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Research article

Spectral parameter power series method for Kurzweil–Henstock integrable functions

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Abstract: In this paper, the convergence of the spectral parameter power series method, proposed by Kravchenko, is performed for the Sturm–Liouville equation with Kurzweil–Henstock integrable coefficients. Numerical simulations of some examples are also presented to validate the performance of the method.

Keywords: Kurzweil–Henstock integral; Sturm–Liouville equation; spectral parameter power series
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1. Introduction

The spectral parameter power series (SPPS) method, introduced in [1], expresses the solution of the Sturm–Liouville equation

$$(\rho y')' + qy = \lambda y$$

as a power series with respect to the spectral parameter λ, where the coefficients are given in terms of a particular solution of the homogeneous equation

$$(\rho y')' + qy = 0.$$

In [2], the SPPS method was used for solving spectral problems for Sturm–Liouville equations. This method is an important and efficient tool for solving a variety of problems involving Sturm–Liouville equations. In most publications devoted to the SPPS method, the coefficients of the differential equations are assumed to be continuous. In [3], it is shown that the SPPS method is valid for Sturm–Liouville equations with coefficients in the Lebesgue space $L([a, b])$.

There are problems that are described by Sturm–Liouville differential equations with highly oscillatory coefficients, and the Lebesgue integral is not enough to integrate some coefficients of this type. The Kurzweil–Henstock integral is more general than the Lebesgue integral, and it is well known that it integrates highly oscillating functions, so in this work we study the SPPS method for solving spectral problems for Sturm–Liouville equations with coefficients in the space of Kurzweil–Henstock integrable functions.

2. Preliminaries

The Kurzweil–Henstock integral was discovered independently by J. Kurzweil in the context of differential equations and R. Henstock, who made a systematic study. It is an integral whose definition is as simple as the Riemann integral but is more general than the Lebesgue integral. It has good convergence criteria and the advantage that it is not necessary to introduce improper integrals. A particular difference with the Lebesgue integral is that it is not an absolute integral.

We denote by $\mathrm{KH}([a,b])$ the space of Kurzweil–Henstock integral functions. The Alexiewicz seminorm for this space is given by

$$\|f\|_{[a,b]} = \sup\left\{\left|\int_c^d f\right| : [c,d] \subseteq [a,b]\right\}.$$

The space $\mathrm{KH}([a,b])$ with the seminorm $\|\cdot\|_{[a,b]}$ is not complete, contrary to what happens with the Lebesgue space $\mathrm{L}([a,b])$ with its norm $\|f\|_1 = \int_a^b |f|$. The space of all functions of bounded variation on $[a,b]$ is denoted by $\mathrm{BV}([a,b])$, and the variation of a function $\varphi \in \mathrm{BV}([a,b])$ is denoted by $V_{[a,b]}\varphi$. It is well known that the multipliers of KH integrable functions are functions of bounded variation, i.e., if $f \in \mathrm{KH}([a,b])$ and $g \in \mathrm{BV}([a,b])$, then

$$fg \in \mathrm{KH}([a,b]). \tag{2.1}$$

Moreover, even though the integral is not absolute, we have an estimate for the integral of this product:

$$\left|\int_a^b fg\right| \leq \left|\int_a^b f\right| \inf_{[a,b]} |g| + \|f\|_{[a,b]} V_{[a,b]} g. \tag{2.2}$$

This is a Holder-type inequality for the Kurzweil–Henstock integral. See [4, Lemma 24].

A useful result, given in [5, Theorem 7.5], which we will use later, says that if $f \in \mathrm{L}([a,b])$, then F, defined by $F(x) = \int_a^x f$, is of bounded variation on $[a,b]$, and

$$V_{[a,b]}F = \int_a^b |f|. \tag{2.3}$$

The space of absolutely continuous functions (respectively, generalized absolutely continuous functions in the restricted sense) on $[a,b]$ is denoted by $\mathrm{AC}([a,b])$, respectively, by $\mathrm{ACG}_*([a,b])$. See [6]. In these spaces, a fundamental theorem of calculus is stated in its general form.

Theorem 2.1. *[6, Fundamental Theorem of Calculus] Let $f, F : [a,b] \to \mathbb{C}$ be functions, and let $x_0 \in [a,b]$.*

a) If $f \in \mathrm{KH}([a,b])$ (resp. $f \in \mathrm{L}([a,b])$) and $F(x) = \int_{x_0}^{x} f$ for all $x \in [a,b]$, then $F \in \mathrm{ACG}_*([a,b])$, (resp. $F \in \mathrm{AC}([a,b])$) and $F' = f$ almost everywhere on $[a,b]$. In particular, if f is continuous at $x \in [a,b]$, then $F'(x) = f(x)$.
b) $F \in \mathrm{ACG}_*([a,b])$ (resp. $F \in \mathrm{AC}([a,b])$) if and only if F' exists almost everywhere on $[a,b]$ and $\int_{x_0}^{x} F' = F(x) - F(x_0)$ for all $x \in [a,b]$. In the case of absolutely continuous functions, the Lebesgue integral is used.

In the study of differential equations, Sobolev spaces become relevant. The classical Sobolev space is defined by

$$W^{1,1}([a,b]) = \left\{ u \in \mathrm{L}([a,b]) : \exists g \in \mathrm{L}([a,b]) \text{ such that } \int_a^b u\varphi' = -\int_a^b g\varphi, \forall \varphi \in C_c^1(a,b) \right\}, \tag{2.4}$$

where $C_c^1(a,b)$ is the space of continuously differentiable functions on (a,b) with compact support in (a,b). In [7], the authors introduce a generalization of this space using the KH-integral; this space is known as the KH-Sobolev space, and it is defined by

$$W_{\mathrm{KH}}([a,b]) = \left\{ u \in \mathrm{KH}([a,b]) : \exists g \in \mathrm{KH}([a,b]) \text{ such that } \int_a^b u\varphi' = -\int_a^b g\varphi, \forall \varphi \in V \right\}, \tag{2.5}$$

where V is a suitable test function space (for additional details, see [7]). The function g given in (2.4) and (2.5) is known as the weak derivative of the function u and is denoted by $\dot{u}$. It is well known (see [7, Theorem 3.6]) that if $u = v$ a.e on $[a,b]$ and $v \in \mathrm{AC}([a,b])$ (resp. $v \in \mathrm{ACG}_*([a,b])$), then $u \in W^{1,1}([a,b])$ (resp. $u \in W_{\mathrm{KH}}([a,b])$). Furthermore, the weak derivative coincides pointwise with the classical derivative whenever it exists, i.e.,

$$\dot{u}(x) = v'(x), \tag{2.6}$$

for all x in which $v'(x)$ exists. For example, if u is defined by $u(x) = x\cos\left(\frac{\pi}{x}\right)$ for all $x \in (0,1]$ and $u(0) = 0$, then $u \in \mathrm{ACG}_*([0,1])$. Thus, $u \in W_{\mathrm{KH}}([0,1]) \setminus W^{1,1}([0,1])$, and $\dot{u}(x) = \cos\left(\frac{\pi}{x}\right) + \frac{\pi}{x}\sin\left(\frac{\pi}{x}\right)$ for all $x \in (0,1]$, and $\dot{u}(0) = 0$.

3. SPPS method for the Sturm–Liovuille equation

Blancarte et al. [3] study the SPPS method for functions that are Lebesgue integrable. In this section, the convergence of the SPPS method is performed for the Sturm–Liouville differential equation with Kurzweil–Henstock integrable coefficients. First, let us introduce some concepts that will be necessary to deal with this generalization.

Definition 3.1. Let $f, g : [a,b] \to \mathbb{C}$ be functions, and $t \in [a,b]$. If there exists U_t an open neighborhood in $[a,b]$ of $t \in [a,b]$, where

a) f is continuous, then we say that f is locally continuous at t;
b) f is of class C^1, then we say that f is locally of class C^1 at t;
c) $f(x) = g(x)$ for all $x \in U_t$, then we say that f is locally equal to g at t.

The set of points $x \in [a,b]$, where $\frac{1}{\rho}$ is locally continuous at x, is denoted by E_ρ. While the space of functions that are locally of class C^1 at every point of E_ρ is denoted by C^1_p.

Consider the Sturm–Liouville (S-L) equation

$$(\rho\dot{y})^{\cdot} + qy = \lambda ry \quad \text{a.e. on } [a,b]. \tag{3.1}$$

Through this paper, we assume that q and r are functions such that $q, r \in \mathrm{KH}([a,b])$ and ρ is a function such that $\frac{1}{\rho} \in \mathrm{L}([a,b])$. Also, we assume that $E_\rho \neq \emptyset$. Define the space

$$\mathcal{A} = \left\{ y \in \mathrm{AC}([a,b]) \cap C^1_\rho : \rho\dot{y} = g \text{ a.e. on } [a,b] \text{ for some } g \in \mathrm{ACG}_*([a,b]) \right\}. \tag{3.2}$$

Sánchez–Perales et al. in [8] use this space to guarantee the existence of the solutions of the Sturm–Liouville type differential equations with KH-integrable coefficients. It is clear that if $y \in \mathcal{A}$, then y' exists a.e. on $[a,b]$, especially at every point of E_ρ and $\dot{y} = y'$. Also, since there exists $g \in \mathrm{ACG}_*([a,b])$ such that $\rho\dot{y} = g$ a.e. on $[a,b]$, it follows that $(\rho\dot{y})^{\cdot} = g' \in \mathrm{KH}([a,b])$.

