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Research article

On the upper bounds for the distance between zeros of solutions of a first-order linear neutral differential equation with several delays

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Abstract: This work is devoted to studying the distribution of zeros of a first-order neutral differential equation with several delays

$$[y(t) + a(t)y(t-\sigma)]' + \sum_{j=1}^{n} b_j(t)y\left(t-\mu_j\right) = 0, \qquad t \geq t_0.$$

New estimations for the upper bounds of the distance between successive zeros are obtained. The properties of a positive solution of a first-order differential inequality with several delays in a closed interval are studied, and many results are established. We apply these results to a first-order neutral differential equation with several delays and also to a first-order differential equation with several delays. Our results for the differential equation with several delays not only provide new estimations but also improve many previous ones. Also, the results are formulated in a general way such that they can be applied to any functional differential equation for which studying the distance between zeros is equivalent to studying this property for a first-order differential inequality with several delays. Further, new estimations of the upper bounds for certain equations are given. Finally, a comparison with all previous results is shown at the end of this paper.

Keywords: neutral differential equations; several delays; oscillation; distance between zeros
Mathematics Subject Classification: 34K11, 39A10, 39A99

1. Introduction

Consider the first-order neutral differential equation with several delays

$$[y(t) + a(t)y(t-\sigma)]' + \sum_{j=1}^{n} b_j(t)y\left(t-\mu_j\right) = 0, \qquad t \geq t_0, \tag{E_1}$$

where $a, b_j \in C([t_0,\infty),[0,\infty))$, $0 < \sigma < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, and the first-order differential equation with several delays

$$y'(t) + \sum_{j=1}^{n} b_j(t)y\left(t-\mu_j\right) = 0, \qquad t \geq t_0. \tag{E_2}$$

By a solution of Eq (E_1) on $[t^* + \mu_n, \infty)$, $t^* \geq t_0$, we mean a function $y \in C([t^*,\infty),\mathbb{R})$ such that $y(t) + a(t)y(t-\sigma) \in C^1([t^* + \mu_n - \sigma,\infty),\mathbb{R})$ and satisfies (E_1) for all $t \geq t^* + \mu_n$. The existence and uniqueness of a solution $y(t)$ of Eq (E_1) with an initial function $\phi \in C([t_0 - \mu_n, t_0],\mathbb{R})$ can be proved using the method of steps as in [24, Thereoem 1.1.2]. Also, the existence of a positive solution for some neutral differential equations can be found in [25].

A solution is said to be oscillatory if it is neither eventually negative nor eventually positive; otherwise, it is called non-oscillatory. If all solutions of a differential equation are oscillatory, then it is called oscillatory; otherwise, it is called non-oscillatory.

Recently, neutral differential equations have arisen in many applications; see [1,5,23,24,27–29,34–36]. In this type of equation, the delays appear in both the unknown function and its derivatives. The qualitative properties of neutral differential equations have received a great deal of attention from many mathematicians; see [1, 2, 5–7, 9, 12, 14–16, 19–24, 26–29, 34–38, 40–42]. In dynamical models, delay and oscillation effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [30, 31]. The oscillation of neutral and delay differential equations has been extensively developed and studied in many works; see [2, 5–7, 9, 12–16, 19–22, 24, 26, 37, 38, 40, 41]. On the other hand, the oscillation of first-order differential equations has numerous applications in the study of the oscillatory behavior of higher-order neutral differential equations; see [32, 33]. However, only a few works have been interested in studying the distance between zeros of all solutions for first-order neutral and delay differential equations; see [2–4, 6–8, 10, 11, 15, 17, 18, 37–41]. In this type of study, the interest is not only in the existence of zeros of solutions (i.e., proving the oscillation) but also in determining their locations. Therefore, studying the distance between zeros for first-order linear neutral and delay differential equations can give a deeper understanding of the dynamics of nonlinear systems of neutral differential equations that are used to model many real-life phenomena. This motivates us to study the distance between successive zeros for all solutions of Eqs (E_1) and (E_2).

In the following, we display some results for the distance between zeros for Eqs (E_1) and (E_2):

The distribution of zeros in first-order differential equations with one delay or several delays was studied by [3, 4, 8, 10, 11, 17, 18, 39]. Further, many estimations for the upper bounds of the distance between zeros for first-order neutral differential equations were established by [2,6,7,15,37,38,40,41]. However, there are no results dealing with the distribution of zeros in Eq (E_1).

In this work, we obtain sufficient conditions ensuring that any solution of the first-order differential inequality with several delays

$$Y'(t) + \sum_{j=1}^{n} B_j(t) Y\left(t - \eta_j\right) \leq 0, \qquad t \in [L + \alpha,\ L_0], \tag{1.1}$$

where $L_0 \geq L + \alpha$, $L \geq t_0$, $\alpha \geq 0$ such that $0 < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_n$, and $B_j \in C([L + \alpha, \infty), [0, \infty))$, $j = 1, 2, ..., n$, cannot be positive on certain intervals. By using these results, many new estimations for the upper bounds of the distance between zeros for both Eqs (E_1) and (E_2) are obtained. A new approximation for the distance between successive zeros of a certain differential equation of the form (E_2) is given, while all previous results cannot give this approximation. Since the distribution of zeros in all solutions of Eq (E_1) has never been examined before, our results for Eq (E_1) are obviously new. An illustrative example is given to show the applicability of our results to Eq (E_1).

2. First-order delay differential inequality

In this section, we study the properties of a positive solution to the first-order differential inequality with several delays (1.1).

Let $Y(t)$ be a solution of inequality (1.1) on $[L + \alpha,\ L_0]$, $L_0 \geq L + \alpha$ such that

$$Y(t) > 0 \quad \text{for } t \in [L + \gamma,\ L + \alpha] \quad \text{and} \quad Y'(t) \leq 0 \quad \text{for } t \in [L + \delta,\ L + \beta], \tag{2.1}$$

where $\gamma, \delta \geq 0$ and $\beta = \max\{\alpha, \eta_n + \gamma\}$.

Next, we prove some lemmas that play an important role in establishing the main results of this work.

Assume that $s \in \{1, 2, ..., n\}$ and the sequence of nonnegative real numbers $\{M_l^s\}_{l \geq -1}$ is defined by

$$M_{-1}^s = 0, \quad M_0^s = 1, \qquad \frac{e^{\int_{t-\eta_s}^{t} \sum_{j=1, j \neq s}^{n} M_{l-1}^j B_j(v) dv}}{1 - \int_{t-\eta_s}^{t} B_s(v) dv} \geq M_l^s, \qquad l = 1, 2, \ldots . \tag{2.2}$$

Lemma 2.1. *Assume that $l \in \mathbb{N}_0$, $Y(t)$ is a solution of inequality (1.1) on $[L + \alpha,\ L_0]$ such that (2.1) is satisfied with $L_0 \geq L + \max\{\delta,\ \alpha\} + (l + 1)\eta_n$ and $Y(t) > 0$ on $[L + \alpha, L_0]$. Then*

$$\frac{Y(t - \eta_s)}{Y(t)} \geq M_l^s \qquad \text{for } t \in [L + \max\{\delta,\ \alpha\} + (l + 1)\eta_n, L_0] \tag{2.3}$$

for $s = 1, 2, ..., n$.

