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Research article

The geometry of geodesic invariant functions and applications to Landsberg surfaces

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Abstract: In this paper, for a given spray S on an n-dimensional manifold M, we investigated the geometry of S-invariant functions. For an S-invariant function $\mathcal{P}$, we associated a vertical subdistribution $\mathcal{V}_{\mathcal{P}}$ and found the relation between the holonomy distribution and $\mathcal{V}_{\mathcal{P}}$ by showing that the vertical part of the holonomy distribution is the intersection of all spaces $\mathcal{V}_{\mathcal{F}_S}$ associated with $\mathcal{F}_S$ where $\mathcal{F}_S$ is the set of all Finsler functions that have the geodesic spray S. As an application, we studied the Landsberg Finsler surfaces. We proved that a Landsberg surface with S-invariant flag curvature is Riemannian or has a vanishing flag curvature. We showed that for Landsberg surfaces with non-vanishing flag curvature, the flag curvature is S-invariant if and only if it is constant; in this case, the surface is Riemannian. Finally, for a Berwald surface, we proved that the flag curvature is H-invariant if and only if it is constant.

Keywords: spray; holonomy distribution; S-invariant functions (first integrals); Landsberg surfaces; flag curvature
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1. Introduction

A system of second-order homogeneous ordinary differential equations (SODE), whose coefficients do not depend explicitly on time, can be identified by a special vector field called spray. The solution of the SODE is called the geodesic of the spray. The spray corresponding to the geodesic equation of a Riemannian or Finslerian metric is called the geodesic spray of the corresponding metric.

The concept of geodesic invariant functions (or, equivalently, S-invariant functions or first integrals of S) has various applications not only in Finsler and Riemann geometries, but also in physics. For example, the norm and energy functions are geodesic invariant functions on Finslerian or Riemannian manifolds; on Landsberg surfaces, the main scalar of the surface is S-invariant. Also, in physics, if a geodesic invariant function is given, then this function can be treated as a constant of motion; in other words, these functions are conserved along motion. Geodesic invariant functions can give important information on the geometric structure. See, for example, [3, 14] and references therein.

By [15], for a given spray S on a n-dimensional manifold M, we can associate the so-called holonomy distribution, which is generated by the horizontal vector fields and their successive Lie brackets. The functions on TM that are invariant with respect to the parallel translation are called holonomy invariant functions. These functions are constant along the holonomy distribution [8]. It is easy to see that the holonomy invariant functions are also S-invariant functions, that is, constant along the spray. However, the opposite is not true: not all functions constant along the spray are holonomy invariant. In the literature S-invariant functions are also known as first integrals of the spray S; for example, we refer to [3, 14].

In this paper, we investigate the geometry of distributions associated with homogeneous S-invariant functions of degree $k \neq 0$. A function $\mathcal{P}$ defined on TM is called k-homogeneous, if it satisfies the equation $\mathcal{P}(\lambda v) = \lambda^k \mathcal{P}(v)$ for any $v \in TM$. We show that, to any k-homogeneous S-invariant nontrivial function $\mathcal{P}$, one can associate the decomposition of TTM

$$TTM = \mathcal{H}_{\mathcal{P}} \oplus Span\{S\} \oplus \mathcal{V}_{\mathcal{P}} \oplus Span\{C\}, \tag{1.1}$$

where $\mathcal{H}_{\mathcal{P}}$ and $\mathcal{V}_{\mathcal{P}}$ are $n-1$-dimensional sub-distribution of the horizontal (resp. the vertical) spaces associated with the spray. Moreover, if $\mathcal{P}$ is a holonomy invariant function, then

$$\operatorname{Ker} d\mathcal{P} = \mathcal{H} \oplus \mathcal{V}_{\mathcal{P}}, \tag{1.2}$$

where $\mathcal{H}$ is the horizontal distribution associated to S.

As a special case, for a Finsler manifold (M, F), since F is constant along its geodesic spray S and also along the horizontal distribution $\mathcal{H}$, we focus our attention on the distribution $\mathcal{V}_F$. In [8], the notion of metrizability freedom of sprays was introduced. For a given spray S, m_S shows how many essentially different Finsler functions can be associated to it. The metrizability freedom of a spray can be determined with the help of its holonomy distribution $\mathcal{H}ol$. We prove that $\mathcal{V}_{\mathcal{H}ol}$ and $\mathcal{V}_F$ coincide if and only if the metrizability freedom of S is one. In the case when $m_S \geq 1$, then $\mathcal{V}_{\mathcal{H}ol}$ is a sub-distribution of $\mathcal{V}_F$ and we prove that

$$\mathcal{V}_{\mathcal{H}ol} = \bigcap_{F \in \mathcal{F}_S} \mathcal{V}_F$$

where $\mathcal{F}_S$ denotes the set of Finsler functions associated with the spray S.

As an application, we turn our attention to the Landsberg surfaces. We show that for a Landsberg surface, if the flag curvature is S-invariant, then the surface is Riemannian or has a vanishing flag curvature. Also, for a Landsberg surface with non-vanishing flag curvature K, we establish that K is S-invariant if and only if K is constant. In this case, the surface is Riemannian. Finally, we prove that, for a Berwald surface, the flag curvature is H-invariant if and only if K is constant.

2. Preliminaries

M is an n-dimensional smooth manifold, its tangent bundle (TM, π_M, M), and its subbundle of non-zero tangent vectors $(\mathcal{T}M, \pi, M)$. On the base manifold M, we indicate local coordinates by (x^i), while on TM, the induced coordinates are (x^i, y^i). The natural almost-tangent structure of TM is defined locally by $J = \frac{\partial}{\partial y^i} \otimes dx^i$, which is the vector 1-form J on TM. The canonical or Liouville vector field is the vertical vector field $C = y^i \frac{\partial}{\partial y^i}$ on TM.

