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Research article

Existence of solutions for Kirchhoff-double phase anisotropic variational problems with variable exponents

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Abstract: This paper is devoted to dealing with a kind of new Kirchhoff-type problem in $\mathbb{R}^N$ that involves a general double-phase variable exponent elliptic operator ϕ. Specifically, the operator ϕ has behaviors like $|\tau|^{q(x)-2}\tau$ if $|\tau|$ is small and like $|\tau|^{p(x)-2}\tau$ if $|\tau|$ is large, where $1 < p(x) < q(x) < N$. By applying some new analytical tricks, we first establish existence results of solutions for this kind of Kirchhoff-double-phase problem based on variational methods and critical point theory. In particular, we also replace the classical Ambrosetti–Rabinowitz type condition with four different superlinear conditions and weaken some of the assumptions in the previous related works. Our results generalize and improve the ones in [Q. H. Zhang, V. D. Rădulescu, J. Math. Pures Appl., 118 (2018), 159–203.] and other related results in the literature.

Keywords: double phase; Kirchhoff-type problem; variable exponent, Orlicz–Sobolev spaces; variational methods
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1. Introduction

In this paper, we are concerned with the following Kirchhoff-type problem:

$$\mathfrak{M}(\varPhi_V(x,v))[-\operatorname{div}(\phi(x,\nabla v)) + V(x)|v|^{r(x)-2}v] + h(x)|v|^{\alpha(x)-2}v = \lambda g(x,v) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where the Kirchhoff function $\mathfrak{M} : [0,\infty) \to \mathbb{R}^+$, $\lambda > 0$ is a parameter, $V, h \in C(\mathbb{R}^N, \mathbb{R})$, $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $\alpha, r \in C_+(\mathbb{R}^N) := \left\{m : m \in C(\mathbb{R}^N), m(x) > 1 \text{ for } x \in \mathbb{R}^N\right\}$ with $1 \ll \alpha(x) \ll r(x) \le p(x)$ and $1 \ll r(x) \ll p^*(x)\frac{q'(x)}{p'(x)}$, here $p^*(x) = \frac{Np(x)}{N-p(x)}$, $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1$, $\frac{1}{q'(x)} + \frac{1}{q(x)} = 1$, and the notation $p_1 \ll p_2$ means that $\operatorname*{ess\,inf}_{x\in\mathbb{R}^N}(p_2(x) - p_1(x)) > 0$, $\phi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ admits a potential Φ with respect to its second variable, i.e., $\nabla_\tau \Phi(x,\tau) = \phi(x,\tau)$, $\mathit{\Phi}_V(x,v) := \int_{\mathbb{R}^N} (\Phi(x,\nabla v) + \frac{V(x)}{r(x)}|v|^{r(x)})dx$. We suppose that the potential Φ satisfies the following basic conditions:

(Φ1) The potential $\Phi = \Phi(x,\tau)$ is a continuous function in $\mathbb{R}^N \times \mathbb{R}^N$ with continuous derivative with respect to τ, $\phi = \partial_\tau \Phi(x,\tau)$ and satisfies:
(i) $\Phi(x,0) = 0$ and $\Phi(x,\tau) = \Phi(x,-\tau)$ for all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}^N$;
(ii) $\Phi(x,\cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \mathbb{R}^N$;
(iii) There exist constants $K_1, K_2 > 0$, and variable exponents $p(x)$ and $q(x)$ such that

$$\left.\begin{matrix} K_1|\tau|^{p(x)}, \text{ if } |\tau| > 1 \\ K_1|\tau|^{q(x)}, \text{ if } |\tau| \le 1 \end{matrix}\right\} \le \phi(x,\tau)\cdot\tau \text{ and } |\phi(x,\tau)| \le \begin{cases} K_2|\tau|^{p(x)-1}, \text{ if } |\tau| > 1 \\ K_2|\tau|^{q(x)-1}, \text{ if } |\tau| \le 1 \end{cases}, \tag{1.2}$$

for all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}^N$;
(iv) $1 \ll p(x) \ll q(x) \ll \min\{N, p^*(x)\}$ and $p(x)$, $q(x)$ are Lipschitz continuous in $\mathbb{R}^N$;
(v) $\phi(x,\tau)\cdot\tau \le s(x)\Phi(x,\tau)$ for all $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}^N$, where s satisfying $q(x) \le s(x) \ll p^*(x)$ is Lipschitz continuous.

(Φ2) Φ is uniformly convex, that is, for any $\epsilon \in (0,1)$, there exists $\eta(\epsilon) \in (0,1)$ such that $|w - z| \le \epsilon \max\{|w|,|z|\}$ or $\Phi(x, \frac{w+z}{2}) \le \frac{(1-\eta(\epsilon))}{2}(\Phi(x,w) + \Phi(x,z))$ for any $x, w, z \in \mathbb{R}^N$.

Remark 1.1. The typical example of ϕ is

$$\phi(x,\nabla v) = \begin{cases} |\nabla v|^{p(x)-2}\nabla v, \text{ if } |\nabla v| > 1, \\ |\nabla v|^{q(x)-2}\nabla v, \text{ if } |\nabla v| \le 1. \end{cases}$$

Then,

$$-\mathrm{div}\phi(x,\nabla v) = \begin{cases} -\mathrm{div}(|\nabla v|^{p(x)-2}\nabla v), \text{ if } |\nabla v| > 1, \\ -\mathrm{div}(|\nabla v|^{q(x)-2}\nabla v), \text{ if } |\nabla v| \le 1, \end{cases}$$

and the potential Φ is

$$\Phi(x,\tau) = \begin{cases} \dfrac{1}{p(x)}|\tau|^{p(x)} + \dfrac{1}{q(x)} - \dfrac{1}{p(x)}, & \text{if } |\tau| > 1, \\ \dfrac{1}{q(x)}|\tau|^{q(x)}, & \text{if } |\tau| \le 1. \end{cases}$$

According to [1, Lemma A.2], it is obvious that the potential Φ satisfies conditions (Φ1)-(Φ2) if $1 \ll p \ll q \ll N$ in $\mathbb{R}^N$.

Moreover, we make the following hypotheses:

(H) $0 \le h \in L^{\beta_0(x)}(\mathbb{R}^N)$ with $\mathrm{meas}\Big\{x \in \mathbb{R}^N : h(x) \ne 0\Big\} > 0$ for any $\beta \in C_+(\mathbb{R}^N)$ with $r(x) \le \beta(x) \ll p^*(x)$ for all $x \in \mathbb{R}^N$, where $\beta_0(x) := \frac{\beta(x)}{\beta(x)-\alpha(x)}$ for all $x \in \mathbb{R}^N$.

(V) $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $V(x) \ge V_0 > 0$ in $\mathbb{R}^N$ and $V(x) \to +\infty$ as $|x| \to +\infty$.

(M1) $\mathfrak{M} \in C(\mathbb{R}^+_0, \mathbb{R}^+)$ and there exists $m_0 > 0$ such that $\inf_{\tau\in\mathbb{R}^+_0} \mathfrak{M}(\tau) \ge m_0 > 0$.

(M2) There exists $\theta \in [1, \frac{(p^+)^*}{q^+})$ such that $\theta \mathcal{M}(\tau) = \theta \int_0^\tau \mathfrak{M}(s)ds \geq \mathfrak{M}(\tau)\tau$ for any $\tau \geq 0$.

(G1) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, and $G(x,\tau) := \int_0^\tau g(x,s)ds \geq 0$.

(G2) There exist nonnegative functions κ, ρ with $\kappa \in L^{\eta'(x)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\rho \in L^{\frac{\eta(x)}{\eta(x)-r^-}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$|g(x,\tau)| \leq \kappa(x) + \rho(x)|\tau|^{\eta(x)-1}, \ \forall (x,\tau) \in \mathbb{R}^N \times \mathbb{R},$$

where η is Lipschitz continuous and $q(x) < \eta(x) \ll p^*(x)$.

Remark 1.2. It is well known that the lack of compactness is the main difficulty in the study of elliptic problems in $\mathbb{R}^N$. To overcome this difficulty, Zhang and Rădulescu [1] considered condition (V) to rebuild the required compact embedding theorem. In the present paper, we also need to introduce the condition (H) to deal with the additional $h(x)|v|^{\alpha(x)-2}v$ term. Conditions (M1)-(M2) and (G1)-(G2) are very important for the study of Kirchhoff problems with subcritical growth, which ensure the compactness condition and the geometric properties of the energy functional corresponding to the problem. Under these conditions, we can find many papers; see [2–5].

In the last few years, the study of variational problems involving double phase operators has become a hot topic due to its extensive applications, for example, nonlinear elasticity, strongly anisotropic materials, Lavrentievs phenomenon, and so on [6–9]. The study of this type of operator started in the works of Zhikov [9], who introduced the following energy functional:

$$v \mapsto \int_\Omega (|\nabla v|^p + a(x)|\nabla v|^q)dx. \tag{1.3}$$

According to Marcellinis terminology [10, 11], the functional (1.3) belongs to the category of the so-called functionals with nonstandard growth conditions. In [12–15], Mingione and coworkers have studied the regularity results for local minimizers of functionals like (1.3). In [16], Colasuonno and Squassina investigated an eigenvalue problem in the framework of double-phase variational integrals. Recently, Zhang and Rădulescu [1] dealt with the following elliptic equation with a general double phase operator:

$$-\text{div}\phi(x,\nabla v) + V(x)|v|^{r(x)-2}v = g(x,v) \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where the operator $-\text{div}\phi(x,\nabla v)$ describes the behavior that the $p(x)$-material is present if $|\nabla v| > 1$ and the $q(x)$-material acts if $|\nabla v| \leq 1$. The authors extended some of the results in [17, 18] to the variable exponent case and obtained remarkable existence results for problem (1.4) without the following Ambrosetti–Rabinowitz condition ((AR)-condition for short):

(AR) There exists $d > s^+$ such that $0 < dG(x,\tau) \leq g(x,\tau)\tau$ for all $\tau \in \mathbb{R}_0^+$ and $x \in \mathbb{R}^N$.

Subsequently, Shi et al. [19] considered the following equation:

$$-\text{div}\phi(x,\nabla v) + |v|^{r(x)-2}v = \lambda a(x)|v|^{\delta(x)-2}v + \mu w(x)g(x,v) \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

They used the weighted method to address the lack of compactness and proved existence results of nontrivial solutions for problem (1.5). Very recently, Liu and Pucci [20] established results of solutions for problem (1.4) with an additional weighted term $h(x)|v|^{r(x)-2}v$. We also refer to [21–31] and the references therein for more related results.

Another feature of the problem (1.1) is the presence of a nonlocal Kirchhoff term, which was originally proposed by Kirchhoff [32], who considered the following model:

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\Big|\frac{\partial u(x)}{\partial x}\Big|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.6}$$

where $\rho,\ \rho_0,\ h,\ E,$ and L are constants that represent various physical entities. This type of problem has a profound physical background in the real world–please see [33, 34] and the references therein. Mathematically, the existence of solutions for Kirchhoff-type problems has been widely studied [35–42].

As far as we know, there are only very few recent works considering the Kirchhoff-double phase problems. Such problems actually have various applications in the fields of mathematical physics and biology, such as plasma physics and population density; see [43–45]. In [46], Fiscella and Pinamonti considered the following Kirchhoff-double phase problem:

$$\begin{cases} -M\Big(\displaystyle\int_\Omega(\frac{|\nabla v|^p}{p}+a(x)\frac{|\nabla v|^q}{q})dx\Big)\mathrm{div}\phi(x,\nabla v)=g(x,v), & \text{in } \Omega,\\ v=0, & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\phi(x,\nabla v) = |\nabla v|^{p-2}\nabla v + a(x)|\nabla v|^{q-2}\nabla v$ and g satisfies the classical (AR)-condition. Based on the mountain pass theorem and the fountain theorem, they established existence results of solutions to the problem (1.7). In [47], Arora et al. studied problem (1.7), including a parametric singular term, and obtained existence results by applying the fibering method in the form of the Nehari manifold. In [48], Kim and Winkert extended the problem (1.7) to the variable exponents case and established the existence of infinitely many nontrivial solutions by the abstract critical point theorem. In [49], Cheng and Bai considered existence of multiple solutions for problem (1.7) with Hardy–Sobolev terms. In [50], Sousa investigated a class of fractional Kirchhoff double phase problems and established the existence of a sequence of solutions whose L^∞-norms converge to zero. We also refer to [51–53] for more results in the setting of Kirchhoff-double phase problems.

Inspired by the above results, it is very natural to put forward a series of interesting questions, especially the following ones:

(i) To the best of our knowledge, there is only a small amount of literature considering Kirchhoff-double phase problems, and especially Kirchhoff-type problems involving the general variable exponent double phase operator like $-\mathrm{div}\phi(x,\nabla v)$ have not yet been studied. Can one deal with the combined problem of the nonlocal Kirchhoff function $\mathfrak{M}$ and the operator $-\mathrm{div}\phi(x,\nabla v)$ and establish existence results of solutions for such problems?

(ii) As we can see, the condition $(\widetilde{\mathrm{G}}1)$, i.e., $g(x,v)v = o(|v|^{r(x)})$ as $v\to 0$, is crucial to the existence of solutions for problem (1.4) in [1, 19, 20] . Can one obtain the existence results of solutions for our problem (1.1) without condition $(\widetilde{\mathrm{G}}1)$?