Proof. In view of $Y'(t) \leq 0$ for $t \in [L + \delta, L + \beta]$ and $Y(t) > 0$ for $t \in [L + \gamma, L_0]$, it follows from (1.1) that

$$Y'(t) \leq 0 \qquad \text{for } t \in [L + \delta, L_0]. \tag{2.4}$$

Therefore,

$$\frac{Y(t - \eta_s)}{Y(t)} \geq 1 = M_0^s \qquad \text{for } t \in [L + \delta + \eta_s, L_0] \subseteq [L + \max\{\delta,\ \alpha\} + \eta_n, L_0]\ . \tag{2.5}$$

Integrating (1.1) from $t-\eta_s$ to t, it follows that

$$Y(t)-Y(t-\eta_s)+\int_{t-\eta_s}^{t}\sum_{j=1}^{n}B_j(v)Y(v-\eta_j)dv\le 0 \qquad \text{for } t\in[L+\alpha+\eta_s, L_0].$$

Therefore,

$$Y(t)-Y(t-\eta_s)+\int_{t-\eta_s}^{t}B_s(v)Y(v-\eta_s)dv+\sum_{\substack{j=1\\ j\neq s}}^{n}\int_{t-\eta_s}^{t}B_j(v)Y(v)\frac{Y(v-\eta_j)}{Y(v)}dv\le 0 \tag{2.6}$$

for $t\in[L+\alpha+\eta_n, L_0]$. Dividing (1.1) by $Y(t)$, and integrating the resulting inequality from ζ to μ, $L+\alpha+\eta_n\le\zeta\le\mu\le L_0$, we obtain

$$Y(\zeta)\ge Y(\mu)\mathrm{e}^{\int_\zeta^\mu\sum_{j_1=1}^{n}B_{j_1}(v)\frac{Y(v-\eta_{j_1})}{Y(v)}dv}. \tag{2.7}$$

Substituting into (2.6), we obtain

$$Y(t)-Y(t-\eta_s)+\int_{t-\eta_s}^{t}B_s(v)Y(v-\eta_s)dv+Y(t)\sum_{\substack{j=1\\ j\neq s}}^{n}\int_{t-\eta_s}^{t}B_j(v)\frac{Y(v-\eta_j)}{Y(v)}\mathrm{e}^{\int_v^t\sum_{j_1=1}^{n}B_{j_1}(v_1)\frac{Y(v_1-\eta_{j_1})}{Y(v_1)}dv_1}dv\le 0$$

for $t\in[L+\alpha+\eta_n+\eta_s, L_0]$.

By (2.4), we have

$$Y(t)-Y(t-\eta_s)+Y(t-\eta_s)\int_{t-\eta_s}^{t}B_s(v)dv+Y(t)\int_{t-\eta_s}^{t}\sum_{\substack{j=1\\ j\neq s}}^{n}B_j(v)\frac{Y(v-\eta_j)}{Y(v)}\mathrm{e}^{\int_v^t\sum_{\substack{j_1=1\\ j_1\neq s}}^{n}B_{j_1}(v_1)\frac{Y(v_1-\eta_{j_1})}{Y(v_1)}dv_1}dv\le 0$$

for $t\in[L+\max\{\delta+\eta_s,\ \alpha+\eta_n\}+\eta_s, L_0]$. This leads to

$$Y(t)-Y(t-\eta_s)+Y(t-\eta_s)\int_{t-\eta_s}^{t}B_s(v)dv+Y(t)\left(\mathrm{e}^{\int_{t-\eta_s}^{t}\sum_{\substack{j=1\\ j\neq s}}^{n}B_j(v)\frac{Y(v-\eta_j)}{Y(v)}dv}-1\right)\le 0$$

for $t\in[L+\max\{\delta+\eta_s,\ \alpha+\eta_n\}+\eta_s, L_0]$. That is,

$$Y(t-\eta_s)\left(1-\int_{t-\eta_s}^{t}B_s(v)dv\right)\ge Y(t)\mathrm{e}^{\int_{t-\eta_s}^{t}\sum_{\substack{j=1\\ j\neq s}}^{n}B_j(v)\frac{Y(v-\eta_j)}{Y(v)}dv}$$

for $t\in[L+\max\{\delta+\eta_s,\ \alpha+\eta_n\}+\eta_s, L_0]$. Therefore

$$\frac{Y(t-\eta_s)}{Y(t)}\ge\frac{\mathrm{e}^{\int_{t-\eta_s}^{t}\sum_{\substack{j=1\\ j\neq s}}^{n}B_j(v)\frac{Y(v-\eta_j)}{Y(v)}dv}}{1-\int_{t-\eta_s}^{t}B_s(v)dv}>0 \quad \text{for } t\in[L+\max\{\delta+\eta_s,\ \alpha+\eta_n\}+\eta_s, L_0]. \tag{2.8}$$

In view of (2.5), we obtain

$$\frac{Y(t-\eta_s)}{Y(t)}\ge\frac{\mathrm{e}^{\int_{t-\eta_s}^{t}\sum_{\substack{j=1\\ j\neq s}}^{n}M_0^jB_j(v)dv}}{1-\int_{t-\eta_s}^{t}B_s(v)dv}\ge M_1^s$$

for $t \in [L + \max\{\delta + \eta_s,\ \alpha + \eta_n\} + \eta_n, L_0] \subseteq [L + \max\{\delta,\ \alpha\} + 2\eta_n, L_0]$. Therefor,

$$\frac{Y(v-\eta_j)}{Y(v)} \geq M_1^j, \qquad \text{for } t \in [L + \max\{\delta + \eta_j,\ \alpha + \eta_n\} + \eta_n, L_0], \quad 1 \leq j \leq n.$$

Substituting into (2.8), we have

$$\frac{Y(t-\eta_s)}{Y(t)} \geq \frac{\mathrm{e}^{\int_{t-\eta_s}^{t} \sum_{\substack{j=1\\ j\neq s}}^{n} M_1^j B_j(v)dv}}{1 - \int_{t-\eta_s}^{t} B_s(v)dv} \geq M_2^s$$

for $t \in [L + \max\{\delta,\ \alpha\} + 3\eta_n, L_0]$. By repeating this procedure, we obtain (2.3). The proof is complete.
□

Lemma 2.2. *Assume that $l \in \mathbb{N}_0$, $0 < \epsilon \leq 1$, and $Y(t)$ is a solution of inequality (1.1) on $[L + \alpha,\ L_0]$ such that (2.1) is satisfied with $L_0 \geq L + \max\{\delta,\ \alpha\} + (l + 1 + \epsilon)\eta_n$. If,*

$$\prod_{s=1}^{n} \frac{\left(\prod_{\substack{j=1\\ j\neq s}}^{n} \int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j} \sum_{j_1=1}^{n} B_{j_1}(v_1)M_{l-1}^{j_1}dv_1} dv\right)^{\frac{1}{n-1}}}{1 - \int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s} \sum_{j=1}^{n} B_j(v_1)M_{l-1}^{j}dv_1} dv} \geq \frac{1}{(n-1)^n} \quad \textit{for } t \geq L + 2\eta_n, \tag{2.9}$$

then $Y(t)$ cannot be positive on $[L + \alpha,\ L_0]$.

Proof. Assume the contrary, and let $Y(t) > 0$ on $[L + \alpha,\ L_0]$. This together with (2.1) implies that $Y(t) > 0$ on $[L + \gamma,\ L_0]$. Form (2.4), we have

$$Y'(t) \leq 0 \qquad \text{for } t \in [L + \delta, L_0]. \tag{2.10}$$

Integrating (1.1) from t to $t - \epsilon\eta_s$, $s = 1, 2, \ldots, n$, we obtain

$$Y(t) - Y(t - \epsilon\eta_s) + \int_{t-\epsilon\eta_s}^{t} \sum_{j=1}^{n} B_j(v)Y\left(v - \eta_j\right) dv \leq 0 \qquad \text{for } t \in [L + \alpha + \epsilon\eta_s,\ L_0].$$

Therefore,

$$Y(t) - Y(t - \epsilon\eta_s) + \int_{t-\epsilon\eta_s}^{t} B_s(v)Y(v - \eta_s)\, dv + \int_{t-\epsilon\eta_s}^{t} \sum_{\substack{j=1\\ j\neq s}}^{n} B_j(v)Y\left(v - \eta_j\right) dv \leq 0 \tag{2.11}$$

for $t \in [L + \alpha + \epsilon\eta_s,\ L_0]$. By (2.10), we obtain

$$Y(t) - Y(t - \epsilon\eta_s) + Y(t - \eta_s) \int_{t-\epsilon\eta_s}^{t} B_s(v)dv + \sum_{\substack{j=1\\ j\neq s}}^{n} Y\left(t - \eta_j\right) \int_{t-\epsilon\eta_s}^{t} B_j(v)dv \leq 0$$

for $t \in [L + \max\{\delta,\ \alpha\} + \epsilon\eta_s + \eta_n,\ L_0]$. That is,

$$Y(t - \epsilon\eta_s) \quad \geq \quad Y(t) + Y(t - \epsilon\eta_s) \int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s} \sum_{j=1}^{n} B_j(v_1)M_{-1}^{j}dv_1} dv$$

$$+\sum_{\substack{j=1\\j\neq s}}^{n} Y\left(t-\epsilon\eta_j\right)\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n} B_{j_1}(v_1)M_{-1}^{j_1}dv_1}dv \tag{2.12}$$

for $t\in[L+\max\{\delta,\ \alpha\}+\epsilon\eta_s+\eta_n,\ L_0]$.

