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*Research article*

## Comparative analysis of sterile mosquito release strategies based on a population suppression model

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**Abstract:** In this paper, we proposed a model for suppressing mosquito populations, evaluating various release strategies for sterilized mosquitoes, including constant continuous release, open-loop control, closed-loop control, and composite control strategies. We established release amount thresholds for each strategy to ensure the ultimate extinction of the wild population. Through numerical simulations, we validated our theoretical results and assessed the efficacy of each strategy. The results indicated that the closed-loop control mechanism significantly enhanced population suppression effectiveness. Higher intensity control notably shortened the control duration, and strategies that incorporated shorter release periods and more frequent population assessments markedly reduced the cumulative release quantity. Furthermore, we found that, compared to closed-loop control, the composite control strategy reduced single release amounts but did not show advantages in other aspects at low control intensities. However, it significantly decreased the cumulative release quantity at high control intensities.

**Keywords:** population suppression; periodic release; closed-loop control; extinction

**Mathematics Subject Classification:** 37N25, 92B05, 92D25

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### 1. Introduction

Diseases transmitted by mosquitoes, including dengue, yellow fever, and the Zika virus, have historically represented a significant risk to health worldwide. These viruses, transmitted through mosquito bites, affect millions of people, particularly in tropical and subtropical regions. With the acceleration of global climate change and urbanization, the prevalence of mosquito-borne diseases is expanding, presenting new challenges to public health control. In controlling mosquito-borne diseases, the Sterile Insect Technique (SIT) has shown significant advantages [1–3]. SIT is a biological control method that involves releasing sterile mosquitoes, treated with radiation or other methods, to mate with wild ones. This results in offspring that cannot survive, thereby reducing the mosquito population.

Compared to traditional chemical control methods, this technique is more environmentally friendly and sustainable, and it does not negatively impact the ecosystem.

Currently, the development of mathematical models for controlling mosquito populations using SIT is advancing rapidly. A portion of these models is based on the traditional continuous and discrete dynamical models [4–10]. L. Cai et al. proposed several continuous dynamical models to describe the temporal evolution of mosquito populations under different release patterns of sterile ones [4] and analyzed the long-term dynamics and steady-state behaviors of mosquito populations. Subsequently, J. Li et al. further refined the population characteristics of mosquitoes, taking into account growth stages and the Allee effect in the modeling process [5, 6]. Considering that discrete models are more flexible and accurate in handling mutations and release strategies caused by SIT intervention, some researchers have studied the impact of sterile mosquito release on wild mosquito populations by constructing difference equation models [9, 10].

In SIT control models, the application of impulsive differential equations is crucial. They are capable of simulating the dynamic changes in mosquito populations, particularly during the abrupt shifts caused by SIT interventions. Recent research findings clearly indicate that the frequency and intensity of impulsive releases of sterile mosquitoes significantly influence the effectiveness of population control [11–14]. In SIT impulsive control models, both open-loop and closed-loop control models have been studied [1, 2, 15]. The open-loop control model does not consider the actual response of the mosquito population during implementation. The release plan for sterile ones is predetermined and not adjusted based on the actual population size. This type of research generally focuses on obtaining release thresholds that promote the extinction of wild mosquito populations [1, 12–14]. This approach is advantageous for its simplicity and predictability but lacks flexibility and cannot adapt to environmental changes or natural fluctuations in mosquito populations.

In contrast, closed-loop control models adjust the release of sterile ones based on real-time feedback. In such models, real-time monitoring data of the mosquito population is used to dynamically adjust the release plan, enabling more precise control of population numbers. The strengths of closed-loop control lie in its high adaptability and flexibility, effectively handling environmental changes and unpredictable population dynamics. In recent years, such models have been widely applied in various fields of biological control process research. For example, in the management of agricultural pests [16–19], measures such as spraying insecticides or releasing natural enemies are taken based on the feedback of the pest population. In fisheries management [20, 21], the time for catching or releasing new fish fry is determined based on the monitored fish density data. In addition, closed-loop control has been extensively utilized in disease control and drug management [22, 23]. For the SIT control model, M. Huang et al. developed and analyzed a model for mosquito population management featuring real-time feedback and impulsive releases of sterile mosquitoes in [15]. They verified that the wild population can be maintained below a preset threshold, and initiating the release of sterile mosquitoes early can yield improved control outcomes.

