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Research article

BMO estimates for commutators of the rough fractional Hausdorff operator on grand-variable-Herz-Morrey spaces

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Abstract: In this paper, we study the boundedness of the commutator of the rough fractional Hausdorff operator on grand-variable-Herz-Morrey spaces when the symbol functions belong to bounded mean oscillations (BMO) space.

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1. Introduction

In the literature, one can find a rich history of the Hausdorff operator in harmonic analysis. Let's start our discussion by introducing the high-dimensional rough fractional Hausdorff operator [1]:

$$H^{\beta}_{\Phi,\Omega}g(z) = \int_{\mathbb{R}^n} \frac{\Phi(z|t|^{-1})}{|t|^{n-\beta}}\Omega(t')g(t)dt, \quad 0 \leq \beta < n. \tag{1.1}$$

This is the most general form of Hausdorff operator, as the remaining definitions can be easily obtained from (1.1). For instance, if we take $\Omega = 1$, then we get the high-dimensional fractional Hausdorff operator [2]. Also, $\beta = 0$ provides the n-dimensional rough Hausdorff operator [3]. Similarly, if $\Omega = 1 = n$ and $\beta = 0$, then we get the one-dimensional Hausdorff operator [4–6]. Furthermore, if we choose the parameters correctly, many celebrated integral operators, such as Hardy-type operators, become special cases of the rough fractional Hausdorff operator. Furthermore, the commutator of (1.1):

$$H^{\beta,b}_{\Phi,\Omega}g(z) = b(z)H^{\beta}_{\Phi,\Omega}g(z) - H^{\beta}_{\Phi,\Omega}(bg)(z), \tag{1.2}$$

is just as important as the operator itself.

On the other hand, function spaces with variable exponents are not a mere generalization of classical ones but appear to have a natural application in many real-world phenomena; see [7], for example. The variable-exponent Lebesgue space $L^{q(\cdot)}$ can be traced back to Orlicz [8]. In 1990, however, the authors of [9] incorporated the formal theory of function spaces with variable exponents into its structure. They introduced the variable-exponent Lebesgue and Sobolev spaces. Since the publication of [9], several authors have contributed to the theory of variable exponent function spaces. The works in [10–13], in particular, greatly influence the basic structure and relevant properties of such spaces. Finally, we must cite some recent publications in this direction: [14–18]. Additionally, the structure of grand spaces was first developed in [19, 20] and continued to flourish through the years [21–24]. Recently, Kokilasvili and Meski [25] defined the grand-variable exponent Lebesgue spaces, which gave this field a new direction and attracted the attention of many authors [26–28].

Commutators of integral operators find their applications in the regularity theory of partial differential equations and in the characterization of function spaces [29]. Such applications make their studies more important and valuable [30]. The commutators of various Hausdorff operators hold significant importance and have been a topic of discussion for numerous authors [31–34]. However, no one has tested the boundedness of such commutator operators on variable-exponent function spaces. The purpose of this article is to fill this gap by establishing the boundedness of (1.2) on grand-variable-exponent Herz-Morrey spaces. In a special case, we also obtain the continuity of $H^{\beta,b}_{\Phi,\Omega}$ on grand-variable Herz spaces.

In the next section, we present preliminary results, which will be helpful in establishing our main results. Finally, in the last section, we state our main results and give their proofs.

2. Variable-exponent function spaces and related results

Let us open our discussion by introducing the variable-exponent Lebesgue spaces. Let A be an open subset of $\mathbb{R}^n$ and $q(\cdot)$ be a measurable function on A with values in $[1, \infty)$, and $q'(\cdot)$ denotes the prime index corresponding to $q(\cdot)$, i.e., $q'(\cdot) = q(\cdot)/(q(\cdot) - 1)$. The collection of all functions $q(\cdot)$ that satisfy:

$$1 < q_- \leq q_+ < \infty,$$

where $q_- = \operatorname{essinf}_{z\in A} q(z)$, and $q_+ = \operatorname{esssup}_{z\in A} q(z)$, is denoted by $\mathfrak{P}(A)$. The Lebesgue space with variable exponent $L^{q(\cdot)}(A)$ is defined as the set of all measurable functions $f(z)$ satisfying:

$$\int_A \left(\frac{|f(z)|}{\zeta}\right)^{q(z)} dz < \infty,$$

where the constant $\zeta > 0$. When equipped with the Luxemburg norm, it becomes a Banach function space.

$$\|f\|_{L^{q(\cdot)}(A)} = \inf\left\{\zeta > 0 : \int_A \left(\frac{|f(z)|}{\zeta}\right)^{q(z)} dz \leq 1\right\}.$$

Its local version, $L^{q(\cdot)}_{\text{loc}}(E)$, is defined as:

$$L^{q(\cdot)}_{\text{loc}}(E) = \left\{f : f \in L^{q(\cdot)}(A)\ \forall \text{ compact subset A} \subset \text{E}\right\}.$$

In the study of variable exponent function spaces, an essential operator is the Hardy-Littlewood maximal operator $\mathcal{M}f$. For a measurable function f on $L^{q(\cdot)}(\mathbb{R}^n)$, it can be defined as:

$$\mathcal{M}f(z) = \sup_{r>0} \frac{1}{|B(z,r)|} \int_{B(z,r)} |f(\sigma)| d\sigma.$$

In the remainder of this paper, we use the notation $\mathcal{B}(\mathbb{R}^n)$ to denote the set consisting of $q(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ such that $\mathcal{M}$ is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$.