From (2.7) and (2.11), we have

$$\begin{aligned} Y(t-\epsilon\eta_s) \ &\geq\ Y(t)+Y\left(t-\epsilon\eta_s\right)\int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s}\sum_{j_1=1}^{n} B_{j_1}(v_1)\frac{Y(v_1-\eta_{j_1})}{Y(v_1)}dv_1}dv \\ &\quad+\sum_{\substack{j=1\\j\neq s}}^{n} Y\left(t-\epsilon\eta_j\right)\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n} B_{j_1}(v_1)\frac{Y(v_1-\eta_{j_1})}{Y(v_1)}dv_1}dv \end{aligned}$$

for $t\in[L+\alpha+\epsilon\eta_s+2\eta_n,\ L_0]$. This, together with (2.3) leads to

$$\begin{aligned} Y(t-\epsilon\eta_s) \ &\geq\ Y(t)+Y\left(t-\epsilon\eta_s\right)\int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s}\sum_{j=1}^{n} B_{j}(v_1)M_{l-1}^{j}dv_1}dv \\ &\quad+\sum_{\substack{j=1\\j\neq s}}^{n} Y\left(t-\epsilon\eta_j\right)\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n} B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv \qquad l=1,2,\ldots \end{aligned}$$

for $t\in[L+\max\{\delta,\ \alpha\}+(l+1)\eta_n+\epsilon\eta_s, L_0]$. From this and (2.12), we have

$$\begin{aligned} &Y(t-\epsilon\eta_s)\left(1-\int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s}\sum_{j=1}^{n} B_{j}(v_1)M_{l-1}^{j}dv_1}dv\right) > \\ &\sum_{\substack{j=1\\j\neq s}}^{n} Y\left(t-\epsilon\eta_j\right)\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n} B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv, \qquad l=0,1,2,\ldots \end{aligned}$$

for $t\in[L+\max\{\delta,\ \alpha\}+(l+1)\eta_n+\epsilon\eta_s, L_0]$. Using the arithmetic-geometric mean, we obtain

$$\begin{aligned} &Y(t-\epsilon\eta_s)\left(1-\int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s}\sum_{j=1}^{n} B_{j}(v_1)M_{l-1}^{j}dv_1}dv\right) > \\ &(n-1)\left(\prod_{\substack{j=1\\j\neq s}}^{n} Y\left(t-\epsilon\eta_j\right)\right)^{\frac{1}{n-1}}\left(\prod_{\substack{j=1\\j\neq s}}^{n}\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n} B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv\right)^{\frac{1}{n-1}} \end{aligned}$$

for $t\in[L+\max\{\delta,\ \alpha\}+(l+1)\eta_n+\epsilon\eta_s, L_0]$. Taking the product on both sides,

$$\begin{aligned} &\prod_{s=1}^{n} Y(t-\epsilon\eta_s)\prod_{s=1}^{n}\left(1-\int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s}\sum_{j=1}^{n} B_{j}(v_1)M_{l-1}^{j}dv_1}dv\right) > \\ &(n-1)^n\prod_{s=1}^{n}\left(\prod_{\substack{j=1\\j\neq s}}^{n} Y\left(t-\epsilon\eta_j\right)\right)^{\frac{1}{n-1}}\prod_{s=1}^{n}\left(\prod_{\substack{j=1\\j\neq s}}^{n}\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n} B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv\right)^{\frac{1}{n-1}} \end{aligned}$$

for $t \in [L + \max\{\delta,\ \alpha\} + (l + 1 + \epsilon)\eta_n, L_0]$. Therefore,

$$\prod_{s=1}^{n} Y(t-\epsilon\eta_s) \prod_{s=1}^{n}\left(1-\int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s}\sum_{j=1}^{n}B_j(v_1)M_{l-1}^{j}dv_1}dv\right) >$$

$$(n-1)^n \prod_{s=1}^{n} Y\left(t-\epsilon\eta_s\right) \prod_{s=1}^{n}\left(\prod_{\substack{j=1\\ j\neq s}}^{n}\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv\right)^{\frac{1}{n-1}}$$

for $t \in [L + \max\{\delta,\ \alpha\} + (l + 1 + \epsilon)\eta_n, L_0]$. Then

$$\prod_{s=1}^{n}\frac{\left(\prod_{\substack{j=1\\ j\neq s}}^{n}\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv\right)^{\frac{1}{n-1}}}{1-\int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s}\sum_{j=1}^{n}B_j(v_1)M_{l-1}^{j}dv_1}dv} < \frac{1}{(n-1)^n}$$

for $t \in [L + \max\{\delta,\ \alpha\} + (l + 1 + \epsilon)\eta_n, L_0]$. This contradicts with (2.9). The proof is complete. □

Lemma 2.3. *Assume that $l \in \mathbb{N}_0$, $1 \le k_1 \le k_2 \le n$, $0 < \epsilon \le 1$, and $Y(t)$ is a solution of inequality (1.1) on $[L + \alpha,\ L_0]$ such that (2.1) is satisfied with $L_0 \ge L + \max\{\delta,\ \alpha\} + (l + 1)\eta_n + \epsilon\eta_{k_2}$. If,*

$$\prod_{s=k_1}^{k_2}\frac{\int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s}\sum_{j=1}^{n}B_j(v_1)M_{l-1}^{j}dv_1}dv}{1-\sum_{j=s+1}^{n}\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_s}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv} \ge 1 \quad \text{for } t \ge T + \alpha + \epsilon\eta_{k_2} + \eta_n, \tag{2.13}$$

then $Y(t)$ cannot be positive on $[L + \alpha,\ L_0]$.

Proof. As before, assume that $Y(t) > 0$ on $[L+\alpha,\ L_0]$. Integrating (1.1) from t to $t-\epsilon\eta_s$, $s = 1, 2, \ldots, k_2$, it follows that

$$Y(t) - Y(t-\epsilon\eta_s) + \int_{t-\epsilon\eta_s}^{t}\sum_{j=1}^{n} B_j(v)Y(v-\eta_j)dv \le 0 \qquad \text{for } t \in [L+\alpha+\epsilon\eta_s,\ L_0].$$

Therefore,

$$Y(t) - Y(t-\epsilon\eta_s) + \int_{t-\epsilon\eta_s}^{t}\sum_{j=1}^{s} B_j(v)Y(v-\eta_j)dv + \int_{t-\epsilon\eta_s}^{t}\sum_{j=s+1}^{n} B_j(v)Y(v-\eta_j)dv \le 0$$

for $t \in [L + \alpha + \epsilon\eta_s,\ L_0]$. The same reasoning as in Lemma 2.2 leads to

$$\begin{aligned} Y(t-\epsilon\eta_s) \quad \ge \quad & Y(t) + \sum_{j=1}^{s} Y(t-\epsilon\eta_j)\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv \\ & + Y\left(t-\epsilon\eta_s\right)\sum_{j=s+1}^{n}\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_s}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv \end{aligned}$$

for $t \in [L + \max\{\delta,\ \alpha\} + (l + 1)\eta_n + \epsilon\eta_s, L_0]$. Consequently,

$$Y(t-\epsilon\eta_s)\left(1-\sum_{j=s+1}^{n}\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_s}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv\right) >$$

$$\sum_{j=1}^{s} Y(t-\epsilon\eta_j)\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv$$

for $t\in[L+\max\{\delta,\ \alpha\}+(l+1)\eta_n+\epsilon\eta_s, L_0]$. Taking the product on both sides,