In Example 4.2, ρ is defined by $\rho(x) = \sqrt{\pi + x}$. Note that $\frac{1}{\rho} \in \mathrm{L}([-\pi, 0])$ and $E_\rho = (-\pi, 0]$. An example of an element of $\mathcal{A}$ is the function y defined by $y(x) = \sqrt[4]{(x+\pi)^3}$. Indeed, y is absolutely continuous on $[-\pi, 0]$ and y is locally of class C^1 at every point of $(-\pi, 0]$, so $y \in \mathrm{AC}([-\pi, 0]) \cap C^1_\rho$; moreover, $\rho(x)\dot{y}(x) = \frac{3}{4}\sqrt{x+\pi}$ for all $x \in (-\pi, 0]$, and this function is ACG_* on $[-\pi, 0]$.

By the form of the S-L Eq (3.1), we consider the differential operator $L : \mathcal{A} \to \mathrm{KH}([a,b])$ defined as $L[y] = (\rho\dot{y})^{\cdot} + qy$. The next lemma lets us rewrite the operator L in terms of a non-vanishing solution of the associated homogeneous equation of the S-L equation.

Lemma 3.2 (Polya factorization)**.** *Let $y_0 \in \mathcal{A}$ be a solution of the homogeneous equation $L[y] = 0$ a.e. on $[a,b]$, with $y_0(x) \neq 0$ for all $x \in [a,b]$. Then,*

$$L[y] = \frac{1}{y_0}\left[\rho y_0^2 \left(\frac{y}{y_0}\right)^{\cdot}\right]^{\cdot} \quad \text{a.e. on } [a,b].$$

Proof. Let $y \in \mathcal{A}$. Then, $y, y_0, \rho\dot{y}, \rho\dot{y}_0$ are derivable in the weak sense. Since y_0 is continuous and does not vanish on $[a,b]$, there exists $\alpha > 0$ such that $\alpha \leq |y_0(x)|$ for all $x \in [a,b]$. Therefore, $\frac{y}{y_0}$ also has weak derivative and

$$\left(\frac{y}{y_0}\right)^{\cdot}(x) = \frac{\dot{y}(x)y_0(x) - y(x)\dot{y}_0(x)}{y_0^2(x)}, \qquad \text{for all } x \in [a,b].$$

Moreover, as $L[y_0] = 0$ a.e. on $[a,b]$, there exists a set $B \subset [a,b]$ with measure zero such that for every $x \in [a,b] \setminus B$, $L[y_0](x) = 0$. Let $x \in [a,b] \setminus B$, then

$$\begin{aligned}
\frac{1}{y_0(x)}\left[\rho(x)y_0^2(x)\left(\frac{y}{y_0}\right)^{\cdot}(x)\right]^{\cdot} &= \frac{1}{y_0(x)}\left[\rho(x)y_0^2(x)\left(\frac{\dot{y}(x)y_0(x) - y(x)\dot{y}_0(x)}{y_0^2(x)}\right)\right]^{\cdot} \\
&= \frac{1}{y_0(x)}\Big[(\rho\dot{y})(x)y_0(x) - (\rho\dot{y}_0)(x)y(x)\Big]^{\cdot} = \frac{1}{y_0(x)}\Big[(\rho\dot{y})^{\cdot}(x)y_0(x) - (\rho\dot{y}_0)^{\cdot}(x)y(x)\Big] \\
&= \frac{1}{y_0(x)}\Big[\big((\rho\dot{y})^{\cdot}(x) + q(x)y(x)\big)y_0(x) - \big((\rho\dot{y}_0)^{\cdot}(x) + q(x)y_0(x)\big)y(x)\Big]
\end{aligned}$$

$$= \frac{1}{y_0(x)}\Big[L[y](x)y_0(x) - L[y_0](x)y(x)\Big] = \frac{1}{y_0(x)}\Big[L[y](x)y_0(x) - 0 \cdot y(x)\Big]$$
$$= L[y](x).$$

□

Now, we will define a family of functions that allow us to write the representation of the general solution of the S-L Eq (3.1) as a spectral power series. Let $x_0 \in [a,b]$, and $y_0 \in \mathcal{A}$ be such that $y_0(x) \neq 0$ for all $x \in [a,b]$. Define

$$\tilde{X}^{(0)} \equiv 1, \qquad \tilde{X}^{(n)}(x) = \begin{cases} \displaystyle\int_{x_0}^{x} \tilde{X}^{(n-1)}(s)r(s)y_0^2(s)ds, & \text{if } n \text{ es odd;} \\ \displaystyle\int_{x_0}^{x} \tilde{X}^{(n-1)}(s)\frac{ds}{\rho(s)y_0^2(s)}, & \text{if } n \text{ es even;} \end{cases} \tag{3.3}$$

$$X^{(0)} \equiv 1, \qquad X^{(n)}(x) = \begin{cases} \displaystyle\int_{x_0}^{x} X^{(n-1)}(s)\frac{ds}{\rho(s)y_0^2(s)}, & \text{if } n \text{ is odd;} \\ \displaystyle\int_{x_0}^{x} X^{(n-1)}(s)r(s)y_0^2(s)ds, & \text{if } n \text{ is even.} \end{cases} \tag{3.4}$$

These functions are bound by the coefficients of the series of an exponential, as illustrated by the following proposition:

Proposition 3.3. *Let $x_0 \in [a,b]$ and $y_0 \in \mathcal{A}$ be such that $y_0(x) \neq 0$ for all $x \in [a,b]$. Then, for each $n \in \mathbb{N} \cup \{0\}$,*

$$\tilde{X}^{(2n+1)} \in \mathrm{ACG}_*([a,b]) \quad \text{and} \quad \tilde{X}^{(2n)} \in \mathrm{AC}([a,b]), \tag{3.5}$$

$$X^{(2n+1)} \in \mathrm{AC}([a,b]) \quad \text{and} \quad X^{(2n)} \in \mathrm{ACG}_*([a,b]). \tag{3.6}$$

Furthermore, the following inequalities are satisfied:

$$\left|\tilde{X}^{(2n)}(x)\right| \leq \|ry_0^2\|_{[a,b]}^{n}\frac{\Big(\mathrm{sgn}(x-x_0)Q(x)\Big)^n}{n!}, \qquad \left|\tilde{X}^{(2n+1)}(x)\right| \leq \|ry_0^2\|_{[a,b]}^{n+1}\frac{\Big(\mathrm{sgn}(x-x_0)Q(x)\Big)^n}{n!}, \tag{3.7}$$

$$\left|X^{(2n)}(x)\right| \leq \|ry_0^2\|_{[a,b]}^{n}\frac{\Big(\mathrm{sgn}(x-x_0)Q(x)\Big)^n}{n!}, \ \& \ \left|X^{(2n+1)}(x)\right| \leq \|ry_0^2\|_{[a,b]}^{n}\frac{\Big(\mathrm{sgn}(x-x_0)Q(x)\Big)^{n+1}}{(n+1)!}, \tag{3.8}$$

for all $x \in [a,b]$, where $Q(x) = \int_{x_0}^{x} \frac{ds}{|\rho(s)y_0^2(s)|}$.

Proof. We will only show (3.5) and (3.7). Since $y_0 \in \mathcal{A}$, it follows that $y_0 \in \mathrm{AC}([a,b]) \subseteq \mathrm{BV}([a,b])$. Then by (2.1), $y_0^2 r \in \mathrm{KH}([a,b])$, and hence, by Theorem 2.1, $\tilde{X}^{(1)} = \int_{x_0}^{(\cdot)} y_0^2(s)r(s)ds \in \mathrm{ACG}_*([a,b])$. Now, suppose that $\tilde{X}^{(2n+1)} \in \mathrm{ACG}_*([a,b])$. By hypothesis $\frac{1}{\rho} \in \mathrm{L}([a,b])$, then $\frac{\tilde{X}^{(2n+1)}}{\rho y_0^2} \in \mathrm{L}([a,b])$, which implies by Theorem 2.1 that $\tilde{X}^{(2n+2)} = \int_{x_0}^{(\cdot)} \frac{\tilde{X}^{(2n+1)}(s)}{\rho(s)y_0^2(s)}ds \in \mathrm{AC}([a,b])$. Thus, $y_0^2\tilde{X}^{(2n+2)} \in \mathrm{BV}([a,b])$ and so by (2.1), $\tilde{X}^{(2n+2)}y_0^2r \in \mathrm{KH}([a,b])$. Hence, by Theorem 2.1, $\tilde{X}^{(2n+2+1)} = \int_{x_0}^{(\cdot)} \tilde{X}^{(2n+2)}(s)y_0^2(s)r(s)ds \in \mathrm{ACG}_*([a,b])$. By induction (3.5) holds.