(iii) In the case when the nonlinear term g is superlinear, the (AR)-condition in [19] and the assumption $(\mathcal{H}_f^2)$ in [1, 20] play important roles in guaranteeing the boundedness of the Palais–Smale sequence or Cerami sequence. Can we replace them with a weaker condition or other appropriate hypotheses and then obtain the existence results?

In this article, we focus on the existence of nontrivial solutions for problem (1.1) and are devoted to answering questions (i)–(iii). To state our conclusions, in addition to (Φ1)-(Φ2), (M1)-(M2), (G1)-(G2), (H) and (V), we also need the following conditions:

(G3) $\lim\limits_{|\tau|\to\infty} \dfrac{G(x,\tau)}{|\tau|^{\theta q^+}} = \infty$ uniformly for a.e. $x \in \mathbb{R}^N$, where θ is given in (M2).

(G4) There exist $\delta > \theta s^+$, $R > 0$ and a function ζ with $0 \leq \zeta \in L^{\frac{r(x)}{r(x)-r^-}}(\Omega_1)$ on $\Omega_1 := \{x \in \mathbb{R}^N : r(x) > r^-\}$ and $\zeta(x) \equiv$ positive constant ζ_0 on $\Omega_2 := \{x \in \mathbb{R}^N : r(x) = r^-\}$ such that meas$\{x \in \mathbb{R}^N : \zeta(x) > 0\} \neq 0$ and

$$g(x,\tau)\tau - \delta G(x,\tau) \geq -\zeta(x)|\tau|^{r^-}, \quad \forall x \in \mathbb{R}^N, |\tau| \geq R.$$

(G5) There exist $C_0, M \geq 0$, and $\frac{N}{p^-} < \mu(x) \leq \frac{r^+}{r^+-r^-}$ such that

$$\widetilde{\mathcal{G}}(x,\tau) := \frac{1}{\delta}g(x,\tau)\tau - G(x,\tau) \geq 0, \quad \forall (x,\tau) \in \mathbb{R}^N \times \mathbb{R},$$

and

$$|G(x,\tau)|^{\mu(x)} \leq C_0|\tau|^{\mu(x)r^-}\widetilde{\mathcal{G}}(x,\tau), \quad \forall x \in \mathbb{R}^N, |\tau| \geq M,$$

where δ is given in (G4).

(G6) There exists $\xi \geq 1$ such that

$$\xi\mathfrak{G}(x,\tau) \geq \mathfrak{G}(x,z\tau),$$

for $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}$ and $z \in [0,1]$, where $\mathfrak{G}(x,\tau) = g(x,\tau)\tau - s^+G(x,\tau)$.

Now, the first main results in this paper are stated as follows:

Theorem 1.3. *Assume that conditions* (V), (H), (Φ1)-(Φ2), (M1)-(M2)*, and* (G1)–(G4) *are satisfied. Then there exists a constant* $\lambda_* > 0$ *such that problem* (1.1) *admits one nontrivial solution for any* $\lambda \in (0,\lambda_*]$.

Theorem 1.4. *Assume that conditions* (V), (H), (Φ1)-(Φ2), (M1)-(M2), (G1)–(G3)*, and* (G5) *are satisfied. Then there exists a constant* $\lambda_* > 0$ *such that problem* (1.1) *admits one nontrivial solution for any* $\lambda \in (0,\lambda_*]$.

Theorem 1.5. *Assume that* $\mathfrak{M}$ *is a decreasing function on* $\mathbb{R}_0^+$ *and that conditions* (V), (H), (Φ1)-(Φ2), (M1)-(M2), (G1)–(G3)*, and* (G6) *are satisfied. Then there exists a constant* $\lambda_* > 0$ *such that problem* (1.1) *admits one nontrivial solution for any* $\lambda \in (0,\lambda_*]$.

Remark 1.6. We must point out that conditions (G4)-(G5), and (G6) are initially provided by the works of Lin and Tang in [54] and Jeanjean in [55], respectively. For more applications of these conditions, we refer readers to [56–59]. The condition (G3) implies that problem (1.1) is superlinear at infinity. A classical way to deal with the superlinear elliptic boundary problem is to use the usual (AR)-condition [60]. However, there are many functions that satisfy (G4), (G5), or (G6) but do not satisfy the (AR)-condition; see the example in [54–59].

Remark 1.7. We would like to point out that if the Kirchhoff function $\mathfrak{M}$ of problem (1.1) is removed in Theorem 1.5, the Cerami compactness condition can be easily obtained using condition (G6). However, the additional monotonicity assumption for $\mathfrak{M}$ is essential to establishing the desired results in the current work. As far as we know, there are very few results about Kirchhoff-type problems under condition (G6).

Inspired by [61–64], we replace (G4) with the following weaker local superlinear hypothesis:

(G7) There exists a domain $D \subset \mathbb{R}^N$ such that $\lim\limits_{|\tau|\to\infty} \dfrac{G(x,\tau)}{|\tau|^{\theta q^+}} = \infty$ for a.e. $x \in D$.

Meanwhile, we suppose that

(G8) There exist constants $c_0, C_{10}, C_{12} > 0$, $\kappa_0 \in (0,1)$ and $\mu_0 \in (1, r^-)$ such that $\widetilde{\mathcal{G}}(x,\tau) := \frac{1}{\delta} g(x,\tau)\tau - G(x,\tau) \geq 0$ and if $N \geq 3$

$$\frac{G(x,\tau)}{|\tau|^{r^-}} \geq \frac{m_0 C_{12} \min\{C_{10} r^+, 1\}(1-\kappa_0)}{\lambda \theta r^+ \gamma_{r^-}} \text{ implies } \left|\frac{G(x,\tau)}{|\tau|^{\mu_0}}\right|^{\frac{(p^+)^*}{(p^+)^*-\mu_0}} \leq c_0 \widetilde{\mathcal{G}}(x,\tau),$$

and if $N = 1, 2$, there exists $\varsigma \in (1, \frac{2}{1-\mu_0}]$ such that

$$\frac{G(x,\tau)}{|\tau|^{r^-}} \geq \frac{m_0 C_{12} \min\{C_{10} r^+, 1\}(1-\kappa_0)}{\lambda \theta r^+ \gamma_{r^-}} \text{ implies } \left|\frac{G(x,\tau)}{|\tau|^{\mu_0}}\right|^{\varsigma} \leq c_0 \widetilde{\mathcal{G}}(x,\tau),$$

where $\delta > \theta s^+$, m_0 and θ are given in (M1) and (M2), respectively, and $\gamma_{r^-} > 0$ is the best embedding constant defined by $\|v\|^{r^-}_{L^{r^-}(\mathbb{R}^N)} \leq \gamma_{r^-} \|v\|^{r^-}_E$. In this regard, the following result is obtained:

Theorem 1.8. *If conditions* (V), (H), (Φ1)-(Φ2), (M1)-(M2), (G1)-(G2)*, and* (G7)-(G8) *are satisfied, then there exists a constant* $\lambda_* > 0$ *such that problem* (1.1) *admits one nontrivial solution for any* $\lambda \in (0, \lambda_*]$.

Finally, we work on the study of multiple solutions to the problem (1.1). Suppose that

(G9) $g(x, -\tau) = -g(x,\tau)$, for any $(x,\tau) \in \mathbb{R}^N \times \mathbb{R}$.

Combining condition (G9), the following three results are obtained:

Theorem 1.9. *If conditions* (Φ1)-(Φ2)*,* (V), (H), (M1)-(M2), (G1)–(G4)*, and* (G9) *are satisfied, then problem* (1.1) *has infinitely many nontrivial solutions for any* $\lambda > 0$.

Theorem 1.10. *If conditions* (Φ1)-(Φ2)*,* (V), (H), (M1)-(M2), (G1)–(G3), (G5)*, and* (G9) *are satisfied, then problem* (1.1) *has infinitely many nontrivial solutions for any* $\lambda > 0$.

Theorem 1.11. *Suppose that* $\mathfrak{M}$ *is a decreasing function on* $\mathbb{R}^+_0$ *and conditions* (Φ1)-(Φ2)*,* (V), (H), (M1)-(M2), (G1)–(G3), (G6)*, and* (G9) *are satisfied. Then problem* (1.1) *has infinitely many nontrivial solutions for any* $\lambda > 0$.

Remark 1.12. It is pointed out that there are no results similar to Theorems 1.3–1.11 for problem (1.1), and our results greatly generalize and improve the results in [1, 19, 20] in another direction under four different superlinear conditions.

Now, we introduce the following notations used throughout this paper:

♣ Denote $C_+(\mathbb{R}^N) = \Big\{m : m \in C(\mathbb{R}^N), m(x) > 1 \text{ for } x \in \mathbb{R}^N\Big\}$;

♣ Define $p^+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} p(x)$, $p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} p(x)$, and $p^*(x) = \frac{Np(x)}{N-p(x)}$;

♣ The notation $p_1 \ll p_2$ means that $\operatorname*{ess\,inf}_{x\in\mathbb{R}^N}(p_2(x) - p_1(x)) > 0$;

♣ $C_1, C_2, C_3, \cdots$ are positive constants that may vary in different locations.

The remainder of this paper is organized as follows: In Section 2, we introduce some preliminary knowledge and functional space settings related to problem (1.1). In Section 3, we verify the Cerami compactness condition for energy functionals corresponding to problem (1.1). In Section 4, we work on the proof of Theorems 1.3–1.11.

2. Preliminaries and functional space setting

We first review some basic results of the variable-exponent Lebesgue spaces and Sobolev spaces. For more information, we refer to [1, 65, 66]. Let $A \subset \mathbb{R}^N$ be an open domain, and $P(A)$ be the set of all measurable real-valued functions defined on A. For any $p \in C_+(\overline{A})$, the variable exponent Lebesgue space is given by

$$L^{p(x)}(A) = \Big\{v \in P(A) : \int_A |v(x)|^{p(x)} dx < \infty\Big\},$$

with the following norm

$$\|v\|_{L^{p(x)}(A)} = \inf\Big\{\mu > 0 : \int_A \Big|\frac{v(x)}{\mu}\Big|^{p(x)} \le 1\Big\}.$$

The variable exponent Sobolev space is defined as:

$$W^{1,p(x)}(A) = \Big\{v \in L^{p(x)}(A) : \nabla v \in [L^{p(x)}(A)]^N\Big\},$$

endowed with the norm

$$\|v\|_{W^{1,p(x)}(A)} = \|v\|_{L^{p(x)}(A)} + \|\nabla v\|_{L^{p(x)}(A)}.$$

The spaces $L^{p(x)}(A)$ and $W^{1,p(x)}(A)$ are separable and reflexive Banach spaces. When $A = \mathbb{R}^N$, we drop $\mathbb{R}^N$ in the notation if there is no ambiguity. For instance, we briefly write the space $(L^{p(x)}(\mathbb{R}^N), \|\cdot\|_{L^{p(x)}(\mathbb{R}^N)})$ as $(L^{p(x)}, \|\cdot\|_{L^{p(x)}})$.

Proposition 2.1. [66] *For any $v \in L^{p(x)}(A)$ and $z \in L^{p'(x)}(A)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, the following inequality holds:*

$$\Big|\int_A vz dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)\|v\|_{L^{p(x)}(A)}\|z\|_{L^{p'(x)}(A)} \le 2\|v\|_{L^{p(x)}(A)}\|z\|_{L^{p'(x)}(A)}.$$

Proposition 2.2. [66]. *Define $\rho(v) = \int_A |v|^{p(x)} dx$. Then, for all $v \in L^{p(x)}(A)$, we have*

(1) $\|v\|_{L^{p(x)}(A)} < 1(= 1, > 1)$ *if and only if* $\rho(v) < 1(= 1, > 1)$ *, respectively;*
(2) *If* $\|v\|_{L^{p(x)}(A)} > 1$*, then* $\|v\|^{p^-}_{L^{p(x)}(A)} \leq \rho(v) \leq \|v\|^{p^+}_{L^{p(x)}(A)}$*;*
(3) *If* $\|v\|_{L^{p(x)}(A)} < 1$*, then* $\|v\|^{p^+}_{L^{p(x)}(A)} \leq \rho(v) \leq \|v\|^{p^-}_{L^{p(x)}(A)}$*.*

Consequently*,*

$$\|v\|^{p^-}_{L^{p(x)}(A)} - 1 \leq \rho(v) \leq \|v\|^{p^+}_{L^{p(x)}(A)} + 1.$$

Definition 2.3. *Suppose that* $(\Phi 1)$*-*(iv) *are satisfied. Define the linear space as follows:*

$$L^{p(x)}(A) + L^{q(x)}(A) = \Big\{v : v = z + m, z \in L^{p(x)}(A), m \in L^{q(x)}(A)\Big\},$$

with the norm

$$\|v\|_{L^{p(x)}(A)+L^{q(x)}(A)} = \inf\Big\{\|z\|_{L^{p(x)}(A)} + \|m\|_{L^{q(x)}(A)} : z \in L^{p(x)}(A), m \in L^{q(x)}(A), v = z + m\Big\}.$$

We also define the space

$$L^{p(x)}(A) \cap L^{q(x)}(A) = \Big\{v : v \in L^{p(x)}(A) \text{ and } v \in L^{q(x)}(A)\Big\},$$

with the norm

$$\|v\|_{L^{p(x)}(A)\cap L^{q(x)}(A)} = \max\Big\{\|v\|_{L^{p(x)}(A)}, \|v\|_{L^{q(x)}(A)}\Big\}.$$

In addition, we denote

$$\Lambda_v = \Big\{x \in A : |v(x)| > 1\Big\} \text{ and } \Lambda_v^c = \Big\{x \in A : |v(x)| \leq 1\Big\}.$$