$$\prod_{s=k_1}^{k_2} Y(t-\epsilon\eta_s)\prod_{s=k_1}^{k_2}\left(1-\sum_{j=s+1}^{n}\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_s}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv\right)>$$
$$\prod_{s=k_1}^{k_2}\left(\sum_{j=1}^{s} Y(t-\eta_j)\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_j}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv\right)$$

for $t\in[L+\max\{\delta,\ \alpha\}+(l+1)\eta_n+\epsilon\eta_{k_2}, L_0]$. That is,

$$\prod_{s=k_1}^{k_2} Y(t-\epsilon\eta_s)\prod_{s=k_1}^{k_2}\left(1-\sum_{j=s+1}^{n}\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_s}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv\right)>$$
$$\prod_{s=k_1}^{k_2} Y\left(t-\eta_s\right)\prod_{s=k_1}^{k_2}\left(\int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv\right)$$

for $t\in[L+\max\{\delta,\ \alpha\}+(l+1)\eta_n+\epsilon\eta_{k_2}, L_0]$. Therefore,

$$\prod_{s=k_1}^{k_2}\frac{\int_{t-\epsilon\eta_s}^{t} B_s(v)\mathrm{e}^{\int_{v-\eta_s}^{t-\epsilon\eta_s}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv}{1-\sum_{j=s+1}^{n}\int_{t-\epsilon\eta_s}^{t} B_j(v)\mathrm{e}^{\int_{v-\eta_j}^{t-\epsilon\eta_s}\sum_{j_1=1}^{n}B_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv}<1$$

for $t\in[L+\max\{\delta,\ \alpha\}+(l+1)\eta_n+\epsilon\eta_{k_2}, L_0]$, which contradicts (2.13). The proof is complete. □

3. Main results

Assume that $D_{t\geq t_1}(E)$, $t_1\geq t_0$ is the upper bound of the distance between successive zeros of all solutions of a differential equation E on $[t_1,\ \infty)$.

3.1. The distribution of zeros in a first-order differential equation with several delays

Below, we obtain new approximations for the upper bound of the distance between successive zeros of all solutions of Eq (E_2).

Theorem 3.1. *Let*

$$B_j(t)=b_j(t),\quad \eta_j=\mu_j,\quad j=1,2,\dots,n.$$

Assume that $l\in\mathbb{N}_0$, $0<\epsilon\leq 1$. *If,*

$$\prod_{s=1}^{n}\frac{\left(\prod_{\substack{j=1\\ j\neq s}}^{n}\int_{t-\epsilon\mu_s}^{t} b_j(v)\mathrm{e}^{\int_{v-\mu_j}^{t-\epsilon\mu_j}\sum_{j_1=1}^{n}b_{j_1}(v_1)M_{l-1}^{j_1}dv_1}dv\right)^{\frac{1}{n-1}}}{\left(1-\int_{t-\epsilon\mu_s}^{t} b_s(v)\mathrm{e}^{\int_{v-\mu_s}^{t-\epsilon\mu_s}\sum_{j=1}^{n}b_j(v_1)M_{l-1}^{j}dv_1}dv\right)}\geq\frac{1}{(n-1)^n}\quad \textit{for } t\geq t_0+2\mu_n,$$

then $D_{t_1}(E_2)\leq(l+2+\epsilon)\mu_n$ *and every solution of Eq* (E_2) *is oscillatory.*

Proof. Assume the contrary, and let $y(t)$ be a solution of Eq (E_2) such that $y(t) > 0$ on $[L,\, L_0]$, $L \geq t_0$, $L_0 \geq L + (l + 2 + \epsilon)\mu_n$. Therefore, $y'(t) \leq 0$ for $t \in [L + \mu_n,\, L_0]$. Then, all assumptions of Lemma 2.2 are satisfied with $\alpha = \gamma = 0$, $\delta = \beta = \mu_n$, $B_j(t) = b_j(t)$, and $\eta_j = \mu_j$, $j = 1, 2, \ldots, n$. Therefore, $y(t)$ cannot be positive on $[L,\, L + (l + 2 + \epsilon)\,\mu_n] \subseteq [L,\, L_0]$. This contradiction completes the proof. □

Using Lemma 2.3 instead of Lemma 2.2 in the proof of the preceding theorem, one can prove the following result, and hence the proof is omitted.

Theorem 3.2. *Let*

$$B_j(t) = b_j(t), \quad \eta_j = \mu_j, \quad j = 1, 2, \ldots, n.$$

Assume that $l \in \mathbb{N}_0$, $1 \leq k_1 \leq k_2 \leq n$, $0 < \epsilon \leq 1$. *If,*

$$\prod_{s=k_1}^{k_2} \frac{\int_{t-\epsilon\mu_s}^{t} b_s(v) e^{\int_{v-\mu_s}^{t-\epsilon\mu_s} \sum_{j_1=1}^{n} b_{j_1}(v_1) M_{l-1}^{j_1} dv_1} dv}{\left(1 - \sum_{j=s+1}^{n} \int_{t-\epsilon\mu_s}^{t} b_j(v) e^{\int_{v-\mu_j}^{t-\epsilon\mu_s} \sum_{j_1=1}^{n} b_{j_1}(v_1) M_{l-1}^{j_1} dv_1} dv\right)} \geq 1 \quad \text{for } t \geq t_0 + \alpha + \epsilon\mu_{k_2} + \mu_n, \tag{3.1}$$

then $D_{t_1}(E_2) \leq (l + 2)\mu_n + \epsilon\mu_{k_2}$ *and every solution of Eq* (E_2) *is oscillatory.*

3.2. The distribution of zeros in a first-order neutral differential equation with several delays

We obtain many upper bounds for the distance between zeros of all solutions of Eq (E_1) with the following assumptions:

(H1) Let $R \in C^1[[t_* + \sigma, \infty),\, [0, \infty)]$, $t_* \geq t_0$, $b_j(t)a(t - \mu_j) \leq b_j(t - \sigma)a(t - \sigma)$, $j = 1, 2, \ldots, n$, and $R(t) \geq a(t - \sigma)$, $t \geq t_1 + \sigma$ for some $t_1 \geq t_*$.

(H2) Let $N \in C^1[[t_* + \mu_n, \infty),\, [0, \infty)]$, $t_* \geq t_0$, $b_j(t) > 0$, $j = 1, 2, \ldots, n$, $N(t) \geq \sum_{j=1}^{n} \frac{b_j(t)a(t-\mu_j)}{b_j(t-\sigma)}$, $t \geq t_1 + \mu_n$ for some $t_1 \geq t_*$.

Lemma 3.1. *Assume that (H1) holds,* $R'(t) \leq 0$ *for* $t \geq t_1 + \sigma$. *Let* $y(t)$ *be a solution of Eq* (E_1) *such that* $y(t) > 0$ *on* $[L,\, L_0]$, $L \geq L_0 + 2\mu_n$, $L \geq t_1$. *Then, there exists a function* $z(t)$ *that satisfies* $z(t) > 0$ *on* $[L + 2\sigma,\, L_0]$ *and* $z'(t) \leq 0$ *on* $[L + \sigma + \mu_n,\, L_0]$, *and*

$$z'(t) + \sum_{j=1}^{n} \frac{b_j(t)}{1 + R(t)} z(t + \sigma - \mu_j) \leq 0 \qquad \text{for } t \in [L + 2\mu_n\, L_0]. \tag{3.2}$$

Proof. Letting $u(t) = y(t) + a(t)y\,(t - \sigma)$, so $u(t) > 0$ for $[L + \sigma,\, L_0]$ and

$$u'(t) = -\sum_{j=1}^{n} b_j(t) y\left(t - \mu_j\right) \leq 0 \qquad \text{for } t \in [L + \mu_n,\, L_0].$$

Therefore $u'(t) \leq 0$ for $t \in [L + \mu_n,\, L_0]$. Note that

$$y(t - \mu_j) = u(t - \mu_j) - a(t - \mu_j) y\left(t - \sigma - \mu_j\right).$$

Then

$$u'(t) = -\sum_{j=1}^{n} b_j(t) u(t - \mu_j) + \sum_{j=1}^{n} b_j(t) a(t - \mu_j) y\left(t - \sigma - \mu_j\right) \text{ for } t \in [L + \mu_n,\, L_0]. \tag{3.3}$$