To prove (3.7), first note that for each $x \in [a, x_0)$ and $n \in \mathbb{N}$,

$$\left|\tilde{X}^{(2n+1)}(x)\right| \leq \|ry_0^2\|_{[a,b]} \int_x^{x_0} \left|\frac{\tilde{X}^{(2n-1)}(s)}{\rho(s)y_0^2(s)}\right| ds. \tag{3.9}$$

Indeed, by the inequality (2.2),

$$\begin{aligned}\left|\tilde{X}^{(2n+1)}(x)\right| &= \left|\int_{x_0}^{x} \tilde{X}^{(2n)}(s)r(s)y_0^2(s)ds\right| = \left|\int_x^{x_0} \tilde{X}^{(2n)}(s)r(s)y_0^2(s)ds\right| \\ &\leq \left|\int_x^{x_0} r(s)y_0^2(s)ds\right| \inf_{t\in[x,x_0]}\left|\tilde{X}^{(2n)}(t)\right| + \|ry_0^2\|_{[x,x_0]}V_{[x,x_0]}\tilde{X}^{(2n)} = \|ry_0^2\|_{[x,x_0]}V_{[x,x_0]}\tilde{X}^{(2n)}. \end{aligned}\tag{3.10}$$

Since $\tilde{X}^{(2n)}(x) = \int_x^{x_0} -\frac{\tilde{X}^{(2n-1)}(s)}{\rho(s)y_0^2(s)}\, ds$, it follows from the equality (2.3) that

$$V_{[x,x_0]}\tilde{X}^{(2n)} = \int_x^{x_0} \left|\frac{\tilde{X}^{(2n-1)}(s)}{\rho(s)y_0^2(s)}\right| ds. \tag{3.11}$$

Substituting (3.11) into Eq (3.10), we obtain that the inequality (3.9) holds.

Now, we show again, by induction, that

$$\left|\tilde{X}^{(2n)}(x)\right| \leq \|ry_0^2\|_{[a,b]}^n \frac{\left(-Q(x)\right)^n}{n!}, \quad \left|\tilde{X}^{(2n+1)}(x)\right| \leq \|ry_0^2\|_{[a,b]}^{n+1} \frac{\left(-Q(x)\right)^n}{n!}, \ \ \forall\, x < x_0. \tag{3.12}$$

For $n = 0$, we have that for every $x \in [a, x_0)$,

$$\left|\tilde{X}^{(1)}(x)\right| = \left|\int_{x_0}^{x} r(s)y_0^2(s)ds\right| \leq \|ry_0^2\|_{[a,b]}. \tag{3.13}$$

By the induction hypothesis, we assume that the second inequality given in (3.12) is valid for the natural n. Let $x < x_0$ and observe that

$$\begin{aligned}\left|\tilde{X}^{(2n+2)}(x)\right| &= \left|\int_x^{x_0} \frac{\tilde{X}^{(2n+1)}(s)}{\rho(s)y_0^2(s)}\right| ds \leq \int_x^{x_0} \left|\frac{\tilde{X}^{(2n+1)}(s)}{\rho(s)y_0^2(s)}\right| ds \leq \int_x^{x_0} \frac{\|ry_0^2\|_{[a,b]}^{n+1}}{n!} \frac{\left(-Q(s)\right)^n}{|\rho(s)y_0^2(s)|} ds \\ &= (-1)^n \|ry_0^2\|_{[a,b]}^{n+1} \int_x^{x_0} \frac{Q^n(s)}{n!} \frac{1}{|\rho(s)y_0^2(s)|} ds. \end{aligned}\tag{3.14}$$

On the other hand, $\left[\frac{Q^{n+1}(s)}{n!(n+1)}\right]' = \frac{(n+1)Q^n(s)Q'(s)}{n!(n+1)} = \frac{Q^n(s)}{n!}\frac{1}{|\rho(s)y_0^2(s)|}$ for almost all $s \in [x, x_0]$. Therefore,

$$\int_x^{x_0} \frac{Q^n(s)}{n!} \frac{1}{|\rho(s)y_0^2(s)|} ds = -\frac{Q^{n+1}(x)}{(n+1)!}.$$

Consequently,

$$\left|\tilde{X}^{(2(n+1))}(x)\right| \leq \int_x^{x_0} \left|\frac{\tilde{X}^{(2n+1)}(s)}{\rho(s)y_0^2(s)}\right| ds \leq (-1)^{n+1} \|ry_0^2\|_{[a,b]}^{n+1} \frac{Q^{n+1}(x)}{(n+1)!}. \tag{3.15}$$

This proves the first inequality in (3.12) for the natural $n+1$. The second inequality in (3.12) for the natural $n+1$ is obtained by (3.9) and (3.15):

$$\left|\tilde{X}^{(2(n+1)+1)}(x)\right| \leq \|ry_0^2\|_{[a,b]} \int_x^{x_0} \left|\frac{\tilde{X}^{(2n+1)}(s)}{\rho(s)y_0^2(s)}\right| ds \leq (-1)^{n+1}\|ry_0^2\|_{[a,b]}^{n+1+1}\frac{Q^{n+1}(x)}{(n+1)!}. \quad (3.16)$$

This completes the induction process. Estimates for the case where $x > x_0$ and for the function $X^{(n)}$, with $n \in \mathbb{N}$, are shown similarly. □

The next result shows us that we can bound the functions $\tilde{X}^{(n)}$ and $X^{(n)}$ at any point by constants.

Corollary 3.4. *Under the conditions of Proposition 3.3, the functions* $\tilde{X}^{(2n)}, \tilde{X}^{(2n+1)}, X^{(2n)}$*, and* $X^{(2n+1)}$ *are bounded by the following constants:*

$$\left|\tilde{X}^{(2n)}(x)\right| \leq \frac{C_1^n C_2^n}{n!}, \quad \left|\tilde{X}^{(2n+1)}(x)\right| \leq \frac{C_1^{n+1} C_2^n}{n!}, \quad \left|X^{(2n)}(x)\right| \leq \frac{C_1^n C_2^n}{n!}, \quad \& \quad \left|X^{(2n+1)}(x)\right| \leq \frac{C_1^n C_2^{n+1}}{(n+1)!}, \quad (3.17)$$

where $C_1 = \left\|ry_0^2\right\|_{[a,b]}$ and $C_2 = \left\|\frac{1}{\rho y_0^2}\right\|_1$.

Proof. Without loss of generality, we assume that $a \leq x < x_0$. Then,

$$0 < -Q(x) = -\int_{x_0}^{x} \frac{ds}{|\rho(s)y_0^2(s)|} = \int_x^{x_0} \frac{ds}{|\rho(s)y_0^2(s)|} \leq \int_a^b \frac{ds}{|\rho(s)y_0^2(s)|} = \left\|\frac{1}{\rho y_0^2}\right\|_1 = C_2.$$

Thus,

$$0 < [-Q(x)]^n \leq C_2^n \quad (3.18)$$

for all $n \in \mathbb{N} \cup \{0\}$. By Proposition 3.3 and the inequalities in (3.18), we have that

$$\left|\tilde{X}^{(2n)}(x)\right| \leq \frac{\|ry_0^2\|_{[a,b]}^n(-Q(x))^n}{n!} \leq \frac{C_1^n C_2^n}{n!}, \qquad \left|\tilde{X}^{(2n+1)}(x)\right| \leq \frac{\|ry_0^2\|_{[a,b]}^{n+1}(-Q(x))^n}{n!} \leq \frac{C_1^{n+1} C_2^n}{n!},$$

$$\left|X^{(2n)}(x)\right| \leq \frac{\|ry_0^2\|_{[a,b]}^n(-Q(x))^n}{n!} \leq \frac{C_1^n C_2^n}{n!}, \quad \& \quad \left|X^{(2n+1)}(x)\right| \leq \frac{\|ry_0^2\|_{[a,b]}^n(-Q(x))^{n+1}}{n!} \leq \frac{C_1^n C_2^{n+1}}{n!}.$$

□

In the following result, we define several functions (related to the representation of the solution of the S-L Eq (3.1) as a power spectral series), the space to which these functions belong, and some aspects related to their derivative.