Lemma 2.4. [1, Proposition 3.2] *Suppose that condition* $(\Phi 1)$*-(iv) holds. Let* $A \subset \mathbb{R}^N$ *and* $v \in L^{p(x)}(A)+L^{q(x)}(A)$*. Then the following properties hold:*
(i) $|\Lambda_v| < +\infty$*;*
(ii) $v \in L^{p(x)}(\Lambda_v) \cap L^{q(x)}(\Lambda_v^c)$*;*
(iii) *If* $B \subset A$*, then* $\|v\|_{L^{p(x)}(A)+L^{q(x)}(A)} \leq \|v\|_{L^{p(x)}(B)+L^{q(x)}(B)} + \|v\|_{L^{p(x)}(A/B)+L^{q(x)}(A/B)}$*;*
(iv) *The following inequality holds:*

$$\begin{aligned}&\max\Big\{\frac{1}{1+2|\Lambda_v|^{\frac{1}{p(\eta)}-\frac{1}{q(\eta)}}}\|v\|_{L^{p(x)}(\Lambda_v)}, c\min\{\|v\|_{L^{q(x)}(\Lambda_v^c)}, \|v\|^{\frac{q(\eta)}{p(\eta)}}_{L^{q(x)}(\Lambda_v^c)}\}\Big\}\\&\leq \|v\|_{L^{p(x)}(A)+L^{q(x)}(A)} \leq \|v\|_{L^{p(x)}(\Lambda_v)} + \|v\|_{L^{q(x)}(\Lambda_v^c)} \leq 2\max\{\|v\|_{L^{p(x)}(\Lambda_v)}, \|v\|_{L^{q(x)}(\Lambda_v^c)}\},\end{aligned}$$

where $\eta \in \mathbb{R}^N$ *and* c *is a small positive constant.*

Conditions $(\Phi 1)$-(i) and (ii) imply that

$$\Phi(x,\tau) \leq \phi(x,\tau)\cdot\tau \text{ for all } (x,\tau) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{2.1}$$

By $(\Phi 1)$-(i) and (iii), it follows that

$$\Phi(x,\tau) = \int_0^1 \frac{d}{dt}\Phi(x,t\tau)dt = \int_0^1 \frac{1}{t}\phi(x,t\tau)\cdot t\tau dt \geq \begin{cases} c_1|\tau|^{p(x)}, |\tau| > 1,\\ c_1|\tau|^{q(x)}, |\tau| \leq 1.\end{cases}$$

Combining this with relations (1.2) and (2.1) yields

$$\left.\begin{matrix} c_1|\tau|^{p(x)}, |\tau| > 1 \\ c_1|\tau|^{q(x)}, |\tau| \leq 1 \end{matrix}\right\} \leq \Phi(x,\tau) \leq \phi(x,\tau)\cdot\tau \leq \begin{cases} c_2|\tau|^{p(x)}, |\tau| > 1 \\ c_2|\tau|^{q(x)}, |\tau| \leq 1 \end{cases}, \quad \forall (x,\tau) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{2.2}$$

where $c_1, c_2 > 0$ are constants.

Define $E = \left\{v \in L_V^{r(x)} : \nabla v \in (L^{p(x)} + L^{q(x)})^N\right\}$ with the following norm:

$$\|v\|_E = \|v\|_{L_V^{r(x)}} + \|\nabla v\|_{L^{p(x)}+L^{q(x)}},$$

where

$$L_V^{r(x)} = \left\{v \in P(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|v(x)|^{r(x)}dx < \infty\right\},$$

with the norm

$$\|v\|_{L_V^{r(x)}} = \inf\left\{\mu > 0 : \int_{\mathbb{R}^N} V(x)\left|\frac{v(x)}{\mu}\right|^{r(x)} \leq 1\right\}.$$

Lemma 2.5. [1, Lemma 2.5] *If* (V) *is satisfied and* $1 < r^- \leq r^+ < \infty$, *then* $L_V^{r(x)}$ *is a separable uniformly convex Banach space.*

Proposition 2.6. [1, Propositions 3.10-3.11] *Assume that* (Φ1)-*(iv) and* (V) *are satisfied. Then E is a reflexive, uniformly convex Banach space.*

Theorem 2.7. [1, Theorem 3.12] *Suppose that* (Φ1)-*(iv) and* (V) *are satisfied and*

$$1 \ll r(x) \ll p^*(x)\frac{N-1}{N}, \;\; 1 \ll r(x) \leq p^*(x)\frac{q'(x)}{p'(x)}.$$

Then, we have the continuous embedding $E \hookrightarrow L^{p^*(x)}$.

Theorem 2.8. [1, Theorem 3.14] *Suppose that all the conditions in Theorem* 2.7 *hold.*

(i) *For any* $r(x) \leq t(x) \leq p^*(x)$, *the embedding* $E \hookrightarrow L^{t(x)}$ *is continuous.*

(ii) *For any bounded subset* $A \subset \mathbb{R}^N$, *the embedding* $E(A) \hookrightarrow\hookrightarrow L^{t(x)}(A)$ *is compact.*

(iii) *If* $t \in C(\mathbb{R}^N)$ *is Lipschitz continuous and satisfies*

$$r(x) \leq t(x) \ll p^*(x) \;\text{ in } \mathbb{R}^N.$$

Then the embedding $E \hookrightarrow\hookrightarrow L^{t(x)}$ *is compact.*

Definition 2.9. *We claim that* $v \in E$ *is a weak solution to problem* (1.1), *if*

$$\mathfrak{M}(\Phi_V(x,v)) \int_{\mathbb{R}^N} \left(\phi(x,\nabla v)\cdot\nabla z + V(x)|v|^{r(x)-2}vz\right)dx + \int_{\mathbb{R}^N} h(x)|v|^{\alpha(x)-2}vzdx = \lambda \int_{\mathbb{R}^N} g(x,v)zdx,$$

for all $z \in E$.

The energy functional corresponding to problem (1.1) is given by

$$J_\lambda(v) = \mathcal{M}(\Phi_V(x,v)) + \int_{\mathbb{R}^N} \frac{h(x)}{\alpha(x)}|v|^{\alpha(x)}dx - \lambda \int_{\mathbb{R}^N} G(x,v)dx.$$

Denote the derivative operator $I = \Phi_V' : E \to E^*$ with

$$\langle I(v), z\rangle = \int_{\mathbb{R}^N} \phi(x,\nabla v)\cdot \nabla z dx + \int_{\mathbb{R}^N} V(x)|v|^{r(x)-2}vzdx \ \ \forall v, z \in E.$$

Lemma 2.10. [1, Lemma 4.7] *If conditions* $(\Phi1)$ *and* (V) *hold, then the derivative operator I has the following properties:*

(i) *$I : E \to E^*$ is a continuous, bounded, strictly monotone operator.*

If $(\Phi2)$ *also holds, then we have*

(ii) *I is a mapping of type (S_+), that is, if $v_n \rightharpoonup v$ in E and $\overline{\lim}_{n\to\infty}(I(v_n), v_n - v) \le 0$, then $v_n \to v$ in E.*

(iii) *$I : E \to E^*$ is a homeomorphism.*

Definition 2.11. *Let E be a real Banach space, and $J_\lambda \in C^1(E,\mathbb{R})$. We say J_λ satisfies the Cerami compactness condition if any sequence $\{v_n\} \subset E$ such that*

$$J_\lambda(v_n) \text{ is bounded and } \|J_\lambda'(v_n)\|_{E^*}(1 + \|v_n\|_E) \to 0 \text{ as } n \to \infty, \tag{2.3}$$

has a strongly convergent subsequence.

3. Compactness condition for the energy functional

In this subsection, we will prove that the functional J_λ satisfies the Cerami compactness condition ((C)-condition for short) under four different superlinear conditions.

Lemma 3.1. *If conditions* (V), (H), $(\Phi1)$-$(\Phi2)$, (M1)-(M2), *and* (G1)–(G4) *are satisfied, then the functional J_λ satisfies the* (C)*-condition for any $\lambda > 0$.*

Proof. Let $\{v_n\} \subset E$ be a sequence satisfying (2.3), then it follows that

$$\sup_{n\in\mathbb{N}} |J_\lambda(v_n)| \le C_* \text{ and } \langle J_\lambda'(v_n), v_n\rangle \to 0, \text{ as } n \to \infty, \tag{3.1}$$

where $C_* > 0$ is a constant. We first verify that the sequence $\{v_n\}$ is bounded in E. To this end, towards a contradiction, assume that $\{v_n\}$ is unbounded in E, namely, $\|v_n\|_E \to \infty$ as $n \to \infty$. Let $z_n = \frac{v_n}{\|v_n\|_E}$. Obviously, $\{z_n\} \subset E$, and $\|z_n\|_E = 1$. Thus, combining with Theorem 2.8 (iii), up to a subsequence, we have

$$z_n \rightharpoonup z \text{ in } E, \ \ z_n \to z \text{ a.e. in } \mathbb{R}^N, \ \ z_n \to z \text{ in } L^{t(x)}, \tag{3.2}$$

where $r(x) \le t(x) \ll p^*(x)$.

According to conditions (G2) and (V), we claim that

$$
\begin{aligned}
&m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx - \widetilde{C} \int_{|v_n|\le R} [|v_n|^{r(x)} + \kappa(x)|v_n| + \rho(x)|v_n|^{\eta(x)}] dx \\
&\ge \frac{m_0}{2} \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx - M_0,
\end{aligned} \tag{3.3}
$$

where $\widetilde{C}$ is any positive constant and M_0 is some positive constant. In fact, it follows from condition (G2) and Young's inequality that

$$
\begin{aligned}
&m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx - \widetilde{C} \int_{|v_n|\le R} |v_n|^{r(x)} + \kappa(x)|v_n| + \rho(x)|v_n|^{\eta(x)} dx \\
&\ge m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \\
&\quad - \widetilde{C} \int_{|v_n|\le R} |v_n|^{r(x)} + \epsilon(\kappa(x))^{\eta'(x)} + C(\epsilon)|v_n|^{\eta(x)} + \rho(x)|v_n|^{\eta(x)} dx \\
&\ge \frac{m_0}{2}\Big[\int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx + \int_{|v_n|\le R} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \Big] \\
&\quad - \epsilon\widetilde{C}(1 + \|\kappa\|_{L^{\eta'(x)}}^{(\eta')^+}) - \widetilde{C} \int_{|v_n|\le 1} |v_n|^{r(x)} + C(\epsilon)|v_n|^{\eta(x)} + \rho(x)|v_n|^{\eta(x)} dx \\
&\quad - \widetilde{C} \int_{1<|v_n|\le R} |v_n|^{r(x)} + C(\epsilon)|v_n|^{\eta(x)} + \rho(x)|v_n|^{\eta(x)} dx \\
&\ge \frac{m_0}{2}\Big[\int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx + \int_{|v_n|\le R} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \Big] \\
&\quad - \widetilde{C}(1 + C(\epsilon) + \|\rho\|_{L^\infty}) \int_{|v_n|\le 1} |v_n|^{r(x)} dx \\
&\quad - \widetilde{C}(1 + C(\epsilon) + \|\rho\|_{L^\infty}) R^{\eta^+ - r^-} \int_{1<|v_n|\le R} |v_n|^{r(x)} dx - C_1 \\
&\ge \frac{m_0}{2}\Big[\int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx + \int_{|v_n|\le R} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \Big] \\
&\quad - C_2 \int_{|v_n|\le R} |v_n|^{r(x)} dx - C_1,
\end{aligned} \tag{3.4}
$$

where $C_1 = \epsilon\widetilde{C}(1 + \|\kappa\|_{L^{\eta'(x)}}^{(\eta')^+})$ and $C_2 = \widetilde{C}(1 + C(\epsilon) + \|\rho\|_{L^\infty}) R^{\eta^+ - r^-}$. According to condition (V), we have $V(x) \to +\infty$ as $|x| \to \infty$. Thus, we can find $x_0 > 0$ such that

$$
V(x) \ge \frac{2C_2\theta\delta r^+}{m_0(\delta - \theta r^+)} \ \text{ for } |x| \ \ge x_0, \tag{3.5}
$$

where $m_0 > 0$ and $\delta > \theta r^+ > 0$ are constants.