2.1. Spray and Finsler manifold

The geometry of sprays and Finsler manifolds has a vast literature. Here, we are using essentially the results and the terminology of [11, 12].

A vector field $S \in \mathfrak{X}(\mathcal{T}M)$ is called a spray if $JS = C$ and $[C, S] = S$. Locally, a spray is expressed as follows

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

where the spray coefficients $G^i = G^i(x, y)$ are 2-homogeneous functions in the $y = (y^1, \ldots, y^n)$ variable. A curve $\sigma : I \to M$ is called regular if $\sigma' : I \to \mathcal{T}M$, where σ' is the tangent lift of σ. A regular curve σ on M is called geodesic of a spray S if $S \circ \sigma' = \sigma''$. Locally, $\sigma(t) = (x^i(t))$ is a geodesic of S if and only if it satisfies the equation

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0. \tag{2.2}$$

A nonlinear connection is described by a supplemental n-dimensional distribution to the vertical distribution, denoted as $\mathcal{H}\colon u \in \mathcal{T}M \to \mathcal{H}_u \subset T_u(\mathcal{T}M)$. For every $u \in \mathcal{T}M$, we have

$$T_u(\mathcal{T}M) = \mathcal{H}_u \oplus \mathcal{V}_u. \tag{2.3}$$

Every spray S induces a canonical nonlinear connection [11] through the corresponding horizontal and vertical projectors,

$$h = \frac{1}{2}(Id + [J, S]), \quad v = \frac{1}{2}(Id - [J, S]). \tag{2.4}$$

Equivalently, the canonical nonlinear connection defined by a spray is expressed as an almost product structure $\Gamma = [J, S] = h - v$. A spray S is horizontal with regard to the induced nonlinear connection; this means that $S = hS$. Moreover, the two projectors, h and v, have the following local expressions

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \qquad v = \frac{\partial}{\partial y^i} \otimes \delta y^i,$$

and the distributions are generated by the vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^j_i(x, y)\frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + G^j_i(x, y)dx^i,$$

where $G^j_i(x, y) = \frac{\partial G^j}{\partial y^i}$. If $X \in \mathfrak{X}(M)$, then $\mathcal{L}_X$ and i_X stand for the Lie derivative with respect to X and the interior product by X, respectively. df represents the differential of $f \in C^\infty(M)$. A skew-symmetric $C^\infty(M)$-linear map $L : (\mathfrak{X}(M))^\ell \longrightarrow \mathfrak{X}(M)$ is a vector ℓ-form on M. Each vector ℓ-form L defines two graded derivations of the Grassmann algebra of M, namely i_L and d_L, as follows

$$i_L f = 0, \quad i_L df = df \circ L \quad (f \in C^\infty(M)),$$

$$d_L := [i_L, d] = i_L \circ d - (-1)^{\ell-1} d i_L.$$

The curvature tensor R of the nonlinear connection is

$$R = -\frac{1}{2}[h, h], \tag{2.5}$$

and the Jacobi endomorphism [12] is defined by

$$\Phi = v \circ [S, h] = R^i{}_j \frac{\partial}{\partial y^i} \otimes dx^j = \left(2\frac{\partial G^i}{\partial x^j} - S(G^i_j) - G^i_k G^k_j\right) \frac{\partial}{\partial y^i} \otimes dx^j.$$

The two curvature tensors are related by

$$3R = [J, \Phi], \quad \Phi = i_S R.$$

For simplicity, we use the notations

$$\delta_i := \frac{\delta}{\delta x^i}, \quad \partial_i := \frac{\partial}{\partial x^i}, \quad \dot{\partial}_i := \frac{\partial}{\partial y^i}.$$

Definition 2.1. *A Finsler manifold of dimension n is a pair (M, F), where M is a smooth manifold of dimension n, and F is a continuous function $F : TM \to \mathbb{R}$ such that:*

a) F is smooth and strictly positive on $\mathcal{T}M$.

b) F is positively homogenous of degree 1 in the directional argument y: $\mathcal{L}_C F = F$.

c) The metric tensor $g_{ij} = \dot{\partial}_i \dot{\partial}_j E$ has rank n on $\mathcal{T}M$, where $E := \frac{1}{2}F^2$ is the energy function.

Since the 2-form $dd_J E$ is non-degenerate, the Euler-Lagrange equation

$$\omega_E := i_S dd_J E - d(E - \mathcal{L}_C E) = 0 \tag{2.6}$$

uniquely determines a spray S on TM. This spray is called the geodesic spray of the Finsler function. The ω_E is called the Euler-Lagrange form associated with S and E.

2.2. Holonomy distribution and metrizability freedom

Definition 2.2. *[15] The holonomy distribution $\mathcal{H}ol$ of a spray S is the distribution on TM generated by the horizontal vector fields and their successive Lie-brackets, that is*

$$\mathcal{H}ol := \left\langle \mathfrak{X}^h(TM) \right\rangle_{Lie} = \left\{ [X_1, [\ldots[X_{m-1}, X_m]\ldots]] \;\middle|\; X_i \in \mathfrak{X}^h(TM) \right\} \tag{2.7}$$

where $\mathfrak{X}^h(TM)$ is the modules of horizontal vector fields.

