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Research article

On a nonlinear time-fractional cable equation

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Abstract: A nonlinear time-fractional cable equation posed on the interval (0, 1) under a homogeneous Dirichlet boundary condition is investigated in this work. The considered equation reflects the anomalous electro-diffusion in nerve cells. Using nonlinear capacity estimates specifically adapted to the considered problem, we establish sufficient conditions for the nonexistence of weak solutions.

Keywords: time-fractional cable equation; weak solution; nonexistence; Caputo fractional derivative
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1. Introduction

In this work, we are concerned with the nonlinear time-fractional cable equation

$$\frac{\partial u}{\partial t} + \frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\beta}{\partial t^\beta}\frac{\partial^2 u}{\partial x^2} = F(x,u), \quad t > 0,\ 0 < x < 1 \tag{1.1}$$

subject to the initial condition

$$u(0,x) = u_0(x), \quad 0 < x < 1 \tag{1.2}$$

and the Dirichlet boundary condition

$$u(t,1) = 0, \quad t > 0. \tag{1.3}$$

Here, $u = u(t,x)$, $0 < \alpha, \beta < 1$, $\frac{\partial^\alpha}{\partial t^\alpha}$ (resp. $\frac{\partial^\beta}{\partial t^\beta}$) is the Caputo fractional derivative of order α (resp. β) with respect to the time-variable t, $F(x,u) = x^{-\sigma}|u|^p$, $\sigma \geq 0$, $p > 1$ is a nonlinear reaction term, and $u_0 \in L^1_{\text{loc}}((0,1])$. Namely, we are interested in the study of the nonexistence of weak solutions to the considered problem.

Fractional derivatives were found to be quite flexible for describing diverse materials and processes presenting memory and hereditary properties. This fact motivated the study of time-fractional evolution equations from both practical and theoretical points of view. The time-fractional cable equation (see [1]) is a generalization of the classical cable equation, which was introduced in [2] as a macroscopic model for electrodiffusion of ions in nerve cells, when molecular diffusion is anomalous subdiffusion due to binding, crowding, or trapping. We can find in the literature several contributions related to the numerical study of the time-fractional cable equation, see, e.g., [3–6] and the references therein.

The study of the nonexistence of solutions to time-fractional evolution equations was first considered by Kirane and his collaborators (see, e.g., [7–10]). Next, this topic was developed by many authors, see, e.g., Tatar [11], Borikhanov, Ruzhansky and Torebek [12], Kassymov, Tokmagambetov, and Torebek [13], Zhang, Sun, et al. [14], He [15], and Jleli [16]. To the best of our knowledge, the study of the time-fractional cable equation was not previously considered in the literature.

The approach used in this work is based on nonlinear capacity estimates specifically adapted to the nonlocal properties of the Caputo fractional derivatives $\frac{\partial^\alpha}{\partial t^\alpha}$ and $\frac{\partial^\beta}{\partial t^\beta}$, the second-order differential operator $\frac{\partial^2}{\partial x^2}$, the domain $(0, 1)$, the initial condition (1.2), and the boundary condition (1.3). The cases $\alpha < \beta$, $\alpha = \beta$, and $\alpha > \beta$ are studied separately. Namely, when $\alpha < \beta$, we show that for suitable initial values u_0, (1.1)–(1.3) admits no weak solution for all $p > 1$. Furthermore, if u_0 satisfies a certain behavior as $x \to 0^+$, then, if $\alpha = \beta$, (1.1)–(1.3) admits no weak solution for all $p > 1$. However, if $\alpha > \beta$ and $\sigma \geq 2$, then there exists a certain range of p where (1.1)–(1.3) admits no weak solution.

The rest of this paper is organized as follows: In Section 2, we briefly recall some basic notions and properties related to the Caputo fractional derivative. In Section 3, we define weak solutions to the considered problem and state our main results. In Section 4, we establish some useful lemmas. We finally prove our main results in Section 5.

Throughout this paper, by $\ell \gg 1$, we mean that ℓ is a sufficiently large real number. By C (or C_i), we mean a positive constant that is independent of the parameters T, R, and the solution u. The value of this constant is not important and is not necessarily the same from one line to another.

2. Some preliminaries on fractional calculus

In this section, we briefly recall some basic notions and properties of fractional calculus. For more details, see, e.g., [17].

Let $T > 0$ be fixed, $\beta > 0$, and $f \in C([0, T])$. The left-sided Riemann-Liouville fractional integral of order β of f is defined by

$$I_0^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\, ds, \quad 0 < t \leq T.$$

The right-sided Riemann-Liouville fractional integral of order β of f is defined by

$$I_T^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} f(s)\, ds, \quad 0 \leq t < T.$$

Here, Γ denotes the Gamma function, that is,

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt, \quad s > 0.$$

It can be easily seen that

$$\lim_{t\to 0^+} I_0^\beta f(t) = \lim_{t\to T^-} I_T^\beta f(t) = 0. \tag{2.1}$$

We have the following integration-by-parts rule: If $\beta > 0$ and $f, g \in C([0,T])$, then

$$\int_0^T g(t) I_0^\beta f(t)\, dt = \int_0^T f(t) I_T^\beta g(t)\, dt. \tag{2.2}$$

