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Research article

Novel identities for elementary and complete symmetric polynomials with diverse applications

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Abstract: This article aims to present novel identities for elementary and complete symmetric polynomials and explore their applications, particularly to generalized Vandermonde and special tri-diagonal matrices. It also extends existing results on Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and introduces an explicit formula based on the zeros of $P_{n-1}^{(\alpha,\beta)}(x)$. Several illustrative examples are included.

Keywords: elementary symmetric polynomials; complete symmetric polynomials; Schur convexity; Vandermonde matrix; tri-diagonal matrix; Jacobi polynomials
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1. Introduction

Symmetric polynomials are significant in various areas of mathematics, including computational linear algebra [1, 2], representation theory [3], combinatorics [4–6], and others. There are several types of symmetric polynomials, including power-sum, monomial, Schur, elementary and complete polynomials. For more information, refer to [7].

According to the fundamental theorem of symmetric polynomials, the elementary symmetric polynomials are distinguished from other symmetric polynomials, as any symmetric polynomial can be uniquely represented in terms of the elementary symmetric polynomials (see [8]). Also, from the Jacobi–Trudi and Nägelsbach–Kostka identities, we see that the elementary and the complete symmetric polynomials are dual to each other (see [9]).

There are numerous studies presenting identities for symmetric polynomials, such as ([10–12]) and others listed in the references. For instance, the authors in [11] introduced some identities for the elementary and complete symmetric polynomials and used them to generalize Stirling numbers, in addition to proving a conjecture proposed in [13]. In [12], the author presented new relationships between elementary and complete symmetric polynomials and used them to provide a new representation for the Gaussian polynomials. Similarly, the author in [10] introduced identities for the elementary symmetric polynomials and used them to present elegant representations for Legendre polynomials. In the current paper, we will present additional identities for elementary and complete symmetric polynomials, supported by some applications. Some of these applications include improving the results presented in [2] related to the Vandermonde determinant, computing determinants for some special cases of tri-diagonal matrices, and generalizing the results presented in [10] related to Legendre polynomials.

To begin with, we will introduce some basic definitions, notations, and well-known results, which we will use in the sequel. For further details, refer to [1,7,10–12,14–16]. Throughout this article, let $n \in \mathbb{N}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$, and we use the notation $\mathbb{R}^n_+ := \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}_+,\ i = 1, 2, \ldots, n\}$. Let us start with the definitions of elementary and complete symmetric polynomials.

Definition 1.1. *The elementary symmetric polynomial (for short, ESP) of degree k, denoted by* $\boldsymbol{\sigma}_k^{(n)}(\mathbf{x})$*, is the sum of all possible products of distinct k variables of* $\{x_1, x_2, \ldots, x_n\}$*, that is,*

$$\boldsymbol{\sigma}_k^{(n)}(\mathbf{x}) = \begin{cases} 0, & \text{if} \quad k > n \text{ or } k < 0, \\ 1, & \text{if} \quad k = 0, \\ \displaystyle\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, & \text{if} \quad k = 1, 2, \ldots, n. \end{cases} \tag{1.1}$$

The complete symmetric polynomial (for short, CSP) of degree k, denoted by $\mathbf{h}_k^{(n)}(\mathbf{x})$*, is defined as follows:*

$$\mathbf{h}_k^{(n)}(\mathbf{x}) = \begin{cases} 0, & \text{if} \quad n < 0 \quad \text{or} \quad k < 0 \text{ or } (n = 0 \text{ with } k \neq 0), \\ 1, & \text{if} \quad k = 0, \\ \displaystyle\sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, & \text{if} \quad k = 1, 2, \ldots. \end{cases} \tag{1.2}$$

For instance, the ESP and CSP of degree 2 for $n = 3$ are given by

$$\begin{aligned} \boldsymbol{\sigma}_2^{(3)}(x_1, x_2, x_3) &= x_1x_2 + x_1x_3 + x_2x_3, \\ \mathbf{h}_2^{(3)}(x_1, x_2, x_3) &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3. \end{aligned}$$

It should be noticed that for a fixed degree k, each $\boldsymbol{\sigma}_k^{(n)}(\mathbf{x})$ involves $\binom{n}{k}$ terms, and each $\mathbf{h}_k^{(n)}(\mathbf{x})$ involves $\binom{n+k-1}{k}$ terms. From the definition of ESP and CSP, we see that they are homogeneous polynomials. Therefore, it is convenient to state the so-called Euler's theorem on homogeneous functions.

Theorem 1.1 (Euler's theorem on homogeneous functions)**.** *Let* $\mathbf{x} \in \mathbb{R}^n$*. If the function* $f : \mathbb{R}^n \to \mathbb{R}$ *is homogeneous of degree m, then*

$$\sum_{i=1}^{n} x_i \frac{\partial f(\mathbf{x})}{\partial x_i} = m\, f(\mathbf{x}).$$

The generating functions for ESP and CSP are given, respectively, by

$$E_n(t; \mathbf{x}) = \prod_{j=1}^{n} (1 + x_j t) = \sum_{k=0}^{n} \boldsymbol{\sigma}_k^{(n)}(\mathbf{x}) t^k, \tag{1.3}$$

and

$$H_n(t;\mathbf{x}) = \prod_{j=1}^{n}(1 - x_j t)^{-1} = \sum_{k=0}^{\infty}\mathbf{h}_k^{(n)}(\mathbf{x})t^k. \tag{1.4}$$

We can rewrite (1.3) and (1.4), respectively, as follows:

$$\sum_{k=0}^{n}\boldsymbol{\sigma}_k^{(n)}(\mathbf{x})t^k = (1 + x_n t)\sum_{k=0}^{n-1}\boldsymbol{\sigma}_k^{(n-1)}(x_1, x_2, \ldots, x_{n-1})t^k, \tag{1.5}$$

and

$$(1 - x_n t)\sum_{k=0}^{\infty}\mathbf{h}_k^{(n)}(\mathbf{x})t^k = \sum_{k=0}^{\infty}\mathbf{h}_k^{(n-1)}(x_1, x_2, \ldots, x_{n-1})t^k. \tag{1.6}$$

From (1.3) and (1.4), we see that $E_n(t;\mathbf{x})H_n(-t;\mathbf{x}) = 1$, consequently [17, Eq (1)]

$$\sum_{k=0}^{n}(-1)^k\boldsymbol{\sigma}_k^{(n)}(\mathbf{x})\,\mathbf{h}_{n-k}^{(n)}(\mathbf{x}) = 0. \tag{1.7}$$

Furthermore, it is important to note that the ESP and CSP of degree k are interconnected through Jacobi–Trudi and Nägelsbach–Kostka identities, receptively,

$$\boldsymbol{\sigma}_k^{(n)}(\mathbf{x}) = \det\left(\left[\mathbf{h}_{1-i+j}^{(n)}(\mathbf{x})\right]_{1\le i,j\le k}\right), \tag{1.8}$$

$$\mathbf{h}_k^{(n)}(\mathbf{x}) = \det\left(\left[\boldsymbol{\sigma}_{1-i+j}^{(n)}(\mathbf{x})\right]_{1\le i,j\le k}\right), \tag{1.9}$$

for any positive integer k (see [9]).

Now, by comparing the coefficients of t^k in (1.5), the ESP satisfies

$$\boldsymbol{\sigma}_k^{(n)}(\mathbf{x}) = \boldsymbol{\sigma}_k^{(n-1)}(x_1, x_2, \ldots, x_{n-1}) + x_n\,\boldsymbol{\sigma}_{k-1}^{(n-1)}(x_1, x_2, \ldots, x_{n-1}). \tag{1.10}$$

Similarly, for the CSP, by comparing the coefficients of t^k in (1.6), we have

$$\mathbf{h}_k^{(n)}(\mathbf{x}) = \mathbf{h}_k^{(n-1)}(x_1, x_2, \ldots, x_{n-1}) + x_n\,\mathbf{h}_{k-1}^{(n)}(\mathbf{x}). \tag{1.11}$$

By using the symmetry property of $\boldsymbol{\sigma}_k^{(n)}(\mathbf{x})$, we see that

$$\boldsymbol{\sigma}_k^{(n)}(\mathbf{x}) = \boldsymbol{\sigma}_k^{(n-1)}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) + x_i\,\boldsymbol{\sigma}_{k-1}^{(n-1)}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

for all $k = 0, 1, 2, \ldots$ and $i = 1, 2, \ldots, n$.

By differentiating the recurrence relation (1.10) with respect to x_i, we obtain

$$\frac{\partial\boldsymbol{\sigma}_k^{(n)}(\mathbf{x})}{\partial x_i} = \boldsymbol{\sigma}_{k,i}^{(n)}(\mathbf{x}) = \boldsymbol{\sigma}_{k-1}^{(n-1)}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), \tag{1.12}$$

for all $k = 0, 1, 2, \ldots$ (see [11]). Moreover, using (1.3) and (1.12) yields

$$\boldsymbol{\sigma}_{k-1}^{(n)}(\mathbf{x}) = \boldsymbol{\sigma}_{k,i}^{(n)}(\mathbf{x}) + x_i\,\boldsymbol{\sigma}_{k-1,i}^{(n)}(\mathbf{x}), \tag{1.13}$$

for all $k = 0, 1, 2, \ldots$ and $i = 1, 2, \ldots, n$. Repeated application of (1.13) gives

$$\boldsymbol{\sigma}_{k,i}^{(n)}(\mathbf{x}) = \sum_{j=1}^{k} (-1)^{j-1} \boldsymbol{\sigma}_{k-j}^{(n)}(\mathbf{x})\, x_i^{j-1}, \tag{1.14}$$

for all $k = 0, 1, 2, \ldots$ and $i = 1, 2, \ldots, n$.