Proposition 2.1. [13,35] Let $A \subset \mathbb{R}^n$ be an open set, and $q(\cdot) \in \mathfrak{P}(A)$ satisfies:

$$|q(\sigma) - q(\zeta)| \le \frac{-C}{\ln(|\sigma - \zeta|)}, \quad \frac{1}{2} \ge |\sigma - \zeta|, \tag{2.1}$$

$$|q(\sigma) - q(\zeta)| \le \frac{C}{\ln(|\sigma| + e)}, \quad |\sigma| \le |\zeta|, \tag{2.2}$$

then $q(\cdot) \in \mathcal{B}(A)$, where C is a positive constant independent of σ and ζ.

Lemma 2.2. [9] Let $q(\cdot) \in \mathfrak{P}(A)$. If $g \in L^{q(\cdot)}(A)$ and $h \in L^{q'(\cdot)}(A)$, then we have

$$\int_A |g(z)h(z)| dz \le r_q \|g\|_{L^{q(\cdot)}(A)} \|h\|_{L^{q'(\cdot)}(A)},$$

where $r_q = 1 + \frac{1}{q_-} - \frac{1}{q_+}$.

Lemma 2.3. [36] If $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exist constants $0 < \delta < 1$ and $C > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \le C \left(\frac{|B|}{|S|} \right)^{\delta}.$$

Lemma 2.4. [11] Define a variable exponent $\tilde{p}(\cdot)$ such that $\frac{1}{q(t)} = \frac{1}{\tilde{p}(t)} + \frac{1}{p}$, $(t \in \mathbb{R}^n)$. Then, we have

$$\|gh\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C \|g\|_{L^{\tilde{p}(\cdot)}(\mathbb{R}^n)} \|h\|_{L^p(\mathbb{R}^n)}.$$

Lemma 2.5. [35] Let $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$ satisfy conditions (2.1) and (2.2) in Proposition 2.1, then

$$\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \approx \begin{cases} |Q|^{\frac{1}{q(x)}} & \text{if } |Q| < 2^n \text{ and } x \in Q, \\ |Q|^{\frac{1}{q(\infty)}} & \text{if } |Q| \ge 1, \end{cases}$$

for all cubes (or balls) $Q \subset \mathbb{R}^n$, where $q(\infty) = \lim_{x \to \infty} q(x)$.

Let $B_k = \{t \in \mathbb{R}^n : |t| \le 2^k\}$, $C_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$. Then, the homogeneous Herz space with variable exponent was first defined in [37,38].

Definition 2.6. Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, and $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions f such that:

$$\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha q} \|f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{\frac{1}{q}}.$$

If $p(\cdot) = p$, then we have the classical Herz space $\dot{K}_p^{\alpha,q}$ studied in [39].

Definition 2.7. [36] Let $\alpha \in \mathbb{R},\ 0 < q < \infty$, $\lambda \in [0, \infty)$, and $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. The space $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions f given by

$$M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \Big\{f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} < \infty\Big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} = \sup_{k_0\in Z} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} \|2^{k\alpha} f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{1/q}.$$

Obviously, $M\dot{K}^{\alpha,0}_{q,p(\cdot)}(\mathbb{R}^n)=\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is the Herz space with variable exponent.

Definition 2.8. [27] Let $\alpha \in \mathbb{R},\ 0 < q < \infty$, $\lambda \in [0, \infty), \theta > 0$, and $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. Then, grand-variable Herz-Morrey space $M\dot{K}^{\alpha,q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)$ is

$$M\dot{K}^{\alpha,q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n) = \Big\{f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha,q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)} < \infty\Big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,q),\theta}_{\lambda,p(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda}\Big(\epsilon^\theta \sum_{j=-\infty}^{j_0} 2^{j\alpha q(1+\epsilon)}\|f\chi_j\|^{q(1+\epsilon)}_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q(1+\epsilon)}}.$$

Taking $\lambda = 0$ in the above definition, we get the definition of grand-variable Herz space defined in [26].

Definition 2.9. Let $\alpha \in \mathbb{R},\ 0 < q < \infty, \theta > 0$, and $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$. Then, grand-variable-Herz space $\dot{K}^{\alpha,q),\theta}_{p(\cdot)}(\mathbb{R}^n)$ is

$$\dot{K}^{\alpha,q),\theta}_{p(\cdot)}(\mathbb{R}^n) = \Big\{f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,q),\theta}_{p(\cdot)}(\mathbb{R}^n)} < \infty\Big\},$$

where

$$\|f\|_{\dot{K}^{\alpha,q),\theta}_{p(\cdot)}(\mathbb{R}^n)} = \sup_{\epsilon>0}\Big(\epsilon^\theta \sum_{j\in\mathbb{Z}} 2^{j\alpha q(1+\epsilon)}\|f\chi_j\|^{q(1+\epsilon)}_{L^{p(\cdot)}(\mathbb{R}^n)}\Big)^{\frac{1}{q(1+\epsilon)}}.$$

Definition 2.10. Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then b is said to belong to the bounded mean oscillation space $\mathrm{BMO}(\mathbb{R}^n)$ if $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} < \infty$, where

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |b(x) - b_B|dx,$$

and supremum is taken over all the balls $B \subset \mathbb{R}^n$ with $b_B = |B|^{-1}\int_B b(y)dy$.