The parallel translation along curves with respect to the canonical nonlinear connection associated with a spray S can be introduced through horizontal lifts. Let $c\colon [0, 1] \to M$ be a piecewise smooth curve such that $c(0) = p$ and $c(1) = q$, and let c^h be a horizontal lift of the curve c (that is, $\pi \circ c^h = c$ and $\dot{c}^h(t) \in \mathcal{H}_{c^h(t)}$). The parallel translation $\tau\colon T_pM \to T_qM$ along c is defined as follows: If $c^h(0) = v$ and $c^h(1) = w$, then $\tau(v) = w$.

Definition 2.3. *Let S be a spray. A function $E \in C^\infty(TM)$ is called a holonomy invariant function if it is invariant with respect to the parallel translation induced by the associated canonical nonlinear connection to S. That is, we have $E(\tau(v)) = E(v)$, where $v \in TM$ and τ is any parallel translation. The set of holonomy invariant functions is denoted by $C^\infty_{\mathcal{H}ol}$.*

Since the parallel translations can be interpreted as travelling along the horizontal lift of curves [8], one can characterize the element of $C^\infty_{\mathcal{H}ol}$ as functions with vanishing horizontal derivatives. It follows that

$$C^\infty_{\mathcal{H}ol} = \{E \in C^\infty(\mathcal{T}M) \mid \mathcal{L}_X E = 0,\ X \in \mathcal{H}ol\}\,. \tag{2.8}$$

Definition 2.4. *Suppose S is a spray on a manifold M. If there is a Finsler function F such that its geodesic spray is S, then S is called Finsler metrizable.*

Let us denote by $\mathcal{F}_S$ the set of Finsler function F generating S as a geodesic spray. Then, we have

$$F \in \mathcal{F}_S \qquad \Longleftrightarrow \qquad E = \tfrac{1}{2}F^2 \in C^\infty_{\mathcal{H}ol} \tag{2.9}$$

meaning that F is a Finsler function of S if and only if the energy function associated is a 2-homogenous regular element of $C^\infty_{\mathcal{H}ol}$.

The questions of how many essentially different Finsler metrics can be associated with a spray, and how to determine this number in terms of geometric quantities were considered in [8]. In the case when the holonomy distribution (2.7) of a spray S is regular, then the metrizability freedom $m_S (\in \mathbb{N})$ can be calculated by the following

Theorem. *([8, Theorem 4.4]) Let S be a metrizable spray with regular holonomy distribution $\mathcal{H}ol$. Then, the metrizability freedom can be calculated as $m_S = \operatorname{codim}(\mathcal{H}ol)$.*

In the case when the metrizability freedom of S is $m_S \geq 1$, then for every $v_0 \in \mathcal{T}M$ there exists a neighborhood $U \subset \mathcal{T}M$ and functionally independent element $E_1, \dots, E_{m_s}$ of $C^\infty_{\mathcal{H}ol}$ on U such that any $E \in C^\infty_{\mathcal{H}ol}$ can be expressed as

$$E(v) = \varphi(E_1(v), \dots, E_{m_s}(v)), \qquad \forall\, v \in U,$$

with some function $\varphi\colon \mathbb{R}^{m_s} \to \mathbb{R}$. We also remark that in that case, since $\mathcal{H}ol$ is generated by horizontal vector fields and their Lie brackets, it contains $\mathcal{H}$, therefore

$$\mathcal{H}ol = \mathcal{H} \oplus \mathcal{V}_{\mathcal{H}ol}, \tag{2.10}$$

where $\mathcal{V}_{\mathcal{H}ol}$ denotes the vertical part of $\mathcal{H}ol$. Since $\dim(\mathcal{H}) = n$, we get

$$\dim \mathcal{V}_{\mathcal{H}ol} = n - m_S. \tag{2.11}$$

3. Geodesic invariant functions

Definition 3.1. *Let S be a spray on M. Then, $\mathcal{P} \in C^\infty(\mathcal{T}M)$ is called a geodesic invariant function, if for any geodesics $c(t)$ of S it satisfies $P(c'(t)) \equiv const.$*

Obviously, for a given spray S, the function $P \in C^\infty(\mathcal{T}M)$ is a geodesic invariant function if and only if

$$\mathcal{L}_S \mathcal{P} = 0, \tag{3.1}$$

that is, $\mathcal{P}$ is a first integral of S [3]. In that spirit, we can call such a function an S-invariant function, referring also to the spray determining the geodesic structure. We remark that $\mathcal{P}$ is constant along S if and only if the dynamical covariant derivative of $\mathcal{P}$ vanishes; see for example [4].

As the results of [4, 9] show, certain geometric distributions associated with sprays and their deformation can play a central role in the investigation of their metrizability property. This is why, motivated by [9], for further computation and analysis, we introduce a decomposition of the horizontal (resp. the vertical) distributions adapted to an S-invariant function $\mathcal{P}$, homogeneous of degree $k \neq 0$; we introduce the endomorphisms

$$h_{\mathcal{P}} = h - \frac{d_J \mathcal{P}}{k\mathcal{P}} \otimes S, \qquad v_{\mathcal{P}} = v - \frac{d_v \mathcal{P}}{k\mathcal{P}} \otimes C, \tag{3.2}$$

and we set

$$\mathcal{H}_{\mathcal{P}} := Im\, h_{\mathcal{P}}, \qquad \mathcal{V}_{\mathcal{P}} := Im\, v_{\mathcal{P}}. \tag{3.3}$$

We have the following

Lemma 3.2.