Let $0 < \beta < 1$ and $f \in C^1([0,T])$. The Caputo fractional derivative of order β of f is defined by

$$^C D_0^\beta f(t) = I_0^{1-\beta} \frac{df}{dt}(t), \quad 0 < t \le T,$$

that is,

$$^C D_0^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{df}{dt}(s)\, ds, \quad 0 < t \le T.$$

Let $w = w(t,x) : [0,T] \times \mathbb{I} \to \mathbb{R}$, where $\mathbb{I} \subset \mathbb{R}$. We denote by $I_0^\beta w$, $\beta > 0$, the left-sided Riemann-Liouville fractional integral of order β of w with respect to the variable t, that is,

$$I_0^\beta w(t,x) = I_0^\beta w(\cdot, x)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} w(s,x)\, ds, \quad 0 < t \le T,\ x \in \mathbb{I}.$$

We denote by $I_T^\beta w$ the right-sided Riemann-Liouville fractional integral of order β of w with respect to the variable t, that is,

$$I_T^\beta w(t,x) = I_T^\beta w(\cdot, x)(t) = \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} w(s,x)\, ds, \quad 0 \le t < T,\ x \in \mathbb{I}.$$

Let $0 < \beta < 1$. We denote by $\frac{\partial^\beta}{\partial t^\beta}$ the Caputo fractional derivative of order β of w with respect to the variable t, that is,

$$\begin{aligned} \frac{\partial^\beta w}{\partial t^\beta}(t,x) &= {}^C D_0^\beta w(\cdot, x)(t) \\ &= I_0^{1-\beta} \frac{\partial w}{\partial t}(t,x) \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial w}{\partial t}(s,x)\, ds, \quad 0 < t \le T,\ x \in \mathbb{I}. \end{aligned}$$

3. Main results

Before stating our main results, we need to define weak solutions to the considered problem.

For all $T > 0$, let

$$S_T = [0,T] \times (0,1].$$

We introduce a set of functions

$$\Psi_T = \left\{ \psi = \psi(t,x) \in C^3(S_T) : \psi \ge 0,\ \mathrm{supp}_x(\psi) \subset\subset (0,1],\ \psi(\cdot, 1) \equiv 0,\ \psi(T, \cdot) \equiv 0 \right\},$$

where by $\mathrm{supp}_x(\psi) \subset\subset (0,1]$, we mean that ψ is uniformly compactly supported on $(0,1]$ with respect to the variable x.

A weak solution to (1.1)–(1.3) is defined as follows:

Definition 3.1. *We say that $u \in L^p_{\text{loc}}([0,\infty)\times(0,1])$ is a weak solution to* (1.1)-(1.2)-(1.3)*, if*

$$\begin{aligned}&\int_{S_T} x^{-\sigma}|u|^p\psi\,dx\,dt+\int_0^1 u_0(x)\left(\psi(0,x)+I_T^{1-\alpha}\psi(0,x)-I_T^{1-\beta}\frac{\partial^2\psi}{\partial x^2}(0,x)\right)dx\\&=-\int_{S_T}u\frac{\partial\psi}{\partial t}\,dx\,dt-\int_{S_T}u\frac{\partial I_T^{1-\alpha}\psi}{\partial t}\,dx\,dt+\int_{S_T}u\frac{\partial^2}{\partial x^2}\left(\frac{\partial I_T^{1-\beta}\psi}{\partial t}\right)dx\,dt\end{aligned}\tag{3.1}$$

for all $T>0$ and $\psi\in\Psi_T$.

It can be easily seen that any classical solution to (1.1)–(1.3) is a weak solution in the sense of Definition 3.1. Namely, for all $T>0$, multiplying (1.1) by $\psi\in\Psi_T$, integrating by parts over S_T, using (1.2), (1.3), properties (2.1) and (2.2), we obtain (3.1).

We are now in a position to state our main results. We first consider the case where $\beta>\alpha$.

Theorem 3.1. *Let $0<\alpha<\beta<1$, $\sigma\geq 0$, and $u_0\in L^1((0,1))$. If*

$$\int_0^1 u_0(x)(1-x)\,dx>0,\tag{3.2}$$

then for all $p>1$, (1.1)–(1.3) *admits no weak solution.*

We next consider the case where $\beta\leq\alpha$.

Theorem 3.2. *Let $0<\beta\leq\alpha<1$, $\sigma\geq 0$, and $u_0\in C([0,1])$. Assume that u_0 satisfies* (3.2) *and*

$$|u_0(x)|\sim x^\delta \ \text{ as } \ x\to 0^+,\tag{3.3}$$

where $\delta>1$.

(i) If $\alpha=\beta$, then for all $p>1$, (1.1)–(1.3) *admits no weak solution.*
(ii) If $\alpha>\beta$, $\sigma\geq 2$, and

$$1+\frac{\sigma-2}{\delta}<p<1+\frac{\sigma-2}{\delta}+\frac{\beta(\delta-1)}{\delta(\alpha-\beta)},\tag{3.4}$$

then (1.1)–(1.3) *admits no weak solution.*

4. Auxiliary results

Some useful lemmas are established in this section.
For $\ell\gg 1$ and $T>0$, let

$$\eta_T(t)=T^{-\ell}(T-t)^\ell,\quad 0\leq t\leq T.\tag{4.1}$$

The following properties can be found in [17, Property 2.1, p 71].

