (1-\gamma_{k_i})\|y_{k_i} - x_{k_i}\| \\ &\le \gamma_{k_i}\|f(\overline{x}) - x_{k_i}\| + (1-\gamma_{k_i})(\|y_{k_i} - w_{k_i}\| + \|w_{k_i} - x_{k_i}\|) \\ &\le \gamma_{k_i}\|f(\overline{x}) - x_{k_i}\| + (1-\gamma_{k_i})\|w_{k_i} - x_{k_i}\| \\ &\quad + (1-\gamma_{k_i})(1-\beta_{k_i})\|\Gamma_{k_i}z_{k_i} - w_{k_i}\|. \end{aligned}\tag{3.29}$$

From (3.29), using the boundedness of $(x_k)_{k\ge1}$, the condition 3.1, and (3.17) and (3.19), we obtain that

$$\lim_{i\to+\infty}\|x_{k_i+1} - x_{k_i}\| = 0. \tag{3.30}$$

Combining (3.30) and (3.28), we infer that

$$\limsup_{i\to\infty}\langle \overline{x} - f(\overline{x}), \overline{x} - x_{k_{i+1}}\rangle \le 0.$$

Hence, $\limsup_{i\to\infty}\sigma_{k_i} \le 0$. By Lemma 2.2, we observe that $\lim_{k\to\infty} S_k = 0$, that is $x_k \to \overline{x}$ as $k \to \infty$. We thus complete the proof. □

Remark 3.2. The condition (ii) in Theorem 3.1 is satisfied when we set θ_k such that $0 \le \theta_k \le \overline{\theta}_k$, where

$$\overline{\theta}_k = \begin{cases}\min\left\{\theta, \frac{\varepsilon_k}{\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \ne x_{k-1}, \\ \theta, & \text{otherwise,}\end{cases}$$

and $(\varepsilon_k)_{k\ge1}$ is a positive sequence such that $\lim_{k\to\infty}\frac{\varepsilon_k}{\gamma_k} = 0$.

4. Applications

In this section, we delve into the practical applications of our proposed method as outlined in this paper, focusing on its utility in convex minimization problems and image restoration problems.

4.1. Convex minimization problems

Consider a convex and differentiable function $h : \mathcal{H} \to \mathbb{R}$ and a convex, lower-semicontinuous function $g : \mathcal{H} \to \mathbb{R}$. To solve the following convex minimization problem: Find $\overline{x} \in \mathcal{H}$ such that

$$h(\overline{x}) + g(\overline{x}) = \min_{x \in \mathcal{H}}\{h(x) + g(x)\}. \tag{4.1}$$

By using Fermat's rule, the problem (4.1) can be written in the form of the following problem as: Find $\overline{x} \in \mathcal{H}$ such that

$$0 \in \nabla h(\overline{x}) + \partial g(\overline{x}),$$

where ∇h is a gradient of h and ∂g is a subdifferential of g.

Remark 4.1. [38] If a function $K : \mathcal{H} \to \mathcal{H}$ is $(1/L)$-Lipschitz continuous, then K is L-inverse strongly monotone.

Remark 4.2. [39] If a function $P : \mathcal{H} \to \mathbb{R}$ is a convex lower-semicontinuous, then ∂P is maximal monotone.

By applying Theorem 3.1 and set $A = \nabla h$ and $B = \partial g$, we can obtain the following result.