Lemma 2.11. [38] Let $p(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$, then for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and all $l, m \in \mathbb{Z}$ with $l > m$, we have

$$C^{-1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \le \sup_{B:Ball} \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b - b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

$$\|(b - b_{B_m})\chi_{B_l}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C(l - m)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Remark 2.12. If $p_1(\cdot)$, $p_2(\cdot)$,$p'(\cdot)$, $p'_1(\cdot)$ are variable exponents and $p_1(\cdot)$, $p_2(\cdot)$ belong to $\mathfrak{P}(\mathbb{R}^n)$, satisfying conditions in Proposition 2.1, then $p'(\cdot), p'_1(\cdot)$, and $p_2(\cdot)$ belong to $\mathcal{B}(\mathbb{R}^n)$. By using Lemma 2.2, there exist constants $\delta_1 \in (0, \frac{1}{(p'_1)_+})$, $\delta_2 \in (0, \frac{1}{(p_2)_+})$, such that the inequalities:

$$\frac{\|\chi_S\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'_1(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

hold for all balls $B \subset \mathbb{R}^n$ and $S \subset B$.

3. Main results and proofs

In this section, our primary goal is to investigate the boundedness properties of the Hausdorff operator's commutators on grand-variable Herz-type spaces. The constant $C_{\Phi,s}$ will appear frequently in the proof of our main results, as defined by:

$$C_{\Phi,s} = \left(\int_0^\infty |\Phi(r)|^s r^{(n-\beta)s-n} \frac{dr}{r}\right)^{\frac{1}{s}}.$$

Theorem 3.1. Let $0 \leq \beta < n$, $1 < q_1 \leq q_2 < \infty$, $\theta > 0$, $\Omega \in L^s(S^{n-1})$, $p_1(\cdot), p_2(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$, and satisfy the conditions in Proposition 2.1 with $\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{\beta}{n}$, $p_1(\cdot) < \frac{n}{\beta}$, and $p'_1(\cdot) < s$. Suppose $\delta_1, \delta_2 \in (0, 1)$, $\frac{n}{s'} - n\delta_2 - \beta < \alpha < n\delta_1 - \frac{n}{s}$, $0 \leq \lambda < \alpha + \beta + n\delta_2 - \frac{n}{s'}$, $C_{\Phi,s} < \infty$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. If Φ is a radial function, then $H^{\beta,b}_{\Phi,\Omega}$ is bounded on the grand-variable Herz-Morrey space and satisfies:

$$\|H^{\beta,b}_{\Phi,\Omega} f\|_{M\dot{K}^{\alpha,q_2),\theta}_{\lambda,p_2(\cdot)}(\mathbb{R}^n)} \leq C C_{\Phi,s} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\alpha,q_1),\theta}_{\lambda,p_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. Since $q_1 \leq q_2$, by definition of grand-variable Herz-Morrey space:

$$\begin{aligned}
&\|H^{\beta,b}_{\Phi,\Omega} f\|^{q_1(1+\epsilon)}_{M\dot{K}^{\alpha,q_2),\theta}_{\lambda,p_2(\cdot)}(\mathbb{R}^n)} \\
&\leq \sup_{\epsilon>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^\theta \sum_{j=-\infty}^{j_0} 2^{j\alpha q_1(1+\epsilon)} \|(H^{\beta,b}_{\Phi,\Omega} f)\chi_j\|^{q_1(1+\epsilon)}_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\leq \sup_{\epsilon>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^\theta \sum_{j=-\infty}^{j_0} 2^{j\alpha q_1(1+\epsilon)} \left(\sum_{l=-\infty}^{\infty} \|(H^{\beta,b}_{\Phi,\Omega}(f\chi_l))\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\epsilon)} \\
&\leq C \sup_{\epsilon>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^\theta \sum_{j=-\infty}^{j_0} 2^{j\alpha q_1(1+\epsilon)} \left(\sum_{l=-\infty}^{j-1} \|(H^{\beta,b}_{\Phi,\Omega}(f\chi_l))\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\epsilon)} \\
&+ C \sup_{\epsilon>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^\theta \sum_{j=-\infty}^{j_0} 2^{j\alpha q_1(1+\epsilon)} \left(\sum_{l=j-1}^{j+1} \|(H^{\beta,b}_{\Phi,\Omega}(f\chi_l))\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\epsilon)} \\
&+ C \sup_{\epsilon>0} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^\theta \sum_{j=-\infty}^{j_0} 2^{j\alpha q_1(1+\epsilon)} \left(\sum_{l=j+1}^{\infty} \|(H^{\beta,b}_{\Phi,\Omega}(f\chi_l))\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\epsilon)} \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