By (H1), we obtain

$$u'(t) \le -\sum_{j=1}^{n} b_j(t)u(t-\mu_j) + \sum_{j=1}^{n} b_j(t-\sigma)a(t-\sigma)y\left(t-\sigma-\mu_j\right) \text{ for } t \in [L+\mu_n+\sigma,\ L_0].$$

Clearly,

$$a(t-\sigma)u'(t-\sigma) = -\sum_{j=1}^{n} b_j(t-\sigma)a(t-\sigma)y\left(t-\mu_j-\sigma\right).$$

Therefore,

$$u'(t) \le -\sum_{j=1}^{n} b_j(t)u(t-\mu_j) - a(t-\sigma)u'(t-\sigma) \quad \text{for } t \in [L+\mu_n+\sigma,\ L_0].$$

Consequently,

$$u'(t) + R(t)u'(t-\sigma) + \sum_{j=1}^{n} b_j(t)u(t-\mu_j) \le 0 \quad \text{for } t \in [L+\mu_n+\sigma,\ L_0].$$

Assume that $v(t) = u(t) + R(t)u(t-\sigma)$, so $v(t) > 0$ on $[L+2\sigma,\ L_0]$. Then

$$v'(t) = u'(t) + R(t)u'(t-\sigma) + R'(t)u(t-\sigma) \le -\sum_{j=1}^{n} b_j(t)u(t-\mu_j) + R'(t)u(t-\sigma) \le 0 \tag{3.4}$$

for $t \in [L+\mu_n+\sigma,\ L_0]$, and so $v'(t) \le 0$ on $[L+\mu_n+\sigma,\ L_0]$. Consequently,

$$v'(t) - R'(t)u(t-\sigma) + \sum_{j=1}^{n} b_j(t)u(t-\mu_j) \le 0 \qquad \text{for } t \in [L+\mu_n+\sigma,\ L_0]. \tag{3.5}$$

In view of $u'(t) \le 0$ on $[L+\mu_n,\ L_0]$, it follows that

$$v(t) = u(t) + R(t)u(t-\sigma) \le (1+R(t))\, u(t-\sigma) \text{ for } t \in [L+\mu_n+\sigma,\ L_0].$$

Therefore,

$$u(t-\sigma) \ge \frac{v(t)}{1+R(t)} \qquad \text{for } t \in [L+\mu_n+\sigma,\ L_0] \tag{3.6}$$

and

$$u(t-\mu_j) \ge \frac{v(t+\sigma-\mu_j)}{1+R(t+\sigma-\mu_j)} \quad \text{for } t \in [L+\mu_n+\mu_j,\ L_0-\sigma+\mu_j],\ \ j=1,2,\ldots,n. \tag{3.7}$$

Substituting into (3.5), we obtain

$$v'(t) - \frac{R'(t)}{1+R(t)}v(t) + \sum_{j=1}^{n} \frac{b_j(t)}{1+R(t+\sigma-\mu_j)}v(t+\sigma-\mu_j) \le 0 \qquad \text{for } t \in [L+2\mu_n,\ L_0].$$

Let $z(t) = \frac{v(t)}{1+R(t)}$. Then $z(t) > 0$ on $[L+2\sigma,\ L_0]$ and

$$z'(t) + \sum_{j=1}^{n} \frac{b_j(t)}{1+R(t)}z(t+\sigma-\mu_j) \le 0 \qquad \text{for } t \in [L+2\mu_n,\ L_0].$$

Also, by using (3.4) and (3.6), we obtain

$$z'(t)=\frac{v'(t)\,(1+R(t))-v(t)R'(t)}{(1+R(t))^2}\le\frac{-\sum_{j=1}^{n}b_j(t)v(t-\mu_j)}{1+R(t)}\le 0$$

for $t\in[L+\sigma+\mu_n,\,L_0]$. The proof is complete. □

Lemma 3.2. *Assume that (H2) holds, and $N'(t)\le 0$ for $t\ge t_1+\sigma$. Let $y(t)$ be a solution of Eq (E_1) such that $y(t)>0$ on $[L,\,L_0]$, $L\ge L_0+2\mu_n$, $L\ge t_1$. Then, there exists a function $z(t)$ that satisfies $z(t)>0$ on $[L+2\sigma,\,L_0]$ and $z'(t)\le 0$ on $[L+\sigma+\mu_n,\,L_0]$, and*

$$z'(t)+\sum_{j=1}^{n}\frac{b_j(t)}{1+N(t)}z(t+\sigma-\mu_j)\le 0\qquad \textit{for } t\in[L+2\mu_n,\,L_0].$$

Proof. By the same method as in the proof of Lemma 2.1 we obtain (see (3.3))

$$u'(t)=-\sum_{j=1}^{n}b_j(t)u(t-\mu_j)+\sum_{j=1}^{n}b_j(t)a(t-\mu_j)y\left(t-\sigma-\mu_j\right)\text{ for } t\in[L+\mu_n,\,L_0],\tag{3.8}$$

where $u(t)=y(t)+a(t)y\,(t-\sigma)$, $u(t)>0$ for $[L+\sigma,\,L_0]$ and $u'(t)\le 0$ for $t\in[L+\mu_n,\,L_0]$. Note that

$$u'(t)=-\sum_{j=1}^{n}b_j(t)y\left(t-\mu_j\right)\le -b_j(t)y\left(t-\mu_j\right)\qquad\text{for } t\in[L+\mu_n,\,L_0].$$

Therefore,

$$y\left(t-\sigma-\mu_j\right)\le\frac{-1}{b_j(t-\sigma)}u'(t-\sigma)\qquad\text{for } t\in[L+\mu_n+\sigma,\,L_0].$$

This together with (3.8) implies that

$$u'(t)\le-\sum_{j=1}^{n}b_j(t)u(t-\mu_j)-u'(t-\sigma)\sum_{j=1}^{n}\frac{b_j(t)a(t-\mu_j)}{b_j(t-\sigma)}\qquad\text{for } t\in[L+\mu_n+\sigma,\,L_0].$$

In view of (H2), it follows that

$$u'(t)+\sum_{j=1}^{n}b_j(t)u(t-\mu_j)+N(t)u'(t-\sigma)\le 0\qquad\text{for } t\in[L+\mu_n+\sigma,\,L_0].\tag{3.9}$$

By using the same method as in Lemma 2.1, we obtain

$$z'(t)+\sum_{j=1}^{n}\frac{b_j(t)}{1+N(t)}z(t+\sigma-\mu_j)\le 0\qquad\text{for } t\in[L+2\mu_n,\,L_0],$$

where $z(t)>0$ on $[L+2\sigma,\,L_0]$, $z'(t)\le 0$ on $t\in[L+\mu_n+\sigma,\,L_0]$, $z(t)=\frac{v(t)}{1+N(t)}$ and $v(t)=u(t)+N(t)u(t-\sigma)$, $u(t)=y(t)+a(t)y\,(t-\sigma)$. The proof is complete. □

Lemma 3.3. *Assume that (H1) holds and $b_r(t) \geq |R'(t)|$, $r \in \{1, 2, \ldots, n\}$. Let $y(t)$ be a solution of Eq (E_1) such that $y(t) > 0$ for $t \in [L, L_0]$, $L \geq L_0 + \mu_n + \mu_r$, $L \geq t_1$. Then, there exists a function $v(t)$ that satisfies $v(t) > 0$ for $t \in [L + 2\sigma, L_0]$ and $v'(t) \leq 0$ for $t \in [L + \mu_n + \mu_r, L_0]$, and*

$$v'(t) + \frac{b_r(t) - |R'(t)|}{1 + R(t + \sigma - \mu_r)} v(t + \sigma - \mu_r) + \sum_{\substack{j=1 \\ j \neq r}}^{n} \frac{b_j(t)}{1 + R(t + \sigma - \mu_j)} v(t + \sigma - \mu_j) \leq 0 \tag{3.10}$$

for $t \in [L + \mu_n + \mu_r, L_0]$.