Let $B_{x_0} := \{x \in \mathbb{R}^N : |x| < x_0\}$. From $V(x) \in L^1_{loc}(\mathbb{R}^N)$, we obtain

$$
\int_{\{|v_n|\le R\}\cap B_{x_0}} V(x)|v_n|^{r(x)} dx \le C_3 \ \text{ and } \ \int_{\{|v_n|\le R\}\cap B_{x_0}} |v_n|^{r(x)} dx \le C_4. \tag{3.6}
$$

Thus, combining relations (3.4)–(3.6), we deduce that

$$
\begin{aligned}
& m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx - \widetilde{C} \int_{|v_n|\le R} |v_n|^{r(x)} + \kappa(x)|v_n| + \rho(x)|v_n|^{\eta(x)} dx \\
\ge & \frac{m_0}{2} \Big[\int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx + \int_{\{|v_n|\le R\}\cap B_{x_0}} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \\
& + \int_{\{|v_n|\le R\}\cap B_{x_0}^c} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \Big] \\
& - C_2 \Big[\int_{\{|v_n|\le R\}\cap B_{x_0}} |v_n|^{r(x)} dx + \int_{\{|v_n|\le R\}\cap B_{x_0}^c} |v_n|^{r(x)} dx \Big] - C_1 \\
\ge & \frac{m_0}{2} \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \\
& + \int_{\{|v_n|\le R\}\cap B_{x_0}^c} \Big(\frac{m_0}{2} (\frac{1}{\theta r^+} - \frac{1}{\delta}) V(x) - C_2 \Big) |v_n|^{r(x)} dx - C_2 C_4 - C_1 \\
\ge & \frac{m_0}{2} \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx - M_0,
\end{aligned} \tag{3.7}
$$

which shows that relation (3.3) holds true. Combining this with conditions (Φ1)-(v), (M1), (M2), (G2), (G4), relations (2.2), (3.1), Propositions 2.1-2.2, and Lemma 2.4-(iv), we deduce that

$$
\begin{aligned}
C_* + 1 \ge & J_\lambda(v_n) - \frac{1}{\delta} \langle J'_\lambda(v_n), v_n \rangle \\
= & \mathcal{M}(\varPhi_V(x, v_n)) + \int_{\mathbb{R}^N} \frac{h(x)}{\alpha(x)} |v_n|^{\alpha(x)} dx - \lambda \int_{\mathbb{R}^N} G(x, v_n) dx \\
& - \frac{1}{\delta} \mathfrak{M}(\varPhi_V(x, v_n)) \int_{\mathbb{R}^N} \Big(\phi(x, \nabla v_n) \cdot \nabla v_n + V(x)|v_n|^{r(x)} \Big) dx \\
& - \frac{1}{\delta} \int_{\mathbb{R}^N} h(x)|v_n|^{\alpha(x)} dx + \frac{\lambda}{\delta} \int_{\mathbb{R}^N} g(x, v_n) v_n dx \\
\ge & \mathfrak{M}(\varPhi_V(x, v_n)) \Big(\int_{\mathbb{R}^N} (\frac{1}{\theta} - \frac{s(x)}{\delta}) \Phi(x, \nabla v_n) dx + \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \Big) \\
& + \int_{\mathbb{R}^N} (\frac{1}{\alpha(x)} - \frac{1}{\delta}) h(x)|v_n|^{\alpha(x)} dx + \lambda \int_{\mathbb{R}^N} \frac{1}{\delta} g(x, v_n) v_n - G(x, v_n) dx \\
\ge & m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta} - \frac{s(x)}{\delta}) \Phi(x, \nabla v_n) dx + m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \\
& + \lambda \int_{|v_n|>R} \frac{1}{\delta} g(x, v_n) v_n - G(x, v_n) dx + \lambda \int_{|v_n|\le R} \frac{1}{\delta} g(x, v_n) v_n - G(x, v_n) dx \\
\ge & m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta} - \frac{s(x)}{\delta}) \Phi(x, \nabla v_n) dx + m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \\
& - \frac{\lambda}{\delta} \int_{|v_n|\ge R} \zeta(x)|v_n|^{r^-} dx - \lambda \int_{|v_n|\le R} (1 + \frac{1}{\delta}) \kappa(x)|v_n| + (\frac{1}{\eta^-} + \frac{1}{\delta}) \rho(x)|v_n|^{\eta(x)} dx \\
\ge & m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta} - \frac{s(x)}{\delta}) \Phi(x, \nabla v_n) dx + m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta}) V(x)|v_n|^{r(x)} dx \\
& - \frac{\lambda}{\delta} \int_{|v_n|\ge R} \zeta(x)|v_n|^{r^-} dx - \lambda (1 + \frac{1}{\delta}) \int_{|v_n|\le R} |v_n|^{r(x)} + \kappa(x)|v_n| + \rho(x)|v_n|^{\eta(x)} dx
\end{aligned}
$$

$$
\begin{aligned}
&\geq m_0 \int_{\mathbb{R}^N} (\frac{1}{\theta} - \frac{s(x)}{\delta})\Phi(x,\nabla v_n)dx + \frac{m_0}{2}\int_{\mathbb{R}^N} (\frac{1}{\theta r(x)} - \frac{1}{\delta})V(x)|v_n|^{r(x)}dx \\
&\quad - \frac{\lambda}{\delta}\int_{\mathbb{R}^N} \zeta(x)|v_n|^{r^-}dx - M_0 \\
&\geq m_0 \min\Big\{(\frac{1}{\theta} - \frac{s^+}{\delta}), \frac{1}{2}(\frac{1}{\theta r^+} - \frac{1}{\delta})\Big\}\int_{\mathbb{R}^N} \Phi(x,\nabla v_n) + V(x)|v_n|^{r(x)}dx \\
&\quad - \frac{\lambda}{\delta}\Big(\int_{\Omega_1} \zeta(x)|v_n|^{r^-}dx + \int_{\Omega_2} \zeta(x)|v_n|^{r^-}dx\Big) - M_0 \\
&\geq m_0 \min\Big\{(\frac{1}{\theta} - \frac{s^+}{\delta}), \frac{1}{2}(\frac{1}{\theta r^+} - \frac{1}{\delta})\Big\}\Big(c_1\int_{\mathbb{R}^N\cap\Lambda_{\nabla v_n}} |\nabla v_n|^{p(x)}dx + c_1\int_{\mathbb{R}^N\cap\Lambda^c_{\nabla v_n}} |\nabla v_n|^{q(x)}dx \\
&\quad + \int_{\mathbb{R}^N} V(x)|v_n|^{r(x)}dx\Big) - \frac{\lambda}{\delta}\Big(\int_{\Omega_1} \zeta(x)|v_n|^{r^-}dx + \int_{\Omega_2} \zeta(x)|v_n|^{r^-}dx\Big) - M_0 \\
&\geq C_5\|v_n\|_E^{r^-} - \frac{2\lambda}{\delta}\|\zeta\|_{L^{\frac{r(x)}{r(x)-r^-}}(\Omega_1)}\|v_n\|^{r^-}_{L^{r(x)}(\Omega_1)} - \frac{\lambda\zeta_0}{\delta}\|v_n\|^{r^-}_{L^{r(x)}(\Omega_2)} - C_6 \\
&\geq C_5\|v_n\|_E^{r^-} - \frac{\lambda}{\delta}\Big(2\|\zeta\|_{L^{\frac{r(x)}{r(x)-r^-}}(\Omega_1)} + \zeta_0\Big)\|v_n\|^{r^-}_{L^{r(x)}} - C_6,
\end{aligned}
\tag{3.8}
$$

where C_5, C_6 represent some positive constants. Since $\|v_n\|_E \to \infty$ as $n \to \infty$, it follows from relations (3.2) and (3.8) that

$$
1 \leq \frac{\lambda}{C_5\delta}\Big(2\|\zeta\|_{L^{\frac{r(x)}{r(x)-r^-}}(\Omega_1)} + \zeta_0\Big)\limsup_{n\to\infty}\|z_n\|^{r^-}_{L^{r(x)}} = \frac{\lambda}{C_5\delta}\Big(2\|\zeta\|_{L^{\frac{r(x)}{r(x)-r^-}}(\Omega_1)} + \zeta_0\Big)\|z\|^{r^-}_{L^{r(x)}}. \tag{3.9}
$$

Hence, it follows that $z \neq 0$. Let $\Theta := \{x \in \mathbb{R}^N : z(x) \neq 0\}$. Then, we have $|v_n(x)| = |z_n(x)|\|v_n\|_E \to \infty$ for a.e. $x \in \Theta$ as $n \to \infty$. From condition (M2), we deduce that

$$
\mathcal{M}(\tau) = \int_0^\tau \mathfrak{M}(s)ds \leq \mathcal{M}(1),\ \forall \tau \in [0,1) \ \text{ and } \ \mathcal{M}(\tau) \leq \mathcal{M}(1)\tau^\theta,\ \forall \tau \in [1,+\infty), \tag{3.10}
$$

which implies that

$$
\mathcal{M}(\tau) \leq \mathcal{M}(1)(1+\tau^\theta) \ \text{ for } \tau \in \mathbb{R}^+_0. \tag{3.11}
$$

Combining this with relation (2.2), Propositions 2.1-2.2, Lemma 2.4, Theorem 2.8, and condition (H), we deduce that

$$
\begin{aligned}
J_\lambda(v_n) &= \mathcal{M}(\varPhi_V(x,v_n)) + \int_{\mathbb{R}^N} \frac{h(x)}{\alpha(x)}|v_n|^{\alpha(x)}dx - \lambda\int_{\mathbb{R}^N} G(x,v_n)dx \\
&\leq \mathcal{M}(1)\Big(1 + (\int_{\mathbb{R}^N} (\Phi(x,\nabla v_n) + \frac{V(x)}{r(x)}|v_n|^{r(x)})dx)^\theta\Big) + \int_{\mathbb{R}^N} \frac{h(x)}{\alpha(x)}|v_n|^{\alpha(x)}dx - \lambda\int_{\mathbb{R}^N} G(x,v_n)dx \\
&\leq \mathcal{M}(1)\Big[1 + \Big(c_2\int_{\Lambda_{\nabla v_n}} |\nabla v_n|^{p(x)}dx + c_2\int_{\Lambda^c_{\nabla v_n}} |\nabla v_n|^{q(x)}dx + \frac{1}{r^-}\int_{\mathbb{R}^N} V(x)|v_n|^{r(x)}dx\Big)^\theta\Big] \\
&\quad + \int_{\mathbb{R}^N} \frac{h(x)}{\alpha(x)}|v_n|^{\alpha(x)}dx - \lambda\int_{\mathbb{R}^N} G(x,v_n)dx \\
&\leq C_7\|v_n\|_E^{\theta q^+} + \frac{2\|h\|_{L^{\beta_0(x)}}}{\alpha^-}\max\{\|v_n\|^{\alpha^+}_{L^{\beta(x)}}, \|v_n\|^{\alpha^-}_{L^{\beta(x)}}\} - \lambda\int_{\mathbb{R}^N} G(x,v_n)dx + C_8
\end{aligned}
$$

$$\leq C_7\|v_n\|_E^{\theta q^+} + \frac{2C_9\|h\|_{L^{\beta_0(x)}}}{\alpha^-}\|v_n\|_E^{\alpha^+} - \lambda\int_{\mathbb{R}^N} G(x,v_n)dx + C_8, \tag{3.12}$$

where C_7, C_8 are some positive constants. Consequently, it follows from relations (3.1), (3.12), conditions (G1), (G3), and Fatou's lemma that

$$0 = \lim_{n\to\infty}\frac{J_\lambda(v_n)}{\|v_n\|_E^{\theta q^+}} \leq C_7 - \lambda\liminf_{n\to\infty}\int_\Theta \frac{G(x,v_n)}{|v_n|^{\theta q^+}}|z_n|^{\theta q^+}dx = -\infty, \tag{3.13}$$

which is a contradiction. Thus, the sequence $\{v_n\}$ is bounded in E. Therefore, up to a subsequence, we may assume that

$$v_n \rightharpoonup v \ \text{ in } E, \ \ v_n \to v \ \text{ a.e. in } \mathbb{R}^N \ \text{ and } \ v_n \to v \ \text{ in } L^{t(x)}, \tag{3.14}$$

where $r(x) \leq t(x) \ll p^*(x)$. Combining this with Proposition 2.2 and Hölder's inequality, it follows that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} h(x)|v_n|^{\alpha(x)-2}v_n(v_n-v)dx\Big| \\
&\leq \int_{\mathbb{R}^N} h(x)|v_n|^{\alpha(x)-1}|v_n-v|dx \\
&\leq C_9\|h\|_{L^{\beta_0(x)}}\||v_n|^{\alpha(x)-1}\|_{L^{\frac{\beta(x)}{\alpha(x)-1}}}\|v_n-v\|_{L^{\beta(x)}} \\
&\leq C_9\|h\|_{L^{\beta_0(x)}}\max\{\|v_n\|_{L^{\beta(x)}}^{\alpha^+-1}, \|v_n\|_{L^{\beta(x)}}^{\alpha^--1}\}\|v_n-v\|_{L^{\beta(x)}} \to 0, \ \text{ as } n\to\infty.
\end{aligned} \tag{3.15}$$

In addition, from (G2), Propositions 2.1-2.2, and relation (3.14), it follows that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} g(x,v_n)(v_n-v)dx\Big| \\
&\leq \int_{\mathbb{R}^N} |\kappa(x)+\rho(x)|v_n|^{\eta(x)-1}||v_n-v|dx \\
&\leq 2\|\kappa\|_{L^{\eta'(x)}}\|v_n-v\|_{L^{\eta(x)}} + 2\|\rho\|_{L^\infty}\max\{\|v_n\|_{L^{\eta(x)}}^{\eta^+-1}, \|v_n\|_{L^{\eta(x)}}^{\eta^--1}\}\|v_n-v\|_{L^{\eta(x)}} \to 0,
\end{aligned} \tag{3.16}$$

as $n\to\infty$. Since $\langle J'_\lambda(v_n), v_n-v\rangle \to 0$ as $n\to\infty$, along with relations (3.15)-(3.16), we conclude that

$$\lim_{n\to\infty}\mathfrak{M}(\Phi_V(x,v_n))\langle I(v_n), v_n-v\rangle = 0. \tag{3.17}$$

From condition (M1), we infer that

$$\langle I(v_n), v_n-v\rangle \to 0 \ \text{ as } n\to\infty. \tag{3.18}$$

According to Lemma 2.10-(ii), we yield that $v_n \to v$ in E as $n\to\infty$. □

Lemma 3.2. *Assume that conditions* (V), (H), (Φ1)-(Φ2)*,* (M1)-(M2), (G1)–(G3)*, and* (G5) *are satisfied. Then the functional* J_λ *satisfies the* (C)*-condition for any* $\lambda > 0$.