In order to estimate I_1, we need to estimate the inner norm $\|(H^{\beta,b}_{\Phi,\Omega}(f\chi_l))\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}$. Thus, for $l \leq j-1$, we proceed as below:

$$\begin{aligned}|H^{\beta,b}_{\Phi,\Omega}(f\chi_l)(z)\chi_j(z)| &= \left|\int_{C_l}\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}}\Omega(x')(b(z)-b(x))f(x)dx\right|\chi_j(z)\\ &\leq \left|\int_{C_l}\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}}\Omega(x')(b(z)-b_{B_l})f(x)dx\right|\chi_j(z)\\ &+ \left|\int_{C_l}\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}}\Omega(x')(b(x)-b_{B_l})f(x)dx\right|\chi_j(z)\\ &= |(b(z)-b_{B_l})H^{\beta}_{\Phi,\Omega}(f\chi_l)(z)|\chi_j(z) + |H^{\beta}_{\Phi,\Omega}((b-b_{B_l})f\chi_l)(z)|\chi_j(z)\\ &=: J_1 + J_2. \end{aligned} \tag{3.1}$$

Now, we proceed with the J_1 approximation and deduce that

$$\begin{aligned}|H^{\beta}_{\Phi,\Omega}(f\chi_l)(z)| &\leq \int_{C_l}\left|\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}}\Omega(x')f(x)\right|dx\\ &\leq \left\|\left(\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}}\Omega(x')\right)\chi_l\right\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

The condition $s > p_1'(\cdot)$ implies that there exists a $p(\cdot)$ such that $\frac{1}{p_1'(\cdot)} = \frac{1}{s} + \frac{1}{p(\cdot)}$. Hence, Lemma 2.4 gives us

$$|H^{\beta}_{\Phi,\Omega}(f\chi_l)(z)| \leq \left\|\left(\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}}\Omega(x')\right)\chi_l\right\|_{L^{s}(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \tag{3.2}$$

By polar decomposition, we see that

$$\begin{aligned}\left\|\left(\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}}\Omega(x')\right)\chi_l\right\|^s_{L^{s}(\mathbb{R}^n)} &= \int_{C_l}\left|\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}}\Omega(x')\right|^s dx\\ &= \int_{2^{l-1}}^{2^l}\int_{S^{n-1}}\left|\frac{\Phi(|z|r^{-1})}{r^{n-\beta}}\right|^s|\Omega(x')|^s d\mu(x')r^n\frac{dr}{r},\end{aligned}$$

where $\mu(x')$ is the normalized Lebesgue measure on the unit sphere S^{n-1}. We get the following inequality by change of variable:

$$\begin{aligned}\left\|\left(\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}}\Omega(x')\right)\chi_l\right\|^s_{L^{s}(\mathbb{R}^n)} &= \int_{S^{n-1}}|\Omega(x')|^s d\mu(x')\int_{\frac{|z|}{2^l}}^{\frac{|z|}{2^{l-1}}}|\Phi(t)|^s(|z|t^{-1})^{n-(n-\beta)s}\frac{dt}{t}\\ &\leq \|\Omega\|^s_{L^s(S^{n-1})}|z|^{n-(n-\beta)s}\int_0^\infty|\Phi(t)|^s t^{(n-\beta)s-n}\frac{dt}{t}\\ &= C^s_{\Phi,s}\|\Omega\|^s_{L^s(S^{n-1})}|z|^{s\beta+n-ns}.\end{aligned}$$

Thus,

$$\left\|\left(\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}}\Omega(x')\right)\chi_l\right\|_{L^{s}(\mathbb{R}^n)} \leq CC_{\Phi,s}|z|^{\beta-\frac{n}{s'}}. \tag{3.3}$$

Furthermore, when $x \in B_l$, $|B_l| \le 2^n$, then from $\frac{1}{p_1'(x)} = \frac{1}{s} + \frac{1}{p(x)}$ and Lemma 2.5, we get

$$\|\chi_{B_l}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |B_l|^{\frac{1}{p(\cdot)}} \approx |B_l|^{-\frac{1}{s}} \|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}.$$

When $|B_l| \ge 1$,

$$\|\chi_{B_l}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |B_l|^{\frac{1}{p(\infty)}} \approx |B_l|^{-\frac{1}{s}} \|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}.$$

Hence, we get

$$\|\chi_{B_l}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \approx |B_l|^{-\frac{1}{s}} \|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}. \tag{3.4}$$

Using results from (3.3) and (3.4) into (3.2), we get

$$|H^{\beta}_{\Phi,\Omega}(f\chi_l)(z)| \le C C_{\Phi,s} |z|^{\beta-\frac{n}{s'}} |B_l|^{-\frac{1}{s}} \|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \tag{3.5}$$

In light of this inequality, J_1 takes the following form:

$$J_1 \le C C_{\Phi,s} |B_j|^{\frac{\beta}{n}-\frac{1}{s'}} |B_l|^{-\frac{1}{s}} \|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} |(b(z) - b_{B_l})\chi_j(z)|, \tag{3.6}$$

which gives us:

$$\|J_1\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \le C C_{\Phi,s} |B_j|^{\frac{\beta}{n}-\frac{1}{s'}} |B_l|^{-\frac{1}{s}} \|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|(b - b_{B_l})\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

Since $l < j$, by Lemma 2.11, we get

$$\|J_1\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \le C C_{\Phi,s} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} (j-l) |B_j|^{\frac{\beta}{n}-\frac{1}{s'}} |B_l|^{-\frac{1}{s}} \|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}. \tag{3.7}$$