In a similar manner, the CSP satisfies

$$\frac{\partial \mathbf{h}_k^{(n)}(\mathbf{x})}{\partial x_i} = \mathbf{h}_{k,i}^{(n)}(\mathbf{x}) = \mathbf{h}_{k-1}^{(n)}(\mathbf{x}) + x_i\, \mathbf{h}_{k-1,i}^{(n)}(\mathbf{x}), \tag{1.15}$$

and repeated application of (1.15), we obtain

$$\mathbf{h}_{k,i}^{(n)}(\mathbf{x}) = \sum_{j=1}^{k} \mathbf{h}_{k-j}^{(n)}(\mathbf{x})\, x_i^{j-1}, \tag{1.16}$$

for all $k = 0, 1, 2, \ldots$ and $i = 1, 2, \ldots, n$.

The structure of the remaining sections of this article is outlined as follows: In Section 2, we introduce new identities for the ESP and the CSP. Section 3 includes applications of these identities, supplemented with numerical examples. Finally, Section 4 presents the conclusion of the article.

2. Main results

In this section, we are going to introduce novel identities for the ESP and the CSP. We begin with the following result, which comes directly from [15, Theorem 1.1].

Corollary 2.1. *Let n and m be any positive integers. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then*

$$\mathbf{h}_r^{(n+m)}(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{r} \mathbf{h}_k^{(n)}(\mathbf{x})\, \mathbf{h}_{r-k}^{(m)}(\mathbf{y}), \tag{2.1}$$

for all $r = 0, 1, 2, \ldots$.

As a direct consequence of Corollary 2.1, we can infer that the CSP adheres to the following identity:

$$\mathbf{h}_k^{(n)}(\mathbf{x}) = \sum_{j=0}^{k} \mathbf{h}_j^{(i)}(x_1, \ldots, x_i)\, \mathbf{h}_{k-j}^{(n-i)}(x_{i+1}, \ldots, x_n), \tag{2.2}$$

for a non-negative integer k. The following result is a particular case of Corollary 2.1.

Corollary 2.2. *For any positive integer n and any non-negative integer k, we have*

$$\mathbf{h}_k^{(2n)}(\mathbf{x}, -\mathbf{x}) = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ \mathbf{h}_{k/2}^{(n)}\left(x_1^2, x_2^2, \ldots, x_n^2\right), & \text{if } k \text{ is even.} \end{cases} \tag{2.3}$$

The following is a more general result of Corollary 2.2.

Proposition 2.1. *For any positive integer n and any non-negative integers k and r such that* $0 \le r \le \lceil \frac{n}{2} \rceil$, *we have*

$$\mathbf{h}_k^{(n)}(x_1, \ldots, x_r, -x_1, \ldots, -x_r, y_1, \ldots, y_{n-2r}) = \sum_{\substack{\ell=0 \\ \ell \equiv 0 (\mathrm{mod}\ 2)}}^{k} \mathbf{h}_{\ell/2}^{(r)}\left(x_1^2 \ldots, x_r^2\right)\, \mathbf{h}_{k-\ell}^{(n-2r)}(y_1, \ldots, y_{n-2r}), \tag{2.4}$$

for all $k = 0, 1, 2, \ldots$, *where* $\lceil \cdot \rceil$ *represents the ceiling function.*

Proof. Using (1.4), we have

$$\sum_{k=0}^{\infty} \mathbf{h}_k^{(n)}(x_1, \ldots, x_r, -x_1, \ldots, -x_r, y_1, \ldots, y_{n-2r})\, t^k = \left(\prod_{i=1}^{r}\left(1 - x_i^2 t^2\right)^{-1}\right)\left(\prod_{i=1}^{n-2r}(1 - y_i t)^{-1}\right).$$

By applying Corollary 2.2 to the first term on the right-hand side of the previous equation, we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} \mathbf{h}_k^{(n)}(x_1, \ldots, x_r, -x_1, \ldots, -x_r, y_1, \ldots, y_{n-2r})\, t^k \\
&= \left(\sum_{\substack{\ell=0 \\ \ell \equiv 0 (\mathrm{mod}\ 2)}}^{\infty} \mathbf{h}_{\ell/2}^{(r)}\left(x_1^2 \ldots, x_r^2\right) t^\ell\right)\left(\sum_{\ell=0}^{\infty} \mathbf{h}_\ell^{(n-2r)}(y_1, \ldots, y_{n-2r}) t^\ell\right) \\
&= \sum_{k=0}^{\infty}\left(\sum_{\substack{\ell=0 \\ \ell \equiv 0 (\mathrm{mod}\ 2)}}^{k} \mathbf{h}_{\ell/2}^{(r)}\left(x_1^2 \ldots, x_r^2\right)\, \mathbf{h}_{k-\ell}^{(n-2r)}(y_1, \ldots, y_{n-2r})\right) t^k. \qquad (2.5)
\end{aligned}$$

Equation (2.5) can be obtained by utilizing the Cauchy product of two infinite series. This concludes the proof. □

The following identities hold for the ESP:

$$\boldsymbol{\sigma}_r^{(n+m)}(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{r} \boldsymbol{\sigma}_k^{(n)}(\mathbf{x})\, \boldsymbol{\sigma}_{r-k}^{(m)}(\mathbf{y}), \tag{2.6}$$

and

$$\boldsymbol{\sigma}_k^{(2n+1)}(\mathbf{x}, 0, -\mathbf{x}) = \boldsymbol{\sigma}_k^{(2n)}(\mathbf{x}, -\mathbf{x}) = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ (-1)^{k/2} \boldsymbol{\sigma}_{k/2}^{(n)}\left(x_1^2, x_2^2, \ldots, x_n^2\right), & \text{if } k \text{ is even,} \end{cases} \tag{2.7}$$

for any non-negative integer k and any positive integers n and m (see [10]). The formula (2.6) reduces to

$$\boldsymbol{\sigma}_k^{(n)}(\mathbf{x}) = \sum_{j=0}^{k} \boldsymbol{\sigma}_j^{(i)}(x_1, \ldots, x_i)\, \boldsymbol{\sigma}_{k-j}^{(n-i)}(x_{i+1}, \ldots, x_n),$$

for any non-negative integer k when $\mathbf{y}$ is missing (see [15]). It should be noticed that, from (1.3) and (1.4), we conclude that $\mathbf{h}_k^{(n)}(-\mathbf{x}) = (-1)^k\, \mathbf{h}_k^{(n)}(\mathbf{x})$ and $\boldsymbol{\sigma}_k^{(n)}(-\mathbf{x}) = (-1)^k\, \boldsymbol{\sigma}_k^{(n)}(\mathbf{x})$. Now, we are ready to extend Lemma 2 presented in [10] as follows:

Proposition 2.2. *For any positive integer n and any non-negative integers k and r such that* $0 \le r \le \lceil \frac{n}{2} \rceil$, *we have*

$$\boldsymbol{\sigma}_k^{(n)}(x_1,\dots,x_r,-x_1,\dots,-x_r,y_1,\dots,y_{n-2r}) = \sum_{\substack{\ell=0\\ \ell\equiv 0 (\mathrm{mod}\ 2)}}^{k} (-1)^{\ell/2}\boldsymbol{\sigma}_{\ell/2}^{(r)}\left(x_1^2\dots,x_r^2\right)\, \boldsymbol{\sigma}_{k-\ell}^{(n-2r)}(y_1,\dots,y_{n-2r}), \tag{2.8}$$

for all $k = 0,1,2,\dots$.

Proof. Using (1.3), we have

$$\begin{aligned}
&\sum_{k=0}^{n}\boldsymbol{\sigma}_k^{(n)}(x_1,\dots,x_r,-x_1,\dots,-x_r,y_1,\dots,y_{n-2r})\, t^k\\
&= \prod_{i=1}^{r}\left(1-x_i^2t^2\right)\prod_{i=1}^{n-2r}(1+y_it)\\
&= \sum_{\substack{\ell=0\\ \ell\equiv 0 (\mathrm{mod}\ 2)}}^{r} (-1)^{\ell/2}\boldsymbol{\sigma}_{\ell/2}^{(r)}\left(x_1^2\dots,x_r^2\right) t^\ell \sum_{\ell=0}^{n-2r}\boldsymbol{\sigma}_\ell^{(n-2r)}(y_1,\dots,y_{n-2r})t^\ell\\
&= \sum_{k=0}^{n}\left(\sum_{\substack{\ell=0\\ \ell\equiv 0 (\mathrm{mod}\ 2)}}^{k} (-1)^{\ell/2}\boldsymbol{\sigma}_{\ell/2}^{(r)}\left(x_1^2\dots,x_r^2\right)\boldsymbol{\sigma}_{k-\ell}^{(n-2r)}(y_1,\dots,y_{n-2r})\right)t^k.
\end{aligned} \tag{2.9}$$

Comparing the coefficients of t^k on both sides, the required result follows. □

For $n = 2r+1$, Proposition 2.2 gives

$$\boldsymbol{\sigma}_k^{(2n+1)}(\mathbf{x},-\mathbf{x},y_1) = \begin{cases} (-1)^{\frac{k-1}{2}}y_1\boldsymbol{\sigma}_{\frac{k-1}{2}}^{(n)}\left(x_1^2,x_2^2,\dots,x_n^2\right), & \text{if } k \text{ is odd},\\[2ex] (-1)^{\frac{k}{2}}\boldsymbol{\sigma}_{\frac{k}{2}}^{(n)}\left(x_1^2,x_2^2,\dots,x_n^2\right), & \text{if } k \text{ is even}. \end{cases} \tag{2.10}$$

It is worth pointing out that if we set $y_1 = 0$ in Eq (2.10), we essentially arrive at Lemma 2 as presented in [10]. The following results may be obtained by using Corollary 2.1 together with Corollary 2.2.