Theorem 3.5. *Let* $y_0 \in \mathcal{A}$ *be such that* $y_0(x) \neq 0$ *for all* $x \in [a,b]$*. If* $\tilde{X}^{(n)}$ *and* $X^{(n)}$ *are the functions given in* (3.3)–(3.4)*, and* $u, v, w, z : [a,b] \to \mathbb{C}$ *are defined as*

$$u = \sum_{n=0}^{\infty} \lambda^n \tilde{X}^{(2n)}, \quad v = \sum_{n=0}^{\infty} \lambda^{n+1} \tilde{X}^{(2n+1)}, \quad w = \sum_{n=0}^{\infty} \lambda^n X^{(2n+1)}, \quad \& \quad z = \sum_{n=0}^{\infty} \lambda^n X^{(2n)}. \quad (3.19)$$

Then,

a) $u, w \in \mathrm{AC}([a,b])$, and

$$\dot{u} = \frac{v}{\rho y_0^2} \quad \& \quad \dot{w} = \frac{z}{\rho y_0^2} \quad \text{a.e. on } [a,b]. \tag{3.20}$$

Moreover, the equalities

$$\dot{u} = \sum_{k=0}^{\infty} \lambda^k \left[\tilde{X}^{(2k)}\right]^{\cdot} = \frac{v}{\rho y_0^2} \quad \text{and} \quad \dot{w} = \sum_{k=0}^{\infty} \lambda^k \left[X^{(2k+1)}\right]^{\cdot} = \frac{z}{\rho y_0^2}, \tag{3.21}$$

are locally satisfied at every point of E_ρ, and so, $u, w \in C^1_\rho$.

b) $v, z \in \mathrm{ACG}_*([a,b])$, and

$$\dot{v} = \sum_{n=0}^{\infty} \lambda^{(n+1)} \left[\tilde{X}^{(2n+1)}\right]^{\cdot} = \lambda r y_0^2 u \quad \text{and} \quad \dot{z} = \sum_{n=0}^{\infty} \lambda^n \left[X^{(2n)}\right]^{\cdot} = \lambda r y_0^2 w \tag{3.22}$$

a.e. on $[a,b]$.

Proof. (a) First, by Corollary 3.4, we have that $\sum_{n=0}^{\infty} \left|\lambda^n \tilde{X}^{(2n)}\right| \le \sum_{n=0}^{\infty} \frac{(|\lambda| C_1 C_2)^n}{n!} < \infty$. Note that the right-hand side of this inequality is the expansion in power series of the function $\exp(|\lambda| C_1 C_2)$. Then, by the Weiersstras M-test, we can actually see that the series $\sum_{n=0}^{\infty} \lambda^n \tilde{X}^{(2n)} \xrightarrow{u} u$. Similarly, this occurs with the functions v, w, and z.

The next part of this proof relies on [3, Proposition 3]. We have a sequence $(\lambda^n \tilde{X}^{(2n)})$ in $\mathrm{AC}([a,b])$ such that $\sum_{n=0}^{\infty} \lambda^n \tilde{X}^{(2n)}$ converges uniformly to u on $[a,b]$. Now, we show that $\sum_{n=0}^{\infty} \lambda^n \tilde{X}^{(2n)}$ converges to the function $\frac{v}{\rho y_0^2}$ in $\mathrm{L}([a,b])$ with the norm $\|\cdot\|_1$. By Theorem 2.1 and the definition of $X^{(2n)}$, we have that $\left(\tilde{X}^{(2n)}\right)^{\cdot} = \frac{\tilde{X}^{(2n-1)}}{\rho y_0^2}$ a.e. on $[a,b]$. Then,

$$\left\| \sum_{n=0}^{N} \left(\lambda^n \tilde{X}^{(2n)}\right)^{\cdot} - \frac{v}{\rho y_0^2} \right\|_1 = \int_a^b \left| \sum_{n=1}^{N} \lambda^n \frac{\tilde{X}^{(2n-1)}}{\rho y_0^2} - \frac{v}{\rho y_0^2} \right| = \int_a^b \left| \frac{1}{\rho y_0^2} \right| \cdot \left| \sum_{n=1}^{N} \lambda^n \tilde{X}^{(2n-1)} - v \right|.$$

$$\le \sup_{[a,b]} \left| \sum_{n'=0}^{N-1} \lambda^{n'+1} \tilde{X}^{(2n'+1)} - v \right| C_2 \to 0,$$

as $N \to \infty$. Thus, by [3, Proposition 3], there exists a function $F \in AC([a,b])$ such that $\sum_{n=0}^{\infty} \lambda^n \tilde{X}^{(2n)}$ converges uniformly to F and $\dot{F} = \frac{v}{\rho y_0^2}$ a.e. on $[a,b]$. But as mentioned earlier, $\sum_{n=0}^{\infty} \lambda^n \tilde{X}^{(2n)}$ converges uniformly to u, so by uniqueness of the limits, $u = F$, which means that $u \in AC([a,b])$ and $\dot{u} = \frac{v}{\rho y_0^2}$ a.e. on $[a,b]$. This proves one of the equalities in Eq (3.20).

Now, let $x \in E_\rho$, then the functions $\frac{1}{\rho}, \frac{\tilde{X}^{(2n-1)}}{\rho y_0^2}$ are locally continuous at x, which means that there exist $\delta_x > 0$ such that $\frac{1}{\rho}, \frac{\tilde{X}^{(2n-1)}}{\rho y_0^2}$ are continuous on $[x - \delta_x, x + \delta_x] \cap [a,b]$. Then, by Theorem 2.1,

$$\left[\tilde{X}^{(2n)}\right]^{\cdot}(s) = \frac{\tilde{X}^{(2n-1)}(s)}{\rho(s) y_0^2(s)} \quad \text{for all s} \in [x - \delta_x, x + \delta_x] \cap [a,b]. \tag{3.23}$$

Let $U_m = \sum_{n=0}^{m} \lambda^n \tilde{X}^{(2n)}$. Then, $U_m \to u$ over $[x - \delta_x, x + \delta_x] \cap [a,b]$, and $\dot{U}_m(s) = \sum_{n=1}^{m} \lambda^n \frac{\tilde{X}^{(2n-1)}(s)}{\rho(s) y_0^2(s)}$ for all $s \in [x - \delta_x, x + \delta_x] \cap [a,b]$. Observe that

$$\left| \dot{U}_m(s) - \frac{v(s)}{\rho(s) y_0^2(s)} \right| = \left| \sum_{n=1}^{m} \lambda^n \frac{\tilde{X}^{(2n-1)}(s)}{\rho(s) y_0^2(s)} - \frac{v(s)}{\rho(s) y_0^2(s)} \right| = \left| \frac{1}{\rho(s) y_0^2(s)} \right| \cdot \left| \sum_{n=1}^{m} \lambda^n \tilde{X}^{(2n-1)}(s) - v(s) \right|$$

$$\leq \max_{s\in[x-\delta_x,x+\delta_x]\cap[a,b]} \left|\frac{1}{\rho(s)y_0^2(s)}\right| \cdot \left|\sum_{n'=0}^{m-1} \lambda^{(n'+1)}\tilde{X}^{(2n'+1)}(s) - v(s)\right| \xrightarrow{u} 0 \tag{3.24}$$

as $m \to \infty$. Thus, $\dot{U}_m$ converges uniformly to $\frac{v}{\rho y_0^2}$ on $[x-\delta_x, x+\delta_x] \cap [a,b]$, i.e.,

$$\sum_{n=0}^{\infty} \lambda^n \Big[\tilde{X}^{(2n)}\Big]^{\cdot}(s) = \frac{v(s)}{\rho(s)y_0^2(s)} \tag{3.25}$$

for all $s \in [x-\delta_x, x+\delta_x] \cap [a,b]$. Consequently, by [9, Theorem 7.17] and (3.25),

$$\dot{u}(s) = \sum_{n=0}^{\infty} \lambda^n \Big[\tilde{X}^{(2n)}\Big]^{\cdot}(s) = \frac{v(s)}{\rho(s)y_0^2(s)} \tag{3.26}$$

for all $s \in [x-\delta_x, x+\delta_x] \cap [a,b]$. From (3.26) and being arbitrary $x \in E_\rho$, we obtain that $u \in C_\rho^1$.