Next, let us estimate J_2. The condition $s > p_1'(\cdot)$ gives that $1 = \frac{1}{p_1(\cdot)} + \frac{1}{s} + \frac{1}{p(\cdot)}$.
Thus,

$$\begin{aligned} |H^{\beta}_{\Phi,\Omega}((b - b_{B_l})f\chi_l)(z)| &\le \int_{C_l} \left| \frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}} \Omega(x')(b(x) - b_{B_l}) f(x) \right| dx \\ &\le \left\| \left(\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}} \Omega(x') \right) \chi_l \right\|_{L^s(\mathbb{R}^n)} \|(b(\cdot) - b_{B_l})\chi_{B_l}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.8}$$

Lemma 2.11 and the inequality (3.3) assist us in achieving

$$J_2 \le C C_{\Phi,s} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} |z|^{\beta-\frac{n}{s'}} \|\chi_{B_l}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \chi_j(z). \tag{3.9}$$

Finally, the obtained inequality (3.4) is quite beneficial for us in getting

$$\|J_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \le C C_{\Phi,s} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} |B_j|^{\frac{\beta}{n}-\frac{1}{s'}} |B_l|^{-\frac{1}{s}} \|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}. \tag{3.10}$$

Adding (3.7) and (3.10) into (3.1), we obtain

$$\begin{aligned} &\|H^{\beta,b}_{\Phi,\Omega}(f\chi_l)\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\ &\le C C_{\Phi,s} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} (j-l) |B_j|^{\frac{\beta}{n}-\frac{1}{s'}} |B_l|^{-\frac{1}{s}} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Since $l < j$, from Remark 2.12, we get the following inequality:

$$\begin{aligned}
&\|H^{\beta,b}_{\Phi,\Omega}(f\chi_l)\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\
&\leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(j-l)|B_j|^{\frac{\beta}{n}-1}2^{n(l-j)(\delta_1-\frac{1}{s})}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \qquad (3.11)
\end{aligned}$$

By virtue of Lemma 2.5 and the condition $\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{\beta}{n}$, we obtain

$$\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \approx |B_j|^{\frac{1}{p_2(\cdot)}+\frac{1}{p_1'(\cdot)}} = |B_j|^{1-\frac{\beta}{n}}. \qquad (3.12)$$

So, inequality (3.11) assumes the following form:

$$\|H^{\beta,b}_{\Phi,\Omega}(f\chi_l)\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(j-l)2^{n(l-j)(\delta_1-\frac{1}{s})}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \qquad (3.13)$$

This completes the estimation of inner norm $\|(H^{\beta,b}_{\Phi,\Omega}(f\chi_l))\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}$.

Next, we approximate I_1. So, by using (3.13), we obtain

$$\begin{aligned}
I_1 &\leq C\sup_{\epsilon>0} C^{q_1(1+\epsilon)}_{\Phi,s}\|b\|^{q_1(1+\epsilon)}_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)}\epsilon^{\theta}\sum_{j=-\infty}^{j_0}2^{j\alpha q_1(1+\epsilon)}\left(\sum_{l=-\infty}^{j-1}(j-l)2^{n(l-j)(\delta_1-\frac{1}{s})}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\epsilon)}\\
&\leq C\sup_{\epsilon>0} C^{q_1(1+\epsilon)}_{\Phi,s}\|b\|^{q_1(1+\epsilon)}_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)}\epsilon^{\theta}\sum_{j=-\infty}^{j_0}\left(\sum_{l=-\infty}^{j-1}(j-l)2^{(j-l)(\alpha-n\delta_1+\frac{n}{s})}2^{l\alpha}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\epsilon)}.
\end{aligned}$$

Since $\alpha < n(\delta_1 - \frac{1}{s})$, for $1 < q_1 < \infty$, we use the Hölder inequality to get

$$\begin{aligned}
I_1 &\leq C\sup_{\epsilon>0} C^{q_1(1+\epsilon)}_{\Phi,s}\|b\|^{q_1(1+\epsilon)}_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)}\epsilon^{\theta}\sum_{j=-\infty}^{j_0}\sum_{l=-\infty}^{j-1}\\
&\times 2^{\frac{q_1(1+\epsilon)}{2}(j-l)(\alpha-n\delta_1+\frac{n}{s})}2^{l\alpha q_1(1+\epsilon)}\|f\chi_l\|^{q_1(1+\epsilon)}_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left(\sum_{l=-\infty}^{j-1}(j-l)^{q_1'(1+\epsilon)}2^{\frac{q_1'(1+\epsilon)}{2}(j-l)(\alpha-n\delta_1+\frac{n}{s})}\right)^{\frac{q_1(1+\epsilon)}{q_1'(1+\epsilon)}}\\
&\leq C\sup_{\epsilon>0} C^{q_1(1+\epsilon)}_{\Phi,s}\|b\|^{q_1(1+\epsilon)}_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)}\epsilon^{\theta}\sum_{j=-\infty}^{j_0}\sum_{l=-\infty}^{j-1}2^{\frac{q_1(1+\epsilon)}{2}(j-l)(\alpha-n\delta_1+\frac{n}{s})}2^{l\alpha q_1(1+\epsilon)}\|f\chi_l\|^{q_1(1+\epsilon)}_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\
&= C\sup_{\epsilon>0} C^{q_1(1+\epsilon)}_{\Phi,s}\|b\|^{q_1(1+\epsilon)}_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)}\epsilon^{\theta}\sum_{l=-\infty}^{j_0-1}2^{l\alpha q_1(1+\epsilon)}\|f\chi_l\|^{q_1(1+\epsilon)}_{L^{p_1(\cdot)}(\mathbb{R}^n)}\sum_{j=l+1}^{j_0}2^{\frac{q_1(1+\epsilon)}{2}(j-l)(\alpha-n\delta_1+\frac{n}{s})}\\
&\leq C\sup_{\epsilon>0} C^{q_1(1+\epsilon)}_{\Phi,s}\|b\|^{q_1(1+\epsilon)}_{\mathrm{BMO}(\mathbb{R}^n)}\epsilon^{\theta}\|f\|^{q_1(1+\epsilon)}_{M\dot{K}^{\alpha,\lambda}_{q_1(1+\epsilon),p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Now, we estimate I_2. For $l = j$, we follow the same steps as for I_1. So, we write