Lemma 4.1. *Let $0<\kappa<1$. For all $t\in[0,T]$, we have*

$$I_T^\kappa\eta_T(t)=\frac{\Gamma(\ell+1)}{\Gamma(\ell+1+\kappa)}T^{-\ell}(T-t)^{\ell+\kappa},\tag{4.2}$$

$$\frac{d}{dt}I_T^\kappa\eta_T(t)=-\frac{\Gamma(\ell+1)}{\Gamma(\ell+\kappa)}T^{-\ell}(T-t)^{\ell+\kappa-1}.\tag{4.3}$$

Lemma 4.2. *Let $m > 1$ and $0 < \kappa < 1$. We have*

$$\int_0^T \eta_T^{\frac{-1}{m-1}} \left|\frac{d\eta_T}{dt}\right|^{\frac{m}{m-1}} dt = CT^{1-\frac{m}{m-1}}, \tag{4.4}$$

$$\int_0^T \eta_T^{\frac{-1}{m-1}} \left|\frac{dI_T^{\kappa}\eta_T}{dt}\right|^{\frac{m}{m-1}} dt = CT^{1-\frac{(1-\kappa)m}{m-1}}. \tag{4.5}$$

Proof. From (4.1) and (4.3), for all $t \in (0, T)$, we have

$$\begin{aligned} \eta_T^{\frac{-1}{m-1}} \left|\frac{d\eta_T}{dt}\right|^{\frac{m}{m-1}} &= C\left[T^{-\ell}(T-t)^{\ell}\right]^{\frac{-1}{m-1}} \left[T^{-\ell}(T-t)^{\ell-1}\right]^{\frac{m}{m-1}} \\ &= CT^{-\ell}(T-t)^{\ell-\frac{m}{m-1}} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} \eta_T^{\frac{-1}{m-1}} \left|\frac{dI_T^{\kappa}\eta_T}{dt}\right|^{\frac{m}{m-1}} &= C\left[T^{-\ell}(T-t)^{\ell}\right]^{\frac{-1}{m-1}} \left[T^{-\ell}(T-t)^{\ell+\kappa-1}\right]^{\frac{m}{m-1}} \\ &= CT^{-\ell}(T-t)^{\ell+\frac{(\kappa-1)m}{m-1}}. \end{aligned} \tag{4.7}$$

Integrating (4.6) (resp. (4.7)) over $(0, T)$, we obtain (4.4) (resp. (4.5)). □

We now introduce the function

$$L(x) = 1 - x, \quad 0 < x \leq 1. \tag{4.8}$$

We also need a cut-off function $\xi \in C^{\infty}([0, \infty))$ satisfying

$$0 \leq \xi \leq 1, \ \xi \equiv 0 \text{ in } \left[0, \frac{1}{2}\right], \ \xi \equiv 1 \text{ in } [1, \infty).$$

For $\ell, R \gg 1$, let

$$\xi_R(x) = L(x)\xi^{\ell}(Rx), \quad 0 < x \leq 1,$$

that is,

$$\xi_R(x) = \begin{cases} 0 & \text{if} \quad 0 < x \leq \frac{1}{2R}, \\ L(x)\xi^{\ell}(Rx) & \text{if} \quad \frac{1}{2R} \leq x \leq \frac{1}{R}, \\ L(x) & \text{if} \quad \frac{1}{R} \leq x \leq 1. \end{cases} \tag{4.9}$$

Lemma 4.3. *Let $a \geq 0$ and $m > 1$. We have*

$$\int_0^1 x^{\frac{a}{m-1}} \xi_R(x)\, dx \leq 1. \tag{4.10}$$

Proof. From (4.8), (4.9) and the properties of ξ (namely $0 \leq \xi \leq 1$), we have

$$\begin{aligned} \int_0^1 x^{\frac{a}{m-1}} \xi_R(x)\, dx &= \int_{\frac{1}{2R}}^1 x^{\frac{a}{m-1}} (1-x)\xi^{\ell}(Rx)\, dx \\ &\leq \int_{\frac{1}{2R}}^1 x^{\frac{a}{m-1}}\, dx \\ &\leq 1 - \frac{1}{2R} \\ &\leq 1, \end{aligned}$$

which proves (4.10). □

Lemma 4.4. *Let $a \geq 0$ and $m > 1$. We have*

$$\int_0^1 x^{\frac{a}{m-1}} \xi_R^{\frac{-1}{m-1}}(x) \left|\frac{d^2\xi_R}{dx^2}(x)\right|^{\frac{m}{m-1}} dx \leq CR^{\frac{2m-a}{m-1}-1}. \tag{4.11}$$