Proof. Let $\{v_n\} \subset E$ be a (C)-sequence satisfying (2.3). As in the proof of Lemma 3.1, we simply need to demonstrate that $\{v_n\}$ is bounded in E. Towards a contradiction, assume that $\|v_n\|_E \to \infty$ as $n \to \infty$. Let $z_n = \frac{v_n}{\|v_n\|_E}$. Obviously, $\{z_n\} \subset E$ and $\|z_n\|_E = 1$. Thus, we yield that relation (3.2) holds. From Proposition 2.2, Lemma 2.4-(iv), conditions (M1)-(M2), (H), and relations (2.2), (3.1), we conclude that

$$\begin{aligned}
C_* &\geq J_\lambda(v_n) \\
&= \mathcal{M}(\Phi_V(x, v_n)) + \int_{\mathbb{R}^N} \frac{h(x)}{\alpha(x)} |v_n|^{\alpha(x)} dx - \lambda \int_{\mathbb{R}^N} G(x, v_n) dx \\
&\geq \frac{1}{\theta} \mathfrak{M}(\Phi_V(x, v_n)) \Phi_V(x, v_n) - \lambda \int_{\mathbb{R}^N} G(x, v_n) dx \\
&\geq \frac{m_0}{\theta} \Big[c_1 \Big(\int_{\Lambda_{\nabla v_n}} |\nabla v_n|^{p(x)} dx + \int_{\Lambda^c_{\nabla v_n}} |\nabla v_n|^{q(x)} dx \Big) + \frac{1}{r^+} \int_{\mathbb{R}^N} V(x) |v_n|^{r(x)} dx \Big] - \lambda \int_{\mathbb{R}^N} G(x, v_n) dx \\
&\geq \frac{m_0}{\theta} \Big(C_{10} \|\nabla v_n\|^{p^-}_{L^{p(x)} + L^{q(x)}} + \frac{1}{r^+} \|v_n\|^{r^-}_{L^{r(x)}_V} - C_{11} \Big) - \lambda \int_{\mathbb{R}^N} G(x, v_n) dx \\
&\geq \frac{m_0 \min\{C_{10} r^+, 1\} C_{12}}{\theta r^+} \|v_n\|^{r^-}_E - \lambda \int_{\mathbb{R}^N} G(x, v_n) dx - \frac{m_0 (C_{10} + C_{11})}{\theta},
\end{aligned} \tag{3.19}$$

where C_{10}, C_{11} and C_{12} are some positive constants. Thus, together with $\|v_n\|_E \to \infty$, we have

$$0 < \frac{m_0 \min\{C_{10} r^+, 1\} C_{12}}{\lambda \theta r^+} \leq \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{G(x, v_n)}{\|v_n\|^{r^-}_E} dx + o_n(1). \tag{3.20}$$

Moreover, we deduce from conditions (Φ1)-(v), (M2), and (G5) and the fact that $\alpha(x) \ll \theta r(x) \ll \theta s(x) \leq \theta s^+ < \delta$ that

$$\begin{aligned}
C_* + 1 &\geq J_\lambda(v_n) - \frac{1}{\delta} \langle J'_\lambda(v_n), v_n \rangle \\
&= \mathcal{M}(\Phi_V(x, v_n)) - \frac{1}{\delta} \mathfrak{M}(\Phi_V(x, v_n)) \int_{\mathbb{R}^N} \Big(\phi(x, \nabla v_n) \cdot \nabla v_n + V(x) |v_n|^{r(x)} \Big) dx \\
&\quad + \int_{\mathbb{R}^N} \Big(\frac{1}{\alpha(x)} - \frac{1}{\delta} \Big) h(x) |v_n|^{\alpha(x)} dx + \lambda \int_{\mathbb{R}^N} \frac{1}{\delta} g(x, v_n) v_n - G(x, v_n) dx \\
&\geq \mathfrak{M}(\Phi_V(x, v_n)) \Big(\int_{\mathbb{R}^N} \Big(\frac{1}{\theta} - \frac{s(x)}{\delta} \Big) \Phi(x, \nabla v_n) dx + \int_{\mathbb{R}^N} \Big(\frac{1}{\theta r(x)} - \frac{1}{\delta} \Big) V(x) |v_n|^{r(x)} dx \Big) \\
&\quad + \int_{\mathbb{R}^N} \Big(\frac{1}{\alpha(x)} - \frac{1}{\delta} \Big) h(x) |v_n|^{\alpha(x)} dx + \lambda \int_{\mathbb{R}^N} \frac{1}{\delta} g(x, v_n) v_n - G(x, v_n) dx \\
&\geq \lambda \int_{\mathbb{R}^N} \widetilde{\mathcal{G}}(x, v_n) dx.
\end{aligned} \tag{3.21}$$

If $z \neq 0$, by the same arguments as in (3.10)–(3.13), we get a contradiction. Thus, we have $z(x) = 0$ for almost all $x \in \mathbb{R}^N$. Taking into account relation (3.2), it is obtained that $z_n \to 0$ in $L^{\eta(x)}$. Let $\Lambda_{v_n}(r_1, r_2) := \{x \in \mathbb{R}^N : r_1 \leq |v_n(x)| < r_2\}$ for $r_1 \geq 0$. Since $z_n \to 0$ in $L^{\eta(x)}$ and $\|v_n\|_E \to \infty$ as $n \to \infty$, we can conclude from condition (G2) and Propositions 2.1-2.2 that

$$\begin{aligned}
\int_{\Lambda_{v_n}(0,M)} \frac{G(x,v_n)}{\|v_n\|_E^{r^-}}dx &\le \frac{1}{\|v_n\|_E^{r^-}}\int_{\Lambda_{v_n}(0,M)} \kappa(x)|v_n|dx + \frac{1}{\eta^-}\int_{\Lambda_{v_n}(0,M)} \frac{\rho(x)|v_n|^{\eta(x)}}{\|v_n\|_E^{r^-}}dx \\
&\le \frac{2\|\kappa\|_{L^{\eta'(x)}}\|v_n\|_{L^{\eta(x)}}}{\|v_n\|_E^{r^-}} + \frac{1}{\eta^-}\int_{\Lambda_{v_n}(0,M)} \rho(x)|v_n|^{\eta(x)-r^-}|z_n|^{r^-}dx \\
&\le \frac{2C_{13}\|\kappa\|_{L^{\eta'(x)}}}{\|v_n\|_E^{r^- -1}} + \frac{2M^{\eta_0 - r^-}}{\eta^-}\|\rho\|_{L^{\frac{\eta(x)}{\eta(x)-r^-}}}\|z_n\|_{L^{\eta(x)}}^{r^-} \to 0 \text{ as } n\to\infty, \qquad (3.22)
\end{aligned}$$

where η_0 is either η^+ or η^-. Set $\mu'(x) = \frac{\mu(x)}{\mu(x)-1}$. It is not difficult to check that $r(x) \le \mu'(x)r^- \ll p^*(x)$. Thus, combining this with relations (3.2), (3.21), condition (G5), and Proposition 2.1, we obtain

$$\begin{aligned}
\int_{\Lambda_{v_n}(M,\infty)} \frac{G(x,v_n)}{\|v_n\|_E^{r^-}}dx &= \int_{\Lambda_{v_n}(M,\infty)} \frac{G(x,v_n)}{|v_n|^{r^-}}|z_n|^{r^-}dx \\
&\le 2\left\|\frac{G(x,v_n)}{|v_n|^{r^-}}\right\|_{L^{\mu(x)}(\Lambda_{v_n}(M,\infty))}\left\||z_n|^{r^-}\right\|_{L^{\mu'(x)}(\Lambda_{v_n}(M,\infty))} \\
&\le 2\|(C_0\widetilde{\mathcal{G}}(x,v_n))^{\frac{1}{\mu(x)}}\|_{L^{\mu(x)}(\Lambda_{v_n}(M,\infty))}\|z_n\|^{r^-}_{L^{\mu'(x)r^-}(\Lambda_{v_n}(M,\infty))} \\
&\le 2C_0^{\frac{1}{\mu_0}}\|\widetilde{\mathcal{G}}(x,v_n))\|_{L^1}^{\frac{1}{\mu_0}}\|z_n\|^{r^-}_{L^{\mu'(x)r^-}(\Lambda_{v_n}(M,\infty))} \\
&\le 2C_0^{\frac{1}{\mu_0}}\Big(\frac{C_*+1}{\lambda}\Big)^{\frac{1}{\mu_0}}\|z_n\|^{r^-}_{L^{\mu'(x)r^-}} \to 0, \text{ as } n\to\infty, \qquad (3.23)
\end{aligned}$$

where μ_0 is either μ^+ or μ^-. Thus, from relations (3.20), (3.22), and (3.23), it follows that

$$\frac{m_0\min\{C_{10}r^+,1\}C_{12}}{\lambda\theta r^+} \le \limsup_{n\to\infty}\int_{\Lambda_{v_n}(0,M)}\frac{G(x,v_n)}{\|v_n\|_E^{r^-}}dx + \limsup_{n\to\infty}\int_{\Lambda_{v_n}(M,\infty)}\frac{G(x,v_n)}{\|v_n\|_E^{r^-}}dx = 0. \qquad (3.24)$$

which is a contradiction. Hence, we conclude that $\{v_n\}$ is bounded in E. □

Lemma 3.3. *Assume that* $\mathfrak{M}$ *is a decreasing function on* $\mathbb{R}_0^+$ *and conditions* (Φ1)-(Φ2), (V), (H), (M1)-(M2), (G1)–(G3), *and* (G6) *are satisfied. Then the functional* J_λ *satisfies the* (C)*-condition for any* $\lambda > 0$.

Proof. Let $\{v_n\} \subset E$ be a (C)-sequence satisfying (2.3). In the same way as the proof of Lemma 3.1, we simply need to demonstrate that $\{v_n\}$ is bounded in E. To this end, towards a contradiction, assume that $\|v_n\|_E \to \infty$ as $n \to \infty$. Let $z_n = \frac{v_n}{\|v_n\|_E}$. Clearly, $\{z_n\} \subset E$ and $\|z_n\|_E = 1$, then we have (3.2) holds. if $z \neq 0$, as in (3.10)–(3.13), we get a contradiction. Hence, we have $z(x) = 0$ for almost all $x \in \mathbb{R}^N$. As $J_\lambda(tv_n)$ is continuous in $t \in [0,1]$, there exists a sequence $t_n \in [0,1]$ such that

$$J_\lambda(t_nv_n) = \max_{t\in[0,1]} J_\lambda(tv_n).$$

Let

$$T := \max\Big\{1, \Big(\frac{\theta(\xi C_* + 1 + C_{15})}{m_0C_{14}}\Big)^{1/r^-}\Big\}.$$

Because $\|v_n\|_E \to \infty$ as $n \to \infty$, we yield that $0 \leq \frac{T}{\|v_n\|_E} \leq 1$ for sufficiently large n. Since $z_n \to 0$ in $L^{t(x)}$ for $r(x) \leq t(x) \ll p^*(x)$, we infer from condition (M1)-(M2), (H), (G1), (G2), (2.2), Propositions 2.1-2.2 and Lemma 2.4-(iv) that for large $n \in \mathbb{N}$,

$$\begin{aligned}
J_\lambda(t_n v_n) &\geq J_\lambda(\frac{T}{\|v_n\|_E} v_n) = J_\lambda(T z_n) \\
&= \mathcal{M}(\Phi_V(x, T z_n)) + \int_{\mathbb{R}^N} \frac{h(x)}{\alpha(x)} |T z_n|^{\alpha(x)} dx - \lambda \int_{\mathbb{R}^N} G(x, T z_n) dx \\
&\geq \frac{1}{\theta} \mathfrak{M}(\Phi_V(x, T z_n)) \Phi_V(x, T z_n) - \lambda \int_{\mathbb{R}^N} \kappa(x) |T z_n| dx - \lambda \int_{\mathbb{R}^N} \frac{\rho(x)}{\eta(x)} |T z_n|^{\eta(x)} dx \\
&\geq \frac{m_0}{\theta} \Big(c_1 \big(\int_{\Lambda_{\nabla T z_n}} |\nabla T z_n|^{p(x)} dx + \int_{\Lambda^c_{\nabla T z_n}} |\nabla T z_n|^{q(x)} dx \big) + \frac{1}{r^+} \int_{\mathbb{R}^N} V(x) |T z_n|^{r(x)} dx \Big) \\
&\quad - \lambda \int_{\mathbb{R}^N} \kappa(x) |T z_n| dx - \lambda \int_{\mathbb{R}^N} \frac{\rho(x)}{\eta(x)} |T z_n|^{\eta(x)} dx \\
&\geq \frac{m_0 C_{14}}{\theta} \|T z_n\|_E^{r^-} - 2\lambda \|\kappa\|_{L^{\eta'(x)}} \|T z_n\|_{L^{\eta(x)}} - \frac{\lambda \|\rho\|_{L^\infty}}{\eta^-} \|T z_n\|^{\eta_0}_{L^{\eta(x)}} - C_{15} \\
&\geq \frac{m_0 C_{14}}{\theta} T^{r^-} + o_n(1) - C_{15} \\
&\geq \max \Big\{ \frac{m_0 C_{14}}{\theta} - C_{15}, \xi C_* + 1 \Big\} + o_n(1),
\end{aligned} \tag{3.25}$$