(b) Define the functions V and Z by

$$V(x) = \int_{x_0}^{x} \lambda r y_0^2 u \qquad \text{and} \qquad Z(x) = \int_{x_0}^{x} \lambda r y_0^2 w.$$

Then, $V, Z \in ACG_*([a,b])$, $v = V$ and $z = Z + 1$. We will only show that $V = v$. Let $x \in [a, x_0)$. Note that $U_m \in \mathrm{BV}([x, x_0])$, $\dot{U}_m = \sum_{n=1}^{m} \lambda^n \frac{\tilde{X}^{(2n-1)}}{\rho y_0^2}$ a.e. on $[a,b]$, and $\dot{U}_m \in \mathrm{L}([a,b])$. Consequently, by (2.3) and from Corollary 3.4, we have that

$$\begin{aligned} V_{[x,x_0]}U_m &\leq \int_x^{x_0} \left|\dot{U}_m(s)\right| ds \leq \int_x^{x_0} \sum_{n=1}^{m} |\lambda^n| \left|\frac{\tilde{X}^{(2n-1)}(s)}{\rho(s)y_0^2(s)}\right| ds \\ &= \sum_{n=1}^{m} |\lambda|^n \int_x^{x_0} \left|\frac{\tilde{X}^{(2(n-1)+1)}(s)}{\rho(s)y_0^2(s)}\right| ds \leq \sum_{n=1}^{m} |\lambda|^n \int_x^{x_0} \frac{C_1^n C_2^{n-1}}{(n-1)!|\rho(s)y_0^2(s)|} ds \\ &\leq \sum_{n=1}^{m} |\lambda|^n \frac{C_1^n C_2^n}{(n-1)!} ds < \infty. \end{aligned} \tag{3.27}$$

Therefore, (U_m) is a uniformly bounded variation on $[x, x_0]$. Moreover, $U_m \xrightarrow{u} u$ on $[x, x_0]$ and $r y_0 \in \mathrm{KH}([x, x_0])$. Consequently, by [10, Corollary 3.2], it follows that

$$\int_x^{x_0} r(s)y_0^2(s) \sum_{n=0}^{m} \lambda^n \tilde{X}^{(2n)}(s) ds \to \int_x^{x_0} r(s)y_0^2(s)u(s) ds, \tag{3.28}$$

when $m \to \infty$. Therefore,

$$\begin{aligned} \left|v(x) - V(x)\right| &= \left|\lim_{m\to\infty} \sum_{n=0}^{m} \lambda^{(n+1)}\tilde{X}^{(2n+1)}(x) - \int_{x_0}^{x} \lambda r(s)y_0^2(s)u(s) ds\right| \\ &= |\lambda| \left|\lim_{m\to\infty} \sum_{n=0}^{m} \lambda^n \int_{x_0}^{x} \tilde{X}^{(2n)}(s) r(s) y_0^2(s) ds - \int_{x_0}^{x} r(s)y_0^2(s)u(s) ds\right| \end{aligned}$$

$$= |\lambda| \left| \lim_{m\to\infty} \int_x^{x_0} r(s)y_0^2(s) \sum_{n=0}^{m} \lambda^n \tilde{X}^{(2n)}(s)ds - \int_x^{x_0} r(s)y_0^2(s)u(s) \right| \to 0. \tag{3.29}$$

The same occurs for $x \geq x_0$. Thus, $v = V$ on $[a,b]$, and so $v \in ACG_*([a,b])$. By Theorem 2.1 (a),

$$\dot{v} = \lambda r y_0^2 u \tag{3.30}$$

a.e. on $[a,b]$. Now, let $V_m = \lambda^{(m+1)}\tilde{X}^{(2m+1)}$. Then, $\dot{V}_m = \lambda^{(m+1)}\tilde{X}^{(2m)} r y_0^2$ a.e. on $[a,b]$. For each $m \in \mathbb{N}$, take E_m a set of measure zero, such that for every $x \in [a,b] \setminus E_m$, $\dot{V}_m(x) = \lambda^{(m+1)}\tilde{X}^{(2m)}(x)r(x)y_0^2(x)$. Define $E = \bigcup_{m\in\mathbb{N}} E_m$, then $m(E) = 0$, and for every $x \in [a,b] \setminus E$,

$$\begin{aligned} \left| \sum_{n=0}^{m} \dot{V}_n(x) - \lambda r(x)y_0^2(x)u(x) \right| &= \left| \sum_{n=0}^{m} \lambda^{(n+1)}\tilde{X}^{(2n)}(x)r(x)y_0^2(x) - \lambda r(x)y_0^2(x)u(x) \right| \\ &= \left|\lambda\right| \cdot \left|r(x)y_0^2(x)\right| \cdot \left| \sum_{n=0}^{m} \lambda^n \tilde{X}^{(2n)}(x) - u(x) \right| \to 0, \end{aligned} \tag{3.31}$$

when $m \to \infty$. Thus,

$$\lambda r y_0^2 u = \sum_{n=0}^{\infty} \lambda^{(n+1)} \left[\tilde{X}^{(2n+1)}\right]^{\cdot} \tag{3.32}$$

a.e. on $[a,b]$. From Eqs (3.30) and (3.32), we conclude Eq (3.22). For the function z, the proof follows similarly. □

Theorem 3.6. *Let $x_0 \in E_\rho$. If $y_0 \in \mathcal{A}$ is a solution of the homogeneous equation*

$$(\rho\dot{y})^{\cdot} + qy = 0 \qquad \text{a.e. on } [a,b] \tag{3.33}$$

with $y_0(x) \neq 0$ for all $x \in [a,b]$. Then, the general solution of the equation

$$(\rho\dot{y})^{\cdot} + qy = \lambda r y \qquad \text{a.e. on } [a,b] \tag{3.34}$$

has the form

$$y = c_1 y_1 + c_2 y_2, \tag{3.35}$$

where c_1, c_2 are arbitrary constants, and

$$y_1 = y_0 \sum_{n=0}^{\infty} \lambda^n \tilde{X}^{(2n)} = y_0 u \qquad \text{and} \qquad y_2 = y_0 \sum_{n=0}^{\infty} \lambda^n X^{(2n+1)} = y_0 w. \tag{3.36}$$

Proof. Let $\mathcal{A}_*$ be the set of all solutions to Eq (3.34). This is a linear space, because

$$\mathcal{A}_* = \left\{ y \in \mathcal{A} : (\rho\dot{y})^{\cdot} + (q - \lambda r)y = 0 \right\}.$$

From conditions $x_0 \in E_\rho$, $q - \lambda r \in \mathrm{KH}([a,b])$, and $\frac{1}{\rho} \in \mathrm{L}([a,b])$, we have by [8, Corollary 3.3] that $\dim(\mathcal{A}_*) = 2$. On the other hand, $y_0 \in \mathrm{AC}([a,b]) \cap C_\rho^1$, and there exists $g_0 \in \mathrm{ACG}_*([a,b])$ such that

$$\rho\dot{y}_0 = g_0 \ \text{ a.e. on } [a,b].$$

By Theorem 3.5, $u \in \mathrm{AC}([a,b]) \cap C^1_\rho$, $\dot{u} = \frac{v}{\rho y_0^2}$ a.e. on $[a,b]$, and $v \in \mathrm{ACG}_*([a,b])$. Therefore, $y_1 = y_0 u \in \mathrm{AC}([a,b]) \cap C^1_\rho$, and

$$\rho \dot{y}_1 = \rho (y_0 u)^{\cdot} = \rho \left[\dot{y}_0 u + \frac{v}{\rho y_0^2} y_0 \right] = (\rho \dot{y}_0) u + \frac{v}{y_0} = g_0 u + \frac{v}{y_0}, \tag{3.37}$$

a.e. on $[a,b]$. Since $y_0(x) \neq 0$ for all $x \in [a,b]$, it follows that $\frac{1}{y_0} \in \mathrm{AC}([a,b])$, thus $g_0 u + \frac{v}{y_0} \in \mathrm{ACG}_*([a,b])$. Consequently, $y_1 \in \mathcal{A}$. In the same way, it is shown that $y_2 \in \mathcal{A}$. Now, as y_0 is a non-vanishing solution of the homogeneous equation, we can apply Lemma 3.2 and Theorem 3.5, and obtain that

$$L[y_1] = \frac{1}{y_0} \left[\rho y_0^2 \left(\frac{y_0 u}{y_0} \right)^{\cdot} \right]^{\cdot} = \frac{1}{y_0} \left[\rho y_0^2 \frac{v}{\rho y_0^2} \right]^{\cdot} = \frac{1}{y_0} \dot{v} = \lambda r y_0 u = \lambda r y_1, \tag{3.38}$$

a.e. on $[a,b]$. For y_2, the proof follows similarly. Thus, $y_1, y_2 \in \mathcal{A}_*$. Now, we will verify that y_1, y_2 are linearly independent. From the definitions of the functions $\tilde{X}^{(2n)}$ and $X^{(2n+1)}$ observe that

$$\begin{aligned}
y_1(x_0) &= y_0(x_0) u(x_0) = y_0(x_0) \sum_{n=0}^{\infty} \lambda^n \tilde{X}^{(2n)}(x_0) = y_0(x_0) \cdot 1 = y_0(x_0), \\
w(x_0) &= \sum_{n=0}^{\infty} \lambda^n X^{(2n+1)}(x_0) = 0, \\
y_2(x_0) &= y_0(x_0) w(x_0) = y_0(x_0) \cdot 0 = 0, \text{ and} \\
z(x_0) &= \sum_{n=0}^{\infty} \lambda^n X^{(2n)}(x_0) = 1.
\end{aligned}$$

Also, as $x_0 \in E_\rho$, then by Theorem 3.5, $\dot{w}(x_0) = \frac{z(x_0)}{\rho(x_0) y_0^2(x_0)}$. Therefore,

$$\dot{y}_2(x_0) = \Big(\dot{y}_0(x_0) w(x_0) + \dot{w}(x_0) y_0(x_0) \Big) = \frac{z(x_0)}{\rho(x_0) y_0^2(x_0)} y_0(x_0) = \frac{1}{\rho(x_0) y_0(x_0)}. \tag{3.39}$$

Then, the generalized Wronskian at x_0 is

$$[\rho W(y_1, y_2)](x_0) = \rho(x_0) \Big(y_1(x_0) \dot{y}_2(x_0) - y_2(x_0) \dot{y}_1(x_0) \Big) = \rho(x_0) \Big(y_0(x_0) \dot{y}_2(x_0) \Big) = 1 \tag{3.40}$$

Therefore, y_1 and y_2 are linearly independent. Moreover, since $\dim \mathcal{A}_* = 2$, then $\{y_1, y_2\}$ is a basis for $\mathcal{A}_*$. □

As we can see in Eq (3.36), y_1 and y_2 are given by the infinite series, but only the terms $\tilde{X}^{(2n)}$ and $X^{(2n+1)}$ are needed. Now, the following result will allow us to avoid computing the unnecessary terms $\tilde{X}^{(2n+1)}$ and $X^{(2n)}$, respectively. This will make the code more numerically efficient.