Proof. For all $x \in (0,1)$, we obtain by the definition of ξ_R that

$$\begin{aligned}\frac{d^2\xi_R}{dx^2}(x) &= \frac{d^2}{dx^2}\left[(1-x)\xi^\ell(Rx)\right]\\ &= \xi^\ell(Rx)\frac{d^2}{dx^2}(1-x) + (1-x)\frac{d^2}{dx^2}[\xi^\ell(Rx)] + 2\frac{d}{dx}(1-x)\frac{d}{dx}[\xi^\ell(Rx)]\\ &= (1-x)\frac{d^2}{dx^2}[\xi^\ell(Rx)] - 2\frac{d}{dx}[\xi^\ell(Rx)],\end{aligned}$$

which yields

$$\frac{\frac{d^2\xi_R}{dx^2}(x)}{L(x)} = \frac{d^2}{dx^2}[\xi^\ell(Rx)] - 2\frac{\frac{d}{dx}[\xi^\ell(Rx)]}{L(x)}. \tag{4.12}$$

Then, by the properties of ξ, we obtain

$$\operatorname{supp}\left(\frac{d^2\xi_R}{dx^2}\right) \subset \left[\frac{1}{2R}, \frac{1}{R}\right], \tag{4.13}$$

which implies that

$$\int_0^1 x^{\frac{a}{m-1}} \xi_R^{\frac{-1}{m-1}}(x) \left|\frac{d^2\xi_R}{dx^2}(x)\right|^{\frac{m}{m-1}} dx = \int_{\frac{1}{2R}}^{\frac{1}{R}} x^{\frac{a}{m-1}} \xi_R^{\frac{-1}{m-1}}(x) \left|\frac{d^2\xi_R}{dx^2}(x)\right|^{\frac{m}{m-1}} dx. \tag{4.14}$$

On the other hand, by the definition of ξ_R, for all $x \in \left(\frac{1}{2R}, \frac{1}{R}\right)$, we have

$$\begin{aligned}x^{\frac{a}{m-1}} \xi_R^{\frac{-1}{m-1}}(x) \left|\frac{d^2\xi_R}{dx^2}(x)\right|^{\frac{m}{m-1}} &= x^{\frac{a}{m-1}} \xi^{\frac{-\ell}{m-1}}(Rx)(1-x)\left|\frac{\frac{d^2\xi_R}{dx^2}(x)}{L(x)}\right|^{\frac{m}{m-1}}\\ &\leq x^{\frac{a}{m-1}} \xi^{\frac{-\ell}{m-1}}(Rx)\left|\frac{\frac{d^2\xi_R}{dx^2}(x)}{L(x)}\right|^{\frac{m}{m-1}},\end{aligned}$$

which implies by (4.14) that

$$\int_0^1 x^{\frac{a}{m-1}} \xi_R^{\frac{-1}{m-1}}(x) \left|\frac{d^2\xi_R}{dx^2}(x)\right|^{\frac{m}{m-1}} dx \leq \int_{\frac{1}{2R}}^{\frac{1}{R}} x^{\frac{a}{m-1}} \xi^{\frac{-\ell}{m-1}}(Rx) \left|\frac{\frac{d^2\xi_R}{dx^2}(x)}{L(x)}\right|^{\frac{m}{m-1}} dx. \tag{4.15}$$

Furthermore, by (4.12) and the properties of ξ, for all $x \in \left(\frac{1}{2R}, \frac{1}{R}\right)$ (with $R \gg 1$), we obtain

$$\begin{aligned}\left|\frac{\frac{d^2\xi_R}{dx^2}(x)}{L(x)}\right| &\leq \left|\frac{d^2}{dx^2}[\xi^\ell(Rx)]\right| + 2\left|\frac{\frac{d}{dx}[\xi^\ell(Rx)]}{L(x)}\right|\\ &\leq C\left(R^2\xi^{\ell-2}(Rx) + R^{-1}\xi^{\ell-1}(Rx)\right)\\ &\leq CR^2\xi^{\ell-2}(Rx),\end{aligned}$$

which yields

$$x^{\frac{a}{m-1}}\xi^{\frac{-\ell}{m-1}}(Rx)\left|\frac{\frac{d^2\xi_R}{dx^2}(x)}{L(x)}\right|^{\frac{m}{m-1}} \le CR^{\frac{2m}{m-1}}x^{\frac{a}{m-1}}\xi^{\ell-\frac{2m}{m-1}}(Rx)$$
$$\le CR^{\frac{2m}{m-1}}x^{\frac{a}{m-1}}.$$

Then, using (4.15) and integrating the above estimate over $\left(\frac{1}{2R},\frac{1}{R}\right)$, we obtain

$$\int_0^1 x^{\frac{a}{m-1}}\xi_R^{\frac{-1}{m-1}}(x)\left|\frac{d^2\xi_R}{dx^2}(x)\right|^{\frac{m}{m-1}}\,dx \le CR^{\frac{2m}{m-1}}\int_{\frac{1}{2R}}^{\frac{1}{R}} x^{\frac{a}{m-1}}\,dx$$
$$\le CR^{\frac{2m}{m-1}}R^{-\left(\frac{a}{m-1}+1\right)},$$

which proves (4.11). □

For $\ell, T, R \gg 1$, let us introduce the function

$$\psi(t,x)=\eta_T(t)\xi_R(x),\quad (t,x)\in S_T. \tag{4.16}$$

The following lemma follows immediately from (4.1), (4.9), and (4.16).