where $\xi \geq 1$ and η_0 is either η^+ or η^-. Since $J_\lambda(0) = 0$ and $|J_\lambda(v_n)| \leq C_*$ as $n \to \infty$, we derive that $t_n \in (0, 1)$ and $\langle J'_\lambda(t_n v_n), t_n v_n \rangle = 0$. Hence, we deduce from the monotonicity of the function $\mathfrak{M}$ and the condition (G6) that for all n large enough,

$$\begin{aligned}
\frac{1}{\xi} J_\lambda(t_n v_n) &= \frac{1}{\xi} J_\lambda(t_n v_n) - \frac{1}{\xi s^+} \langle J'_\lambda(t_n v_n), t_n v_n \rangle + o_n(1) \\
&= \frac{1}{\xi} \mathcal{M}(\Phi_V(x, t_n v_n)) + \frac{1}{\xi} \int_{\mathbb{R}^N} \frac{h(x)}{\alpha(x)} |t_n v_n|^{\alpha(x)} dx - \frac{\lambda}{\xi} \int_{\mathbb{R}^N} G(x, t_n v_n) dx \\
&\quad - \frac{1}{\xi s^+} \mathfrak{M}(\Phi_V(x, t_n v_n)) \int_{\mathbb{R}^N} \Big(\phi(x, \nabla t_n v_n) \cdot \nabla t_n v_n + V(x) |t_n v_n|^{r(x)} \Big) dx \\
&\quad - \frac{1}{\xi s^+} \int_{\mathbb{R}^N} h(x) |t_n v_n|^{\alpha(x)} dx + \frac{\lambda}{\xi s^+} \int_{\mathbb{R}^N} g(x, t_n v_n) t_n v_n dx + o_n(1) \\
&\leq \frac{1}{\xi} \mathcal{M}(\Phi_V(x, t_n v_n)) - \frac{1}{\xi s^+} \mathfrak{M}(\Phi_V(x, t_n v_n)) \int_{\mathbb{R}^N} \Big(\phi(x, \nabla t_n v_n) \cdot \nabla t_n v_n + V(x) |t_n v_n|^{r(x)} \Big) dx \\
&\quad + \frac{1}{\xi} \int_{\mathbb{R}^N} \Big(\frac{1}{\alpha(x)} - \frac{1}{s^+} \Big) h(x) |t_n v_n|^{\alpha(x)} dx + \frac{\lambda}{\xi s^+} \int_{\mathbb{R}^N} \mathfrak{G}(x, t_n v_n) dx + o_n(1) \\
&\leq \mathcal{M}(\Phi_V(x, v_n)) - \frac{1}{s^+} \mathfrak{M}(\Phi_V(x, v_n)) \int_{\mathbb{R}^N} \Big(\phi(x, \nabla v_n) \cdot \nabla v_n + V(x) |v_n|^{r(x)} \Big) dx \\
&\quad + \int_{\mathbb{R}^N} \Big(\frac{1}{\alpha(x)} - \frac{1}{s^+} \Big) h(x) |v_n|^{\alpha(x)} dx + \frac{\lambda}{s^+} \int_{\mathbb{R}^N} \mathfrak{G}(x, v_n) dx + o_n(1) \\
&= J_\lambda(v_n) - \frac{1}{s^+} \langle J'_\lambda(v_n), v_n \rangle + o_n(1) \leq C_*,
\end{aligned} \tag{3.26}$$

which contradicts (3.25). Therefore, $\{v_n\}$ is bounded in E. □

Lemma 3.4. *If conditions* (V), (H), (Φ1)-(Φ2), (M1)-(M2), (G1)-(G2)*, and* (G7)-(G8) *hold, then the functional J_λ satisfies the* (C)*-condition for any $\lambda > 0$.*

Proof. We consider here only the case $N \geq 3$, since $N = 1, 2$ can be treated in the same way. Using the same proof as in Lemma 3.1, we simply need to demonstrate that the sequence $\{v_n\}$ is bounded in E. Towards a contradiction, assume that $\|v_n\|_E \to \infty$ as $n \to \infty$. Let $z_n = \frac{v_n}{\|v_n\|_E}$. Then $\{z_n\} \subset E$ and $\|z_n\|_E = 1$. Let

$$\mathcal{D}_n := \Big\{x \in \mathbb{R}^N : \frac{G(x, v_n)}{|v_n|^{r^-}} \leq \frac{m_0 C_{12} \min\{C_{10} r^+, 1\}(1 - \kappa_0)}{\lambda \theta r^+ \gamma_{r^-}}\Big\}. \tag{3.27}$$

Since $\|z_n\|_{L^{r^-}}^{r^-} \leq \gamma_{r^-} \|z_n\|_E^{r^-} = \gamma_{r^-}$, then we have

$$\begin{aligned}\int_{\mathcal{D}_n} \frac{G(x, v_n)}{\|v_n\|_E^{r^-}} dx &= \int_{\mathcal{D}_n} \frac{G(x, v_n)}{|v_n|^{r^-}} |z_n|^{r^-} dx \\ &\leq \frac{m_0 C_{12} \min\{C_{10} r^+, 1\}(1 - \kappa_0)}{\lambda \theta r^+ \gamma_{r^-}} \|z_n\|_{L^{r^-}}^{r^-} \\ &\leq \frac{m_0 C_{12} \min\{C_{10} r^+, 1\}(1 - \kappa_0)}{\lambda \theta r^+}. \end{aligned} \tag{3.28}$$

In addition, it follows from condition (G8), relation (3.21), Proposition 2.1, and Theorem 2.7 that

$$\begin{aligned}\int_{\mathbb{R}^N \setminus \mathcal{D}_n} \frac{G(x, v_n)}{\|v_n\|_E^{r^-}} dx &= \frac{1}{\|v_n\|_E^{r^- - \mu_0}} \int_{\mathbb{R}^N \setminus \mathcal{D}_n} \frac{G(x, v_n)}{|v_n|^{\mu_0}} |z_n|^{\mu_0} dx \\ &\leq \frac{2}{\|v_n\|_E^{r^- - \mu_0}} \Big(\int_{\mathbb{R}^N \setminus \mathcal{D}_n} \Big| \frac{G(x, v_n)}{|v_n|^{\mu_0}} \Big|^{\frac{(p^+)^*}{(p^+)^* - \mu_0}} dx \Big)^{\frac{(p^+)^* - \mu_0}{(p^+)^*}} \|z_n\|_{L^{(p^+)^*}}^{\mu_0} \\ &\leq \frac{2}{\|v_n\|_E^{r^- - \mu_0}} \Big(\int_{\mathbb{R}^N \setminus \mathcal{D}_n} c_0 \widetilde{\mathcal{G}}(x, v_n) dx \Big)^{\frac{(p^+)^* - \mu_0}{(p^+)^*}} \|z_n\|_{L^{(p^+)^*}}^{\mu_0} \\ &\leq \frac{2 C_{16}}{\|v_n\|_E^{r^- - \mu_0}} \Big(\frac{c_0 (1 + C_*)}{\lambda} \Big)^{\frac{(p^+)^* - \mu_0}{(p^+)^*}} \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.29}$$

Combining this with relations (3.20) and (3.28), we yield that

$$\begin{aligned}\frac{m_0 C_{12} \min\{C_{10} r^+, 1\}}{\lambda \theta r^+} &\leq \limsup_{n \to \infty} \int_{\mathcal{D}_n} \frac{G(x, v_n)}{\|v_n\|_E^{r^-}} dx + \limsup_{n \to \infty} \int_{\mathbb{R}^N \setminus \mathcal{D}_n} \frac{G(x, v_n)}{\|v_n\|_E^{r^-}} dx \\ &\leq \frac{m_0 C_{12} \min\{C_{10} r^+, 1\}(1 - \kappa_0)}{\lambda \theta r^+}, \end{aligned} \tag{3.30}$$

which is a contradiction. Hence, we deduce that $\{v_n\}$ is bounded in E. □

4. Proof of the theorems

In this subsection, our goal is to give the proof of Theorems 1.3–1.11. We first show the geometric properties of the functional J_λ.

Lemma 4.1. *If conditions* (Φ1)*,* (V)*,* (H)*,* (M1)-(M2)*,* (G1)-(G2)*, and* (G7) *are satisfied, then the functional J_λ possesses the following geometric properties:*

(i) *There exist* $\vartheta, \epsilon > 0$ *such that* $J_\lambda(\varphi_0) \geq \vartheta$ *for any* $\varphi_0 \in E$ *with* $\|\varphi_0\|_E = \epsilon$.

(ii) *There exists* $v \in E$ *such that* $J_\lambda(v) < 0$ *if* $\|v\|_E > \epsilon$.

Proof. (i) Take $\varphi_0 \in E$ with $\|\varphi_0\|_E \leq 1$. Since $\|\varphi_0\|_E \leq 1$, we deduce $\|\nabla\varphi_0\|_{L^{p(x)}+L^{q(x)}} \leq 1$ and $\|\varphi_0\|_{L_V^{r(x)}} \leq 1$. By conditions (M1)-(M2), (H), (G2), relation (2.2), Propositions 2.1-2.2, Lemma 2.4-(iv), and Theorem 2.8, it follows that

$$\begin{aligned}
J_\lambda(\varphi_0) &= \mathcal{M}(\Phi_V(x,\varphi_0)) + \int_{\mathbb{R}^N} \frac{h(x)}{\alpha(x)}|\varphi_0|^{\alpha(x)}dx - \lambda\int_{\mathbb{R}^N} G(x,\varphi_0)dx \\
&\geq \frac{1}{\theta}\mathfrak{M}(\Phi_V(x,\varphi_0))\Phi_V(x,\varphi_0) - \lambda\int_{\mathbb{R}^N}\kappa(x)|\varphi_0|dx - \frac{\lambda}{\eta^-}\int_{\mathbb{R}^N}\rho(x)|\varphi_0|^{\eta(x)}dx \\
&\geq \frac{m_0}{\theta}\Big(c_1\max\Big\{\int_{\Lambda_{\nabla\varphi_0}}|\nabla\varphi_0|^{p(x)}dx, \int_{\Lambda^c_{\nabla\varphi_0}}|\nabla\varphi_0|^{q(x)}dx\Big\} + \frac{1}{r^+}\int_{\mathbb{R}^N}V(x)|\varphi_0|^{r(x)}dx\Big) \\
&\quad - \lambda\int_{\mathbb{R}^N}\kappa(x)|\varphi_0|dx - \frac{\lambda}{\eta^-}\int_{\mathbb{R}^N}\rho(x)|\varphi_0|^{\eta(x)}dx \\
&\geq \frac{m_0c_1}{\theta}\max\Big\{\min\{\|\nabla\varphi_0\|^{p^+}_{L^{p(x)}(\Lambda_{\nabla\varphi_0})}, \|\nabla\varphi_0\|^{p^-}_{L^{p(x)}(\Lambda_{\nabla\varphi_0})}\}, \\
&\quad \min\{\|\nabla\varphi_0\|^{q^+}_{L^{q(x)}(\Lambda^c_{\nabla\varphi_0})}, \|\nabla\varphi_0\|^{q^-}_{L^{q(x)}(\Lambda^c_{\nabla\varphi_0})}\}\Big\} + \frac{m_0}{\theta r^+}\min\{\|\varphi_0\|^{r^+}_{L_V^{r(x)}}, \|\varphi_0\|^{r^-}_{L_V^{r(x)}}\} \\
&\quad - 2\lambda\|\kappa\|_{L^{\eta'(x)}}\|\varphi_0\|_{L^{\eta(x)}} - \frac{\lambda}{\eta^-}\|\rho\|_{L^\infty}\max\Big\{\|\varphi_0\|^{\eta^+}_{L^{\eta(x)}}, \|\varphi_0\|^{\eta^-}_{L^{\eta(x)}}\Big\} \\
&\geq \frac{m_0c_1}{2^{q^+}\theta}\min\{\|\nabla\varphi_0\|^{p^+}_{L^{p(x)}+L^{q(x)}}, \|\nabla\varphi_0\|^{p^-}_{L^{p(x)}+L^{q(x)}}, \|\nabla\varphi_0\|^{q^+}_{L^{p(x)}+L^{q(x)}}, \|\nabla\varphi_0\|^{q^-}_{L^{p(x)}+L^{q(x)}}\} \\
&\quad + \frac{m_0}{\theta r^+}\min\{\|\varphi_0\|^{r^+}_{L_V^{r(x)}}, \|\varphi_0\|^{r^-}_{L_V^{r(x)}}\} - 2\lambda C_{17}\|\kappa\|_{L^{\eta'(x)}}\|\varphi_0\|_E - \frac{\lambda C_{18}\|\rho\|_{L^\infty}}{\eta^-}\|\varphi_0\|^{\eta^-}_E \\
&\geq \frac{m_0C_{19}}{\theta}\min\{\frac{c_1}{2^{q^+}}, \frac{1}{r^+}\}\|\varphi_0\|^{q^+}_E - 2\lambda C_{17}\|\kappa\|_{L^{\eta'(x)}}\|\varphi_0\|_E - \frac{\lambda C_{18}\|\rho\|_{L^\infty}}{\eta^-}\|\varphi_0\|^{\eta^-}_E.
\end{aligned} \tag{4.1}$$

As $q^+ < \eta^-$, we can choose $\|\varphi_0\|_E = \epsilon$ sufficiently small such that

$$C_\epsilon = \frac{m_0C_{19}}{\theta}\min\{\frac{c_1}{2^{q^+}}, \frac{1}{r^+}\}\epsilon^{q^+} - \frac{\lambda C_{18}\|\rho\|_{L^\infty}\epsilon^{\eta^-}}{\eta^-} > 0. \tag{4.2}$$

Let $\lambda_* = \frac{C_\epsilon}{4C_{17}\|\kappa\|_{L^{\eta'(x)}}\epsilon}$. Thus, for any $\lambda \in (0, \lambda_*]$, we have $J_\lambda(\varphi_0) \geq \frac{C_\epsilon}{2} = \vartheta > 0$.