Corollary 2.3. *For* $s = 0,1,2,\dots$, *the following identities are satisfied:*

(1) $\displaystyle\sum_{i=-s}^{s+1}(-1)^i\, \mathbf{h}_{s+i}^{(n)}(\mathbf{x})\, \mathbf{h}_{s-i+1}^{(n)}(\mathbf{x}) = 0$;

(2) $\displaystyle\sum_{i=-s}^{s}(-1)^i\, \mathbf{h}_{s+i}^{(n)}(\mathbf{x})\, \mathbf{h}_{s-i}^{(n)}(\mathbf{x}) = \mathbf{h}_s^{(n)}\left(x_1^2,x_2^2,\dots,x_n^2\right)$.

Likewise, regarding ESP, the authors in [18] demonstrated that the following identities hold true for $s = 0, 1, 2, \ldots$

$$\sum_{i=-s}^{s+1} (-1)^i \, \boldsymbol{\sigma}_{s+i}^{(n)}(\mathbf{x}) \, \boldsymbol{\sigma}_{s-i+1}^{(n)}(\mathbf{x}) = 0, \tag{2.11}$$

and

$$\sum_{i=-s}^{s} (-1)^i \, \boldsymbol{\sigma}_{s+i}^{(n)}(\mathbf{x}) \, \boldsymbol{\sigma}_{s-i}^{(n)}(\mathbf{x}) = \boldsymbol{\sigma}_{s}^{(n)} \left(x_1^2, x_2^2, \ldots, x_n^2\right). \tag{2.12}$$

It should be noticed that both $\boldsymbol{\sigma}_k^{(n)}(\mathbf{x})$ and $\mathbf{h}_k^{(n)}(\mathbf{x})$ are homogeneous polynomials of degree k. The following identities are satisfied by using Theorem 1.1.

Corollary 2.4. *For $k = 0, 1, 2, \ldots, n$, the ESP and the CSP satisfy,*

(1) $\sum_{i=1}^{n} x_i \, \boldsymbol{\sigma}_{k,i}^{(n)}(\mathbf{x}) = k \, \boldsymbol{\sigma}_k^{(n)}(\mathbf{x})$*;*

(2) $\sum_{i=1}^{n} x_i \, \mathbf{h}_{k,i}^{(n)}(\mathbf{x}) = k \, \mathbf{h}_k^{(n)}(\mathbf{x})$.

The following result introduces some novel additional identities concerning the ESP and the CSP.

Theorem 2.5. *For any positive integer n and any non-negative integer k, the ESP and the CSP satisfy the following identities:*

(1) $\boldsymbol{\sigma}_k^{(k+1)}(x_1, x_2, \ldots, x_{k+1}) = (x_k + x_{k+1}) \, \boldsymbol{\sigma}_{k-1}^{(k)}(x_1, x_2, \ldots, x_k) - x_k^2 \, \boldsymbol{\sigma}_{k-2}^{(k-1)}(x_1, x_2, \ldots, x_{k-1})$*;*

(2) $\mathbf{h}_k^{(2)}(x_1, x_2) = (x_1 + x_2) \, \mathbf{h}_{k-1}^{(2)}(x_1, x_2) - x_1 x_2 \, \mathbf{h}_{k-2}^{(2)}(x_1, x_2)$*;*

(3) $\mathbf{h}_k^{(n)}(\mathbf{x}) = \sum_{i=1}^{n} x_i \, \mathbf{h}_{k-1}^{(i)}(x_1, \ldots, x_i)$*;*

(4) If $x_1, x_2, \ldots, x_n$ are distinct non-zero variables, then

$$\boldsymbol{\sigma}_k^{(n)} \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n} \right) = \frac{\boldsymbol{\sigma}_{n-k}^{(n)}(\mathbf{x})}{\boldsymbol{\sigma}_n^{(n)}(\mathbf{x})},$$

for all $i = 1, 2, \ldots, n$;

(5) If $x_1, x_2, \ldots, x_n$ are distinct non-zero variables, then

$$\mathbf{h}_k^{(n)} \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n} \right) = \sum_{i=1}^{n} x_i^{-k} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_j}{(x_j - x_i)}; \tag{2.13}$$

(6) $\sum_{i=1}^{n} x_i^2 \, \boldsymbol{\sigma}_{k,i}^{(n)}(\mathbf{x}) = \boldsymbol{\sigma}_1^{(n)}(\mathbf{x}) \, \boldsymbol{\sigma}_k^{(n)}(\mathbf{x}) - (k+1) \, \boldsymbol{\sigma}_{k+1}^{(n)}(\mathbf{x})$*;*

(7) $\sum_{i=1}^{n} x_i^2 \, \mathbf{h}_{k,i}^{(n)}(\mathbf{x}) = (k+1) \, \mathbf{h}_{k+1}^{(n)}(\mathbf{x}) - \mathbf{h}_1^{(n)}(\mathbf{x}) \, \mathbf{h}_k^{(n)}(\mathbf{x})$*;*

(8) $\sum_{i=1}^{n} \boldsymbol{\sigma}_{k+1,i}^{(n)}(\mathbf{x}) = (n-k) \, \boldsymbol{\sigma}_k^{(n)}(\mathbf{x})$*;*

(9) $\sum_{i=1}^{n} \mathbf{h}_{k+1,i}^{(n)}(\mathbf{x}) = (n+k)\, \mathbf{h}_{k}^{(n)}(\mathbf{x})$;

(10) $\sum_{i=1}^{n} x_i\, \boldsymbol{\sigma}_{k,i,j}^{(n)}(\mathbf{x}) = (k-1)\, \boldsymbol{\sigma}_{k,j}^{(n)}(\mathbf{x}), \quad j = 0, 1, 2, \ldots, n$;

(11) $\sum_{i=1}^{n} x_i\, \mathbf{h}_{k,i,j}^{(n)}(\mathbf{x}) = (k-1)\, \mathbf{h}_{k,j}^{(n)}(\mathbf{x}), \quad j = 0, 1, 2, \ldots, n$;

(12) $\boldsymbol{\sigma}_{k,i}^{(n)}(\mathbf{x}) - \boldsymbol{\sigma}_{k,j}^{(n)}(\mathbf{x}) = (x_j - x_i)\, \boldsymbol{\sigma}_{k,i,j}^{(n)}(\mathbf{x}), \quad i, j = 0, 1, 2, \ldots, n.$

Proof.

- To prove Theorem 2.5 (1), we rewrite the right-hand side as follows:

$$\begin{aligned}(x_k+x_{k+1})\, &\boldsymbol{\sigma}_{k-1}^{(k)}(x_1, x_2, \ldots, x_k) - x_k^2\, \boldsymbol{\sigma}_{k-2}^{(k-1)}(x_1, x_2, \ldots, x_{k-1}) \\ &= x_k\, (\boldsymbol{\sigma}_{k-1}^{(k)}(x_1, x_2, \ldots, x_k) - x_k\, \boldsymbol{\sigma}_{k-2}^{(k-1)}(x_1, x_2, \ldots, x_{k-1})) + x_{k+1}\, \boldsymbol{\sigma}_{k-1}^{(k)}(x_1, x_2, \ldots, x_k).\end{aligned}$$

  By applying the recurrence relation (1.10), this completes the proof of Theorem 2.5 (1).
- To prove Theorem 2.5 (2), we express the right-hand side in the following manner:

$$(x_1+x_2)\, \mathbf{h}_{k-1}^{(2)}(x_1, x_2) - x_1 x_2\, \mathbf{h}_{k-2}^{(2)}(x_1, x_2) = x_1\, (\mathbf{h}_{k-1}^{(2)}(x_1, x_2) - x_2\, \mathbf{h}_{k-2}^{(2)}(x_1, x_2)) + x_2\, \mathbf{h}_{k-1}^{(2)}(x_1, x_2).$$

  By using the recurrence relation given in Eq (1.11), we have now completed the proof of Theorem 2.5 (2).
- To prove Theorem 2.5 (3), we directly apply and repeatedly use the recurrence relation (1.11).
- To prove Theorem 2.5 (4), we replace x_j by $\frac{1}{x_j}$ on the generating function of the elementary symmetric polynomial (1.3). Hence, we obtain

$$\sum_{k=0}^{n} \boldsymbol{\sigma}_{k}^{(n)}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right) t^k = \frac{\prod_{j=1}^{n}(t+x_j)}{x_1 x_2 \cdots x_n} = \frac{\sum_{k=0}^{n} \boldsymbol{\sigma}_{n-k}^{(n)}(\mathbf{x})\, t^k}{\boldsymbol{\sigma}_{n}^{(n)}(\mathbf{x})}.$$

  Note that the numerator of the last term above comes from Vieta's theorem (see [19]). Furthermore, by comparing the coefficients of t^k, the proof of Theorem 2.5 (4) is complete.
- To prove Theorem 2.5 (5), since $x_1, x_2, \ldots, x_n$ are distinct, then by partial fraction decomposition, we have

$$\sum_{k=0}^{\infty} \mathbf{h}_{k}^{(n)}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right) t^k = \prod_{i=1}^{n} \frac{1}{\left(1 - \frac{t}{x_i}\right)} = \sum_{i=1}^{n} \lambda_i \left(1 - \frac{t}{x_i}\right)^{-1},$$

  where

$$\lambda_i = \prod_{\substack{j=1 \\ j\neq i}}^{n} \left(1 - \frac{t}{x_j}\right)^{-1} \Bigg|_{t=x_i} = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x_j}{(x_j - x_i)},$$

  for all $i = 1, 2, \ldots, n$. Consequently,

$$\sum_{k=0}^{\infty} \mathbf{h}_{k}^{(n)}\left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right) t^k = \sum_{i=1}^{n} \lambda_i \sum_{k=0}^{\infty} x_i^{-k}\, t^k = \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} \lambda_i x_i^{-k}\right) t^k$$

$$= \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} x_i^{-k} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_j}{(x_j - x_i)} \right) t^k,$$

the proof of Theorem 2.5 (5) is complete.