Theorem 4.3. *Let $\mathcal{H}$ be a real Hilbert space. Let $h : \mathcal{H} \to \mathbb{R}$ be a convex differentiable function with a $(1/L)$-Lipschitz continuous gradient ∇h and $g : \mathcal{H} \to \mathbb{R}$ be a convex lower-semicontinuous such that $(\nabla h + \partial g)^{-1}(0) \neq \emptyset$. Let $f : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with constant $c \in (0, 1)$. Let $(x_k)_{k\geq 1}$ be generated by $x_0, x_1 \in \mathcal{H}$*

$$\begin{cases} w_k = x_k + \theta_k(x_k - x_{k-1}), \\ z_k = \alpha_k w_k + (1 - \alpha_k)J^{\partial g}_{\lambda_k}(w_k - \lambda_k \nabla h w_k), \\ y_k = \beta_k w_k + (1 - \beta_k)J^{\partial g}_{\lambda_k}(z_k - \lambda_k \nabla h z_k), \\ x_{k+1} = \gamma_k f(x_n) + (1 - \gamma_k)y_k, \qquad \text{for all } k \geq 1. \end{cases} \tag{4.2}$$

Assume that the following conditions hold:

(i) $\lim_{k\to\infty} \gamma_k = 0$ *and* $\sum_{k\geq 1} \gamma_k = +\infty$.
(ii) $\lim_{k\to\infty} \frac{\theta_k}{\gamma_k}\|x_k - x_{k-1}\| = 0$.
(iii) $0 < \liminf_{k\to+\infty} \lambda_k \leq \limsup_{k\to+\infty} \lambda_k < 2L$.
(iv) $\liminf_{k\to+\infty}(1 - \alpha_k)(1 - \beta_k) > 0$.

Then, the sequence $(x_k)_{k\geq 1}$ converges strongly to $\overline{x} := \mathbf{proj}_{(\nabla h+\partial g)^{-1}(0)}(f(\overline{x}))$.

Next, we present some comparisons among three algorithms: Our proposed algorithm, Kitkuan et al.'s algorithm (2019) (1.6), as presented in [32], and Tan's algorithm (2024), as described in [31, Algorithm 1.3].

Example 4.4. Let $\mathbf{K} \in \mathbb{R}^{l\times s}$ and $\mathbf{b} \in \mathbb{R}^l$ with $l > s$. Let $g : \mathbb{R}^s \to \mathbb{R}$ be defined by $g(x) = \|x\|_1$ for all $x \in \mathbb{R}^s$, and $h : \mathbb{R}^s \to \mathbb{R}$ be defined by $h(x) = \frac{1}{2}\|\mathbf{K}x - \mathbf{b}\|_2^2$ for all $x \in \mathbb{R}^s$. To find the solution of the minimization problem as follows:

$$\begin{aligned} &\text{minimize } \frac{1}{2}\|\mathbf{K}x - \mathbf{b}\|_2^2 + \|x\|_1, \\ &\text{subject to } x \in \mathbb{R}^s. \end{aligned} \tag{4.3}$$

By setting this, we obtain that for each $x = (x^1, x^2, ..., x^s) \in \mathbb{R}^s$

$$J^{\partial g}_{\lambda_k}(x) = (\max\{0, 1 - \frac{\lambda_k}{|x^1|}\}x_1, \max\{0, 1 - \frac{\lambda_k}{|x^2|}\}x_2, ..., \max\{0, 1 - \frac{\lambda_k}{|x^s|}\}x_s),$$

$\nabla h(x) = \mathbf{K}^T(\mathbf{K}x - \mathbf{b})$ and ∇h is $\|\mathbf{K}\|^2$-Lipschitz continuous, where $\mathbf{K}^T$ is a transpose of $\mathbf{K}$.

To begin, we randomly select vectors $x_0, x_1 \in \mathbb{R}^s$, along with $\mathbf{b} \in \mathbb{R}^l$ and the matrix $\mathbf{K} \in \mathbb{R}^{l\times s}$. Subsequently, we set $f(x) = \frac{x}{6}$ for all $x \in \mathbb{R}^s$ and choose the parameters in this example as follows: $\alpha_k = \frac{1}{100k+1}$, $\beta_k = \frac{1}{k+1}$, $\gamma_k = \frac{1}{100k+1}$, $\lambda_k = \frac{1}{\|\mathbf{K}\|^2+1}$ and

$$\theta_k = \begin{cases} \min\left\{\frac{1}{2}, \frac{1}{(k+1)^2\|x_k - x_{k-1}\|}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases} \tag{4.4}$$