Proposition 3.7. *Let $x_0 \in [a,b]$ and $y_0 \in \mathcal{A}$ be such that $y_0(x) \neq 0$ for all $x \in [a,b]$. If $P(x) = \int_{x_0}^{x} \frac{1}{\rho y_0^2}$ and $\tilde{X}^{(2n)}$, $X^{(2n+1)}$ are the functions defined in* (3.3)–(3.4)*, then, for each $n \in \mathbb{N}$, it is satisfied that*

$$\tilde{X}^{(2n)}(x) = \int_{x_0}^{x} [P(x) - P(t)] y_0^2(t) r(t) \tilde{X}^{(2n-2)}(t) dt, \tag{3.41}$$

and

$$X^{(2n+1)}(x) = \int_{x_0}^{x} [P(x) - P(t)] y_0^2(t) r(t) \tilde{X}^{(2n-1)}(t) dt. \tag{3.42}$$

Proof. Without loss of generality, let $x \geq x_0$. Note that

$$\begin{aligned} \tilde{X}^{(2n)}(x) &= \int_{x_0}^{x} \frac{1}{\rho(s) y_0^2(s)} \tilde{X}^{(2n-1)}(s) ds \\ &= \int_{x_0}^{x} \frac{1}{\rho(s) y_0^2(s)} \int_{x_0}^{s} r(t) y_0^2(t) \tilde{X}^{(2n-2)}(t) dt\, ds. \end{aligned} \tag{3.43}$$

Define

$$g(t,s) = \begin{cases} \dfrac{1}{\rho(s) {y_0}^2(s)}, & \text{if } x_0 \leq t \leq s \leq x \\ 0, & \text{if } x \geq t > s \geq x_0 \end{cases}, \tag{3.44}$$

and

$$f(t) = y_0^2(t) r(t) \tilde{X}^{(2n-2)}(t). \tag{3.45}$$

Then, $f \in \mathrm{KH}([a,b])$ and

$$V_{[x_0,x]} g(\cdot, s) = \frac{1}{\rho(s) y_0^2(s)} \tag{3.46}$$

for all $s \in [x, x_0]$. Since $\frac{1}{\rho y_0^2} \in \mathrm{L}([a,b])$, it follows from [11, Theorem 57] that

$$\int_{x_0}^{x} \int_{x_0}^{x} f(t) g(t,s) dt\, ds = \int_{x_0}^{x} \int_{x_0}^{x} f(t) g(t,s) ds\, dt. \tag{3.47}$$

With the left-hand side of this equality, we obtain that

$$\begin{aligned} \int_{x_0}^{x} \int_{x_0}^{x} f(t) g(t,s) dt\, ds &= \int_{x_0}^{x} \int_{x_0}^{s} f(t) \frac{1}{\rho(s) y_0^2(s)} dt\, ds = \int_{x_0}^{x} \frac{1}{\rho(s) y_0^2(s)} \int_{x_0}^{s} f(t) dt\, ds \\ &= \int_{x_0}^{x} \frac{1}{\rho(s) y_0^2(s)} \int_{x_0}^{s} y_0^2(t) r(t) \tilde{X}^{(2n-2)}(t) dt\, ds. \end{aligned} \tag{3.48}$$

Whereas on the right-hand side of Eq (3.47), we have that

$$\begin{aligned} \int_{x_0}^{x} \int_{x_0}^{x} f(t) g(t,s) ds\, dt &= \int_{x_0}^{x} \int_{t}^{x} f(t) \frac{1}{\rho(s) y_0^2(s)}\, ds\, dt = \int_{x_0}^{x} f(t) \int_{t}^{x} \frac{ds}{\rho(s) y_0^2(s)}\, dt. \\ &= \int_{x_0}^{x} [P(x) - P(t)] y_0^2(t) r(t) \tilde{X}^{(2k-2)}(t) dt. \end{aligned} \tag{3.49}$$

Therefore, by Eqs (3.43), (3.48), and (3.49), we can conclude that Eq (3.41) is satisfied. Equality (3.42) is proved in an analogous way. □

Remark 3.8. Note that almost all results depend on the existence of y_0, a solution of the homogeneous Eq (3.33), such that $y_0(x) \neq 0$ for all $x \in [a,b]$. The construction of this function does not represent any difficulty. As a matter of fact, if $P_h(x) = \int_{x_0}^{x} \frac{1}{\rho(s)} ds$ and

$$\tilde{X}_h^{(0)} \equiv 1, \qquad \tilde{X}_h^{(2n)}(x) = \int_{x_0}^{x} \tilde{X}_h^{(2n-2)}(t) q(t) [P_h(t) - P_h(x)] dt; \tag{3.50}$$

$$X_h^{(1)} = P_h, \quad X_h^{(2n+1)}(x) = \int_{x_0}^{x} X_h^{(2n-1)}(t)q(t)[P_h(t) - P_h(x)]dt; \tag{3.51}$$

then,

$$y = c_1 \sum_{n=0}^{\infty} \tilde{X}_h^{(2n)} + c_2 \sum_{n=0}^{\infty} X_h^{(2n+1)} \tag{3.52}$$

is the general solution of the homogeneous Eq (3.33), and

$$y_0 = \sum_{n=0}^{\infty} \tilde{X}_h^{(2n)} + i \sum_{n=0}^{\infty} X_h^{(2n+1)} \tag{3.53}$$

is a particular solution of the homogeneous equation such that $y_0(x) \neq 0$ for all $x \in [a, b]$. Indeed, let us first rewrite the homogeneous Eq (3.33) as $(\rho\dot{\psi})^{\cdot} = 1(-q)\psi$. Note that $\psi_0 \equiv 1$ is a non-vanishing solution of the homogeneous equation $(\rho\dot{\psi})^{\cdot} = 0$. Then, by Theorem 3.6, the general solution of the equation $(\rho\dot{\psi})^{\cdot} = 1(-q)\psi$ has the form

$$\psi = c_1 \left[\psi_0 \sum_{n=0}^{\infty} 1^n \tilde{\Psi}^{(2n)}\right] + c_2 \left[\psi_0 \sum_{n=0}^{\infty} 1^n \Psi^{(2n+1)}\right] = c_1 \sum_{n=0}^{\infty} \tilde{\Psi}^{(2n)} + c_2 \sum_{n=0}^{\infty} \Psi^{(2n+1)}, \tag{3.54}$$

where

$$\tilde{\Psi}^{(0)} \equiv 1, \ \tilde{\Psi}^{(2n)}(x) = \int_{x_0}^{x} \tilde{\Psi}^{(2n-1)} \frac{1}{\rho(t)\psi_0^2(t)} dt, \ \Psi^{(1)} = R, \ X^{(2n+1)}(x) = \int_{x_0}^{x} \tilde{\Psi}^{(2n)} \frac{dt}{\rho(t)\psi_0^2(t)} \tag{3.55}$$

and $R(x) = \int_{x_0}^{x} \frac{1}{\rho{\psi_0}^2} = \int_{x_0}^{x} \frac{1}{\rho}$.

Moreover, by Proposition 3.7,

$$\tilde{\Psi}^{(2n)}(x) = \int_{x_0}^{x} [R(x) - R(t)]{\psi_0}^2(t)(-q)(t)\tilde{\Psi}^{(2n-2)}(t)dt = \int_{x_0}^{x} [R(t) - R(x)]q(t)\tilde{\Psi}^{(2n-2)}(t)dt,$$
$$X^{(2n+1)}(x) = \int_{x_0}^{x} [R(x) - R(t)]{\psi_0}^2(t)(-q)(t)\tilde{\Psi}^{(2n-1)}(t)dt = \int_{x_0}^{x} [R(t) - R(x)]q(t)\tilde{\Psi}^{(2n-1)}(t)dt.$$

Notice that actually $R = P_h$, $\Psi^{(2n)} = \tilde{X}_h^{(2n)}$, and $\Psi^{(2n+1)} = X_h^{(2n+1)}$. Therefore, $y = c_1 \sum_{n=0}^{\infty} \tilde{X}^{(2n)} + c_2 \sum_{n=0}^{\infty} X^{(2n+1)}$ is the general solution in $\mathcal{A}$ of the homogeneous equation $(\rho\dot{y})^{\cdot} + qy = 0$.