$$\begin{aligned}|H^{\beta,b}_{\Phi,\Omega}(f\chi_j)(z)\chi_j(z)| &= |(b(z)-b_{B_j})H^{\beta}_{\Phi,\Omega}(f\chi_j)(z)|\chi_j(z) + |H^{\beta}_{\Phi,\Omega}((b(x)-b_{B_j})f\chi_j)(z)|\chi_j(z)\\ &= K_1 + K_2. \end{aligned} \tag{3.14}$$

Replacing l by j in the inequality (3.6), constructed for J_1, we obtain K_1:

$$K_1 \leq CC_{\Phi,s}|B_j|^{\frac{\beta}{n}-1}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}|(b(z)-b_{B_j})\chi_j(z)|.$$

From Lemma 2.11 and the inequality (3.12), we deduce that

$$\begin{aligned}\|K_1\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}|B_j|^{\frac{\beta}{n}-1}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\ &\leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.15}$$

However, by replacing l with j, the K_2 estimation is obtained from the inequalities (3.8)–(3.10) established for J_2.

$$\begin{aligned}\|K_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}|B_j|^{\frac{\beta}{n}-1}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\ &\leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14), we obtain

$$\|H^{\beta,b}_{\Phi,\Omega}(f\chi_j)\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}. \tag{3.17}$$

Thus,

$$\begin{aligned}I_2 &\leq C\sup_{\epsilon>0} C^{q_1(1+\epsilon)}_{\Phi,s}\|b\|^{q_1(1+\epsilon)}_{\mathrm{BMO}(\mathbb{R}^n)}\sup_{j_0\in\mathbb{Z}}2^{-j_0\lambda q_1(1+\epsilon)}\epsilon^{\theta}\sum_{j=-\infty}^{j_0}2^{j\alpha q_1(1+\epsilon)}\|f\chi_j\|^{q_1(1+\epsilon)}_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\ &\leq C\sup_{\epsilon>0} C^{q_1(1+\epsilon)}_{\Phi,s}\|b\|^{q_1(1+\epsilon)}_{\mathrm{BMO}(\mathbb{R}^n)}\epsilon^{\theta}\|f\|^{q_1(1+\epsilon)}_{M\dot{K}^{\alpha,\lambda}_{q_1(1+\epsilon),p_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

Next, we estimate I_3. For $l \geq j+1$, with a small adjustment in the first step of the inequality (3.1), we write

$$\begin{aligned}|H^{\beta,b}_{\Phi,\Omega}(f\chi_l)(z)\chi_j(z)| &= |(b(z)-b_{B_j})H^{\beta}_{\Phi,\Omega}(f\chi_l)(z)|\chi_j(z) + |H^{\beta}_{\Phi,\Omega}((b(x)-b_{B_j})f\chi_l)(z)|\chi_j(z)\\ &= L_1 + L_2. \end{aligned} \tag{3.18}$$

As with the estimation of J_1, we approximate L_1 using the inequalities (3.3)–(3.5) to obtain

$$L_1 \leq CC_{\Phi,s}|B_j|^{\frac{\beta}{n}-\frac{1}{s'}}|B_l|^{-\frac{1}{s}}\|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}|(b(z)-b_{B_j})\chi_j(z)|.$$

Taking the $L^{p_2(\cdot)}$ norm of the above inequality

$$\|L_1\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq CC_{\Phi,s}|B_j|^{\frac{\beta}{n}-\frac{1}{s'}}|B_l|^{-\frac{1}{s}}\|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|(b-b_{B_j})\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)},$$

and using Lemma 2.11, we obtain

$$\|L_1\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}|B_j|^{\frac{\beta}{n}-\frac{1}{s'}}|B_l|^{-\frac{1}{s}}\|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}. \tag{3.19}$$

To estimate L_2, we replace the factor $(b - b_{B_l})$ with $(b - b_{B_j})$ in the inequality (3.8):

$$|H^{\beta}_{\Phi,\Omega}((b - b_{B_j})f\chi_l)(z)| \leq \left\| \left(\frac{\Phi(z|x|^{-1})}{|x|^{n-\beta}} \Omega(x') \right) \chi_l \right\|_{L^s(\mathbb{R}^n)} \|(b(\cdot) - b_{B_j})\chi_{B_l}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