We compare our proposed algorithm with Kitkuan et al.'s algorithm (2019) (1.6), as presented in [32], and Tan's algorithm (2024), as described in [31, Algorithm 1.3]. For Tan's algorithm (2024), we choose the following parameter values: $\zeta_k = \theta_k$, $\delta = 1.5, \varphi = \frac{1}{20}$, and $\chi_k = \lambda_k$. We evaluate all three algorithms and record the number of iterations (k) and the CPU times (seconds) by using the stopping criteria: $\|x_k - x_{k-1}\| \leq 10^{-6}$.

Table 1 shows the performance of three algorithms in solving problem (4.3) with different sizes of matrix $\mathbf{K}$. Our algorithm consistently achieves optimality tolerance in the shortest CPU time across all cases. Additionally, it is notable that our algorithm requires fewer iterations compared to Kitkuan et al.'s algorithm (2019) and Tan's algorithm (2024) for each matrix size $\mathbf{K}$.

Table 1. The comparison of three algorithms with different sizes of matrix **K**.

(s, l)	Our algorithm		Kitkuan et al.'s algorithm (2019)		Tan's algorithm (2024)	
	CPU time (s)	Iterations	CPU time (s)	Iterations	CPU time (s)	Iterations
(20,500)	0.1991	8113	0.3707	25476	0.7224	23631
(50,500)	0.5676	7095	0.8174	17998	1.1194	9539
(300,500)	0.6786	3757	1.2733	12185	8.9313	35344
(20,1000)	0.3004	8475	0.5241	22350	0.6597	12788
(50,1000)	0.5820	4968	0.7979	13085	2.6585	18461
(300,1000)	1.2100	4577	1.7316	11568	26.6996	72012
(500,1000)	1.7890	4705	2.5680	12714	58.4649	106069
(20,2000)	0.6905	5459	0.7423	10751	3.7730	23994
(50,2000)	1.0284	6016	1.0665	13636	8.7545	42958
(300,2000)	2.0282	4260	3.3280	7027	100.4799	129957
(500,2000)	3.6868	4829	3.7700	9385	201.6677	183047
(1000,2000)	5.2149	3979	6.1491	6603	793.1800	317432

4.2. Image restoration problems

In this subsection, we showcase the efficacy of the proposed algorithm by employing it to tackle image restoration problems, specifically focusing on deblurring and denoising images. The image restoration problem can be defined as the inversion of the following degradation model:

$$y = \mathbf{H}x + w, \tag{4.5}$$

where y, $\mathbf{H}$, x, and w denote the degraded image, degradation or blurring operator, original image, and noise operator, respectively.

To approximate the reconstructed image by solving the regularized least-squares minimization problem:

$$\min_{x}\left\{\frac{1}{2}\|\mathbf{H}x-y\|_2^2+\mu\phi(x)\right\}, \tag{4.6}$$

where $\mu > 0$ is the regularization parameter and $\phi(\cdot)$ represents the regularizer. The l_1 norm serves as a regularization functional, commonly utilized to eliminate noise in restoration problems, known as Tikhonov regularization [40]. The problem (4.6) can be reformulated as follows:

$$\text{find } x \in \arg\min_{x\in\mathbb{R}^s}\left\{\frac{1}{2}\|\mathbf{H}x-y\|_2^2+\mu\|x\|_1\right\}, \tag{4.7}$$