- To prove Theorem 2.5 (6), we rewrite the left-hand side as follows:

$$\sum_{i=1}^{n} x_i^2 \, \boldsymbol{\sigma}_{k,i}^{(n)}(\mathbf{x}) = \sum_{i=1}^{n} x_i \, \left(x_i \, \boldsymbol{\sigma}_{k,i}^{(n)}(\mathbf{x}) \right).$$

Now, by using the recurrences (1.13) and the identity (1) in Corollary 2.4, we obtain Theorem 2.5 (6).

- To prove Theorem 2.5 (7), we use the recurrence relation (1.15), and applying the identity (2) from Corollary 2.4, we establish Theorem 2.5 (7).
- To prove Theorem 2.5 (8), we will rewrite (1.13) as

$$\boldsymbol{\sigma}_{k}^{(n)}(\mathbf{x}) = \boldsymbol{\sigma}_{k+1,i}^{(n)}(\mathbf{x}) + x_i \, \boldsymbol{\sigma}_{k,i}^{(n)}(\mathbf{x}).$$

Then, by summing both sides in the above identity over i from 1 to n and using the identity (1) in Corollary 2.4. Thus, we obtain

$$n \, \boldsymbol{\sigma}_{k}^{(n)}(\mathbf{x}) = \sum_{i=1}^{n} \boldsymbol{\sigma}_{k+1,i}^{(n)}(\mathbf{x}) + k \, \boldsymbol{\sigma}_{k}^{(n)}(\mathbf{x}).$$

- To prove Theorem 2.5 (9), similarly, we will rewrite (1.15) as

$$\mathbf{h}_{k+1,i}^{(n)}(\mathbf{x}) = \mathbf{h}_{k}^{(n)}(\mathbf{x}) + x_i \, \mathbf{h}_{k,i}^{(n)}(\mathbf{x}).$$

Then, by summing both sides over i from 1 to n and using the identity (2) in Corollary 2.4. Thus, we directly obtain the identity Theorem 2.5 (9).

- To prove Theorem 2.5 (10), we rewrite the identity (1) in Corollary 2.4 as follows:

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_i \, \boldsymbol{\sigma}_{k,i}^{(n)}(\mathbf{x}) + x_j \, \boldsymbol{\sigma}_{k,j}^{(n)}(\mathbf{x}) = k \, \boldsymbol{\sigma}_{k}^{(n)}(\mathbf{x}).$$

Differentiate both sides with respect to x_j gives

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_i \, \boldsymbol{\sigma}_{k,i,j}^{(n)}(\mathbf{x}) + \boldsymbol{\sigma}_{k,j}^{(n)}(\mathbf{x}) = k \, \boldsymbol{\sigma}_{k,j}^{(n)}(\mathbf{x}).$$

Since $\boldsymbol{\sigma}_{k,j,j}^{(n)}(\mathbf{x}) = 0$, we directly get the identity Theorem 2.5 (10).

- To prove Theorem 2.5 (11), in a similar way, by using (2) in Corollary 2.4 and noting that $\mathbf{h}_{k,j,j}^{(n)}(\mathbf{x}) \neq 0$, we see that Theorem 2.5 (11) is satisfied.

- To prove Theorem 2.5 (12), we rewrite the recurrence relation (1.13) as

$$\boldsymbol{\sigma}^{(n)}_{k,i}(\mathbf{x}) = \boldsymbol{\sigma}^{(n)}_{k-1}(\mathbf{x}) - x_i\, \boldsymbol{\sigma}^{(n)}_{k-1,i}(\mathbf{x}).$$

  By partial differentiation with respect to x_j, then we have

$$\boldsymbol{\sigma}^{(n)}_{k,i,j}(\mathbf{x}) = \boldsymbol{\sigma}^{(n)}_{k-1,j}(\mathbf{x}) - x_i\, \boldsymbol{\sigma}^{(n)}_{k-1,i,j}(\mathbf{x}). \tag{2.14}$$

  Moreover, we conclude that

$$\boldsymbol{\sigma}^{(n)}_{k,j,i}(\mathbf{x}) = \boldsymbol{\sigma}^{(n)}_{k-1,i}(\mathbf{x}) - x_j\, \boldsymbol{\sigma}^{(n)}_{k-1,j,i}(\mathbf{x}). \tag{2.15}$$

  Since $\boldsymbol{\sigma}^{(n)}_{k}(\mathbf{x})$ is symmetric, then we complete the proof by subtracting (2.15) from (2.14).

□

Additionally, the authors in [16] showed that the CSP satisfies the following identity

$$\mathbf{h}^{(n)}_{k,i}(\mathbf{x}) - \mathbf{h}^{(n)}_{k,j}(\mathbf{x}) = (x_i - x_j)\, \mathbf{h}^{(n)}_{k,i,j}(\mathbf{x}), \tag{2.16}$$

for $i, j, k = 1, 2, \ldots, n$.

Based on the Schur-concavity of $\boldsymbol{\sigma}^{(n)}_{k}(\mathbf{x})$ on $\mathbb{R}^n_+$ and the identity (12) in Theorem 2.5, we see that $\boldsymbol{\sigma}^{(n)}_{k,i,j}(\mathbf{x}) \geq 0$ holds true for all $i, j, k = 1, 2, \ldots, n$ and $\mathbf{x} \in \mathbb{R}^n_+$ (see [20]). While the Schur-convexity of $\mathbf{h}^{(n)}_{k}(\mathbf{x})$ for even degree on $\mathbb{R}^n$, combined with the identity (2.16), results in $\mathbf{h}^{(n)}_{k,i,j}(\mathbf{x}) \geq 0$ for all $i, j = 1, 2, \ldots, n$ and k being an even positive integer (see [21]).

The complete symmetric polynomial can be written as a rational function, as shown by Jacobi (see [4]). The author in [22] provided proof of this fact using matrix decomposition. The current paper gives the proof by using partial fractions.

Theorem 2.6. *For a positive integer n and a set of distinct variables $x_1, x_2, \ldots, x_n$, then*

$$\mathbf{h}^{(n)}_{k}(\mathbf{x}) = \sum_{i=1}^{n} \frac{x_i^{n+k-1}}{\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i - x_j)}. \tag{2.17}$$

Proof. Since the variables $x_1, x_2, \ldots, x_n$ are all distinct from each other, we can use partial fraction decomposition to express the following:

$$H(t) = \prod_{i=1}^{n} \frac{1}{(1 - x_i t)} = \sum_{i=1}^{n} \frac{a_i}{(1 - x_i t)},$$

where for all $i = 1, 2, \ldots, n$, a_i is defined as

$$a_i = \left. \frac{1}{\prod_{\substack{j=1\\ j\neq i}}^{n}(1 - x_j t)} \right|_{t=1/x_i} = \frac{x_i^{n-1}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(x_i - x_j)}.$$

Hence,

$$\begin{aligned}\sum_{k=0}^{\infty}\mathbf{h}_k^{(n)}(\mathbf{x})\, t^k &= \sum_{i=1}^{n}a_i\sum_{k=0}^{\infty}x_i^k\, t^k = \sum_{k=0}^{\infty}\left(\sum_{i=1}^{n}a_ix_i^k\right) t^k \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=1}^{n}\frac{x_i^{n+k-1}}{\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i-x_j)}\right) t^k.\end{aligned}$$

The required result follows. □

3. Applications and illustrative examples

The main objective of the current section is to demonstrate three potential applications. Firstly, we will concentrate on the inversion of a generalized Vandermonde matrix. Secondly, we will explore specific applications concerning the determinant of two special tri-diagonal matrices. Lastly, we will discuss the representation of Jacobi polynomials in terms of their zeros.

3.1. On the inverse of a generalized Vandermonde matrix

The Vandermonde matrix is an example of such matrices, and it finds applications in various fields, including mathematics ([23–25]), engineering ([26, 27]), and natural science ([28–30]).