Closed-loop control with real-time feedback is complex and relies on dense data collection and real-time feedback to adjust the release of sterile mosquitoes. However, in practical applications, frequent and extensive data collection is both costly and impractical [24]. In contrast, models with sparse state feedback only require intermittent collection of key data on mosquito populations. This approach significantly reduces the need for data collection while maintaining effective control over mosquito population dynamics [2]. In such models, sparse data is utilized to guide the release strategy

of sterile mosquitoes, making the control process more economical and efficient. Although this method may reduce the fine-tuning capabilities of the control model, it offers a practical solution for resource-limited environments by lightening the load of monitoring and data processing.

In this study, we aim to build a new type of mosquito population suppression model based on previous research, examining both open-loop control and closed-loop control featuring sparse state feedback for the wild population under impulsive releases of sterile ones.

The structure of the paper is as follows: Section 2 introduces a new mosquito suppression model and propose three distinct control strategies. We then study the constant continuous release and open-loop control in Section 3, identifying the release threshold required to ultimately eradicate the wild mosquito population, laying the groundwork for subsequent research on closed-loop control with sparse state feedback. In Section 4, we explore how to determine the impulsive release amount of sterile ones under different sparsities of state feedback, ensuring the eradication of the wild mosquito populations. Additionally, we combine open-loop and closed-loop controls to establish the minimum release amount threshold. In Section 5, we present a series of numerical simulations to validate the theoretical results. Finally, we provide a brief conclusion in Section 6.

## 2. Model formulation

In [4], L. Cai and colleagues developed a mathematical model to describe the interaction between two types of mosquitoes as follows:

$$\begin{cases} \frac{dW(t)}{dt} = \frac{aW^2(t)}{W(t) + g(t)} - (\mu_1 + \xi_1(W(t) + g(t))) W(t), \\ \frac{dg(t)}{dt} = b(\cdot) - g(t)(\mu_2 + \xi_2(W(t) + g(t))). \end{cases} \quad (2.1)$$

Here,  $W(t)$  and  $g(t)$  represent the population density of wild and sterile mosquitoes at time  $t$ , respectively.  $a$  reflects the reproduction rate of the wild population, while  $\xi_i$  and  $\mu_i$ ,  $i = 1, 2$ , are parameters that measure the density dependent and independent death rates, respectively.  $b(\cdot)$  stands the release rate of sterile ones.

Building on this model, a substantial number of researchers have popularized and extended it, developing new models, studying different release modes, and exploring the mechanisms of wild mosquito population suppression [7, 10, 13, 14]. In model (2.1), we note the presence of intraspecific competition among both wild and sterile mosquitoes. The growth of wild mosquitoes encompasses several stages, including eggs, larvae, and pupae, all of which occur in aquatic environments. During these stages, significant intraspecific competition occurs due to limited living space. Once the pupae transform into adult mosquitoes and leave the aquatic environment, there is competition among adult mosquitoes for breeding sites (such as puddles). However, this form of competition is absent among adult sterile males. Hence, this aspect should be distinctly addressed when constructing the model.

In addition, the authors in [2] investigated a SIT control model with sex structure:

$$\begin{cases} \frac{dM}{dt} = r\rho \frac{FM}{M+\gamma M_s(t)} e^{-\beta(F+M)} - \delta_M M, \\ \frac{dF}{dt} = (1-r)\rho \frac{FM}{M+\gamma M_s(t)} e^{-\beta(F+M)} - \delta_F F, \\ \frac{dM_s}{dt} = \Lambda(\cdot) - \delta_s M_s, \end{cases} \quad (2.2)$$

where  $M$ ,  $F$  respectively represent the population of adult males and females, while  $M_s(t)$  denotes sterile males.  $\rho$  stands for the egg-laying rate,  $r$  is the proportion of males to females at birth, and  $\gamma$  quantifies the competitive ability of sterile males in mating scenarios.  $e^{-\beta(F+M)}$  describes intraspecific competition among adult mosquitoes, and  $\delta_i, i = M, F, s$ , are the natural death rates.  $\Lambda(\cdot)$  denotes the release of sterile males.