where y denotes the degraded image, and $\mathbf{H}$ represents a bounded linear operator. We can see that problem (4.7) can be formed in the problem (1.1) by setting $B = \partial\|\cdot\|_1, \mu = 0.001$ and $A = \nabla L(\cdot)$ where $L(x) = \frac{1}{2}\|\mathbf{H}x-y\|_2^2$. By using this, we observe that $A(x) = \nabla L(x) = \mathbf{H}^T(\mathbf{H}x-y)$. First, we degrade image by adding random noise and different types of blurring. The Gaussian blur (size 20 by 20 with the standard deviation 20), the average blur (size 10 by 10), and the motion blur (the linear motion of a camera by 20 pixels with an angle of 40 degrees). Next, we solve problem (4.7) using our algorithm in Theorem 4.3 and putting $f(x) = \frac{x}{2}$ for all $x \in \mathbb{R}^s$, $\alpha_k = \frac{1}{k+1}$, $\beta_k = \frac{1}{k+1}$, $\gamma_k = \frac{1}{100k+1}$, $\lambda_k = 0.7$ and θ_k is defined as (4.4).

The comparisons of the performance among our proposed algorithm, Kitkuan et al.'s algorithm (2019), and Tan's algorithm (2024) are presented. In the case of the Kitkuan et al.'s algorithm (2019) (1.6) was presented in [32], we set $f(x) = \frac{x}{2}$ for all $x \in \mathbb{R}^s$, $\gamma_k = \frac{1}{100k+1}$, $\lambda_k = 0.7$ and θ_k is defined as (4.4). For the Tan's algorithm (2024) presented in [31, Algorithm 1.3], we choose the following parameter values: $\zeta_k = \theta_k$, $\delta = 1.5, \varphi = \frac{1}{20}$, and $\chi_k = \lambda_k$. The reconstructed image's quality is evaluated using the signal-to-noise ratio (SNR) formula:

$$\mathbf{SNR}(k) = 20\log_{10}\frac{\|x\|_2^2}{\|x-x_k\|_2^2},$$

where x represents the original image, while x_k stands for the image restored at iteration k.

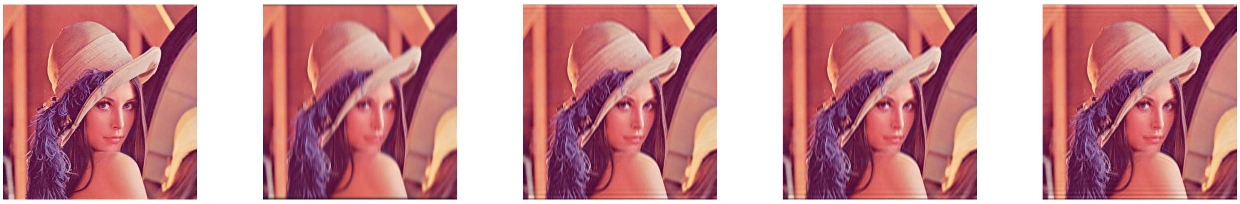
The effectiveness of image restoration using our proposed algorithm, Kitkuan et al.'s algorithm (2019), and Tan's algorithm (2024) is depicted in Figures 1–3.

The comparisons among our proposed algorithm, Kitkuan et al.'s algorithm (2019), and Tan's algorithm (2024) in image restoration problems are illustrated in Figure 4.

The experiments were carried out using MATLAB 9.19 (R2022b) and were performed on a MacBook Pro 14-inch 2021 model, which is equipped with an Apple M1 Pro processor and 16 GB of memory.

(a) (b) (c) (d) (e)

Figure 1. Figure (a) illustrates the original pirate image; Figure (b) depicts the images degraded by Gaussian blur and random noise; Figure (c) showcases the reconstructed images using Kitkuan et al.'s algorithm (2019); Figure (d) showcases the reconstructed images using Tan's algorithm (2024); and Figure (e) presents the reconstructed images using our algorithm as described in (4.2).