Lemma 4.5. *The function ψ belongs to Ψ_T.*

We now introduce the nonlinear capacity terms

$$J(a,m,0,\psi) = \int_{S_T} x^{\frac{a}{m-1}}\psi^{\frac{-1}{m-1}}\left|\frac{\partial\psi}{\partial t}\right|^{\frac{m}{m-1}}\,dx\,dt, \tag{4.17}$$

$$J(a,m,\kappa,\psi) = \int_{S_T} x^{\frac{a}{m-1}}\psi^{\frac{-1}{m-1}}\left|\frac{\partial I_T^\kappa\psi}{\partial t}\right|^{\frac{m}{m-1}}\,dx\,dt, \tag{4.18}$$

$$K(a,m,\kappa,\psi) = \int_{S_T} x^{\frac{a}{m-1}}\psi^{\frac{-1}{m-1}}\left|\frac{\partial^2}{\partial x^2}\left(\frac{\partial I_T^\kappa\psi}{\partial t}\right)\right|^{\frac{m}{m-1}}\,dx\,dt, \tag{4.19}$$

where $a\ge 0$, $m>1$, and $0<\kappa<1$.

Lemma 4.6. *Let $a\ge 0$ and $m>1$. We have*

$$J(a,m,0,\psi)\le CT^{1-\frac{m}{m-1}}. \tag{4.20}$$

Proof. By (4.16) and (4.17), we have

$$J(a,m,0,\psi)=\left(\int_0^T \eta_T^{\frac{-1}{m-1}}\left|\frac{d\eta_T}{dt}\right|^{\frac{m}{m-1}}\,dt\right)\left(\int_0^1 x^{\frac{a}{m-1}}\xi_R(x)\,dx\right).$$

Then, using (4.4) and Lemma 4.3, we obtain (4.20). □

Similarly, by (4.16), (4.18), (4.5), and Lemma 4.3, we obtain the following estimate:

Lemma 4.7. *Let $a\ge 0$, $m>1$, and $0<\kappa<1$. We have*

$$J(a,m,\kappa,\psi)\le CT^{1-\frac{(1-\kappa)m}{m-1}}.$$

We now use (4.16), (4.19), (4.5), and Lemma 4.4 to obtain the following estimate:

Lemma 4.8. *Let $a\ge 0$, $m>1$, and $0<\kappa<1$. We have*

$$K(a,m,\kappa,\psi)\le CT^{1-\frac{(1-\kappa)m}{m-1}}R^{\frac{2m-a}{m-1}-1}.$$

5. Proofs of the main results

This section is devoted to the proofs of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. Let us suppose that $u \in L^p_{\text{loc}}([0,\infty)\times(0,1])$ is a weak solution to (1.1)-(1.2)-(1.3). By (3.1) and Lemma 4.5, for all $\ell, T, R \gg 1$, we have

$$\begin{aligned}&\int_{S_T} x^{-\sigma}|u|^p\psi\,dx\,dt + \int_0^1 u_0(x)\left(\psi(0,x) + I_T^{1-\alpha}\psi(0,x) - I_T^{1-\beta}\frac{\partial^2\psi}{\partial x^2}(0,x)\right)dx\\ &\leq \int_{S_T}|u|\left|\frac{\partial\psi}{\partial t}\right|dx\,dt + \int_{S_T}|u|\left|\frac{\partial I_T^{1-\alpha}\psi}{\partial t}\right|dx\,dt + \int_{S_T}|u|\left|\frac{\partial^2}{\partial x^2}\left(\frac{\partial I_T^{1-\beta}\psi}{\partial t}\right)\right|dx\,dt,\end{aligned} \tag{5.1}$$

where ψ is the function defined by (4.16). On the other hand, because of Young's inequality, we have

$$\begin{aligned}\int_{S_T}|u|\left|\frac{\partial\psi}{\partial t}\right|dx\,dt &= \int_{S_T}\left(x^{\frac{-\sigma}{p}}|u|\psi^{\frac{1}{p}}\right)\left(x^{\frac{\sigma}{p}}\psi^{\frac{-1}{p}}\left|\frac{\partial\psi}{\partial t}\right|\right)dx\,dt\\ &\leq \frac{1}{3}\int_{S_T}x^{-\sigma}|u|^p\psi\,dx\,dt + C\int_{S_T}x^{\frac{\sigma}{p-1}}\psi^{\frac{-1}{p-1}}\left|\frac{\partial\psi}{\partial t}\right|^{\frac{p}{p-1}},\end{aligned}$$

that is,

$$\int_{S_T}|u|\left|\frac{\partial\psi}{\partial t}\right|dx\,dt \leq \frac{1}{3}\int_{S_T}x^{-\sigma}|u|^p\psi\,dx\,dt + CJ(\sigma,p,0,\psi), \tag{5.2}$$

where $J(\sigma,p,0,\psi)$ is given by (4.17) with $a=\sigma$ and $m=p$. Similarly, we obtain

$$\int_{S_T}|u|\left|\frac{\partial I_T^{1-\alpha}\psi}{\partial t}\right|dx\,dt \leq \frac{1}{3}\int_{S_T}x^{-\sigma}|u|^p\psi\,dx\,dt + CJ(\sigma,p,1-\alpha,\psi), \tag{5.3}$$