For $l > j$, from Lemma 2.11 and the inequality (3.3), we deduce that

$$L_2 \leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(l - j)|z|^{\beta-\frac{n}{s'}}\|\chi_{B_l}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\chi_j(z),$$

from which, by virtue of (3.4), we obtain the following inequality:

$$\|L_2\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(l - j)|B_j|^{\frac{\beta}{n}-\frac{1}{s'}}|B_l|^{-\frac{1}{s}}\|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}. \tag{3.20}$$

Substitute (3.19) and (3.20) into (3.18), and we get

$$\begin{aligned}
&\|H^{\beta,b}_{\Phi,\Omega}(f\chi_l)\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(l - j)|B_j|^{\frac{\beta}{n}-\frac{1}{s'}}|B_l|^{-\frac{1}{s}}\|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Since $l > j$, again with the help of Remark 2.12, we obtain

$$\begin{aligned}
&\|H^{\beta,b}_{\Phi,\Omega}(f\chi_l)\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \\
&\leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(l - j)|B_j|^{\frac{\beta}{n}-\frac{1}{s'}}|B_l|^{-\frac{1}{s}}\|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}2^{n(j-l)\delta_2}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{3.21}$$

Using condition $\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{\beta}{n}$ and Lemma 2.5, we get

$$\|\chi_{B_l}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\|\chi_{B_l}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \approx |B_l|^{\frac{1}{p_2(\cdot)}+\frac{1}{p_1'(\cdot)}} = |B_l|^{1-\frac{\beta}{n}}.$$

Therefore, (3.21) becomes

$$\begin{aligned}
\|H^{\beta,b}_{\Phi,\Omega}(f\chi_l)\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\leq CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(l - j)|B_j|^{\frac{\beta}{n}-\frac{1}{s'}}|B_l|^{\frac{1}{s'}-\frac{\beta}{n}}2^{n(j-l)\delta_2}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \\
&= CC_{\Phi,s}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(l - j)2^{(j-l)(\beta+n\delta_2-\frac{n}{s'})}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, we can express I_3 as:

$$\begin{aligned}
I_3 &\leq C\sup_{\epsilon>0} C^{q_1(1+\epsilon)}_{\Phi,s}\|b\|^{q_1(1+\epsilon)}_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)}\epsilon^{\theta}\sum_{j=-\infty}^{j_0}\left(\sum_{l=j+1}^{\infty}(l - j)2^{(j-l)(\alpha+\beta+n\delta_2-\frac{n}{s'})}2^{l\alpha}\|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right)^{q_1(1+\epsilon)}.
\end{aligned}$$

Since $\alpha + \beta + n\delta_2 - \frac{n}{s'} > 0$ and $\lambda < \alpha + \beta + n\delta_2 - \frac{n}{s'}$, so, we can select a constant $\sigma > 1$ such that

$\lambda - \frac{1}{\sigma}(\alpha + \beta + n\delta_2 - \frac{n}{s'}) < 0$. Hence, for $1 < q_1 < \infty$, Hölder's inequality gives

$$\begin{aligned}
I_3 \leq & C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon} \\
& \times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^{\theta} \sum_{j=-\infty}^{j_0} \sum_{l=j+1}^{\infty} 2^{\frac{q_1(1+\epsilon)}{\sigma}(j-l)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{l\alpha q_1(1+\epsilon)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \\
& \times \Big(\sum_{l=j+1}^{\infty} (l-j)^{q_1'(1+\epsilon)} 2^{q_1'(1+\epsilon)\frac{(\sigma-1)}{\sigma}(j-l)(\alpha+\beta+n\delta_2-\frac{n}{s'})} \Big)^{\frac{q_1(1+\epsilon)}{q_1'(1+\epsilon)}} \\
\leq & C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \\
& \times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^{\theta} \sum_{j=-\infty}^{j_0} \sum_{l=j+1}^{\infty} 2^{\frac{q_1(1+\epsilon)}{\sigma}(j-l)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{l\alpha q_1(1+\epsilon)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \\
\leq & C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \\
& \times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^{\theta} \sum_{j=-\infty}^{j_0} \sum_{l=j+1}^{j_0-1} 2^{\frac{q_1(1+\epsilon)}{\sigma}(j-l)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{l\alpha q_1(1+\epsilon)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \\
& + C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \\
& \times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^{\theta} \sum_{j=-\infty}^{j_0} \sum_{l=j_0}^{\infty} 2^{\frac{q_1(1+\epsilon)}{\sigma}(j-l)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{l\alpha q_1(1+\epsilon)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \\
=: & M_1 + M_2.
\end{aligned}$$

By using $\frac{n}{s'} - n\delta_2 - \beta < \alpha$, M_1 can be approximated as

$$\begin{aligned}
M_1 \leq & C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \\
& \times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^{\theta} \sum_{l=-\infty}^{j_0-1} 2^{l\alpha q_1(1+\epsilon)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \sum_{j=-\infty}^{l-1} 2^{\frac{q_1(1+\epsilon)}{\sigma}(j-l)(\alpha+\beta+n\delta_2-\frac{n}{s'})} \\
\leq & C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^{\theta} \sum_{l=-\infty}^{j_0-1} 2^{l\alpha q_1(1+\epsilon)} \|f\chi_l\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)}.
\end{aligned}$$