We show now that $y_0(x) \neq 0$ for all $x \in [a, b]$. Let $\psi_1 = \sum_{n=0}^{\infty} \tilde{\Psi}^{(2n)}$, and $\psi_2 = \sum_{n=0}^{\infty} \Psi^{(2n+1)}$. Then, $y_0 = \psi_1 + i\psi_2$. Since $\psi_1, \psi_2 \in \mathcal{A}$, there exist functions $g_1, g_2 \in \mathrm{ACG}_*([a, b])$ such that $\rho\dot{\psi}_1 = g_1$, and $\rho\dot{\psi}_2 = g_2$ a.e. on $[a, b]$, specially at every point of E_ρ. Therefore, as $x_0 \in E_\rho$, we have that

$$\rho(x_0)\dot{\psi}_1(x_0) = g_1(x_0) \quad \& \quad \rho(x_0)\dot{\psi}_2(x_0) = g_2(x_0).$$

The functions $\frac{1}{\rho}\tilde{\Psi}^{(2n-1)}$ and $\frac{1}{\rho}\Psi^{(2n)}$ are locally continuous at x_0, thus by (3.55) and Theorem 2.1,

$$\left[\tilde{\Psi}^{(2n)}\right]^{\cdot}(x_0) = \frac{1}{\rho(x_0)}\tilde{\Psi}^{(2n-1)}(x_0) = 0 \ \ \forall n \in \mathbb{N},$$

and

$$\left[\Psi^{(2n+1)}\right]^{\cdot}(x_0) = \frac{1}{\rho(x_0)}\Psi^{(2n)}(x_0) = \begin{cases} \frac{1}{\rho(x_0)}, & \text{if } n = 0 \\ 0, & \text{if } n \in \mathbb{N}. \end{cases}$$

Therefore,

$$\dot{\psi}_1(x_0) = \sum_{n=0}^{\infty}\left[\Psi^{(2n)}\right]^{\cdot}(x_0) = 0 \quad \text{and} \quad \dot{\psi}_2(x_0) = \sum_{n=0}^{\infty}\left[\Psi^{(2n+1)}\right]^{\cdot}(x_0) = \frac{1}{\rho(x_0)}.$$

Note also that $\psi_1(x_0) = 1$ and $\psi_2(x_0) = 0$. Then,

$$[\psi_1(x_0)g_2(x_0) - \psi_2(x_0)g_1(x_0)] = g_2(x_0) = \rho(x_0)\dot{\psi}_2(x_0) = \rho(x_0)\frac{1}{\rho(x_0)} = 1.$$

It s clear that $\psi_1, \psi_2 \in \mathrm{AC}([a,b]) \subset \mathrm{ACG}_*([a,b])$, therefore, $\psi_1 g_2 - \psi_2 g_1 \in \mathrm{ACG}_*([a,b])$. Let $x \in [a,b]$. Then,

$$\begin{aligned}
\psi_1(x)g_2(x) - \psi_2(x)g_1(x) - 1 &= \psi_1(x)g_2(x) - \psi_2(x)g_1(x) - [\psi_1(x_0)g_2(x_0) - \psi_2(x_0)g_1(x_0)] \\
&= \int_{x_0}^{x}[\psi_1 g_2 - \psi_2 g_1]^{\cdot} = \int_{x_0}^{x}[\psi_1(\rho\dot{\psi}_2) - \psi_2(\rho\dot{\psi}_1)]^{\cdot} \\
&= \int_{x_0}^{x}[\psi_1(\rho\dot{\psi}_2)^{\cdot} - \psi_2(\rho\dot{\psi}_1)^{\cdot}] = \int_{x_0}^{x}[\psi_1(\rho\dot{\psi}_2)^{\cdot} - \psi_2(\rho\dot{\psi}_1)^{\cdot}] + \psi_1 q\psi_2 - \psi_2 q\psi_1 \\
&= \int_{x_0}^{x}[\psi_1[(\rho\dot{\psi}_2)^{\cdot} + q\psi_2] - \psi_2[(\rho\dot{\psi}_1)^{\cdot} + q\psi_1]] \\
&= \int_{x_0}^{x}[\psi_1 \cdot 0 - \psi_2 \cdot 0] = \int_{x_0}^{x} 0 = 0, \quad \text{for all } x \in [a,b].
\end{aligned}$$

Thus, for every $x \in [a,b]$,

$$\psi_1(x)g_2(x) - \psi_2(x)g_1(x) = 1.$$

Then, for each $x \in [a,b]$, $\psi_1(x) \neq 0$ or $\psi_2(x) \neq 0$. Consequently, $y_0(x) \neq 0$ for all $x \in [a,b]$.

4. Numerical solution to Sturm–Liouville problems

The representation of the solution of the S-L equation (see (3.35)) allows us to solve spectral problems in a simple way. In this section, we present some aspects related to the numerical implementation of the SPPS method, as well as some examples illustrating how we can apply it to find the eigenvalues of spectral problems.

Example 4.1. Consider the equation $(\rho\dot{y})^{\cdot} + qy = \lambda ry$ a.e. on $[0,\pi]$, where

$$\rho(x) = -1, \quad q(x) = \frac{2\pi}{x}\sin\left(\frac{\pi}{x^2}\right) \quad \text{and} \quad r(x) = 1$$

with boundary conditions

$$y(0) = y(\pi) = 0.$$

The function q is highly oscillating, and $q \in \mathrm{KH}([0,\pi]) \setminus \mathrm{L}([0,\pi])$. This example clearly shows that the results of this paper cover wider cases than those results using the Lebesgue integral. Let

$$\mathcal{D} = \{y \in \mathcal{A} : y(0) = 0 = y(\pi)\},$$

and define $L : \mathcal{D} \to \mathrm{KH}([0,\pi])$ as $L(y) = (\rho\dot{y})^{\cdot} + qy$. We will find an approximation for the point spectrum $\sigma_p(L)$.

From Theorem 3.6, the general solution of the equation $(\rho\dot{y})^{\cdot} + qy = \lambda ry$ has the form

$$y = c_1\left[y_0 \sum_{s=0}^{\infty} \lambda^s \tilde{X}^{(2s)}\right] + c_2\left[y_0 \sum_{s=0}^{\infty} \lambda^s X^{(2s+1)}\right], \tag{4.1}$$

where $\tilde{X}^{(2s)}$ and $X^{(2s+1)}$ are defined as in (3.41) by taking $x_0 = 0$. Using the boundary condition $y(0) = 0$ in (4.1), we obtain that

$$0 = c_1 y_0(0) + c_2 y_0(0) \cdot 0.$$

Therefore, $c_1 = 0$. Now, using the boundary condition $y(\pi) = 0$, we find that

$$0 = c_2 y_0(\pi) \sum_{s=0}^{\infty} \lambda^s X^{(2s+1)}(\pi).$$

Since the homogeneous solution satisfies $y_0(\pi) \neq 0$, it follows that

$$\Omega(\lambda) := \sum_{s=0}^{\infty} \lambda^s X^{(2s+1)}(\pi) = 0.$$

The zeros of Ω form the point spectrum of L. To find the zeros of this function, we truncate this series after m terms. Then, we have to find the zeros of

$$\Omega_m(\lambda) := \sum_{s=0}^{m} \lambda^s X^{(2s+1)}(\pi).$$

According to Theorem 3.8, to find the values of $X^{(2s+1)}(\pi)$, we first need to find the values of y_0 over $[0,\pi]$. Then, in order to build the non-vanishing homogeneous solution y_0, as Remark 3.8 states, we must find the values of $\tilde{X}_h^{(2s)}$ and $X_h^{(2s+1)}$ over $[0,\pi]$. From Eq (3.50), we have that

$$\tilde{X}_h^{(2s)}(x) = \int_{x_0}^{x} [P_h(t) - P_h(x)]q(t)\tilde{X}^{(2s-2)}(t)dt,$$
$$X_h^{(2s+1)}(x) = \int_{x_0}^{x} X_h^{(2s-1)}(t)q(t)[P_h(t) - P_h(x)]dt.$$