(ii) Let $0 < \psi_0 \in C_c^\infty(\Omega)$ with $\|\psi_0\|_E = 1$. According to relations (2.2), (3.11), conditions (M2), (G1), and the fact that $1 \ll r(x) \leq p(x) \ll q(x)$ and $\alpha^+ < \theta q^+$, we deduce that for $t \geq 1$

$$\begin{aligned}
J_\lambda(t\psi_0) &= \mathcal{M}(\Phi_V(x,t\psi_0)) + \int_{\mathbb{R}^N}\frac{h(x)}{\alpha(x)}|t\psi_0|^{\alpha(x)}dx - \lambda\int_{\mathbb{R}^N}G(x,t\psi_0)dx \\
&\leq \mathcal{M}(1)\Big(1 + (\int_{\mathbb{R}^N}(\Phi(x,\nabla t\psi_0) + \frac{V(x)}{r(x)}|t\psi_0|^{r(x)})dx)^\theta\Big) \\
&\quad + \int_{\mathbb{R}^N}\frac{h(x)}{\alpha(x)}|t\psi_0|^{\alpha(x)}dx - \lambda\int_{\mathbb{R}^N}G(x,t\psi_0)dx \\
&\leq \mathcal{M}(1)\Big(1 + t^{\theta q^+}(\int_{\mathbb{R}^N}(\Phi(x,\nabla\psi_0) + \frac{V(x)}{r(x)}|\psi_0|^{r(x)})dx)^\theta\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{t^{\alpha^+}}{\alpha^-}\int_{\mathbb{R}^N} h(x)|\psi_0|^{\alpha(x)}dx-\lambda\int_{\mathbb{R}^N} G(x,t\psi_0)dx\\
&\le t^{\theta q^+}\Big[\mathcal{M}(1)\Big(1+\big(\int_{\mathbb{R}^N}(\Phi(x,\nabla\psi_0)+\frac{V(x)}{r(x)}|\psi_0|^{r(x)})dx\big)^{\theta}\Big)\\
&+\frac{t^{\alpha^+-\theta q^+}}{\alpha^-}\int_{\mathbb{R}^N} h(x)|\psi_0|^{\alpha(x)}dx-\lambda\int_D \frac{G(x,t\psi_0)}{t^{\theta q^+}}dx\Big]. \qquad (4.3)
\end{aligned}
$$

Thus, combining this with conditions (G1), (G7), and Fatou's lemma, it follows that $J_\lambda(t\psi_0)\to-\infty$ as $t\to\infty$. Hence, we can find a $t_0>1$ sufficiently large such that $\|t_0\psi_0\|_E=\|v\|_E>\epsilon$ and $J_\lambda(v)<0$. □

Proof of Theorems 1.3–1.8. By Lemma 4.1 and $J_\lambda(0)=0$, the functional J_λ satisfies the mountain pass geometry. Noting that (G7) implies (G3), Theorems 1.3, 1.4, 1.5, and 1.8 can be obtained from Lemmas 3.1, 3.2, 3.3, and 3.4, respectively, using the mountain pass theorem.

Next, we work on the proof of Theorems 1.9–1.11. Since E is a separable and reflexive real Banach space (see [67, Section 17, Theorems 2-3]), there exist $\{e_n\}\subset E$ and $\{f_n^*\}\subset E^*$ such that

$$E=\overline{\mathrm{span}\{e_n:n=1,2,\dots\}},\ \ E^*=\overline{\mathrm{span}\{f_n^*:n=1,2,\dots\}},$$

and

$$\langle f_i^*,e_j\rangle=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

Define

$$E_n=\mathrm{span}\{e_n\},\ Y_k=\oplus_{n=1}^k E_n,\ Z_k=\overline{\oplus_{n=k}^\infty E_n}.$$

Lemma 4.2. *Denote* $\zeta_k=\sup\{\|v\|_{L^{\eta(x)}}:v\in Z_k,\ \|v\|_E=1\}$. *Then* $\lim_{k\to\infty}\zeta_k=0$.

Proof. In view of Theorem 2.8, the proof is essentially similar to that of [68, Lemma 4.9]. □

Lemma 4.3. [69] *Let E be a real reflexive Banach space, $J_\lambda\in C^1(E,\mathbb{R})$ satisfies the (C)-condition and is even functional. If for each sufficiently large $k\in\mathbb{N}$, there exists $\xi_k>\delta_k>0$ such that*

(i) $h_k:=\inf\{J_\lambda(v):v\in Z_k,\|v\|_E=\delta_k\}\to+\infty$ *as* $k\to+\infty$,

(ii) $m_k:=\max\{J_\lambda(v):v\in Y_k,\|v\|_E=\xi_k\}\le 0$.

Then J_λ has a sequence of critical points $\{v_n\}$ such that $J_\lambda(v_n)\to+\infty$.

Lemma 4.4. *If conditions* (Φ1), (V), (H), (M1)-(M2), (G1)–(G3) *are satisfied, then relations* (i) *and* (ii) *in* Lemma 4.3 *hold true.*

Proof. (i) For any $v\in Z_k$ with $\|v\|_E=\delta_k>1$, we deduce from conditions (M1)-(M2), (H), (G2), relation (2.2), Proposition 2.1-2.2, Lemma 2.4-(iv), Theorem 2.8, and Lemma 4.2 that

$$
\begin{aligned}
J_\lambda(v)&=\mathcal{M}(\mathit{\Phi}_V(x,v))+\int_{\mathbb{R}^N}\frac{h(x)}{\alpha(x)}|v|^{\alpha(x)}dx-\lambda\int_{\mathbb{R}^N}G(x,v)dx\\
&\ge\frac{1}{\theta}\mathfrak{M}(\mathit{\Phi}_V(x,v))\mathit{\Phi}_V(x,v)-\lambda\int_{\mathbb{R}^N}\kappa(x)|v|dx-\frac{\lambda}{\eta^-}\int_{\mathbb{R}^N}\rho(x)|v|^{\eta(x)}dx
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{m_0}{\theta}\Big(c_1\Big(\int_{\Lambda_{\nabla v}}|\nabla v|^{p(x)}dx+\int_{\Lambda^c_{\nabla v}}|\nabla v|^{q(x)}dx\Big)+\frac{1}{r^+}\int_{\mathbb{R}^N}V(x)|v|^{r(x)}dx\Big)\\
&\quad-\lambda\int_{\mathbb{R}^N}\kappa(x)|v|dx-\frac{\lambda}{\eta^-}\int_{\mathbb{R}^N}\rho(x)|v|^{\eta(x)}dx\\
&\geq C_{20}\|v\|_E^{r^-}-2\lambda C_{21}\|k\|_{L^{\eta'(x)}}\|v\|_E-\frac{\lambda}{\eta^-}\int_{\mathbb{R}^N}\rho(x)|v|^{\eta(x)}dx-C_{22}\\
&\geq\begin{cases}C_{20}\|v\|_E^{r^-}-2\lambda C_{21}\|k\|_{L^{\eta'(x)}}\|v\|_E-\dfrac{\lambda\|\rho\|_{L^\infty}}{\eta^-}-C_{22}, & \|v\|_{L^{\eta(x)}}\leq 1\\[2ex] C_{20}\|v\|_E^{r^-}-2\lambda C_{21}\|k\|_{L^{\eta'(x)}}\|v\|_E-\dfrac{\lambda\|\rho\|_{L^\infty}}{\eta^-}\zeta_k^{\eta^+}\|v\|_E^{\eta^+}-C_{22}, & \|v\|_{L^{\eta(x)}}>1\end{cases}\\
&\geq C_{20}\|v\|_E^{r^-}-2\lambda C_{21}\|k\|_{L^{\eta'(x)}}\|v\|_E-\frac{\lambda\|\rho\|_{L^\infty}}{\eta^-}\zeta_k^{\eta^+}\|v\|_E^{\eta^+}-\frac{\lambda\|\rho\|_{L^\infty}}{\eta^-}-C_{22}.
\end{aligned}
$$

Choosing $\delta_k=(\lambda\|\rho\|_{L^\infty}\zeta_k^{\eta^+}\eta^+/C_{20}\eta^-)^{\frac{1}{r^--\eta^+}}$, then we have

$$
J_\lambda(v)\geq\delta_k^{r^-}\Big(C_{20}(1-\frac{1}{\eta^+})-2\lambda C_{21}\|k\|_{L^{\eta'(x)}}\delta_k^{1-r^-}\Big)-\frac{\lambda\|\rho\|_{L^\infty}}{\eta^-}-C_{22}.
$$

Because $1<r^-<\eta^+$ and $\delta_k\to+\infty$, it follows that

$$
\inf_{v\in Z_k,\ \|v\|_E=\delta_k}J_\lambda(v)\to+\infty\ \text{as}\ k\to+\infty.
$$

(ii) Suppose that relation (ii) does not hold for some given k. Then there exists a sequence $\{v_n\}\subset Y_k$ such that $\|v_n\|_E\to\infty$ and $J_\lambda(v_n)\geq 0$. Let $z_n=\frac{v_n}{\|v_n\|_E}$. Obviously, $\{z_n\}\subset E$ and $\|z_n\|_E=1$. Since $Y_k<\infty$, up to a subsequence, there exists $z\in Y_k\setminus\{0\}$ such that

$$
\|z_n-z\|_E\to 0\ \text{in}\ E,\ \ \text{and}\ \ z_n(x)\to z(x)\ \ \text{a.e.}\ x\in\mathbb{R}^N. \tag{4.4}
$$

If $z(x)\neq 0$ then $|v_n(x)|\to\infty$ as $n\to\infty$. In the same way as the proof of relations (3.10)–(3.13), we get a contradiction. Thus, relation (ii) holds true. □

Proof of Theorems 1.9–1.11. According to (Φ1)-(i) and (G9), J_λ is even functional. Using Lemmas 4.3-4.4, we merely need to show that the functional J_λ fulfills the (C)-condition. Therefore, Theorems 1.9, 1.10, and 1.11 follow from Lemmas 3.1, 3.2, and 3.3, respectively.

5. Conclusions

In this paper, we establish multiple results of solutions to the Kirchhoff-double phase problem using the mountain pass theorem and the fountain theorem. To the best of our knowledge, the present paper is the first attempt to deal with the combined problem of the Kirchhoff term and the general variable exponent double phase operator. Especially, our assumptions on the nonlinear term g are different from the previous related works and do not satisfy the classical (AR)-condition. Therefore, we need to use some new analytical tricks to ensure the boundedness of the Cerami sequence. Our results in this article improve and generalize the related ones in the literature. In addition, condition (Φ1)-(iv) means that our results are established in a subcritical setting. Therefore, a new research direction closely related to problem (1.1) is to replace (Φ1)-(iv) with the following critical condition: $1\ll p(x)\ll q(x)$ and $A=\Big\{x\in\mathbb{R}^N: q(x)=\min\{N,p^*(x)\}\Big\}\neq\emptyset$.

Author contributions

W. Ma: Writing–original draft; Q. F. Zhang: Supervision, Writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. Q. H. Zhang, V. D. Rădulescu, Double phase anisotropic variational problems and combined effects of reaction and absorption terms, *J. Math. Pures Appl.*, **118** (2018), 159–203. http://doi.org/10.1016/j.matpur.2018.06.015
2. G. W. Dai, R. F. Hao, Existence of solutions for a $p(x)$-Kirchhoff-type equation, *J. Math. Anal. Appl.*, **359** (2009), 275–284. http://doi.org/10.1016/j.jmaa.2009.05.031
3. J. Lee, J. M. Kim, Y. H. Kim, Existence and multiplicity of solutions for Kirchhoff-Schrödinger type equations involving $p(x)$-Laplacian on the entire space $\mathbb{R}^N$, *Nonlinear Anal.-Real World Appl.*, **45** (2019), 620–649. http://doi.org/10.1016/j.nonrwa.2018.07.016
4. X. C. Hu, H. B. Chen, Multiple positive solutions for a $p(x)$-Kirchhoff problem with singularity and critical exponent, *Mediterr. J. Math.*, **20** (2023), 200. http://doi.org/10.1007/s00009-023-02314-4
5. Y. P. Zhang, D. D. Qin, Existence of solutions for a critical Choquard-Kirchhoff problem with variable exponents, *J. Geom. Anal.*, **33** (2023), 200. http://doi.org/10.1007/s12220-023-01266-1
6. V. V. Zhikov, On Lavrentiev's phenomenon, *Russ. J. Math. Phys.*, **3** (1995), 249–269.
7. V. Bögelein, F. Duzaar, P. Marcellini, Parabolic equations with p, q-growth, *J. Math. Pures Appl.*, **100** (2013), 535–563. http://doi.org/10.1016/j.matpur.2013.01.012
8. V. V. Zhikov, On some variational problems, *Russ. J. Math. Phys.*, **5** (1997), 105–116.
9. V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Math. USSR Izvestiya*, **29** (1987), 33–66. https://doi.org/10.1070/im1987v029n01abeh000958
10. P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Differ. Equations*, **90** (1991), 1–30. https://doi.org/10.1016/0022-0396(91)90158-6

11. P. Marcellini, Regularity of minimizers of integrals of the calculus of variations with nonstandard growth conditions, *Arch. Ration. Mech. Anal.*, **105** (1989), 267–284. https://doi.org/10.1007/BF00251503