Let $p \in \mathbb{R}$. A generalized Vandermonde matrix denoted by $\mathbb{V}_{n,p}(x_1, x_2, \ldots, x_n)$ (for short, $\mathbb{V}_{n,p}$) and defined as $\mathbb{V}_{n,p} = \left[x_j^{p+i-1}\right]_{i,j=1}^{n}$ for distinct nodes $x_1, x_2, \ldots, x_n \in \mathbb{C}$. Here, we assume that $\mathbb{V}_{n,p}$ is an invertible matrix. It is clear that the classical Vandermonde matrix is a special case of $\mathbb{V}_{n,p}(x_1, x_2, \ldots, x_n)$ with $p = 0$. Following [1], the explicit formula of the determinant for a generalized Vandermonde matrix $\mathbb{V}_{n,p}$, is given by

$$\det\left(\mathbb{V}_{n,p}\right) = x_1^p\prod_{i=2}^{n}x_i^p\prod_{j=1}^{i-1}\left(x_i - x_j\right).$$

In their recent work [2], concise and rigorous proofs were presented for the determinant and inverse formulas of a generalized Vandermonde matrix. For the convenience of the reader, we mention the following result:

Theorem 3.1 ([2])**.** *Consider a generalized Vandermonde matrix,* $\mathbb{V}_{n,p}$ *with distinct nodes* $x_1, x_2, \ldots, x_n \in \mathbb{C}$*. Then, we have* $\mathbb{V}_{n,p}^{-1} = \left[\frac{\mathrm{N}_{ij}}{\mathrm{D}(x_i)}\right]_{i,j=1}^{n}$*, where*

$$\mathrm{N}_{ij} = \sum_{\ell=0}^{n-j}\varrho_\ell\, x_i^{n-j-\ell}, \tag{3.1}$$

$$\mathrm{D}(x_i) = \sum_{\ell=1}^{n}\ell\,\varrho_{n-\ell}\, x_i^{p+\ell-1}, \tag{3.2}$$

and

$$\varrho_\ell = (-1)^\ell\boldsymbol{\sigma}_\ell^{(n)}(x_1, x_2, \ldots, x_n). \tag{3.3}$$

It is advantageous to note, according to formula (3.2), that for every Vandermonde node x_i, there exists a corresponding denominator $\mathrm{D}(x_i)$. So, the denominators $\mathrm{D}(x_i)$ in any row of $\mathbb{V}_{n,p}^{-1}$ remain consistent. Moreover, in $\mathbb{V}_{n,p}^{-1}\left(x_1, x_2, \ldots, x_{\frac{n}{2}}, -x_1, -x_2, \ldots, -x_{\frac{n}{2}}\right)$, we can infer that

$$\mathrm{D}(-x_i) = (-1)^{p+1}\mathrm{D}(x_i). \tag{3.4}$$

Due to the relationship described in Eq (3.4), this will lead to a reduction in the computational cost for inverting the Vandermonde matrix $\mathbb{V}_{n,p}\left(x_1, x_2, \ldots, x_{\frac{n}{2}}, -x_1, -x_2, \ldots, -x_{\frac{n}{2}}\right)$.

Proposition 2.2 enables us to introduce the following result, which encompasses Corollaries 1 and 2 in [2] and provides a generalization of them for computing the inverse of $\mathbb{V}_{n,p}$ with distinct nodes $x_1, x_2, \ldots, x_r, -x_1, -x_2, \ldots, -x_r, y_1, y_2, \ldots, y_{n-2r} \in \mathbb{C}$, where r be a non-negative integer such that $0 \leq r \leq \lceil \frac{n}{2} \rceil$.

Corollary 3.2. *Consider the generalized Vandermonde matrix $\mathbb{V}_{n,t}$ with distinct nodes $x_1, x_2, \ldots, x_r$, $-x_1, -x_2 \ldots, -x_r, y_1, y_2, \ldots, y_{n-2r} \in \mathbb{C}$, where r be a non-negative integer such that $0 \leq r \leq \lceil \frac{n}{2} \rceil$. Then we have*

$$\mathbb{V}_{n,p}^{-1} = \left[\frac{\mathrm{N}_{ij}}{\mathrm{D}(x_i)}\right]_{i,j=1}^{n},$$

where

$$\mathrm{N}_{ij} = \sum_{\ell=0}^{n-j} \varrho_\ell \, x_i^{n-j-\ell}, \tag{3.5}$$

$$\mathrm{D}(x_i) = \sum_{\ell=1}^{n} \ell \, \varrho_{n-\ell} \, x_i^{p+\ell-1}, \tag{3.6}$$

and

$$\varrho_\ell = \sum_{\substack{\kappa=0 \\ \kappa\equiv 0 (\mathrm{mod}\ 2)}}^{\ell} (-1)^{(\kappa+2\ell)/2} \boldsymbol{\sigma}_{\kappa/2}^{(r)}(x_1^2 \ldots, x_r^2)\, \boldsymbol{\sigma}_{\ell-\kappa}^{(n-2r)}(y_1, \ldots, y_{n-2r}). \tag{3.7}$$

Notice that, when $r = \frac{n}{2}$ in Corollary 3.2, we obtain Corollary 1 in [2], which involves computing the inverse of $\mathbb{V}_{n,0}\left(x_1, x_2, \ldots, x_{\frac{n}{2}}, -x_1, -x_2, \ldots, -x_{\frac{n}{2}}\right)$. The special case $r = \frac{n-1}{2}$ and $y_1 = 0$ gives Corollary 2 in [2], which entails computing the inverse of $\mathbb{V}_{n,0}\left(x_1, x_2, \ldots, x_{\frac{n-1}{2}}, 0, -x_1, -x_2, \ldots, -x_{\frac{n-1}{2}}\right)$, with ones as the first-row inputs. The benefit of Corollary 3.2 is to reduce the computational cost of computing the inverse of $\mathbb{V}_{n,p}$ with distinct nodes $x_1, x_2, \ldots, x_r, -x_1, -x_2, \ldots, -x_r, y_1, y_2, \ldots, y_{n-2r} \in \mathbb{C}$, as the number of ESPs $\boldsymbol{\sigma}_k^{(n)}$ evaluations in (3.7) will decrease by approximately r times.

Example 3.3. *Consider the Vandemonde matrix*

$$\mathbb{V}_{5,\frac{1}{2}}(-2,-1,2,1,3) = \begin{bmatrix} \sqrt{-2} & \sqrt{-1} & \sqrt{2} & 1 & \sqrt{3} \\ (\sqrt{-2})^3 & (\sqrt{-1})^3 & (\sqrt{2})^3 & 1 & (\sqrt{3})^3 \\ (\sqrt{-2})^5 & (\sqrt{-1})^5 & (\sqrt{2})^5 & 1 & (\sqrt{3})^5 \\ (\sqrt{-2})^7 & (\sqrt{-1})^7 & (\sqrt{2})^7 & 1 & (\sqrt{3})^7 \\ (\sqrt{-2})^9 & (\sqrt{-1})^9 & (\sqrt{2})^9 & 1 & (\sqrt{3})^9 \end{bmatrix}.$$

For this matrix, we have $n = 5, x_1 = -2, x_2 = -1, x_3 = 2, x_4 = 1, x_5 = 3,$ $p = \frac{1}{2}$ *and* $r = 2$. *As stated in [2], through the implementation of the* VMIEA *algorithm and employing formula (3.7), we can infer that* $\varrho_0 = 1, \varrho_1 = -3, \varrho_2 = -5, \varrho_3 = 15, \varrho_4 = 4,$ *and* $\varrho_5 = -12$. *Furthermore, utilizing formula (3.6), we find that* $\mathrm{D}(-2) = 60\sqrt{2}\boldsymbol{i}, \mathrm{D}(-1) = -24\boldsymbol{i}, \mathrm{D}(2) = -12\sqrt{2}, \mathrm{D}(1) = 12$ *and* $\mathrm{D}(3) = 40\sqrt{3}$, *where* $\boldsymbol{i} = \sqrt{-1}$. *Thus, the inverse of* $\mathbb{V}_{5,\frac{1}{2}}(-2,-1,2,1,3)$ *can be expressed as*

$$\mathbb{V}^{-1}_{5,\frac{1}{2}} = \begin{bmatrix} \frac{-6}{60\sqrt{2}\boldsymbol{i}} & \frac{5}{60\sqrt{2}\boldsymbol{i}} & \frac{5}{60\sqrt{2}\boldsymbol{i}} & \frac{-5}{60\sqrt{2}\boldsymbol{i}} & \frac{1}{60\sqrt{2}\boldsymbol{i}} \\ \frac{-12}{-24\boldsymbol{i}} & \frac{16}{-24\boldsymbol{i}} & \frac{-1}{-24\boldsymbol{i}} & \frac{-4}{-24\boldsymbol{i}} & \frac{1}{-24\boldsymbol{i}} \\ \frac{6}{-12\sqrt{2}} & \frac{1}{-12\sqrt{2}} & \frac{-7}{-12\sqrt{2}} & \frac{-1}{-12\sqrt{2}} & \frac{1}{-12\sqrt{2}} \\ \frac{12}{12} & \frac{8}{12} & \frac{-7}{12} & \frac{-2}{12} & \frac{1}{12} \\ \frac{4}{40\sqrt{3}} & \frac{0}{40\sqrt{3}} & \frac{-5}{40\sqrt{3}} & \frac{0}{40\sqrt{3}} & \frac{1}{40\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{20}\boldsymbol{i} & -\frac{\sqrt{2}}{24}\boldsymbol{i} & -\frac{\sqrt{2}}{24}\boldsymbol{i} & \frac{\sqrt{2}}{24}\boldsymbol{i} & -\frac{\sqrt{2}}{120}\boldsymbol{i} \\ -\frac{1}{2}\boldsymbol{i} & \frac{2}{3}\boldsymbol{i} & -\frac{1}{24}\boldsymbol{i} & -\frac{1}{6}\boldsymbol{i} & \frac{1}{24}\boldsymbol{i} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{12\sqrt{2}} & \frac{7}{12\sqrt{2}} & \frac{1}{12\sqrt{2}} & -\frac{1}{12\sqrt{2}} \\ 1 & \frac{2}{3} & -\frac{7}{12} & -\frac{1}{6} & \frac{1}{12} \\ \frac{1}{10\sqrt{3}} & 0 & -\frac{1}{8\sqrt{3}} & 0 & \frac{1}{40\sqrt{3}} \end{bmatrix}.$$

3.2. Evaluating the determinant of two special tri-diagonal matrices

The tri-diagonal matrix is defined as $T = \left[t_{ij}\right]_{i,j=1}^{n}$ with $t_{ij} = 0$ for $|i-j| \geq 2$. This matrix is a common occurrence in various scientific and engineering fields, such as algebra [31], physics [32], parallel computing [33], and engineering [34].