Proof. Letting $u(t) = y(t) + a(t)y(t - \sigma)$ and $v(t) = u(t) + R(t)u(t - \sigma)$, so $u'(t) \leq 0$ for $t \in [L + \mu_n, L_0]$ and $v(t) > 0$ for $[L + 2\sigma, L_0]$. By (3.5), we have

$$v'(t) \leq R'(t)u(t - \sigma) - \sum_{j=1}^{n} b_j(t)u(t - \mu_j) \qquad \text{for } t \in [L + \mu_n + \sigma, L_0].$$

Since $u'(t) \leq 0$ for $t \in [L + \mu_n, L_0]$, then

$$\begin{aligned} v'(t) &\leq R'(t)u(t - \sigma) - \sum_{j=1}^{n} b_j(t)u(t - \mu_j) \leq |R'(t)|u(t - \sigma) - \sum_{j=1}^{n} b_j(t)u(t - \mu_j) \\ &\leq |R'(t)|u(t - \mu_r) - \sum_{j=1}^{n} b_j(t)u(t - \mu_j) \leq 0 \end{aligned}$$

for $t \in [L + \mu_n + \mu_r, L_0]$. Therefore, $v'(t) \leq 0$ for $t \in [L + \mu_n + \mu_r, L_0]$, and

$$v'(t) + \left(b_r(t) - |R'(t)|\right) u(t - \mu_r) + \sum_{\substack{j=1 \\ j \neq r}}^{n} b_j(t)u(t - \mu_j) \leq 0 \quad \text{for } t \in [L + \mu_n + \mu_r, L_0].$$

Using (3.7), we obtain

$$v'(t) + \frac{b_r(t) - |R'(t)|}{1 + R(t + \sigma - \mu_r)} v(t + \sigma - \mu_r) + \sum_{\substack{j=1 \\ j \neq r}}^{n} \frac{b_j(t)}{1 + R(t + \sigma - \mu_j)} v(t + \sigma - \mu_j) \leq 0$$

for $t \in [L + \mu_n + \mu_r, L_0]$, where $v(t) > 0$ for $[L + 2\sigma, L_0]$ and $v'(t) \leq 0$ for $t \in [L + \mu_n + \mu_r, L_0]$. The proof is complete. □

Lemma 3.4. *Assume that (H2) holds and $b_r(t) \geq |N'(t)|$, $r \in \{1, 2, \ldots, n\}$. Let $y(t)$ be a solution of Eq (E_1) such that $y(t) > 0$ for $t \in [L, L_0]$, $L_0 \geq L + \mu_n + \mu_r$, $L \geq t_1$. Then, there is a function $v(t)$ that satisfies $v(t) > 0$ for $t \in [L + 2\sigma, L_0]$ and $v'(t) \leq 0$ for $t \in [L + \mu_n + \mu_r, L_0]$, and*

$$v'(t) + \frac{b_r(t) - |N'(t)|}{1 + N(t + \sigma - \mu_r)} v(t + \sigma - \mu_r) + \sum_{\substack{j=1 \\ j \neq r}}^{n} \frac{b_j(t)}{1 + N(t + \sigma - \mu_j)} v(t + \sigma - \mu_j) \leq 0$$

for $t \in [L + \mu_n + \mu_r, L_0]$.

Proof. Using the same method as in the proof of Lemma 3.2, we obtain (see (3.9))

$$u'(t) + \sum_{j=1}^{n} b_j(t)u(t-\mu_j) + N(t)u'(t-\sigma) \le 0 \qquad \text{for } t \in [L+\mu_n+\sigma,\ L_0],$$

where $u(t) = y(t) + a(t)y(t-\sigma)$, $u(t) > 0$ for $t \in [L+\sigma,\ L_0]$ and $u'(t) \le 0$ for $t \in [L+\mu_n,\ L_0]$. Letting $v(t) = u(t) + N(t)u(t-\sigma)$, then

$$v'(t) = u'(t) + N(t)u'(t-\sigma) + N'(t)u(t-\sigma) \le -\sum_{j=1}^{n} b_j(t)u(t-\mu_j) + N'(t)u(t-\sigma)$$

for $t \in [L+\mu_n+\sigma,\ L_0]$. Therefore,

$$v'(t) \le -\sum_{j=1}^{n} b_j(t)u(t-\mu_j) + N'(t)u(t-\sigma) \le -\sum_{j=1}^{n} b_j(t)u(t-\mu_j) + |N'(t)|u(t-\sigma)$$

for $t \in [L+\mu_n+\sigma,\ L_0]$. That is,

$$v'(t) \le -\sum_{j=1}^{n} b_j(t)u(t-\mu_j) + |N'(t)|u(t-\sigma) \le 0 \qquad \text{for } t \in [L+\mu_n+\sigma,\ L_0].$$

By using $u'(t) \le 0$ for $t \in [L+\mu_n,\ L_0]$, we obtain

$$v'(t) \le -\left(b_r(t) - |N'(t)|\right)u(t-\mu_r) - \sum_{\substack{j=1\\ j\ne r}}^{n} b_j(t)u(t-\mu_j) \le 0 \qquad \text{for } t \in [L+\mu_n+\mu_r,\ L_0], \tag{3.11}$$

and

$$v(t) = u(t) + N(t)u(t-\sigma) \le (1+N(t))\,u(t-\sigma) \text{ for } t \in [L+\mu_n+\sigma,\ L_0].$$

Therefore,

$$u(t-\mu_j) \ge \frac{v(t+\sigma-\mu_j)}{1+N(t+\sigma-\mu_j)} \qquad \text{for } t \in [L+\mu_n+\mu_j,\ L_0+\mu_j-\sigma].$$

From this and (3.11), we have

$$v'(t) + \frac{b_r(t) - |N'(t)|}{1+N(t+\sigma-\mu_r)}v(t+\sigma-\mu_r) + \sum_{\substack{j=1\\ j\ne r}}^{n} \frac{b_j(t)}{1+N(t+\sigma-\mu_j)}v(t+\sigma-\mu_j) \le 0$$

for $t \in [L+\mu_n+\mu_r,\ L_0]$, where $v(t) > 0$ for $[L+2\sigma,\ L_0]$ and $v'(t) \le 0$ for $t \in [L+\mu_n+\mu_r,\ L_0]$. The proof is complete. □

Theorem 3.3. *Let*

$$B_j(t) = \frac{b_j(t)}{1+R(t)}, \quad \eta_j = \mu_j - \sigma, \quad j = 1, 2, \ldots, n. \tag{3.12}$$

Assume that (H1) holds, $l \in \mathbb{N}_0$, $0 < \epsilon \le 1$, $R'(t) \le 0$ for $t \ge t_1 + \sigma$. If condition (2.9) is satisfied, then $D_{t_1}(E_1) \le (l+3+\epsilon)\mu_n - (l+1+\epsilon)\sigma$ and every solution of Eq (E_2) is oscillatory.