12. P. Baroni, M. Colombo, G. Mingione, Harnack inequalities for double phase functionals, *Nonlinear Anal.-Theory Methods Appl.*, **121** (2015), 206–222. https://doi.org/10.1016/j.na.2014.11.001

13. M. Colombo, G. Mingione, Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.*, **215** (2015), 443–496. https://doi.org/10.1007/s00205-014-0785-2

14. P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, *Calc. Var.*, **57** (2018), 62. https://doi.org/10.1007/s00526-018-1332-z

15. P. Baroni, M. Colombo, G. Mingione, Non-autonomous functionals, borderline cases and related function classes, *St. Petersburg Math. J.*, **27** (2016), 347–379. https://doi.org/10.1090/spmj/1392

16. F. Colasuonno, M. Squassina, Eigenvalues for double phase variational integrals, *Ann. Mat. Pura Appl.*, **195** (2016), 1917–1959. https://doi.org/10.1007/s10231-015-0542-7

17. A. Azzollini, P. d'Avenia, A. Pomponio, Quasilinear elliptic equations in $\mathbb{R}^N$ via variational methods and Orlicz-Sobolev embeddings, *Calc. Var.*, **49** (2014), 197–213. https://doi.org/10.1007/s00526-012-0578-0

18. N. Chorfi, V. D. Rădulescu, Standing wave solutions of a quasilinear degenerate Schrödinger equation with unbounded potential, *Electron. J. Qual. Theory Differ. Equ.*, **37** (2016), 1–12. https://doi.org/10.14232/ejqtde.2016.1.37

19. X. Y. Shi, V. D. Rădulescu, D. D. Repovš, Q. H. Zhang, Multiple solutions of double phase variational problems with variable exponent, *Adv. Calc. Var.*, **13** (2020), 385–401. https://doi.org/10.1515/acv-2018-0003

20. J. J. Liu, P. Pucci, Existence of solutions for a double-phase variable exponent equation without the Ambrosetti-Rabinowitz condition, *Adv. Nonlinear Anal.*, **12** (2023), 20220292. https://doi.org/10.1515/anona-2022-0292

21. B. Ge, D. J. Lv, J. F. Lu, Multiple solutions for a class of double phase problem without the Ambrosetti-Rabinowitz conditions, *Nonlinear Anal.-Theory Methods Appl.*, **188** (2019), 294–315. https://doi.org/10.1016/j.na.2019.06.007

22. L. Gasiński, N. S. Papageorgiou, Constant sign and nodal solutions for superlinear double phase problems, *Adv. Calc. Var.*, **14** (2021), 613–626. https://doi.org/10.1515/acv-2019-0040

23. W. L. Liu, G. W. Dai, Existence and multiplicity results for double phase problem, *J. Differ. Equations*, **265** (2018), 4311–4334. https://doi.org/10.1016/j.jde.2018.06.006

24. L. Gasiński, P. Winkert, Sign changing solution for a double phase problem with nonlinear boundary condition via the Nehari manifold, *J. Differ. Equations*, **274** (2021), 1037–1066. https://doi.org/10.1016/j.jde.2020.11.014

25. I. H. Kim, Y. H. Kim, M. W. Oh, S. D. Zeng, Existence and multiplicity of solutions to concave-convex-type double-phase problems with variable exponent, *Nonlinear Anal.-Real World Appl.*, **67** (2022), 103627. https://doi.org/10.1016/j.nonrwa.2022.103627

26. S. D. Zeng, V. D. Rădulescu, P. Winkert, Double phase obstacle problems with variable exponent, *Adv. Differential Equations*, **27** (2022), 611–645. https://doi.org/10.57262/ade027-0910-611

27. Á. Crespo-Blanco, L. Gasiński, P. Harjulehto, P. Winkert, A new class of double phase variable exponent problems:existence and uniqueness, *J. Differ. Equations*, **323** (2022), 182–228. https://doi.org/10.1016/j.jde.2022.03.029

28. F. Vetro, P. Winkert, Constant sign solutions for double phase problems with variable exponents, *Appl. Math. Lett.*, **135** (2023), 108404. https://doi.org/10.1016/j.aml.2022.108404

29. K. Ho, P. Winkert, New embedding results for double phase problems with variable exponents and a priori bounds for corresponding generalized double phase problems, *Calc. Var.*, **62** (2023), 227. https://doi.org/10.1007/s00526-023-02566-8

30. J. Zhang, W. Zhang, V. D. Rădulescu, Double phase problems with competing potentials: concentration and multiplication of ground states, *Math. Z.*, **301** (2022), 4037–4078. https://doi.org/10.1007/s00209-022-03052-1

31. W. Zhang, J. Zhang, V. D. Rădulescu, Concentrating solutions for singularly perturbed double phase problems with nonlocal reaction, *J. Differ. Equations*, **347** (2023), 56–103. https://doi.org/10.1016/j.jde.2022.11.033

32. G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

33. A. Arosio, S. Panizzi, On the well-posedness of the kirchhoff string, *Trans. Amer. Math. Soc.*, **348** (1996), 305–330. https://doi.org/10.1090/S0002-9947-96-01532-2

34. S. Bernstein, Sur une classe d'équations fonctionnelles aux dérivées partielles, *Bull. Acad. Sci. URSS. Sér. Math. [Izv. Akad. Nauk SSSR]*, **4** (1940), 17–26.

35. J. Yang, H. B. Chen, Existence of constant sign and nodal solutions for a class of (p, q)-Laplacian-Kirchhoff problems, *J. Nonlinear Var. Anal.*, **7** (2023), 345–365. https://doi.org/10.23952/jnva.7.2023.3.02

36. X. Hu, Y. Y. Lan, Multiple solutions of Kirchhoff equations with a small perturbations, *J. Nonlinear Funct. Anal.*, **2022** (2022), 1–11. https://doi.org/10.23952/jnfa.2022.19

37. W. Chen, Z. W. Fu, Y. Wu, Positive solutions for nonlinear Schrödinger-Kirchhoff equations in $\mathbb{R}^3$, *Appl. Math. Lett.*, **104** (2020), 106274. https://doi.org/10.1016/j.aml.2020.106274

38. G. Autuori, P. Pucci, M. C. Salvatori, Global nonexistence for nonlinear Kirchhoff systems, *Arch. Rational Mech. Anal.*, **196** (2010), 489–516. https://doi.org/10.1007/s00205-009-0241-x

39. E. Azroul, A. Benkirane, M. Shimi, M. Srati, On a class of fractional $p(x)$-Kirchhoff type problems, *Appl. Anal.*, **100** (2021), 383–402. https://doi.org/10.1080/00036811.2019.1603372

40. M. K. Hamdani, A. Harrabi, F. Mtiri, D. D. Repovš, Existence and multiplicity results for a new $p(x)$-Kirchhoff problem, *Nonlinear Anal.-Theory Methods Appl.*, **190** (2020), 111598. https://doi.org/10.1016/j.na.2019.111598

41. C. S. Chen, J. C. Huang, L. H. Liu, Multiple solutions to the nonhomogeneous p-Kirchhoff elliptic equation with concave-convex nonlinearities, *Appl. Math. Lett.*, **26** (2013), 754–759. https://doi.org/10.1016/j.aml.2013.02.011

42. Q. F. Zhang, H. Xie, Y. R. Jiang, Ground state solutions of Pohožaev type for Kirchhoff type problems with general convolution nonlinearity and variable potential, *Math. Meth. Appl. Sci.*, **46** (2022), 11757–11779. https://doi.org/10.1002/mma.8559

43. V. V. Jikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer, Berlin, 1994. https://doi.org/10.1007/978-3-642-84659-5

44. M. Chipot, J. F. Rodrigues, On a class of nonlocal nonlinear elliptic problems, *ESAIM-M2AN*, **26** (1992), 447–467. https://doi.org/10.1051/m2an/1992260304471

45. M. Chipot, B. Lovat, Some remarks on nonlocal elliptic and parabolic problems, *Nonlinear Anal.-Theory Methods Appl.*, **30** (1997), 4619–4627. https://doi.org/10.1016/S0362-546X(97)00169-7

46. A. Fiscella, A. Pinamonti, Existence and multiplicity results for Kirchhoff type problems on a double phase setting, *Mediterr. J. Math.*, **20** (2023), 33. https://doi.org/10.1007/s00009-022-02245-6

47. R. Arora, A. Fiscella, T. Mukherjee, P. Winkert, On double phase Kirchhoff problems with singular nonlinearity, *Adv. Nonlinear Anal.*, **12** (2023), 20220312. https://doi.org/10.1515/anona-2022-0312

48. K. Ho, P. Winkert, Infinitely many solutions to Kirchhoff double phase problems with variable exponents, *Appl. Math. Lett.*, **145** (2023), 108783. https://doi.org/10.1016/j.aml.2023.108783

49. Y. Cheng, Z. B. Bai, Existence and multiplicity results for parameter Kirchhoff double phase problem with Hardy-Sobolev exponents, *J. Math. Phys.*, **65** (2024), 011506. https://doi.org/10.1063/5.0169972

50. J. V. C. Sousa, Existence of nontrivial solutions to fractional Kirchhoff double phase problems, *Comput. Appl. Math.*, **43** (2024), 93. https://doi.org/10.1007/s40314-024-02599-5

51. A. Fiscella, G. Marino, A. Pinamonti, S. Verzellesi, Multiple solutions for nonlinear boundary value problems of Kirchhoff type on a double phase setting, *Rev. Mat. Complut.*, **37** (2024), 205–236. https://doi.org/10.1007/s13163-022-00453-y

52. T. Isernia, D. D. Repovš, Nodal solutions for double phase Kirchhoff problems with vanishing potentials, *Asymptotic Anal.*, **124** (2021), 371–396. https://doi.org/10.3233/ASY-201648

53. J. X. Cen, C. Vetro, S. D. Zeng, A multiplicity theorem for double phase degenerate Kirchhoff problems, *Appl. Math. Lett.*, **146** (2023), 108803. https://doi.org/10.1016/j.aml.2023.108803

54. X. Y. Lin, X. H. Tang, Existence of infinitely many solutions for p-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.-Theory Methods Appl.*, **92** (2013), 72–81. https://doi.org/10.1016/j.na.2013.06.011

55. L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$, *Proc. R. Soc. Edinb. Sect. A-Math.*, **129** (1999), 787–809. https://doi.org/10.1017/S0308210500013147

56. S. B. Liu, On ground states of superlinear p-Laplacian equations in $\mathbb{R}^N$, *J. Math. Anal. Appl.*, **361** (2010), 48–58. https://doi.org/10.1016/j.jmaa.2009.09.016

57. Z. Tan, F. Fang, On superlinear $p(x)$-Laplacian problems without Ambrosetti and Rabinowitz condition, *Nonlinear Anal.-Theory Methods Appl.*, **75** (2012), 3902–3915. https://doi.org/10.1016/j.na.2012.02.010

58. J. M. Kim, Y. H. Kim, Multiple solutions to the double phase problems involving concave-convex nonlinearities, *AIMS Math.*, **8** (2023), 5060–5079. https://doi.org/10.3934/math.2023254

59. W. H. Xie, H. B. Chen, Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, *Math. Nachr.*, **291** (2018), 2476–2488. https://doi.org/10.1002/mana.201700059
60. A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349–381. https://doi.org/10.1016/0022-1236(73)90051-7
61. X. H. Tang, S. T. Chen, X. Y. Lin, J. S. Yu, Ground state solutions of Nehari-Pankov type for Schrödinger equations with local super-quadratic conditions, *J. Differ. Equations*, **268** (2020), 4663–4690. https://doi.org/10.1016/j.jde.2019.10.041
62. X. H. Tang, X. Y. Lin, J. S. Yu, Nontrivial solutions for Schrödinger equation with local super-quadratic conditions, *J. Dyn. Diff. Equat.*, **31** (2019), 369–383. https://doi.org/10.1007/s10884-018-9662-2
63. S. T. Chen, X. H. Tang, Existence and multiplicity of solutions for Dirichlet problem of $p(x)$-Laplacian type without the Ambrosetti-Rabinowitz condition, *J. Math. Anal. Appl.*, **501** (2021), 123882. https://doi.org/10.1016/j.jmaa.2020.123882
64. Q. F. Zhang, C. L. Gan, T. Xiao, Z. Jia, Some results of nontrivial solutions for Klein-Gordon-Maxwell systems with local super-quadratic conditions, *J. Geom. Anal.*, **31** (2021), 5372–5394. https://doi.org/10.1007/s12220-020-00483-2
65. B. H. Dong, Z. W. Fu, J. S. Xu, Riesz-Kolmogorov theorem in variable exponent Lebesgue spaces and its applications to Riemann-Liouville fractional differential equations, *Sci. China-Math.*, **61** (2018), 1807–1824. https://doi.org/10.1007/s11425-017-9274-0
66. X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424–446. https://doi.org/10.1006/jmaa.2000.7617
67. J. F. Zhao, *Structure theory of Banach spaces (in Chinese)*, Wuhan: Wuhan University Press, 1991.
68. X. L. Fan, Q. H. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.-Theory Methods Appl.*, **52** (2003), 1843–1852. https://doi.org/10.1016/S0362-546X(02)00150-5
69. C. O. Alves, S. B. Liu, On superlinear $p(x)$-Laplacian equations in $\mathbb{R}^N$, *Nonlinear Anal.-Theory Methods Appl.*, **73** (2010), 2566–2579. https://doi.org/10.1016/j.na.2010.06.033