1. *Properties of $v_{\mathcal{P}}$ and $\mathcal{V}_{\mathcal{P}}$:*

   i) $\ker(v_{\mathcal{P}}) = \mathcal{H} \oplus Span\{C\}$,

   ii) $\mathrm{Im}(v_{\mathcal{P}}) = \mathcal{V}_{\mathcal{P}}$ *is an $(n-1)$-dimensional involutive subdistribution of $\mathcal{V}$,*

   iii) any $X \in \mathcal{V}_{\mathcal{P}}$ is an infinitesimal symmetry of $\mathcal{P}$ that is $\mathcal{L}_X \mathcal{P} = 0$,

   iv) the vertical distribution has the decomposition $\mathcal{V} = \mathcal{V}_{\mathcal{P}} \oplus Span\{C\}$.

2. *Properties of $h_{\mathcal{P}}$ and $\mathcal{H}_{\mathcal{P}}$:*

   i) $\ker(h_{\mathcal{P}}) = \mathcal{V} \oplus Span\{S\}$,

   ii) $\mathrm{Im}(h_{\mathcal{P}}) = \mathcal{H}_{\mathcal{P}}$ *is an $(n-1)$-dimensional subdistribution of $\mathcal{H}$,*

   iii) the horizontal distribution has the decomposition $\mathcal{H} = \mathcal{H}_{\mathcal{P}} \oplus Span\{S\}$,

3. $J(\mathcal{H}_{\mathcal{P}}) = \mathcal{V}_{\mathcal{P}}$.

Proof. We prove *(1)* in detail. The computations for *(2)* are similar.

ad i) We note that $\mathcal{H} = \mathrm{Ker}\, v$, therefore $\mathcal{H} \subset \mathrm{Ker}\, v_{\mathcal{P}}$. Moreover, if $V \in \ker v_{\mathcal{P}}$ is vertical, then using $v(V) = V$ we get

$$v_{\mathcal{P}}(V) = 0 \quad \Longleftrightarrow \quad V = \frac{V(\mathcal{P})}{k\mathcal{P}} C,$$

that is $V \in \mathrm{Span}\{C\}$ and we get *i)*.

ad ii) We introduce the simplified notation $\mathcal{P}_i := \dot{\partial}_i \mathcal{P}$ and the vector fields

$$h_i := h_{\mathcal{P}}(\delta_i) = \delta_i - \frac{\mathcal{P}_i}{k\mathcal{P}} S, \tag{3.4a}$$

$$v_i := v_{\mathcal{P}}(\dot{\partial}_i) = \dot{\partial}_i - \frac{\mathcal{P}_i}{k\mathcal{P}}C \tag{3.4b}$$

for $i = 1, \ldots, n$. We get

$$\mathcal{H}_{\mathcal{P}} = \mathrm{Span}\{h_1, \ldots, h_n\}, \tag{3.5a}$$

$$\mathcal{V}_{\mathcal{P}} = \mathrm{Span}\{v_1, \ldots, v_n\}. \tag{3.5b}$$

We note that the vector fields in (3.5a) (resp., in (3.5b)) are not independent since $y^i h_i = 0$ (resp., $y^i v_i = 0$). Because the k-homogeneity property of $\mathcal{P}$ (and the $(k-1)$-homogeneity property of $\mathcal{P}_i$) for any $v_i, v_j \in \mathcal{V}_{\mathcal{P}}$, their Lie bracket is

$$[v_i, v_j] = \left[\dot{\partial}_i - \frac{\mathcal{P}_i}{k\mathcal{P}}y^k\dot{\partial}_k,\ \dot{\partial}_j - \frac{\mathcal{P}_j}{k\mathcal{P}}y^\ell\dot{\partial}_\ell\right] = \frac{\mathcal{P}_i}{k\mathcal{P}}\dot{\partial}_j - \frac{\mathcal{P}_j}{k\mathcal{P}}\dot{\partial}_i = \frac{\mathcal{P}_i}{k\mathcal{P}}v_j - \frac{\mathcal{P}_j}{k\mathcal{P}}v_i$$

and hence, from (3.5b), we get that $[v_i, v_j] \in \mathcal{V}_{\mathcal{P}}$ hence $\mathcal{V}_{\mathcal{P}}$ is involutive.

ad iii) One can check that the generators (3.5b) of the distribution are infinitesimal symmetry of $\mathcal{P}$. Indeed, using Euler's theorem of the homogeneous functions, we get for the k-homogeneous $\mathcal{P}$:

$$\mathcal{L}_C\mathcal{P} = k\mathcal{P}, \tag{3.6}$$

and therefore

$$\mathcal{L}_{v_i}\mathcal{P} = \dot{\partial}_i(\mathcal{P}) - \frac{\mathcal{P}_i}{k\mathcal{P}}C(\mathcal{P}) = \mathcal{P}_i - \frac{\mathcal{P}_i}{k\mathcal{P}}k\mathcal{P} = 0. \tag{3.7}$$

ad iv) Supposing $C \in \mathcal{V}_{\mathcal{P}}$ we get from (3.5b) that $C = C^i v_i$ with some coefficients C^i. Solving this equation, since $C(\mathcal{P}) = k\mathcal{P}$ and $v_i(\mathcal{P}) = 0$, we find that $C(\mathcal{P}) = C^i v_i(\mathcal{P}) = 0$, which is a contradiction.