Proof. Assume the contrary, and let $y(t)$ be a solution of Eq (E_1) such that $y(t) > 0$ on $[L, L_0]$, $L \geq t_0$, $L_0 \geq L + (l + 3 + \epsilon)\,\mu_n - (l + 1 + \epsilon)\,\sigma$. By using Lemma 3.1, then there exists a solution $z(t)$ of (3.2) such that $z(t) > 0$ on $[L + 2\sigma, L_0]$ and $z'(t) \leq 0$ on $[L + \sigma + \mu_n, L_0]$. Clearly

$$\max\{\delta, \alpha\} + (l + 1 + \epsilon)\,\eta_n = (l + 3 + \epsilon)\,\mu_n - (l + 1 + \epsilon)\,\sigma,$$

where $\alpha = 2\mu_n$, $\delta = \sigma + \mu_n$, and $\eta_n = \mu_n - \sigma$. Applying Lemma 2.2 with $\gamma = 2\sigma$, $\alpha = \beta = 2\mu_n$, $\delta = \sigma + \mu_n$, so $z(t)$ cannot be positive on $[L + 2\mu_n, L_0]$. □

Theorem 3.4. *Assume that (H1) holds, $l \in \mathbb{N}_0$, $1 \leq k_1 \leq k_2 \leq n$, $0 < \epsilon \leq 1$, $R'(t) \leq 0$ for $t \geq t_1 + \sigma$. If condition (2.13) is satisfied such that $B_j(t)$, η_j, $j = 1, 2, \ldots, n$, are defined by (3.12), then $D_{t_1}(E_1) \leq (l + 3)\,\mu_n + \epsilon\mu_{k_2} - (l + 1 + \epsilon)\sigma$ and every solution of Eq (E_2) is oscillatory.*

Proof. Assume $y(t)$ is a solution of Eq (E_1) such that $y(t) > 0$ on $[L, L_0]$, $L \geq t_0$, $L_0 \geq L + (l + 3)\,\mu_n + \epsilon\mu_{k_2} - (l + 1 + \epsilon)\sigma$. By using Lemma 3.1, there exists a solution $z(t)$ of (3.2) such that $z(t) > 0$ on $[L + 2\sigma, L_0]$ and $z'(t) \leq 0$ on $[L + \sigma + \mu_n, L_0]$. It is clear that

$$\max\{\delta,\ \alpha\} + (l + 1)\eta_n + \epsilon\eta_{k_2} = (l + 3)\,\mu_n + \epsilon\mu_{k_2} - (l + 1 + \epsilon)\sigma,$$

where $\alpha = 2\mu_n$, $\delta = \sigma + \mu_n$, and $\eta_j = \mu_j - \sigma$. Applying Lemma 2.3 with $\gamma = 2\sigma$, $\alpha = 2\mu_n$, $\delta = \sigma + \mu_n$, so $z(t)$ cannot be positive on $[L + 2\mu_n, L_0]$. □

The following two theorems can be proven using Lemma 3.2 instead of Lemma 3.1 in the proofs of Theorems 3.3 and 3.4, respectively.

Theorem 3.5. *Let*

$$B_j(t) = \frac{b_j(t)}{1 + N(t)}, \quad \eta_j = \mu_j - \sigma, \quad j = 1, 2, \ldots, n. \tag{3.13}$$

Assume that (H2) holds, $l \in \mathbb{N}_0$, $0 < \epsilon \leq 1$, $N'(t) \leq 0$ for $t \geq t_1 + \sigma$. If condition (2.9) is satisfied, then $D_{t_1}(E_1) \leq (l + 3 + \epsilon)\,\mu_n - (l + 1 + \epsilon)\,\sigma$ and every solution of Eq (E_2) is oscillatory.

Theorem 3.6. *Assume that (H2) holds, $l \in \mathbb{N}_0$, $1 \leq k_1 \leq k_2 \leq n$, $0 < \epsilon \leq 1$, $N'(t) \leq 0$ for $t \geq t_1 + \sigma$. If condition (2.13) is satisfied such that $B_j(t)$, η_j, $j = 1, 2, \ldots, n$, are defined by (3.13), then $D_{t_1}(E_1) \leq (l + 3)\,\mu_n + \epsilon\mu_{k_2} - (l + 1 + \epsilon)\sigma$ and every solution of Eq (E_2) is oscillatory.*

Theorem 3.7. *Let $r \in \{1, 2, \ldots, n\}$,*

$$B_j(t) = \begin{cases} \frac{b_j(t) - |R'(t)|}{1 + R(t + \sigma - \mu_j)} & \text{if } j = r \\ \frac{b_j(t)}{1 + R(t + \sigma - \mu_j)} & \text{otherwise,} \end{cases} \qquad \eta_j = \mu_j - \sigma, \quad j = 1, 2, \ldots, n. \tag{3.14}$$

Assume that (H1) holds, $l \in \mathbb{N}_0$, $0 < \epsilon \leq 1$, $b_r(t) \geq |R'(t)|$ for $t \geq t_1 + \sigma$. If condition (2.9) is satisfied, then $D_{t_1}(E_1) \leq (l + 2 + \epsilon)\,\mu_n + \mu_r - (l + 1 + \epsilon)\,\sigma$ and every solution of Eq (E_2) is oscillatory.

Proof. Assume the contrary, and let $y(t)$ be a solution of Eq (E_1) such that $y(t) > 0$ on $[L, L_0]$, $L \geq t_0$, $L_0 \geq L + (l + 2 + \epsilon)\,\mu_n + \mu_r - (l + 1 + \epsilon)\,\sigma$. By using Lemma 3.3, there exists a solution $v(t)$ of (3.10) on $[L_0 + \mu_n + \mu_r, L]$ such that $v(t) > 0$ on $[L + 2\sigma, L_0]$ and $v'(t) \leq 0$ on $[L + \mu_n + \mu_r, L_0]$. Clearly

$$\max\{\delta,\ \alpha\} + (l + 1 + \epsilon)\,\eta_n = (l + 2 + \epsilon)\,\mu_n + \mu_r - (l + 1 + \epsilon)\,\sigma,$$

where $\alpha = \delta = \mu_n + \mu_r$, and $\eta_n = \mu_n - \sigma$. Applying Lemma 2.2 with $\gamma = 2\sigma$, $\alpha = \delta = \beta\mu_n + \mu_r$, so $v(t)$ cannot be positive on $[L + \mu_n + \mu_r, L_0]$. □

Theorem 3.8. *Assume that (H1) holds, $l \in \mathbb{N}_0$, $1 \le k_1 \le k_2 \le n$, $0 < \epsilon \le 1$, $b_r(t) \ge |R'(t)|$ for $t \ge t_1 + \sigma$, $r \in \{1, 2, \dots, n\}$. If condition (2.13) is satisfied with $B_j(t)$ and η_j, $j = 1, 2, \dots, n$, are defined by (3.14), then $D_{t_1}(E_1) \le (l+2)\mu_n + \mu_r + \epsilon\mu_{k_2} - (l+1+\epsilon)\sigma$ and every solution of Eq (E_2) is oscillatory.*

Proof. Assume the contrary, and let $y(t)$ be a solution of Eq (E_1) such that $y(t) > 0$ on $[L, L_0]$, $L \ge t_0$, $L_0 \ge L + (l+2)\mu_n + \mu_r + \epsilon\mu_{k_2} - (l+1+\epsilon)\sigma$. By using Lemma 3.3, then there exists a solution $v(t)$ of (3.10) on $[L_0 + \mu_n + \mu_r, L]$ such that $v(t) > 0$ on $[L + 2\sigma, L_0]$ and $v'(t) \le 0$ on $[L + \mu_n + \mu_r, L_0]$. Clearly

$$\max\{\delta, \alpha\} + (l+1)\eta_n + \epsilon\eta_{k_2} = (l+2)\mu_n + \mu_r + \epsilon\mu_{k_2} - (l+1+\epsilon)\sigma,$$

where $\alpha = \delta = \mu_n + \mu_r$, and $\eta_j = \mu_j - \sigma$, $j = 1, 2, \dots, n$. Applying Lemma 2.3 with $\gamma = 2\sigma$, $\alpha = \delta = \beta = \mu_n + \mu_r$, so $v(t)$ cannot be positive on $[L + \mu_n + \mu_r, L_0]$. □

Using Lemma 3.4 instead of Lemma 3.3 in the proofs of the preceding two theorems, we obtain the following results:

Theorem 3.9. *Let $r \in \{1, 2, \dots, n\}$,*

$$B_j(t) = \begin{cases} \frac{b_j(t) - |N'(t)|}{1 + N(t + \sigma - \mu_j)} & \text{if } j = r \\ \frac{b_j(t)}{1 + N(t + \sigma - \mu_j)} & \text{otherwise,} \end{cases} \qquad \eta_j = \mu_j - \sigma, \quad j = 1, 2, \dots, n. \tag{3.15}$$

Assume that (H2) holds, $l \in \mathbb{N}_0$, $0 < \epsilon \le 1$, $b_r(t) \ge |N'(t)|$ for $t \ge t_1 + \sigma$. If condition (2.9) is satisfied, then $D_{t_1}(E_1) \le (l+2+\epsilon)\mu_n + \mu_r - (l+1+\epsilon)\sigma$ and every solution of Eq (E_2) is oscillatory.