Based on the identities (1) and (2) in Theorem 2.5, we obtain the following result: calculating the determinant of a particular tri-diagonal matrix.

Corollary 3.4. *Consider real symmetric* $n \times n$ *tri-diagonal matrices of the form*

$$A_n = \begin{bmatrix} x_1 + x_2 & x_2 & 0 & \cdots & \cdots & 0 \\ x_2 & x_2 + x_3 & x_3 & \ddots & & \vdots \\ 0 & x_3 & x_3 + x_4 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & x_n \\ 0 & \cdots & \cdots & 0 & x_n & x_n + x_{n+1} \end{bmatrix},$$

and

$$B_n = \begin{bmatrix} a+b & b & 0 & \cdots & \cdots & 0 \\ a & a+b & b & \ddots & & \vdots \\ 0 & a & a+b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & b \\ 0 & \cdots & \cdots & 0 & a & a+b \end{bmatrix},$$

then $\det(A_n) = \boldsymbol{\sigma}_n^{(n+1)}(x_1, x_2, \ldots, x_{n+1})$ *and* $\det(B_n) = \mathbf{h}_n^{(2)}(a, b)$.

Proof. Let $\det(A_n) = \triangle_n$. Write $\triangle_1 = x_1 + x_2 = \boldsymbol{\sigma}_1^{(2)}(x_1, x_2)$. The determinant of A_n can be computed via the three-term recurrence relation, that is,

$$\triangle_i = (x_i + x_{i+1})\triangle_{i-1} - x_i^2 \triangle_{i-2}, \tag{3.8}$$

for $i = 1, 2, \ldots, n$ and $\triangle_0 = 1,\ \triangle_{-1} = 0$ (see [35]). According to identity Theorem 2.5 (1), we can deduce that

$$\triangle_2 = (x_2 + x_3)\, \boldsymbol{\sigma}_1^{(2)}(x_1, x_2) - x_2^2\, \boldsymbol{\sigma}_0^{(1)}(x_1) = \boldsymbol{\sigma}_2^{(3)}(x_1, x_2, x_3).$$

By repeating this process, we get $\triangle_n = \boldsymbol{\sigma}_n^{(n+1)}(x_1, x_2, \ldots, x_{n+1})$.

Similarly, define $\det(B_n) = \hat{\triangle}_n$ and write $\hat{\triangle}_1 = a + b = \mathbf{h}_1^{(2)}(a, b)$. Utilizing the following three-term recurrence relation

$$\hat{\triangle}_i = (a + b)\, \hat{\triangle}_{i-1} - ab\, \hat{\triangle}_{i-2}, \tag{3.9}$$

for $i = 1, 2, \ldots, n$ and $\hat{\triangle}_0 = 1,\ \hat{\triangle}_{-1} = 0$ (see [35]), we can compute the determinant of B_n. Based on the identity Theorem 2.5 (2), we obtain

$$\hat{\triangle}_2 = (a + b)\, \mathbf{h}_1^{(2)}(a, b) - ab\, \mathbf{h}_0^{(2)}(a, b) = \mathbf{h}_2^{(2)}(a, b).$$

By repeating this procedure, we complete the proof. □

The inverse of matrices A_n and B_n can be calculated using the methods described in [36] or the algorithm outlined in [37]. The following corollary presents some specific cases that can be derived from Corollary 3.4. The proof of this corollary is straightforward and will not be included here.

Corollary 3.5. *Consider the tri-diagonal matrices A_n and B_n defined in Corollary 3.4.*

(1) If $a = b = 1$, then $\det(B_n) = n + 1$,
(2) If $a = b = -1$, then $\det(B_n) = (-1)^n(n + 1)$,
(3) If $x_i = i - 1,\ i = 1, 2, \ldots, n + 1$, then $\det(A_n) = n!$,
(4) If $x_i = i,\ i = 1, 2, \ldots, n + 1$, then $\det(A_n) = (n + 1)!\, H_{n+1}$, where H_n is the harmonic numbers,
(5) If $x_i = y,\ i = 1, 2, \ldots, n + 1$, then $\det(A_n) = (n + 1)\, y^n$,
(6) If $x_i = -y,\ i = 1, 2, \ldots, n + 1$, then $\det(A_n) = (-1)^{n+1}(n + 1)y^n$,
(7) If $x_i = q$ for $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil$, and $x_i = -q$ for $i = \lceil \frac{n}{2} \rceil + 1, \ldots, n + 1$, then

$$\det(A_n) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ (-1)^{n/2} q^n & \text{if } n \text{ is even}. \end{cases}$$

3.3. Zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$

In this subsection, we are going to focus on Jacobi polynomials and their zeros. A formula for Legendre polynomials in terms of their zeros was previously presented in [10]. The objective of the present study is to build upon this idea and derive a formula for Jacobi polynomials that expresses them in terms of their zeros.

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are a class of orthogonal polynomials with respect to a weight function $\omega(x) = (1-x)^\alpha(1+x)^\beta$ that are defined as the polynomials of degree n on the interval $[-1, 1]$. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are characterized by the two parameters $\alpha, \beta > -1$. According to formula (4.21.2) in [38], we can deduce that $P_n^{(\alpha,\beta)}(x)$ satisfies the following explicit formula:

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \binom{n + \alpha + \beta + k}{k} \left(\frac{x - 1}{2} \right)^k. \tag{3.10}$$

Following the formula (4.5.7) in [38], the Jacobi polynomials satisfy the subsequent recurrence relation

$$\begin{aligned}(2n+\alpha+\beta)\left(1-x^2\right)\frac{\mathrm{d}}{\mathrm{d}x}P_{n-1}^{(\alpha,\beta)}(x) =&(n+\alpha+\beta)(\alpha-\beta+(2n+\alpha+\beta)x)P_{n-1}^{(\alpha,\beta)}(x)\\ &-2n(n+\alpha+\beta)P_n^{(\alpha,\beta)}(x).\end{aligned} \tag{3.11}$$

There are various special instances of Jacobi polynomials. These include the Legendre polynomials $P_n(x)$ ($\alpha=\beta=0$), the Chebyshev polynomials of the first kind $T_n(x)$ ($\alpha=\beta=-1/2$), the Chebyshev polynomials of the second kind $U_n(x)$ ($\alpha=\beta=1/2$), the Chebyshev polynomials of the third kind $V_n(x)$ ($\alpha=-1/2,\beta=1/2$), the Chebyshev polynomials of the fourth kind $W_n(x)$ ($\alpha=1/2,\beta=-1/2$), and the ultraspherical polynomials (Gegenbauer polynomials) $C_n^{(\lambda)}(x)$ ($\alpha=\beta>-\frac{1}{2}$), where $\lambda=\alpha+\frac{1}{2}$. These polynomials have many applications in mathematics and physics, as demonstrated in works such as [39–42]. An alternate explicit formula for Jacobi polynomials, equivalent to formula (3.10), is provided by the following result:

Corollary 3.6. *Consider the sequence of Jacobi polynomials* $\left\{P_n^{(\alpha,\beta)}(x)\right\}_{n=0}^{\infty}$. *Then the explicit formula (3.10) is equivalent to the following formula:*

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\left[\sum_{\ell=0}^{n-k}\frac{(-1)^{\ell}}{2^{k+\ell}}\binom{n+\alpha}{n-k-\ell}\binom{n+\alpha+\beta+k+\ell}{k+\ell}\binom{k+\ell}{k}\right]x^k. \tag{3.12}$$

Proof. The proof is obtained by applying the binomial theorem to $(x-1)^k$ in formula (3.10), and then applying the associativity and commutativity properties of double summations. □

Following this, we will proceed to introduce a result that allows us to express Jacobi polynomials in terms of their zeros.

Theorem 3.7. *For any positive integer n, consider* $x_1,x_2,\dots,x_n$ *are the zeros of the Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$. *Then*

$$P_n^{(\alpha,\beta)}(x)=2^{-n}\binom{2n+\alpha+\beta}{n}\sum_{k=0}^{n}(-1)^{n-k}\sigma_{n-k}^{(n)}(x_1,x_2,\dots,x_n)\,x^k. \tag{3.13}$$

Proof. Using the formula (3.12), we can deduce that the highest-degree term of $P_n^{(\alpha,\beta)}(x)$ has a coefficient of $2^{-n}\binom{2n+\alpha+\beta}{n}$. Since $x_1,x_2,\dots,x_n$ are the zeros of $P_n^{(\alpha,\beta)}(x)$, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be expressed as follows:

$$P_n^{(\alpha,\beta)}(x)=2^{-n}\binom{2n+\alpha+\beta}{n}Q(x),$$

where $Q(x)$ is a monic polynomial of degree n with zeros $x_1,x_2,...,x_n$. Furthermore, $Q(x)$ can be written as the product of n linear factors, that is, $Q(x)=\prod_{i=1}^{n}(x-x_i)$. Utilizing Vieta's formula [43, Theorem 33.3] on $Q(x)$ allows us to complete the proof and arrive at the desired result. □

By combining Corollary 3.6 and Theorem 3.7, we can derive the following result, whose proof will be omitted.