For *3)*, we note that for the generators (3.4a) of (3.5a) and (3.4b) of (3.5b), we get

$$Jh_i = J\delta_i - \frac{\mathcal{P}_i}{k\mathcal{P}}JS = \dot{\partial}_i - \frac{\mathcal{P}_i}{k\mathcal{P}}C = v_i, \tag{3.8}$$

$i = 1, \ldots, n$, and this proves *3)*. □

From Lemma 3.2 we get the following

Corollary 3.3. *For a given spray S on TM, then any non-trivial S-invariant function $\mathcal{P} \in C^\infty(\mathcal{T}M)$ and homogeneous of degree $k \neq 0$ gives rise to the direct sum decomposition* (1.1). *Moreover, if $\mathcal{P}$ is constant along $\mathcal{H}_{\mathcal{P}}$, then we have also* (1.2).

We have the following

Proposition 3.4. *Let (M, F) be a Finsler manifold with geodesic spray S. If $\mathcal{P}$ is a k-homogeneous holonomy invariant function with $k \neq 0$, then*

$$\mathcal{V}_{\mathcal{H}ol} \subseteq \mathcal{V}_{\mathcal{P}}. \tag{3.9}$$

Proof. Assume that $\mathcal{P}$ is a k-homogeneous holonomy invariant function with $k \neq 0$, then $\mathcal{P} \in C^\infty_{\mathcal{H}ol}$, and according to (2.8), we have $\mathcal{V}_{\mathcal{H}ol} \subseteq \mathcal{H}ol \subseteq \mathrm{Ker}\, d\mathcal{P}$. It follows that

$$\mathcal{V}_{\mathcal{H}ol} \subseteq \mathcal{V} \cap \mathrm{Ker}\, d\mathcal{P} = \mathcal{V}_{\mathcal{P}},$$

where we use the notation (3.3). □

Remark 3.5. Let (M, F) be a Finsler manifold with geodesic spray S. If $\mathcal{P}$ is a k-homogeneous S-invariant (but not necessarily holonomy invariant) function with $k \neq 0$ and $\mathcal{V}_{\mathcal{H}ol} \subseteq \mathcal{V}_{\mathcal{P}}$, then $d_h d_h \mathcal{P} = 0$.

Proof. We note that, since $\mathcal{P}$ is not necessarily a holonomy invariant function, we do not have $d_h \mathcal{P} = 0$. However, the image of the curvature tensor R is in the holonomy distribution. If $\mathcal{V}_{\mathcal{H}ol} \subseteq \mathcal{V}_{\mathcal{P}}$, then $d_R \mathcal{P} = 0$. On the other hand, using (2.5) and the properties $d_{[h,h]} = [d_h, d_h]$ and

$$[d_h, d_h] = d_h d_h - (-1) d_h d_h = 2 d_h d_h,$$

we have

$$d_h d_h \mathcal{P} = \frac{1}{2} d_{[h,h]} \mathcal{P} = -d_R \mathcal{P} = 0,$$

which shows the statement of the remark. □

It should be noted that in the generic case, the holonomy distribution of a spray is the $2n$-dimensional distribution TTM and the metrizability freedom is $m_S = 0$. For $m_S = 1$ we get the following

Theorem 3.6. *Let S be a given spray metrizability freedom $m_S = 1$, that is (essentially) uniquely metrizable by a Finsler function F. Then, for any 1-homogeneous S-invariant function $\mathcal{P}$, we have $\mathcal{V}_{\mathcal{H}ol} = \mathcal{V}_{\mathcal{P}}$ if and only if $F = c\mathcal{P}$ where $c \in \mathbb{R} \setminus \{0\}$.*

Proof. Since the metrizability freedom of S is 1, then by [8] the codimension of $\mathcal{H}ol$ is one. That is, the dimension of $\mathcal{H}ol$ is $2n - 1$ and by the fact that the dimension of $\mathcal{H}_{\mathcal{H}ol}$ is n, we can conclude that the dimension of $\mathcal{V}_{\mathcal{H}ol} = n - 1$.

Assume that $F = c\mathcal{P}$, then $\mathcal{P}$ is holonomy invariant 1-homogenous function. From Proposition 3.4, we have $\mathcal{V}_{\mathcal{H}ol} \subseteq \mathcal{V}_{\mathcal{P}}$. Since the dimension of both spaces is $n - 1$, we get their equality.

Conversely, assume that $\mathcal{V}_{\mathcal{H}ol} = \mathcal{V}_{\mathcal{P}}$, then

$$d_{v_{\mathcal{P}}} F = 0 \Longrightarrow d_v F - \frac{d_v \mathcal{P}}{\mathcal{P}} d_C F = 0.$$

Since $d_C F = F$, then we have

$$d_v F - \frac{d_v \mathcal{P}}{\mathcal{P}} F = 0 \Longrightarrow \frac{d_v F}{F} = \frac{d_v \mathcal{P}}{\mathcal{P}}.$$

Then, there exists a function $a(x)$ on M such that $F = e^{a(x)} \mathcal{P}$. Now, since $\mathcal{P}$ is S-invariant, then $\mathcal{L}_S \mathcal{P} = 0$ and also $\mathcal{L}_S F = 0$; therefore, $\mathcal{L}_S a(x) = 0$. Locally, we obtain that

$$y^i \partial_i a(x) - 2G^i \dot{\partial}_i a(x) = 0 \quad \Longrightarrow \quad y^i \partial_i a(x) = 0.$$

By differentiation with respect to y^j, we get $\partial_j a(x) = 0$, that is $a(x)$ is constant function. Hence, we get $F = c\mathcal{P}$. □