Theorem 3.10. *Assume that (H2) holds, $l \in \mathbb{N}_0$, $1 \le k_1 \le k_2 \le n$, $0 < \epsilon \le 1$, $b_r(t) \ge |N'(t)|$ for $t \ge t_1 + \sigma$, $r \in \{1, 2, \dots, n\}$. If condition (2.13) is satisfied with $B_j(t)$ and $\eta_j = \mu_j - \sigma$, $j = 1, 2, \dots, n$, are defined by (3.15), then $D_{t_1}(E_1) \le (l+2)\mu_n + \mu_r + \epsilon\eta_{k_2} - (l+1+\epsilon)\sigma$ and every solution of Eq (E_2) is oscillatory.*

Remark 3.1. *Following the same techniques used in the proof of our results, several sufficient criteria for the oscillation of both Eqs (E_1) and (E_2) can be obtained. For example, the conditions*

$$\limsup_{t\to\infty} \prod_{s=1}^{n} \frac{\left(\prod_{\substack{j=1 \\ j\neq s}}^{n} \int_{t-\epsilon(\mu_s-\sigma)}^{t} \frac{b_j(v)}{1+R(v)} e^{\int_{v-(\mu_s-\sigma)}^{t-\epsilon(\mu_s-\sigma)} \sum_{j_1=1}^{n} \frac{b_{j_1}(v_1)}{1+R(v_1)} M_{l-1}^{j_1} dv_1} dv\right)^{\frac{1}{n-1}}}{\left(1 - \int_{t-\epsilon(\mu_s-\sigma)}^{t} \frac{b_s(v)}{1+R(v)} e^{\int_{v-(\mu_s-\sigma)}^{t-\epsilon(\mu_s-\sigma)} \sum_{j=1}^{n} \frac{b_j(v_1)}{1+R(v_1)} M_{l-1}^{j} dv_1} dv\right)} > \frac{1}{(n-1)^n},$$

where the sequence $\{M_l^s\}_{l\ge0}$ is defined by (2.2) with $B_j(t) = \frac{b_j(t)}{1+R(t)}$ and $\eta_j = \mu_j - \sigma$, $j = 1, 2, \dots, n$, and

$$\limsup_{t\to\infty} \prod_{s=k_1}^{k_2} \frac{\int_{t-\epsilon\mu_s}^{t} b_s(v) e^{\int_{v-\mu_s}^{t-\epsilon\mu_s} \sum_{j_1=1}^{n} b_{j_1}(v_1) M_{l-1}^{j_1} dv_1} dv}{\left(1 - \sum_{j=s+1}^{n} \int_{t-\epsilon\mu_s}^{t} b_j(v) e^{\int_{v-\mu_j}^{t-\epsilon\mu_s} \sum_{j_1=1}^{n} b_{j_1}(v_1) M_{l-1}^{j_1} dv_1} dv\right)} > 1,$$

where the sequence $\{M_l^s\}_{l\ge0}$ is defined by (2.2) with $B_j(t) = q(t)$ and $\eta_j = \mu_j$, $j = 1, 2, \dots, n$, can be proved using the same proofs of Theorems 3.2 and 3.3, respectively.

4. Examples

Example 4.1. *Consider the first-order differential equation with several delays*

$$y'(t) + b_1(t)y(t-\mu_1) + b_2(t)y(t-\mu_2) = 0, \tag{4.1}$$

where $b_1(t) = \frac{18}{5}$ *and* $b_2(t) = \frac{1}{10}$, *and* $\mu_1 = \delta$, $0 < \delta < \frac{5}{36}$, *and* $\mu_2 = 1$. *Clearly,*

$$\int_{t-\mu_2}^{t} b_2(v) e^{\int_{v-\mu_2}^{t-\mu_2} \sum_{j_1=1}^{2} b_{j_1}(v_1)dv_1} dv > 1.$$

Then, condition (3.1) is satisfied with $l = 1$, $k_2 = k_3 = 3$, *and* $\epsilon = 1$, *and hence* D_{t_1}*(4.1)*$\leq 4\mu_2 = 4$, *and every solution of Eq (4.1) is oscillatory for* $0 < \delta < \frac{5}{36}$. *However,* $\min_{1\leq j\leq 2}\mu_j = \mu_1 = \delta$. *Then,* δ *can be chosen small enough such that all the results of [18] and [4, Theorem 2.2] fail to apply. Also, note that*

$$\int_{t-\mu_2}^{t} b_2(w)dw + \frac{1 + \int_{t-\mu_2}^{t} b_1(w)dw}{1 - \int_{t-\mu_2}^{t} b_2(w)dw} \int_{t-\mu_2}^{t} b_2(w) \int_{t-\mu_2}^{w} b_2(w_1)dw_1 dw < \frac{13}{100},$$

and

$$\int_{t-\mu_1}^{t} b_1(w)dw + \frac{1 + \int_{t-\mu_1}^{t} b_2(w)dw}{1 - \int_{t-\mu_1}^{t} b_1(w)dw} \int_{t-\mu_1}^{t} b_1(w) \int_{t-\mu_1}^{w} b_1(w_1)dw_1 dw = \frac{9}{25} \frac{\delta\left(9\delta^2 - 90\delta + 50\right)}{5 - 18\delta} < 1$$

for sufficiently small δ. *Therefore, [4, Theorem 2.1] cannot give an approximation better than* $4\mu_2$.

Example 4.2. *Consider the first-order neutral differential equation with several delays*

$$[y(t) + a(t)y(t-\sigma)]' + b_1(t)y(t-\mu_1) + b_2(t)y(t-\mu_2) = 0, \qquad t \geq \pi, \tag{4.2}$$

where $a(t) = a^* = \max\{2\pi - \frac{99}{100}, \frac{b^*\pi}{2} - \frac{99}{100}\}$, $b_1(t) = b^* > 0$, *and* $b_2(t) = 2 + \sin(2t)$, $\sigma = \pi$, $\mu_1 = \frac{2\pi}{3}$, *and* $\mu_2 = 2\pi$. *It is clear that condition (H1) is satisfied with* $R(t) = a^*$. *Note that*

$$\begin{aligned}\frac{\left(\int_{t-\frac{\pi}{2}}^{t} \frac{2+sin(2v)}{1+a^*}dv\right)\left(\int_{t-\pi}^{t} \frac{b^*}{1+a^*}dv\right)}{\left(1 - \int_{t-\frac{\pi}{2}}^{t} \frac{b^*}{1+a^*}dv\right)\left(1 - \int_{t-\pi}^{t} \frac{2+sin(2v)}{1+a^*}dv\right)} &= \frac{2b^*\pi\left(1 + \pi - 2\cos^2(t)\right)}{(1 + a^* - 2\pi)(2 + 2a^* - b^*\pi)} \\ &\geq \frac{2b^*\pi(\pi - 1)}{(1 + a^* - 2\pi)(2 + 2a^* - b^*\pi)} > 1\end{aligned}$$

for $b^* > \frac{1}{50}\frac{200\pi+1}{\pi(200\pi-199)}$. *Then, condition (2.9) is satisfied with* $B_1(t) = \frac{b^*}{1+a^*}$, $B_2(t) = \frac{2+\sin(2t)}{1+a^*}$, $\eta_1 = \mu_1 - \sigma = \frac{\pi}{2}$, $\eta_2 = \mu_2 - \sigma = \pi$, $l = 0$ *and* $\epsilon = 1$. *Therefore, all requirements of Theorem 3.3 with* $l = 0$ *are satisfied, and hence* D_π*(4.2)*$\leq 4\mu_2 - 2\sigma = 6\pi$.

5. Conclusions

In this work, we obtained many new estimates for the upper bounds of the distance between adjacent zeros of Eqs (E_1) and (E_2). Our results are established in a general form (for inequality (1.1)), and hence they can be applied to any differential equation that can be transformed into an inequality of the form (1.1). Many new oscillation results for neutral differential equations with several not necessarily monotone delays can be obtained using the methods proposed in this work.

Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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