where $J(\sigma,p,1-\alpha,\psi)$ is given by (4.18) with $a=\sigma$, $m=p$, $\kappa=1-\alpha$, and

$$\int_{S_T}|u|\left|\frac{\partial^2}{\partial x^2}\left(\frac{\partial I_T^{1-\beta}\psi}{\partial t}\right)\right|dx\,dt \leq \frac{1}{3}\int_{S_T}x^{-\sigma}|u|^p\psi\,dx\,dt + CK(\sigma,p,1-\beta,\psi), \tag{5.4}$$

where $K(\sigma,p,1-\beta,\psi)$ is given by (4.19) with $a=\sigma$, $m=p$, and $\kappa=1-\beta$. Then, it follows from (5.1)–(5.4) that

$$\begin{aligned}&\int_0^1 u_0(x)\left(\psi(0,x) + I_T^{1-\alpha}\psi(0,x) - I_T^{1-\beta}\frac{\partial^2\psi}{\partial x^2}(0,x)\right)dx\\ &\leq C\left(J(\sigma,p,0,\psi) + J(\sigma,p,1-\alpha,\psi) + K(\sigma,p,1-\beta,\psi)\right).\end{aligned} \tag{5.5}$$

Furthermore, by (4.16) and (4.2) (with $\kappa\in\{1-\alpha,1-\beta\}$), for all $x\in(0,1)$, we have

$$\psi(0,x) = (1-x)\xi^\ell(Rx),\ \ I_T^{1-\alpha}\psi(0,x) = C_1T^{1-\alpha}(1-x)\xi^\ell(Rx)$$

and

$$I_T^{1-\beta}\frac{\partial^2\psi}{\partial x^2}(0,x) = C_2T^{1-\beta}\frac{d^2}{dx^2}\left[(1-x)\xi^\ell(Rx)\right].$$

We also have by (4.13) that

$$\operatorname{supp}\left(I_T^{1-\beta}\frac{\partial^2\psi}{\partial x^2}(0,x)\right)\subset\left[\frac{1}{2R},\frac{1}{R}\right].$$

Consequently, we obtain

$$\begin{aligned}&\int_0^1 u_0(x)\left(\psi(0,x)+I_T^{1-\alpha}\psi(0,x)-I_T^{1-\beta}\frac{\partial^2\psi}{\partial x^2}(0,x)\right)dx\\&=T^{1-\alpha}\left(T^{\alpha-1}+C_1\right)\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx\\&\quad-C_2T^{1-\beta}\int_{\frac{1}{2R}}^{\frac{1}{R}}u_0(x)\frac{d^2}{dx^2}\left[(1-x)\xi^\ell(Rx)\right]dx.\end{aligned}\tag{5.6}$$

Next, using (5.5), (5.6), Lemma 4.6 (with $a=\sigma$ and $m=p$), Lemma 4.7 (with $a=\sigma$, $m=p$, and $\kappa=1-\alpha$), and Lemma 4.8 (with $a=\sigma$, $m=p$, and $\kappa=1-\beta$), we obtain

$$\begin{aligned}&T^{1-\alpha}\left(T^{\alpha-1}+C_1\right)\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx-C_2T^{1-\beta}\int_{\frac{1}{2R}}^{\frac{1}{R}}u_0(x)\frac{d^2}{dx^2}\left[(1-x)\xi^\ell(Rx)\right]dx\\&\le C\left(T^{1-\frac{p}{p-1}}+T^{1-\frac{\alpha p}{p-1}}+T^{1-\frac{\beta p}{p-1}}R^{\frac{2p-\sigma}{p-1}-1}\right),\end{aligned}$$

that is,

$$\begin{aligned}&\left(T^{\alpha-1}+C_1\right)\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx-C_2T^{\alpha-\beta}\int_{\frac{1}{2R}}^{\frac{1}{R}}u_0(x)\frac{d^2}{dx^2}\left[(1-x)\xi^\ell(Rx)\right]dx\\&\le C\left(T^{\alpha-\frac{p}{p-1}}+T^{\alpha\left(1-\frac{p}{p-1}\right)}+T^{\alpha-\frac{\beta p}{p-1}}R^{\frac{2p-\sigma}{p-1}-1}\right).\end{aligned}\tag{5.7}$$

We now take $T=R^\theta$, where

$$\theta>\max\left\{\frac{2}{\beta-\alpha},\frac{\frac{2p-\sigma}{p-1}+1}{\frac{\beta p}{p-1}-\alpha}\right\}.\tag{5.8}$$

Then, (5.7) reduces to

$$\begin{aligned}&\left(R^{\theta(\alpha-1)}+C_1\right)\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx-C_2R^{\theta(\alpha-\beta)}\int_{\frac{1}{2R}}^{\frac{1}{R}}u_0(x)\frac{d^2}{dx^2}\left[(1-x)\xi^\ell(Rx)\right]dx\\&\le C\left(R^{\lambda_1(\theta)}+R^{\lambda_2(\theta)}+R^{\lambda_3(\theta)}\right),\end{aligned}\tag{5.9}$$

where

$$\begin{aligned}\lambda_1(\theta)&=\theta\left(\alpha-\frac{p}{p-1}\right),\\\lambda_2(\theta)&=\theta\alpha\left(1-\frac{p}{p-1}\right),\\\lambda_3(\theta)&=\theta\left(\alpha-\frac{\beta p}{p-1}\right)+\frac{2p-\sigma}{p-1}-1.\end{aligned}$$