The above model distinguishes between adult wild mosquitoes by sex and uses an exponential term to depict the intraspecific competition among wild mosquitoes. Unlike model (2.1), it differentiates the growth stages of mosquitoes, specifically focusing on the adult mosquito population. However, under closed-loop control, it is necessary to assess and provide feedback on the mosquito population in the environment. According to the depiction in model (2.2), this process requires separate estimations and feedback for the numbers of male and female mosquitoes, which inevitably complicates the actual estimation process. A more streamlined approach would be to estimate the mosquito population without differentiating between males and females. For this purpose, and based on the above two models, we propose a mosquito population suppression model as follows:

$$\begin{cases} \frac{dW(t)}{dt} = \frac{\rho_1 W^2(t)}{W(t) + \gamma_1 M_s(t)} e^{-\beta W(t)} - \delta_W W(t), \\ \frac{dM_s(t)}{dt} = \Lambda(\cdot) - \delta_s M_s(t). \end{cases} \quad (2.3)$$

In this context,  $W$  and  $M_s$  correspond to adult wild mosquitoes and sterile male mosquitoes, respectively.  $\rho_1$  indicates the reproduction rate of the wild ones, while  $\delta_i, i = W, s$ , are natural death rates.  $e^{-\beta(W)}$  describes intraspecific competition among adult mosquitoes.

In this study, we delve into the conditions for the successful suppression of wild mosquito populations based on model (2.3), examining this through the lens of the asymptotic behavior of dynamical models. Our focus is on the release of sterile mosquitoes, and in alignment with practical operational norms, we evaluate three distinct release strategies:

**(R1) Constant Continuous Release Strategy:** this approach involves a steady, ongoing release of sterile mosquitoes.

**(R2) Open-loop Control Strategy:** here, sterile mosquitoes are periodically released, without adjusting for the current state of the mosquito population.

**(R3) Closed-loop Control Strategy with Sparse State Feedback:** this strategy is characterized by periodic releases that vary based on frequent assessments of the mosquito population and corresponding feedback mechanisms.

In the subsequent sections, we will meticulously examine the three release strategies (R1)–(R3) for sterile mosquitoes, utilizing model (2.3) as our analytical framework. Our objective is to ascertain the specific release thresholds required to achieve the ultimate eradication of the wild mosquito population under each of these distinct modes. Furthermore, we will conduct a comparative analysis of these release strategies, aiming to evaluate and highlight their relative effectiveness in controlling mosquito populations.

### 3. Constant continuous release and open-loop control

We consider the population development model of wild mosquitoes in the environment without releasing sterile mosquitoes:

$$\frac{dW(t)}{dt} = \rho_1 W(t)e^{-\beta W(t)} - \delta_W W(t). \quad (3.1)$$

Note that  $\frac{dW}{dt}|_{W=0} = 0$ , we can conclude that  $W(t) \geq 0, t \geq 0$  if  $W(0) \geq 0$ . Additionally, since  $W e^{-\beta W} \leq \frac{1}{\beta e}$  for  $W \geq 0$ , we can deduce that  $\frac{dW}{dt} \leq \frac{\rho_1}{\beta e} - \delta_W W$ , thus we know that

$$\Omega_1 = \{W | 0 \leq W \leq \frac{\rho_1}{\beta e \delta_W}\}$$

is forward-invariant and exhibits global attractivity for model (3.1) in  $R^+ = \{W | W \geq 0\}$ .