Lemma 3.8. *Consider n to be a positive integer and k to be a non-negative integer. If we let $x_1, x_2, \ldots, x_n$ be the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, then*

$$\boldsymbol{\sigma}_k^{(n)}(x_1, x_2, \ldots, x_n) = \frac{2^k}{k!\,(2n+\alpha+\beta)!}\binom{n}{k}\sum_{\ell=0}^{k}(-1)^{\ell+k}\binom{n+\alpha}{k-\ell}\frac{(2n+\alpha+\beta-k+\ell)!}{\ell!}. \tag{3.14}$$

In addition, it can be shown that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be expressed using the zeros of $P_{n-1}^{(\alpha,\beta)}(x)$. The following result will provide this fact.

Theorem 3.9. *For any positive integer $n > 1$. Let $P_n^{(\alpha,\beta)}(x)$ and $P_{n-1}^{(\alpha,\beta)}(x)$ be Jacobi polynomials with zeros $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_{n-1}$, respectively. Then,*

$$P_n^{(\alpha,\beta)}(x) = \frac{2^{-n}}{(2n+\alpha+\beta-1)}\binom{2n+\alpha+\beta}{n}\sum_{k=0}^{n}(-1)^k A_k\, x^{n-k}, \tag{3.15}$$

with

$$A_k = (2n+\alpha+\beta-k-1)\boldsymbol{\sigma}_k^{(n-1)}(\mathbf{y}) - \frac{(n+\alpha+\beta)(\alpha-\beta)}{(2n+\alpha+\beta)}\boldsymbol{\sigma}_{k-1}^{(n-1)}(\mathbf{y}) - (n-k+1)\boldsymbol{\sigma}_{k-2}^{(n-1)}(\mathbf{y}), \tag{3.16}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_{n-1})$.

Proof. We can demonstrate the validity of this proof by using the recurrence relation (3.11) with the modified formula of Jacobi polynomials (3.13). □

If we set $\alpha = \beta$, we obtain the property that $P_n^{(\alpha,\alpha)}(-x) = (-1)^n P_n^{(\alpha,\alpha)}(x)$. By virtue of Theorem 3.7, the following corollaries for $P_n^{(\alpha,\alpha)}(x)$ with even and odd orders can be derived, respectively.

Corollary 3.10. *Let n be an even positive integer, and $x_1, x_2, \ldots, x_{\frac{n}{2}}$ are the positive zeros of the Jacobi polynomial $P_n^{(\alpha,\alpha)}(x)$. Then $P_n^{(\alpha,\alpha)}(x)$ can be expressed as follows:*

$$P_n^{(\alpha,\alpha)}(x) = 2^{-n}\binom{2n+2\alpha}{n}\sum_{k=0}^{\frac{n}{2}}(-1)^k\boldsymbol{\sigma}_k^{(\frac{n}{2})}\left(x_1^2, x_2^2, \ldots, x_{\frac{n}{2}}^2\right)x^{n-2k}. \tag{3.17}$$

Remark 1. *Drawing from Corollary 3.10 and Theorem 2.5, let us delve into the following observations: Let n be an even positive integer and $k = 0, 1, 2, \ldots, \frac{n}{2}$.*

(1) In case $\alpha > -\frac{1}{2}$, consider that $x_1, x_2, \ldots, x_{\frac{n}{2}}$ are the positive zeros of the ultraspherical polynomial $C_n^{(\lambda)}(x)$ with $\lambda = \alpha + \frac{1}{2}$, then

$$\boldsymbol{\sigma}_k^{(\frac{n}{2})}\left(x_1^2, x_2^2, \ldots, x_{\frac{n}{2}}^2\right) = 2^{2n-2k}\binom{n-k}{k}\frac{\binom{n-k+\alpha-\frac{1}{2}}{\alpha-\frac{1}{2}}\binom{n+\alpha}{n}}{\binom{2n+2\alpha}{n}\binom{n+2\alpha}{2\alpha}},$$

$$\boldsymbol{\sigma}_k^{(\frac{n}{2})}\left(\frac{1}{x_1^2}, \frac{1}{x_2^2}, \ldots, \frac{1}{x_{\frac{n}{2}}^2}\right) = 4^k\frac{\binom{\frac{n}{2}+k}{\frac{n}{2}-k}\binom{\frac{n}{2}+k+\alpha-\frac{1}{2}}{\alpha-\frac{1}{2}}}{\binom{\frac{n}{2}+\alpha-\frac{1}{2}}{\alpha-\frac{1}{2}}}.$$

(2) *In case* $\alpha = 0$*, consider that* $x_1, x_2, \ldots, x_{\frac{n}{2}}$ *are the positive zeros of the Legendre polynomial* $P_n(x)$*, then*

$$\boldsymbol{\sigma}_k^{\left(\frac{n}{2}\right)}\left(x_1^2, x_2^2, \ldots, x_{\frac{n}{2}}^2\right) = \frac{\binom{n}{k}\binom{2n-2k}{n}}{\binom{2n}{n}},$$

$$\boldsymbol{\sigma}_k^{\left(\frac{n}{2}\right)}\left(\frac{1}{x_1^2}, \frac{1}{x_2^2}, \ldots, \frac{1}{x_{\frac{n}{2}}^2}\right) = \frac{\binom{n}{\frac{n}{2}-k}\binom{n+2k}{n}}{\binom{n}{\frac{n}{2}}}.$$

(3) *In case* $\alpha = -1/2$*, consider that* $x_1, x_2, \ldots, x_{\frac{n}{2}}$ *are the positive zeros of Chebyshev polynomials of the first kind* $T_n(x)$*, then*

$$\boldsymbol{\sigma}_k^{\left(\frac{n}{2}\right)}\left(x_1^2, x_2^2, \ldots, x_{\frac{n}{2}}^2\right) = \frac{n}{4^k(n-k)}\binom{n-k}{k},$$

$$\boldsymbol{\sigma}_k^{\left(\frac{n}{2}\right)}\left(\frac{1}{x_1^2}, \frac{1}{x_2^2}, \ldots, \frac{1}{x_{\frac{n}{2}}^2}\right) = \frac{4^k n}{(n+2k)}\binom{\frac{n}{2}+k}{2k}.$$

(4) *In case* $\alpha = 1/2$*, consider that* $x_1, x_2, \ldots, x_{\frac{n}{2}}$ *are the positive zeros of Chebyshev polynomials of the second kind* $U_n(x)$*, then*

$$\boldsymbol{\sigma}_k^{\left(\frac{n}{2}\right)}\left(x_1^2, x_2^2, \ldots, x_{\frac{n}{2}}^2\right) = \frac{1}{4^k}\binom{n-k}{k},$$

$$\boldsymbol{\sigma}_k^{\left(\frac{n}{2}\right)}\left(\frac{1}{x_1^2}, \frac{1}{x_2^2}, \ldots, \frac{1}{x_{\frac{n}{2}}^2}\right) = 4^k\binom{\frac{n}{2}+k}{2k}.$$

Corollary 3.11. *Let n be an even positive integer, and* $y_1, y_2, \ldots, y_{\frac{n}{2}}$ *are the positive zeros of the Jacobi Polynomial* $P_{n+1}^{(\alpha,\alpha)}(x)$*. Then* $P_{n+1}^{(\alpha,\alpha)}(x)$ *can be expressed as follows:*

$$P_{n+1}^{(\alpha,\alpha)}(x) = 2^{-(n+1)}\binom{2n+2\alpha+2}{n+1}\sum_{k=0}^{\frac{n}{2}}(-1)^k\boldsymbol{\sigma}_k^{\left(\frac{n}{2}\right)}\left(y_1^2, y_2^2, \ldots, y_{\frac{n}{2}}^2\right)x^{n-2k+1}. \tag{3.18}$$

Remark 2. *Referring to Corollary 3.11 and Theorem 2.5, we now turn our attention to the following insights: Let n be an even positive integer and* $k = 0, 1, 2, \ldots, \frac{n}{2}$*.*

(1) *If we set* $\alpha > -\frac{1}{2}$*, let us assume that* $y_1, y_2, \ldots, y_{\frac{n}{2}}$ *represent the positive zeros of the ultraspherical polynomial* $C_{n+1}^{(\lambda)}(x)$ *with* $\lambda = \alpha + \frac{1}{2}$*. Then,*

$$\boldsymbol{\sigma}_k^{\left(\frac{n}{2}\right)}\left(y_1^2, y_2^2, \ldots, y_{\frac{n}{2}}^2\right) = 2^{2n-2k+2}\binom{n-k+1}{k}\frac{\binom{n-k+\alpha+\frac{1}{2}}{\alpha-\frac{1}{2}}\binom{n+\alpha+1}{n}}{\binom{2n+2\alpha+2}{n+1}\binom{n+2\alpha+1}{2\alpha}},$$

$$\boldsymbol{\sigma}_k^{(\frac{n}{2})}\left(\frac{1}{y_1^2},\frac{1}{y_2^2},\ldots,\frac{1}{y_{\frac{n}{2}}^2}\right)=\frac{2^{2k+1}}{n+2}\frac{\binom{\frac{n}{2}+k+1}{\frac{n}{2}-k}\binom{\frac{n}{2}+k+\alpha+\frac{1}{2}}{\alpha-\frac{1}{2}}}{\binom{\frac{n}{2}+\alpha+\frac{1}{2}}{\alpha-\frac{1}{2}}}.$$