Remark that for all $\theta > 0$, we have

$$\lambda_i(\theta) < 0, \;\; i = 1, 2. \tag{5.10}$$

Moreover, by (5.8), we have

$$\lambda_3(\theta) < 0. \tag{5.11}$$

On the other hand, by the properties of ξ and since $u_0 \in L^1((0,1))$, we obtain by the dominated convergence theorem that

$$\lim_{R\to\infty} \left(R^{\theta(\alpha-1)} + C_1\right) \int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx = C_1 \int_0^1 u_0(x)(1-x)\,dx. \tag{5.12}$$

We also have from the proof of Lemma 4.4 that

$$\begin{aligned} R^{\theta(\alpha-\beta)} \int_{\frac{1}{2R}}^{\frac{1}{R}} |u_0(x)| \left|\frac{d^2}{dx^2}\left[(1-x)\xi^\ell(Rx)\right]\right| dx &\le CR^{\theta(\alpha-\beta)}R^2 \int_{\frac{1}{2R}}^{\frac{1}{R}} |u_0(x)|\xi^{\ell-2}(Rx)\,dx \\ &= CR^{\theta(\alpha-\beta)+2} \int_{\frac{1}{2R}}^{\frac{1}{R}} |u_0(x)|\xi^{\ell-2}(Rx)\,dx \\ &\le CR^{\theta(\alpha-\beta)+2} \int_{\frac{1}{2R}}^{\frac{1}{R}} |u_0(x)|\,dx. \end{aligned}$$

Since $\theta(\alpha-\beta)+2 < 0$ (by (5.8)) and $u_0 \in L^1((0,1))$, we obtain by the dominated convergence theorem that

$$\lim_{R\to\infty} R^{\theta(\alpha-\beta)+2} \int_{\frac{1}{2R}}^{\frac{1}{R}} |u_0(x)|\,dx = 0,$$

which yields

$$\lim_{R\to\infty} R^{\theta(\alpha-\beta)} \int_{\frac{1}{2R}}^{\frac{1}{R}} |u_0(x)| \left|\frac{d^2}{dx^2}\left[(1-x)\xi^\ell(Rx)\right]\right| dx = 0. \tag{5.13}$$

Finally, passing to the limit as $R \to \infty$ in (5.9), using (5.10)–(5.13), we obtain

$$\int_0^1 u_0(x)(1-x)\,dx \le 0,$$

which contradicts (3.2). The proof of Theorem 3.1 is then completed. □

Proof of Theorem 3.2. We also use the contradiction argument. Namely, supposing that $u \in L^p_{\text{loc}}([0,\infty)\times(0,1])$ is a weak solution to (1.1)–(1.3), and following the first steps of the proof of Theorem 3.1, we obtain (5.7), which is equivalent to

$$\begin{aligned} &\left(T^{\alpha-1} + C_1\right) \int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx \\ &\le C\left(T^{\alpha-\frac{p}{p-1}} + T^{\alpha\left(1-\frac{p}{p-1}\right)} + T^{\alpha-\frac{\beta p}{p-1}} R^{\frac{2p-\sigma}{p-1}-1}\right) + C_2 T^{\alpha-\beta} \int_{\frac{1}{2R}}^{\frac{1}{R}} u_0(x) \frac{d^2}{dx^2}\left[(1-x)\xi^\ell(Rx)\right] dx. \end{aligned} \tag{5.14}$$

On the other hand, from (3.3) and the proof of Lemma 4.4, we have (for $R \gg 1$)

$$\begin{aligned}\int_{\frac{1}{2R}}^{\frac{1}{R}} |u_0(x)| \left| \frac{d^2}{dx^2}\left[(1-x)\xi^\ell(Rx)\right]\right| dx &\leq CR^2 \int_{\frac{1}{2R}}^{\frac{1}{R}} |u_0(x)|\xi^{\ell-2}(Rx)\,dx \\ &\leq CR^2 \int_0^{\frac{1}{R}} |u_0(x)|\,dx \\ &= CR^2 \int_0^{\frac{1}{R}} x^\delta\,dx \\ &= CR^{1-\delta},\end{aligned}$$

which implies by (5.14) that

$$\begin{aligned}&\left(T^{\alpha-1}+C_1\right)\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx \\ &\leq C\left(T^{\alpha-\frac{p}{p-1}} + T^{\alpha\left(1-\frac{p}{p-1}\right)} + T^{\alpha-\frac{\beta p}{p-1}}R^{\frac{2p-\sigma}{p-1}-1} + T^{\alpha-\beta}R^{1-\delta}\right).\end{aligned} \tag{5.15}$$

We now take $T = R^\theta$, where $\theta > 0$, and (5.15) reduces to

$$\begin{aligned}&\left(R^{\theta(\alpha-1)}+C_1\right)\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx \\ &\leq C\left(R^{\theta\left(\alpha-\frac{p}{p-1}\right)} + R^{\theta\alpha\left(1-\frac{p}{p-1}\right)} + R^{\theta\left(\alpha-\frac{\beta p}{p-1}\right)+\frac{2p-\sigma}{p-1}-1} + R^{\theta(\alpha-\beta)+1-\delta}\right).\end{aligned} \tag{5.16}$$