As $\lambda < \frac{1}{\sigma}(\alpha + \beta + n\delta_2 - \frac{n}{s'})$ and $\alpha + \beta + n\delta_2 - \frac{n}{s'} > 0$, hence we have

$$M_1 \leq C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \epsilon^{\theta} \|f\|_{M\dot{K}_{q_1(1+\epsilon),p_1(\cdot)}^{\alpha,\lambda}(\mathbb{R}^n)}^{q_1(1+\epsilon)}.$$

Finally, M_2 is approximated as

$$M_2 \le C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \epsilon^\theta \sum_{j=-\infty}^{j_0} \sum_{l=j_0}^{\infty} 2^{\frac{q_1(1+\epsilon)}{\sigma}(j-l)(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{l\lambda q_1(1+\epsilon)}$$
$$\times 2^{-l\lambda q_1(1+\epsilon)} \Big(\sum_{k=-\infty}^{l} 2^{k\alpha q_1(1+\epsilon)} \|f\chi_k\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \Big)$$
$$\le C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon)} 2^{-l\lambda q_1(1+\epsilon)} \epsilon^\theta \Big(\sum_{k=-\infty}^{l} 2^{k\alpha q_1(1+\epsilon)} \|f\chi_k\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \Big)$$
$$\times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} \sum_{j=-\infty}^{j_0} 2^{\frac{q_1(1+\epsilon)}{\sigma} j(\alpha+\beta+n\delta_2-\frac{n}{s'})} \sum_{l=j_0}^{\infty} 2^{q_1(1+\epsilon)l\left(\lambda-\frac{1}{\sigma}(\alpha+\beta+n\delta_2-\frac{n}{s'})\right)}$$
$$\le C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \epsilon^\theta \|f\|_{M\dot{K}^{\alpha,\lambda}_{q_1(1+\epsilon),p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)}$$
$$\times \sup_{j_0\in\mathbb{Z}} 2^{-j_0\lambda q_1(1+\epsilon)} 2^{\frac{q_1(1+\epsilon)}{\sigma} j_0(\alpha+\beta+n\delta_2-\frac{n}{s'})} 2^{q_1(1+\epsilon)j_0\left(\lambda-\frac{1}{\sigma}(\alpha+\beta+n\delta_2-\frac{n}{s'})\right)}$$
$$\le C \sup_{\epsilon>0} C_{\Phi,s}^{q_1(1+\epsilon)} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{q_1(1+\epsilon)} \epsilon^\theta \|f\|_{M\dot{K}^{\alpha,\lambda}_{q_1(1+\epsilon),p_1(\cdot)}(\mathbb{R}^n)}^{q_1(1+\epsilon)}.$$

Finally, when we combine all these estimates, we get

$$\|H^{\beta,b}_{\Phi,\Omega} f\|_{M\dot{K}^{\alpha,q_2),\theta}_{\lambda,p_2(\cdot)}(\mathbb{R}^n)} \le C C_{\Phi,s} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\alpha,q_1),\theta}_{\lambda,p_1(\cdot)}(\mathbb{R}^n)}.$$

□

We end this paper by stating the following theorem:

Theorem 3.2. Let $0 \le \beta < n$, $1 < q_1 \le q_2 < \infty$, $\theta > 0$, $\Omega \in L^s(S^{n-1})$, $p_1(\cdot), p_2(\cdot) \in \mathfrak{P}(\mathbb{R}^n)$, and satisfy the conditions in Proposition 2.1 with $\frac{1}{p_1(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{\beta}{n}$, $p_1(\cdot) < \frac{n}{\beta}$ and $p_1'(\cdot) < s$. Suppose $\delta_1, \delta_2 \in (0, 1)$, $\frac{n}{s'} - n\delta_2 - \beta < \alpha < n\delta_1 - \frac{n}{s}$, $C_{\Phi,s} < \infty$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. If Φ is a radial function, then $H^{\beta,b}_{\Phi,\Omega}$ is bounded on grand-variable-Herz space and satisfies:

$$\|H^{\beta,b}_{\Phi,\Omega} f\|_{\dot{K}^{\alpha,q_2),\theta}_{p_2(\cdot)}(\mathbb{R}^n)} \le C C_{\Phi,s} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{\dot{K}^{\alpha,q_1),\theta}_{p_1(\cdot)}(\mathbb{R}^n)}.$$

Proof. The proof is similar to the proof of Theorem 3.1, so we omit the details. □

4. Conclusions

In this note, we examined the boundedness of the Hausdorff operator's commutators on the variable exponent grand-Herz-Morrey spaces, assuming that the symbol functions originate from BMO spaces. We got affirmative results under certain conditions. The results of this study may stimulate the researchers to establish the same bounds on other function spaces with variable expomnents.

Author contributions

Javeria Younas: Writing – original draft, Software, Methodology, Formal analysis; Amjad Hussain: Validation, Supervision, Funding acquisition, Conceptualization; Hadil Alhazmi: Writing

– review & editing, Visualization, Validation, Supervision; A. F. Aljohani: Investigation, Formal analysis, Conceptualization, Visualization, Validation, Supervision, Resources, Methodology, Funding acquisition, Conceptualization; Ilyas Khan: Methodology, Analysis, Writing – review & editing, Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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