Obviously, model (3.1) always has an extinction equilibrium  $W_0 = 0$ . If  $\frac{\rho_1}{\delta_W} > 1$ , then model (3.1) has a unique positive equilibrium  $W^* = \frac{1}{\beta} \ln \frac{\rho_1}{\delta_W}$ . Denote

$$\mathcal{N}_W = \frac{\rho_1}{\delta_W},$$

and we usually call it the basic offspring number of wild mosquitoes.

For the stability of the equilibria of model (3.1), we have the following conclusion:

**Lemma 1.** Consider model (3.1). We have two cases:

- (i) If  $\mathcal{N}_W \leq 1$ , the extinction equilibrium  $W_0 = 0$  is globally asymptotically stable.
- (ii) If  $\mathcal{N}_W > 1$ , then the extinction equilibrium  $W_0 = 0$  becomes unstable, and there exists a unique positive equilibrium  $W^* = \frac{1}{\beta} \ln \frac{\rho_1}{\delta_W}$ , which is globally asymptotically stable.

*Proof.* (i) When  $\mathcal{N}_W < 1$ , the inequality  $\frac{dW}{dt} \leq \delta_W(\mathcal{N}_W - 1)W$  holds. This implies that  $W_0 = 0$  is locally asymptotically stable and  $\lim_{t \rightarrow \infty} W(t) = 0$ , confirming its global asymptotic stability.

In the scenario where  $\mathcal{N}_W = 1$ , we have  $\frac{dW}{dt} = \rho_1(e^{-\beta W} - 1)W$ . Since  $\frac{dW}{dt}|_{W=0} = 0$  and  $\frac{dW}{dt} < 0$  for  $W > 0$ ,  $W_0 = 0$  is globally attractive. To establish its local stability, let us choose a sufficiently small  $\eta > 0$ . For  $W(t) > \eta$ , it follows that  $\frac{dW}{dt} < \rho_1(e^{-\beta \eta} - 1)W < 0$ , indicating that  $W(t)$  is strictly monotonically decreasing. When  $W \leq \eta$ , we have

$$\frac{dW}{dt} = \rho_1(e^{-\beta W} - 1)W = \rho_1(-\beta W + O(W^2))W = -\beta \rho_1 W^2 + O(W^3),$$

thus,  $W_0 = 0$  is locally asymptotically stable on  $[0, \eta]$ . Overall, for  $\mathcal{N}_W = 1$ ,  $W_0 = 0$  is globally asymptotically stable.

(ii) When  $\mathcal{N}_W > 1$ , considering the linearized model  $\frac{dW}{dt} = (\rho_1(1 - \beta \bar{W})e^{-\beta \bar{W}} - \delta_W)W$  of model (3.1), where  $\bar{W} = W_0$  or  $W^*$ , it is evident that  $W_0 = 0$  is unstable and  $W^*$  is locally stable. We need to further demonstrate that  $\lim_{t \rightarrow \infty} W(t) = W^*$  for any  $W(t)$  with  $W(0) \in (0, \frac{\rho_1}{\beta e \delta_W}]$ . Since  $\Omega_1$  is a forward invariant set, both  $\underline{W} = \lim_{t \rightarrow \infty} \inf W(t) \geq 0$  and  $\bar{W} = \lim_{t \rightarrow \infty} \sup W(t)$  exist.

Choosing a sufficiently small  $\eta > 0$ , and considering  $\frac{dW}{dt} = (\rho_1 e^{-\beta W(t)} - \delta_W)W(t)$ , we find that for  $0 < W(t) \leq \eta$ :

$$\begin{aligned}\frac{dW}{dt} &= (\rho_1(1 - \beta W(t) + O(W^2(t))) - \delta_W)W(t) \\ &= [(\rho_1 - \delta_W) - \rho_1\beta W(t) + O(W^2(t))]W(t) \\ &= (\rho_1 - \delta_W)W(t) + O(W^2(t)).\end{aligned}$$

Given  $\rho_1 > \delta_W$ , it follows that for any  $W(0) > 0$  and  $W(t) \leq \eta$ ,  $W(t)$  is strictly monotonically increasing with respect to  $t$ . Hence, for any  $W(0) > 0$ ,  $\liminf_{t \rightarrow \infty} W(t) = \underline{W} \geq \eta > 0$ .