We first consider

(i) The case $\alpha = \beta$. In this case, (5.16) reduces to

$$\left(R^{\theta(\alpha-1)}+C_1\right)\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx \leq C\left(R^{\mu_1(\theta)} + R^{\mu_2(\theta)} + R^{\mu_3(\theta)} + R^{\mu_4}\right), \tag{5.17}$$

where

$$\mu_1(\theta) = \theta\left(\alpha - \frac{p}{p-1}\right), \tag{5.18}$$

$$\mu_2(\theta) = \theta\alpha\left(1 - \frac{p}{p-1}\right), \tag{5.19}$$

$$\begin{aligned}\mu_3(\theta) &= \theta\alpha\left(1-\frac{p}{p-1}\right) + \frac{2p-\sigma}{p-1} - 1, \\ \mu_4 &= 1-\delta.\end{aligned}$$

Remark that for all $\theta > 0$, we have

$$\mu_i(\theta) < 0, \;\; i = 1, 2. \tag{5.20}$$

We also have (since $\delta > 1$)

$$\mu_4 < 0. \tag{5.21}$$

Furthermore, imposing that

$$\theta > \max\left\{0, \frac{p+1-\sigma}{\alpha}\right\},$$

we obtain

$$\mu_3(\theta) < 0. \tag{5.22}$$

Then, passing to the limit as $R \to \infty$ in (5.17), using the dominated convergence theorem, (5.20)–(5.22), we reach a contradiction with (3.2).

We next consider

(ii) The case $\alpha > \beta$, $\sigma \geq 2$ and p satisfies (3.4). In this case, we take

$$\theta = \frac{\delta(p-1)+2-\sigma}{\beta}.$$

Notice that by (3.4), we have $\theta > 0$. Furthermore, we have

$$\theta\left(\alpha - \frac{\beta p}{p-1}\right) + \frac{2p-\sigma}{p-1} - 1 = \theta(\alpha-\beta) + 1 - \delta = 1 - \delta + \frac{(\alpha-\beta)}{\beta}\left[\delta(p-1)+2-\sigma\right].$$

Thus, (5.16) reduces to

$$\left(R^{\theta(\alpha-1)} + C_1\right)\int_0^1 u_0(x)(1-x)\xi^{\ell}(Rx)\,dx \leq C\left(R^{\mu_1(\theta)} + R^{\mu_2(\theta)} + R^{\mu}\right), \tag{5.23}$$

where $\mu_1(\theta)$ and $\mu_2(\theta)$ are given, respectively, by (5.18) and (5.19), and

$$\mu = 1 - \delta + \frac{(\alpha-\beta)}{\beta}\left[\delta(p-1)+2-\sigma\right].$$

Notice that from (3.4), we have

$$\mu < 0. \tag{5.24}$$

Hence, passing to the limit as $R \to \infty$ in (5.23), using the dominated convergence theorem, (5.20), and (5.24), we reach a contradiction with (3.2). This completes the proof of Theorem 3.2. □

6. Conclusions

Sufficient conditions are obtained for the nonexistence of weak solutions to the nonlinear time-fractional cable equation (1.1), subject to the initial condition (1.2) and the boundary condition (1.3). Two cases are studied. In the first one (see Theorem 3.1), it is assumed that $0 < \alpha < \beta < 1$. If the initial function satisfies

$$\int_0^1 u_0(x)(1-x)\,dx > 0,$$

it is proven that for all $p > 1$, the problem has no weak solution. In the second case (see Theorem 3.2), it is assumed that $0 < \beta \leq \alpha < 1$. If u_0 satisfies the above integral condition and $|u_0(x)| \sim x^{\delta}$ as $x \to 0^+$, where $\delta > 1$, it is proven that the problem has no weak solution if one of the following conditions holds: $\alpha = \beta$; or $\alpha > \beta$, $\sigma \geq 2$ and

$$1 + \frac{\sigma - 2}{\delta} < p < 1 + \frac{\sigma - 2}{\delta} + \frac{\beta(\delta - 1)}{\delta(\alpha - \beta)}.$$

In this paper, we only studied the one-dimensional case. It would be interesting to extend the present study to the N-dimensional case, where $N \geq 2$. Namely, the problem

$$\frac{\partial u}{\partial t} + \frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\beta}{\partial t^\beta}\Delta u = F(x, u), \quad t > 0,\ x \in B(0, 1)$$

subject to the initial condition

$$u(0, x) = u_0(x), \quad x \in B(0, 1)$$

and the boundary condition

$$u(t, x) = 0, \quad t > 0,\ |x| = 1,$$

where Δ is the Laplacian operator in $\mathbb{R}^N$, $B(0, 1)$ is the open unit ball in $\mathbb{R}^N$ and $F(x, u) = |x|^{-\sigma}|u|^p$, $\sigma \geq 0$, $p > 1$.

Author contributions

Both authors contributed equally and significantly in writing this paper. All authors have read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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