(2) *If we set $\alpha = 0$, let us assume that $y_1, y_2, \ldots, y_{\frac{n}{2}}$ represent the positive zeros of the Legendre polynomial $P_{n+1}(x)$. Then,*

$$\boldsymbol{\sigma}_k^{(\frac{n}{2})}\left(y_1^2,y_2^2,\ldots,y_{\frac{n}{2}}^2\right)=\frac{\binom{n+1}{k}\binom{2n-2k+2}{n+1}}{\binom{2n+2}{n+1}},$$

$$\boldsymbol{\sigma}_k^{(\frac{n}{2})}\left(\frac{1}{y_1^2},\frac{1}{y_2^2},\ldots,\frac{1}{y_{\frac{n}{2}}^2}\right)=\frac{\binom{n+1}{\frac{n}{2}-k}\binom{n+2k+2}{n+1}}{(n+2)\binom{n+1}{\frac{n}{2}}}.$$

(3) *If we set $\alpha = -1/2$, let us assume that $y_1, y_2, \ldots, y_{\frac{n}{2}}$ represent the positive zeros of Chebyshev polynomials of the first kind, $T_{n+1}(x)$. Then*

$$\boldsymbol{\sigma}_k^{(\frac{n}{2})}\left(y_1^2,y_2^2,\ldots,y_{\frac{n}{2}}^2\right)=\frac{n+1}{4^k(n-k+1)}\binom{n-k+1}{k},$$

$$\boldsymbol{\sigma}_k^{(\frac{n}{2})}\left(\frac{1}{y_1^2},\frac{1}{y_2^2},\ldots,\frac{1}{y_{\frac{n}{2}}^2}\right)=\frac{4^k}{(2k+1)}\binom{\frac{n}{2}+k}{2k}.$$

(4) *If we set $\alpha = 1/2$, let us assume that $y_1, y_2, \ldots, y_{\frac{n}{2}}$ represent the positive zeros of Chebyshev polynomials of the second kind $U_{n+1}(x)$. Then*

$$\boldsymbol{\sigma}_k^{(\frac{n}{2})}\left(y_1^2,y_2^2,\ldots,y_{\frac{n}{2}}^2\right)=\frac{1}{4^k}\binom{n-k+1}{k},$$

$$\boldsymbol{\sigma}_k^{(\frac{n}{2})}\left(\frac{1}{y_1^2},\frac{1}{y_2^2},\ldots,\frac{1}{y_{\frac{n}{2}}^2}\right)=\frac{2^{2k+1}}{(n+2)}\binom{\frac{n}{2}+k+1}{2k+1}.$$

Clearly, setting $\alpha = 0$ in Corollaries 3.10 and 3.11 allows us to obtain Corollaries 2 and 3, respectively, as derived in [10]. It is clear that we can also define the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in terms of the complete symmetric polynomials of their zeros via the relation (1.8).

Now, we will provide some illustrative examples concerning Theorem 3.7, Theorem 3.9, and Corollary 3.11.

Example 3.12. *Taking $\alpha = 0$ and $\beta = 1$, the zeros of $P_2^{(0,1)}(x)$ are $x_1 = \frac{1}{5} - \frac{\sqrt{6}}{5}$ and $x_2 = \frac{1}{5} + \frac{\sqrt{6}}{5}$. Therefore, we have*

$$\boldsymbol{\sigma}_2^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)=-\frac{1}{5},$$

$$\boldsymbol{\sigma}_1^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)=\frac{2}{5}.$$

Using Theorem 3.7, we can further deduce

$$\begin{aligned}P_2^{(0,1)}(x)&=2^{-2}\binom{5}{2}\sum_{k=0}^{2}(-1)^{2-k}\boldsymbol{\sigma}_{2-k}^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)x^k\\&=\frac{5}{2}\left[-\frac{1}{5}-\frac{2}{5}x+x^2\right]=-\frac{1}{2}-x+\frac{5}{2}x^2.\end{aligned}$$

Consequently, by applying Theorem 3.9, we can formulate the Jacobi polynomial $P_3^{(0,1)}(x)$ *in the following manner:*

$$P_3^{(0,1)}(x)=\frac{2^{-3}}{6}\binom{7}{3}\sum_{k=0}^{3}(-1)^kA_kx^{3-k},$$

where, the values of A_k *for* $k=0,1,2,3$ *are computed using the formula (3.16) as follows:*

$$\begin{aligned}A_0&=6\boldsymbol{\sigma}_0^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)=6,\\A_1&=5\boldsymbol{\sigma}_1^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)+\frac{4}{7}\boldsymbol{\sigma}_0^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)=\frac{18}{7},\\A_2&=4\boldsymbol{\sigma}_2^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)+\frac{4}{7}\boldsymbol{\sigma}_1^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)-2\boldsymbol{\sigma}_0^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)=-\frac{18}{7},\\A_3&=\frac{4}{7}\boldsymbol{\sigma}_2^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)-\boldsymbol{\sigma}_1^{(2)}\left(\frac{1}{5}-\frac{\sqrt{6}}{5},\frac{1}{5}+\frac{\sqrt{6}}{5}\right)=-\frac{18}{35}.\end{aligned}$$

Hence, the Jacobi polynomial $P_3^{(0,1)}(x)$ *is given by*

$$\begin{aligned}P_3^{(0,1)}(x)&=\frac{35}{48}\left[6x^3-\frac{18}{7}x^2-\frac{18}{7}x+\frac{18}{35}\right]\\&=\frac{1}{8}\left(35x^3-15x^2-15x+3\right).\end{aligned}$$

Example 3.13. *Let us consider the case where* $\alpha=\beta=\frac{5}{2}$. *The zeros of* $P_3^{\left(\frac{5}{2},\frac{5}{2}\right)}(x)$ *are* $x_1=-\sqrt{\frac{3}{10}}$, $x_2=0$, *and* $x_3=\sqrt{\frac{3}{10}}$.

Utilizing Corollary 3.11, we obtain

$$P_3^{\left(\frac{5}{2},\frac{5}{2}\right)}(x)=2^{-3}\binom{11}{3}\sum_{k=0}^{1}(-1)^k\boldsymbol{\sigma}_k^{(1)}\left(\frac{3}{10}\right)x^k=\frac{165}{8}\left[-\frac{3}{10}x+x^3\right].$$

Applying formula (4.7.1) in [38], we obtain the normalized ultraspherical polynomial of degree 3 *with* $\lambda=3$ *as*

$$C_3^{(3)}(x)=\frac{\left(2+\frac{1}{2}\right)!\;8!}{5!\;\left(5+\frac{1}{2}\right)!}\,P_3^{\left(\frac{5}{2},\frac{5}{2}\right)}(x)=8x\left(10x^2-3\right).$$

By employing Theorem 3.9, we can express the Jacobi polynomial $P_4^{(\frac{5}{2},\frac{5}{2})}(x)$ *in terms of the zeros of* $P_3^{(\frac{5}{2},\frac{5}{2})}(x)$ *as follows*

$$P_4^{(\frac{5}{2},\frac{5}{2})}(x) = \frac{2^{-4}}{12}\binom{13}{4}\sum_{k=0}^{4}(-1)^k A_k x^{4-k},$$

where the coefficients A_k *are calculated using the formula (3.16) as shown*

$$A_k = (12-k)(-1)^{\frac{k}{2}}\boldsymbol{\sigma}^{(1)}_{\frac{k}{2}}\left(\frac{3}{10}\right) - (5-k)(-1)^{\frac{k-2}{2}}\boldsymbol{\sigma}^{(1)}_{\frac{k-2}{2}}\left(\frac{3}{10}\right), \quad k = 0,1,2,3,4.$$

Therefore, the Jacobi polynomial $P_4^{(\frac{5}{2},\frac{5}{2})}(x)$ *is given by*

$$P_4^{(\frac{5}{2},\frac{5}{2})}(x) = \frac{2^{-4}}{12}\binom{13}{4}\left[12x^4 - 6x^2 + \frac{3}{10}\right].$$

Finally, the normalized ultraspherical polynomial of degree 4 *with* $\lambda = 3$ *can be represented as*

$$C_4^{(3)}(x) = \frac{\left(2+\frac{1}{2}\right)!\ 9!}{5!\ \left(6+\frac{1}{2}\right)!}\ P_4^{(\frac{5}{2},\frac{5}{2})}(x) = 6\left[40x^4 - 20x^2 + 1\right].$$

4. Conclusions

In this article, we presented novel identities for elementary and complete symmetric polynomials and explored their practical implications. These identities helped to increase our understanding of elementary and complete symmetric polynomials, as well as their connections to various fields. For instance, we extended certain results presented in [2], specifically those concerning the inversion of a generalized Vandermonde matrix. Additionally, we applied some of the derived identities to compute the determinant of two symmetric tri-diagonal matrices. Furthermore, we extended the results presented in [10] concerning orthogonal polynomials. Theorem 3.9 shows that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be expressed using the zeros of $P_{n-1}^{(\alpha,\beta)}(x)$.

Author contributions

Both authors A. Arafat and M. El-Mikkawy contributed equally to this work. Both authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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