Corollary 3.7. *Let (M, F) be a Finsler manifold with isotropic non-vanishing curvature. Then, for any 1-homogeneous S-invariant function $\mathcal{P}$, we have $\mathcal{V}_{\mathcal{H}ol} = \mathcal{V}_{\mathcal{P}}$ if and only if $F = c\mathcal{P}$, where c is a non-zero constant.*

Proof. In the case where the Finsler manifold has a non-vanishing isotropic curvature, then by [8], the metrizability freedom of its geodesic spray is 1. Therefore, the result follows by Theorem 3.6. □

The next theorem characterizes $\mathcal{V}_{\mathcal{H}ol}$ and therefore $\mathcal{H}ol$ as the intersection of distributions associated with geodesic invariant functions:

Theorem 3.8. *Let S be a metrizable spray with regular holonomy distribution. Then, we have*

$$\mathcal{V}_{\mathcal{H}ol} = \bigcap_{F \in \mathcal{F}_S} \mathcal{V}_F. \tag{3.10}$$

Proof. Let us assume that S is a metrizable spray with regular holonomy distribution on an n-dimensional manifold M, and its metric freedom is m_S (≥ 1). According to [8, Theorem 4.4], we have $\operatorname{codim}(\mathcal{H}ol) = m_S$, or equivalently,

$$\dim(\mathcal{H}ol) = 2n - m_S, \tag{3.11}$$

and at the neighborhood of any $(x, y) \in TM$, there exists a set $\left\{E_1, \dots E_{m_S}\right\}$ of energy functions associated with S such that any energy function of S can be locally written as a functional combination of $E_1, \dots E_{m_S}$. It follows that the corresponding Finsler functions $\left\{F_1, \dots F_{m_S}\right\}$ are functionally independent, and locally generating the set of Finsler functions of S, that is, every Finsler function F of S can be written as a functional combination

$$F = \phi(F_1, \dots, F_{m_S})$$

with some 1-homogeneous function ϕ. It follows that

$$\bigcap_{F \in \mathcal{F}_S} \operatorname{Ker}(dF) = \bigcap_{\mu=1}^{m_S} \operatorname{Ker}(dF_\mu). \tag{3.12}$$

Since $\{F_1, \dots, F_{m_S}\}$ are functionally independent, their derivatives are linearly independent, therefore $\bigcap_{\mu=1}^{m_S} \operatorname{Ker}(dF_\mu)$ is characterized by m_S linearly independent equations in TTM. It follows that

$$\dim\left(\bigcap_{\mu=1}^{m_S} \operatorname{Ker}(dF_\mu)\right) = \dim(TTM) - m_S = 2n - m_S. \tag{3.13}$$

Moreover, the functions F_μ are all holonomy invariant functions; therefore, $\operatorname{Ker}(dF_\mu)$ contains the holonomy distribution for $\mu = 1, \dots, m_S$, and as a consequence, their intersection $\bigcap_{\mu=1}^{m_S} \operatorname{Ker}(dF_\mu)$ also contains $\mathcal{H}ol$. Since the dimension of the intersection (3.13) and the dimension of the holonomy distribution (3.11) are equal, we get

$$\mathcal{H}ol = \bigcap_{\mu=1}^{m_S} \operatorname{Ker}(dF_\mu). \tag{3.14}$$

Using the vertical projection for (3.14) we get

$$\begin{aligned} \mathcal{V}_{\mathcal{H}ol} = v\,(\mathcal{H}ol) &\overset{(3.14)}{=} v\left(\bigcap_{\mu=1}^{m_S} \operatorname{Ker}(dF_\mu)\right) \overset{(3.12)}{=} \\ &= v\left(\bigcap_{F \in \mathcal{F}_S} \operatorname{Ker}(dF)\right) = \bigcap_{F \in \mathcal{F}_S} v\,(\operatorname{Ker}(dF)) = \bigcap_{F \in \mathcal{F}_S} \mathcal{V}_F \end{aligned}$$

showing the statement of the theorem. □

Corollary 3.9. *Let S be a metrizable spray by a Finsler function F. Then, $\mathcal{V}_{\mathcal{H}ol} = \mathcal{V}_F$ if and only if the metrizability freedom of S is $m_S = 1$.*

Theorem 3.10. *Let F be a Finsler function and S its geodesic spray. Then, if $\mathcal{P}$ is a 1-homogeneous nontrivial $\mathcal{V}_F$-invariant function, then it is regular. Moreover, if $\mathcal{P}$ is S-invariant, then $\mathcal{P} = cF$ with some constant $c \in \mathbb{R}$.*

We remark that the theorem shows that the S-invariant and $\mathcal{V}_F$-invariant properties are essentially characterizing the Finsler function associated with S.

Proof. Let $\mathcal{P}$ be a 1-homogeneous $\mathcal{V}_F$-invariant function. It follows that it satisfies the the system

$$d_X\mathcal{P} = 0, \quad \forall X \in \mathcal{V}_F.$$

Then, we have

$$d_{v_F}\mathcal{P} = d_v\mathcal{P} - \frac{d_vF}{F}\mathcal{P} = 0 \Longrightarrow \frac{d_vF}{F} = \frac{d_v\mathcal{P}}{\mathcal{P}}.$$

Then, there exists a function $a(x)$ on M such that $F = e^{a(x)}\mathcal{P}$. Then $\mathcal{P} = e^{-a(x)}F$, and hence $\mathcal{P}$ inherits its regularity from the Finsler function F.