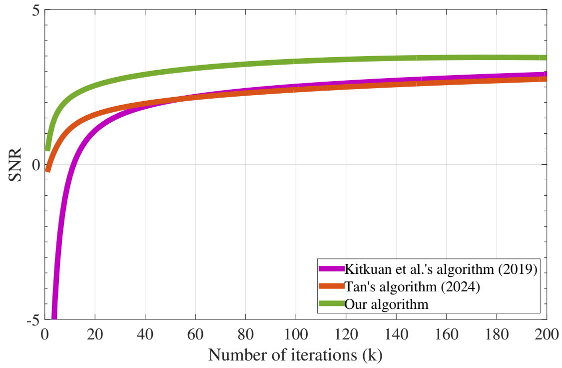


(a) (b) (c) (d) (e)

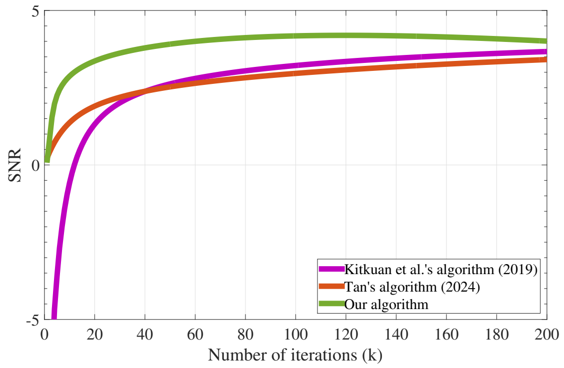
Figure 2. Figure (a) illustrates the original Lena image; Figure (b) depicts the images degraded by average blur and random noise; Figure (c) showcases the reconstructed images using Kitkuan et al.'s algorithm (2019); Figure (d) showcases the reconstructed images using Tan's algorithm (2024); and Figure (e) presents the reconstructed images using our algorithm as described in (4.2).

(a) (b) (c) (d) (e)

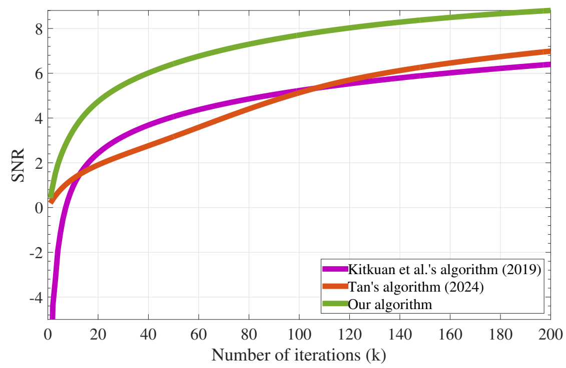
Figure 3. Figure (a) illustrates the original dog image; Figure (b) depicts the images degraded by motion blur and random noise; Figure (c) showcases the reconstructed images using Kitkuan et al.'s algorithm (2019); Figure (d) showcases the reconstructed images using Tan's algorithm (2024); and Figure (e) presents the reconstructed images using our algorithm as described in (4.2).



(a) Pirate



(b) Lena



(c) Dog

Figure 4. (a) The performance of **SNR** for the pirate image using three algorithms shown in Figure 1; (b) The performance of **SNR** for the Lena image using three algorithms displayed in Figure 2; and (c) The performance of **SNR** for the dog image using three algorithms shown in Figure 3.

5. Conclusions

We introduce a novel generalized viscosity forward-backward splitting scheme that incorporates inertial terms aimed at addressing the monotone inclusion problem. We also include a proof of the strong convergence of this algorithm under certain specified conditions for the relevant parameters. Furthermore, we leverage these results to approximate solutions for convex minimization problems. Additionally, we present a numerical example to compare our proposed algorithm with others in convex minimization problems. Finally, we demonstrate the efficacy of our method in solving image restoration problems.

Author Contributions

Kasamsuk Ungchittrakool: Conceptualization, Methodology, Software, Validation, Convergence analysis, Investigation, Writing-original draft preparation, Writing-review and editing, Visualization; Natthaphon Artsawang: Conceptualization, Methodology, Software, Validation, Convergence analysis, Investigation, Writing-original draft preparation, Writing-review and editing, Visualization, Project administration, Funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declare no conflict of interest.

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