These integrals are from Kurzweil–Henstock because $q \in \mathrm{KH}([0,\pi]) \setminus \mathrm{L}([0,\pi])$, so we have to use an appropriate method to estimate the integrals since uniform partitions do not work. Instead, we have to use unequal partitions. For this, we will use the method described by Yang et al. in [12]. As q has a singularity at $x = 0$, and near this point it oscillates quite a bit, we take the following sequence of points that approaches $x = 0$ without reaching it:

$$x_i = \pi\left(5^{-(i-1)}\right), \quad \text{where} \quad i \in \{1, 2, \ldots, t+1\}.$$

Then, on each subinterval $[x_{i+1}, x_i]$, we take an unequal partition of size n generated by the points

$$u_{n,0}^{(i)} = x_{i+1}, \qquad u_{n,k}^{(i)} = x_{i+1} + \sum_{j=1}^{k} a_{n,j}^{(i)}, \tag{4.2}$$

where

$$a_{n,j}^{(i)} = \frac{2(x_i - x_{i+1})j}{n(n+1)} \quad \text{for} \quad j \in \{1, 2, \ldots, n\}. \tag{4.3}$$

Therefore, we have that

$$0 < x_{t+1} = u_{n,0}^{(t)} < u_{n,1}^{(t)} < u_{n,2}^{(t)} < \cdots < u_{n,n}^{(t)} = x_t = u_{n,0}^{(t-1)} < u_{n,1}^{(t-1)} < u_{n,2}^{(t-1)} < \cdots u_{n,n}^{(t-1)} = x_{t-1} < \cdots \tag{4.4}$$

$$\cdots < x_2 = u_{n,0}^{(1)} < u_{n,1}^{(1)} < u_{n,2}^{(1)} < \cdots < u_{n,n}^{(1)} = x_1 = \pi. \tag{4.5}$$

The quadrature that will be used to estimate the value of the integral of a function f over each subinterval $[x_{i+1}, x_i]$ is given by

$$\int_{x_{i+1}}^{x_i} f \approx Q_n^2(f) = \sum_{j=1}^{n} \frac{a_{n,j}^{(i)}}{2}\Big(f(u_{n,j}^{(i)}) + f(u_{n,j-1}^{(i)})\Big). \tag{4.6}$$

This quadrature allows us to estimate the values of $\tilde{X}_h^{(2s)}, X_h^{(2s+1)}$ for $s \in \{0, 1, \ldots, m\}$ at each node $u_{n,j}^{(i)}$. As mentioned in Remark 3.8, we take $y_0 = \sum_{s=0}^{m} \lambda^s \tilde{X}_h^{(2s)} + i \sum_{s=0}^{m} \lambda^s X_h^{(2s+1)}$. This allows us to calculate the values of P and $X^{(2s+1)}$ over $[0, \pi]$, especially for $s = m$ and $x = \pi$, which leads us to the characteristic polynomial $\Omega_m(\lambda)$, where the roots of Ω_m must be the point spectrum $\sigma_p(L)$.

The eigenvalues in this example were calculated with Python using $t = 30$, $n = 3000$, and $m = 100$.

n	λ_n
1	2.361300545482816
2	4.7417217745472335
3	10.825072417633294
4	17.486039265939382
5	25.821116277868896
6	37.210546294478405
7	50.67457532648284
8	65.42292022702168
9	81.93125101877169
10	101.0198225062351

Example 4.2. Consider the equation $(\rho\dot{y})^{\cdot} + qy = \lambda r y$ a.e. on $[-\pi, 0]$, where

$$\rho(x) = \sqrt{\pi + x}, \quad q(x) = -\frac{2x + 2\pi + 1}{2\sqrt{\pi + x}} \quad \text{and} \quad r(x) = \csc\left(\frac{x + \pi}{2}\right)\sin\left(\csc\left(\frac{\pi + x}{2}\right)\right)$$

with boundary conditions

$$y(-\pi) = 0, \qquad \dot{y}(0) = 0.$$

It is clear that $\frac{1}{\rho}, q \in \mathrm{L}([-\pi, 0])$ and have a singularity at $x = -\pi$; also, the function r is highly oscillating, and $r \in \mathrm{KH}([-\pi, 0])$. Let

$$\mathcal{D} = \{y \in \mathcal{A} : y(-\pi) = 0 = \dot{y}(0)\},$$

and define $L : \mathcal{D} \to \mathrm{KH}([-\pi, 0])$ as $L(y) = (\rho\dot{y})^{\cdot} + qy$. We will find an approximation for the point spectrum $\sigma_p(L)$. From Theorem 3.6, the general solution of the equation $(\rho\dot{y})^{\cdot} + qy = \lambda ry$ has the form

$$y = c_1\left[y_0 \sum_{s=0}^{\infty} \lambda^s \tilde{X}^{(2s)}\right] + c_2\left[y_0 \sum_{s=0}^{\infty} \lambda^s X^{(2s+1)}\right] = c_1 y_0 u + c_2 y_0 w, \tag{4.7}$$

where $\tilde{X}^{(2s)}$ and $X^{(2s+1)}$ are defined as in (3.41) by taking $x_0 = 0$. The choice of this point is due to the fact that $0 \in E_\rho$, note that $-\pi \notin E_\rho$. Using the boundary condition $y(-\pi) = 0$ in (4.7), we obtain that

$$0 = c_1\left[y_0(-\pi) \sum_{s=0}^{\infty} \lambda^s \tilde{X}^{(2s)}(-\pi)\right] + c_2\left[y_0(-\pi) \sum_{s=0}^{\infty} \lambda^s X^{(2s+1)}(-\pi)\right]. \tag{4.8}$$

Now, using the boundary condition $\dot{y}(x_0) = 0$ in Eq (4.7), together with the fact that $u(x_0) = 1$, $v(x_0) = 0$, $w(x_0) = 0$, $z(x_0) = 1$, $\dot{u}(x_0) = \frac{v(x_0)}{\rho(x_0)y_0^2(x_0)}$, and $\dot{w}(x_0) = \frac{z(x_0)}{\rho(x_0)y_0^2(x_0)}$, we have that

$$\begin{aligned}
0 = \dot{y}(x_0) &= c_1\Big(\dot{y}_0(x_0)u(x_0) + y_0(x_0)\dot{u}(x_0)\Big) + c_2\Big(\dot{y}_0(x_0)w(x_0) + y_0(x_0)\dot{w}(x_0)\Big) \\
&= c_1\left(\dot{y}_0(x_0) \cdot 1 + y_0(x_0)\frac{v(x_0)}{\rho(x_0)y_0^2(x_0)}\right) + c_2\left(\dot{y}_0(x_0) \cdot 0 + y_0(x_0)\frac{z(x_0)}{\rho(x_0)y_0^2(x_0)}\right) \\
&= c_1\dot{y}_0(x_0) + c_2\frac{1}{\rho(x_0)y_0(x_0)}.
\end{aligned}$$

Then, $c_2 = -\sqrt{\pi}y_0(0)\dot{y}_0(0)c_1$. When we substitute the value of c_2 into Eq (4.8), considering that c_1 and $y_0(-\pi)$ must not be zero, we will arrive at the next equation

$$\left[\sum_{s=0}^{\infty} \lambda^s \tilde{X}^{(2s)}(-\pi)\right] - \sqrt{\pi}y_0(0)\dot{y}_0(0)\left[\sum_{s=0}^{\infty} \lambda^s X^{(2s+1)}(-\pi)\right] = 0.$$

Thus, to find the point spectrum of L, we have to find the zeros of

$$\Omega_m(\lambda) := \sum_{s=0}^{m} \lambda^s \left[\tilde{X}^{(2s)}(-\pi) - \sqrt{\pi}y_0(0)\dot{y}_0(0)X^{(2s+1)}(-\pi)\right].$$

The eigenvalues for this example were calculated using $t = 30$, $n = 3000$, and $m = 100$.

n	λ_n
1	4.524225992700899
2	28.39425092237783
3	57.41629224255273
4	84.21626647213287
5	160.32418088288506
6	179.02925228986282
7	508.1105242245128
8	520.70477362163
9	523.5330777360434

5. Conclusions

In this paper, we show the convergence of the spectral parameter power series method, proposed by Kravchenko, for the Sturm–Liouville equation with Kurzweil–Henstock integrable coefficients. By incorporating the Kurzweil–Henstock integral into the SPPS method, we have significantly expanded the scope and applicability of the method, allowing us to tackle a wider variety of problems, including those containing highly oscillating functions that are not Lebesgue integrable. The result given by Blancarte et al. in [3, Theorem 7] remains a particular case of the results presented here when $q, r \in \mathrm{L}([a, b]) (\subseteq \mathrm{KH}([a, b]))$. The numerical implementation of the method was reasonably tractable and has proven to be a powerful tool for solving Sturm–Liouville problems. This is clearly shown in the examples in Section 4, where we were able to find the point spectrum for problems with Kurzweil–Henstock integrable functions.

Author contributions

All authors, I. A. Cordero-Martínez, S. Sánchez-Perales and F. J. Mendoza-Torres, have contributed equally to this work. The authors have read and accepted the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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