Now, assume that $\mathcal{P}$ is S-invariant; then, we have $\mathcal{L}_S\mathcal{P} = 0$ and using the fact that $\mathcal{L}_SF = 0$, we have

$$\mathcal{L}_SF = \mathcal{L}_Se^{a(x)}\mathcal{P} = e^{a(x)}\mathcal{P}\mathcal{L}_Sa(x) = 0.$$

Then, we obtain that $y^i\partial_ia(x) = 0$. But by differentiating with respect to the y^j variable, we get $\partial_ja(x) = 0$. That is $a(x) = const$. Consequently, we get $F = c\mathcal{P}$.

□

4. Applications to the Landsberg surfaces

Definition 4.1. *A Finsler metric F on a manifold M is called a Berwald metric, if in any standard local coordinate system in $\mathcal{T}M$ the connection coefficients $G^i_j(x, y)$ are linear. A Finsler metric F is called Landsberg metric if Landsberg tensor with the components $L_{ijk} = -\frac{1}{2}FG^h_{ijk}\frac{\partial F}{\partial y^h}$ is identically zero.*

The Berwald- and Landsberg-type Finsler metrics are the most important particular cases in Finler geometry. For Berwald metrics, the associated canonical connection is linear; for Landsberg metrics the parallel transport with respect to the canonical connection preserves the metric [1]. It is well known that all Berwald-type Finsler metrics are also Landsbergian, but there is the long-open, so-called unicorn problem: Is there a Landsberg metric that is not Berwald? In higher dimensions ($n \geq 3$), there are non-regular Landsberg metrics that are not Berwladian; for more details, we refer to [7, 17]. In dimension two, L. Zhou [19] investigated a class of Landsberg surfaces and claimed that this class is not Bewaldian. Later, in [10], it was shown that the class is, in fact, Berwaldian. Up to the best of our knowledge, there is no example of non-Berwaldian Landsberg surfaces.

A Finsler function F with the geodesic spray S is said to be of scalar flag curvature if there exists a function $K \in C^\infty(\mathcal{T}M)$ such that the Jacobi endomorphism Φ of the geodesic spray S is given by

$$\Phi = K(F^2J - Fd_JF \otimes C). \tag{4.1}$$

Since the Jacobi endomorphism Φ of any Finsler surface is in the above form, then it is clear that all Finsler surfaces are of scalar flag curvature $K(x, y)$. Also, since the curvature R of a spray vanishes if and only if the Jacobi endomorphism vanishes, then the curvature of any Finsler surface vanishes if and only if K vanishes.

Whenever the scalar curvature K of the Finsler surface is non-vanishing, we will use the so-called Berwald frame, introduced by Berwald in [6]: It is a frame on $\mathcal{T}M$ canonically associated with a 2-dimensional Finsler manifold and used to investigate projectively flat 2-dimensional Finsler manifolds. We note that when the scalar curvature vanishes, the Berwald frame is not defined. For more details, we refer, for instance, to [18].

Lemma 4.2. *[2] Let (M, F) be a Finslerian surface with the geodesic spray S and of flag curvature $K \neq 0$. Then, the Berwald frame $\{S, H, C, V\}$ satisfies $JH = V$,*

$$[S, H] = KV, \tag{4.2a}$$

$$[S, V] = -H, \tag{4.2b}$$

$$[H, V] = S + IH + S(I)V, \tag{4.2c}$$

and

$$H(F) = V(F) = 0. \tag{4.3}$$

Moreover, the Bianchi's identity is given by [14, Proposition 1.4]

$$S^2(I) + V(K) + IK = 0, \tag{4.4}$$

where K is the flag curvature and I is the main scalar of (M, F).

One can characterize the Berwald- and Landsberg-type Finler metrics in terms of the main scalar:

Lemma 4.3. *[5] A Finsler surface (M, F) is*

1. *Landsberg if and only if $S(I) = 0$.*
2. *Berwald if and only if $S(I) = 0$ and $H(I) = 0$.*

Proposition 4.4. *All Landsberg surfaces with basic flag curvature are either Riemannian or have vanishing flag curvature.*

Proof. Let (M, F) be a Landsberg surface with basic flag curvature, that is, $K = K(x)$ is a function on the manifold M. Then, $V(K) = 0$, and by using the fact that $S(I) = 0$ together with (4.4), we have

$$KI = 0.$$

Then, we have either $K = 0$ or $I = 0$ and this completes the proof. □

Proposition 4.5. *For any Landsberg surface (M, F) with non-vanishing curvature, we have*

$$\beta + I\,V(\beta) + H(I) + V^2(\beta) = 0, \tag{4.5}$$

where $\beta := \frac{S(K_0)}{K_0} - S\left(\int_0^t I(t)dt\right)$, $K_0 \in C^\infty(\mathcal{T}M)$, $V(K_0) = 0$, I is the main scalar of (M, F) and the integration here is taken with respect to V.

Proof. Assume that (M, F) is a Landsberg surface with non-vanishing K. We work on a neighborhood of a point $(x_0, y_0) \in \mathcal{T}M$ where F is regular. Then, from Lemma 4.3, we get that $S(I) = 0$ and hence $S^2(I) = S(S(I)) = 0$. Then, (4.4) has the form

$$V(K) = -IK. \tag{4.6}$$

Since $K \neq 0$, then we can write

$$\frac{V(K)}{K} = -I.$$

Using integration as in [16] we obtain

$$K = K_0 \exp\left(-\int_0^t I(t)dt\right), \tag{4.7}$$

where $K_0 \in C^\infty(\mathcal{T}M)$ and $V(K_0) = 0$. But since K is homogeneous of degree 0 and by the fact that $[C, V] = 0$, then K_0 must be homogeneous of degree 0, that is, $C(K_0) = 0$. That is, $V(K_0) = 0$ and $C(K_0) = 0$, hence $K_0 = K_0(x)$.

Taking the fact that $S(I) = 0$, (4.7) implies

$$S(K) = S(K_0)\exp\left(-\int_0^t I(t)dt\right) + KS\left(-\int_0^t I(t)dt\right) = S(K_0)\frac{K}{K_0} + KS\left(-\int_0^t I(t)dt\right).$$

From which we can write

$$\frac{S(K)}{K} = \frac{S(K_0)}{K_0} + S\left(-\int_0^t I(t)dt\right). \tag{4.8}$$

Then, (4.8) can be written in the form

$$S(K) = \beta K, \tag{4.9}$$

where $\beta = \frac{S(K_0)}{K_0} + S\left(-\int_0^t I(t)dt\right)$. Applying S on (4.6) and using (4.9), we have

$$S(V(K)) = -IS(K) = -\beta IK. \tag{4.10}$$

Applying V on (4.9) and using (4.6), we have

$$V(S(K)) = V(\beta)K + \beta V(K) = V(\beta)K - \beta IK. \tag{4.11}$$

Now, by the property that $[V, S] = H$ (4.2b), (4.10), and (4.11) we have

$$H(K) = V(\beta)K. \tag{4.12}$$

From which, together with (4.6), we get

$$V(H(K)) = V^2(\beta)K + V(\beta)V(K) = V^2(\beta)K - IK\,V(\beta). \tag{4.13}$$

$$H(V(K)) = -H(I)K - IH(K) = -H(I)K - IK\,V(\beta). \tag{4.14}$$

Since $[H, V]K = H(V(K)) - V(H(K))$ then by (4.2c), (4.13), and (4.14), we have

$$S(K) + I\,H(K) = -K\,H(I) - K\,V^2(\beta)$$

from which, together with the fact that $K \neq 0$, and by (4.9), (4.12), we have

$$\beta + I\,V(\beta) + H(I) + V^2(\beta) = 0.$$

This completes the proof. □

As a consequence of the above proposition, we have the following result, which is obtained by [13] and [18], proved in a different way.

Theorem 4.6. *Let (M, F) be a Landsberg surface with non-zero flag curvature. If the flag curvature is S-invariant, then the surface is Riemannian.*

Proof. Let (M, F) be a Landsberg surface with non-vanishing flag curvature K and the property that $S(K) = 0$. Then, by (4.9), we get that $\beta = 0$ and therefore $V(\beta) = V^2(\beta) = 0$. Now, by (4.5), we obtain that $H(I) = 0$ and the surface is Berwaldian. Moreover, by (4.12), we have $H(K) = 0$ and using the fact that $S(K) = 0$, (4.2a) implies

$$K\,V(K) = 0,$$

from which, together with Proposition 4.4, the result follows. □

Theorem 4.7. *Let (M, F) be a Landsberg surface with non-vanishing flag curvature K; then, K is S-invariant if and only if K is constant. In this case, F is Riemannian.*

Proof. Let (M, F) be a surface with non-vanishing flag curvature K. It is obvious that if K is constant, then $S(K) = 0$ and hence K is S-invariant. Now, assume that K is S-invariant, that is, $S(K) = 0$. By (4.9), $\beta = 0$ and then by (4.12) we get that $H(K) = 0$. Since $[S, H] = KV$, then $KV(K) = S(H(K)) - H(S(K)) = 0$, and hence $V(K) = 0$ since $K \neq 0$. Moreover, K is zero homogeneous in y, then $C(K) = 0$. Therefore, we have

$$S(K) = 0, \quad H(K) = 0, \quad V(K) = 0, \quad C(K) = 0$$

which implies that K is constant. Then, F is Riemnnian by Theorem 4.6. □

A smooth function f on $\mathcal{T}M$ is said to be H-invariant if $H(f) = 0$. Let's end this work by the following result.

Theorem 4.8. *Let (M, F) be a Berwald surface with non-vanishing flag curvature. Then, the flag curvature K is H-invariant if and only if K is constant.*

Proof. Let (M, F) be a Berwald surface. If K is constant, then it is clear that $H(K) = 0$ and hence it is H-invariant. Now, assume that $H(K) = 0$. If $K = 0$, then the proof is done. If $K \neq 0$, then by (4.12), $V(\beta) = 0$. Since the surface is Berwaldian, then $H(I) = 0$. Therefore, by (4.5), $\beta = 0$ and by (4.9), we have $S(K) = 0$. Using (4.2a), we get that $V(K) = 0$ since $K \neq 0$. Since $C(K) = 0$, we have

$$S(K) = 0, \quad H(K) = 0, \quad V(K) = 0, \quad C(K) = 0$$

which means that K is constant. □

5. Conclusions

In this work, we have investigated the concept of geodesically invariant functions (or, equivalently, S-invariant functions for a given spray S) and some of its geometric consequences. For a given S-invariant function $\mathcal{P}$ and homogeneous of degree $k \neq 0$, we manged to express the horizontal and vertical subbundles as a direct sum of associated distributions depending on the function $\mathcal{P}$. Moreover, we study the relationship between the holonomy distribution and the kernel distribution of $\mathcal{P}$. Also, we pay some attentions to the role of the metrizability freedom and its effect on the geometry of an S-invariant function. Finally, as an application, we focus on the Berwald- and Landsberg-type surfaces.

Author contributions

Salah G. Elgendi and Zoltán Muzsnay: Conceptualization, methodology, validation, writing-original draft, writing-review & editing. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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