By applying the fluctuation lemma, there exists a monotonically increasing time series  $t_k$  with  $\lim_{k \rightarrow \infty} t_k = +\infty$ , such that  $\lim_{k \rightarrow \infty} W(t_k) = \underline{W}$  and  $W'(t_k) = 0$ . Taking the limit in (3.1) along  $\{t_k\}$  yields  $\rho_1 W(t_k) e^{-\beta W(t_k)} = \delta_W W(t_k)$ , leading to the conclusion that  $\rho_1 \underline{W} e^{-\beta \underline{W}} = \delta_W \underline{W}$ . Since  $\underline{W} > 0$ , we have  $\rho_1 e^{-\beta \underline{W}} = \delta_W$ , implying  $\underline{W} = W^*$ .

Moreover, there exists another monotonically increasing time series  $t_l$ , satisfying  $\lim_{l \rightarrow \infty} t_l = \infty$ , such that  $\lim_{l \rightarrow \infty} W(t_l) = \overline{W}$  and  $W'(t_l) = 0$ . Following a similar argument, we deduce that  $\overline{W} = W^*$ .

From the above analysis, we conclude that  $\liminf_{t \rightarrow \infty} W(t) = \limsup_{t \rightarrow \infty} W(t) = W^*$ , meaning  $\lim_{t \rightarrow \infty} W(t) = W^*$ . Therefore, the positive equilibrium  $W^*$  is globally asymptotically stable. This completes the proof.  $\square$

In subsequent discussions, we shall consistently assume that  $\mathcal{N}_W > 1$ .

### 3.1. Constant continuous release

In the following, we explore the constant continuous release of sterile mosquitoes, a method suitable for scenarios requiring long-term and stable control of mosquito populations. This approach is particularly relevant in environmentally sensitive areas, densely populated regions, or areas with persistent disease transmission risks. At this juncture, the release function  $\Lambda(\cdot) \equiv \Lambda$ , and model (2.3) becomes:

$$\begin{cases} \frac{dW(t)}{dt} = \rho_1 \frac{W^2}{W + \gamma_1 M_s} e^{-\beta W} - \delta_W W, \\ \frac{dM_s(t)}{dt} = \Lambda - \delta_s M_s(t). \end{cases} \quad (3.2)$$

Note that the second equation in model (3.2) operates independently of the wild mosquito population. Over time, the number of sterile mosquitoes in the environment stabilizes at  $M_s^* = \frac{\Lambda}{\delta_s}$ . Consequently, the limit form derived from model (3.2) is

$$\frac{dW}{dt} = \rho_1 \frac{W^2}{W + \gamma_1 M_s^*} e^{-\beta W} - \delta_W W. \quad (3.3)$$

Since our primary objective is to examine the asymptotic behavior of the wild mosquito population, so in the following we focus on studying model (3.3). The solution to model (3.3) remains positive and bounded. The extinction equilibrium point, denoted as  $W_0 = 0$ , persists. To discuss the positive equilibrium point, we examine the existence of a positive root for the following algebraic equation:

$$\mathcal{N}_W \frac{W}{W + \gamma_1 M_s^*} = e^{\beta W}. \quad (3.4)$$

Consider the functions  $f_1(W, \Lambda) = \mathcal{N}_W \frac{W}{W + \gamma_1 M_s^*}$  and  $f_2(W, \Lambda) = e^{\beta W}$ . It is evident that  $f_1(0, \Lambda) = 0$ ,  $f_1(+\infty, \Lambda) = \mathcal{N}_W > 1$ ,  $f_1'(W, \Lambda) > 0$ ,  $f_2(0, \Lambda) = 1$ ,  $f_2(+\infty, \Lambda) = +\infty$ , and  $f_2'(W, \Lambda) > 0$ . Therefore, there must exist a critical value  $\Lambda^{crit} > 0$  such that  $M_s^{*crit} = \frac{\Lambda^{crit}}{\delta_s}$  and at this critical point, the curves of  $f_1(W, \Lambda)$  and  $f_2(W, \Lambda)$  are tangent at a positive value  $W = W_c^*$ . Furthermore, if  $\Lambda > \Lambda^{crit}$ , then these two curves do not intersect for any  $W > 0$ . Conversely, if  $\Lambda < \Lambda^{crit}$ , they intersect at two distinct points  $W = W_1^*$  and  $W = W_2^*$ , indicating the existence of two possible positive equilibria under certain conditions.

To determine the threshold  $\Lambda^{crit}$ , we note that at the tangency point  $W = W_c^* > 0$ , when  $M_s^* = M_s^{*crit} = \frac{\Lambda^{crit}}{\delta_s}$ , the following conditions hold:  $f_1(W_c^*, \Lambda^{crit}) = f_2(W_c^*, \Lambda^{crit})$  and  $f_1'(W_c^*, \Lambda^{crit}) = f_2'(W_c^*, \Lambda^{crit})$ .

Direct calculation leads to

$$W_c^* = \frac{\gamma_1 M_s^{*crit}}{2} \left( -1 + \sqrt{1 + \frac{4}{\beta \gamma_1 M_s^{*crit}}} \right)$$

and

$$\frac{1}{W_c^*} = \frac{\beta}{2} \left( 1 + \sqrt{1 + \frac{4}{\beta \gamma_1 M_s^{*crit}}} \right).$$

From Eq (3.4), it is known that  $1 + \frac{\gamma_1 M_s^{*crit}}{W_c^*} = \mathcal{N}_W e^{-\beta W_c^*}$ . Substituting  $W_c^*$ ,  $\frac{1}{W_c^*}$  and  $M_s^{*crit} = \frac{\Lambda^{crit}}{\delta_s}$  into this equation, we obtain:

$$1 + \frac{\beta \gamma_1 \Lambda^{crit}}{\delta_s} \left( 1 + \sqrt{1 + \frac{4\delta_s}{\beta \gamma_1 \Lambda^{crit}}} \right) = \mathcal{N}_W \exp \left[ -\frac{1}{2} \frac{\beta \gamma_1 \Lambda^{crit}}{\delta_s} \left( 1 + \sqrt{1 + \frac{4\delta_s}{\beta \gamma_1 \Lambda^{crit}}} \right) \right]. \quad (3.5)$$

The function on the left side of Eq (3.5) is monotonically increasing with respect to  $\Lambda^{crit}$  over the range  $[0, \infty)$ , starting from a minimum value of 1 and tending towards infinity as  $\Lambda^{crit}$  increases. The function on the right side, conversely, is monotonically decreasing, with a maximum value of  $\mathcal{N}_W$  and tending towards 0. Given that  $\mathcal{N}_W > 1$ , Eq (3.5) has a unique positive root  $\Lambda^{crit} > 0$ . Thus, we can state the following lemma:

**Lemma 2.** *Given  $\mathcal{N}_W > 1$ , there is a critical threshold  $\Lambda^{crit} > 0$  beyond which model (3.3) supports two positive equilibria  $0 < W_1^* < W_2^*$  for  $0 < \Lambda < \Lambda^{crit}$ , a singular positive equilibrium  $W_c^*$  at  $\Lambda = \Lambda^{crit}$ , and no positive equilibrium for  $\Lambda > \Lambda^{crit}$ . Here,  $\Lambda^{crit}$  is the uniquely positive root of Eq (3.5).*

**Theorem 1.** *Given  $\mathcal{N}_W > 1$ , for model (3.3), we derive the following conclusions:*

- (i) *When  $\Lambda > \Lambda^{crit}$ ,  $W_0 = 0$  is globally asymptotically stable.*
- (ii) *When  $\Lambda < \Lambda^{crit}$ ,  $W_0 = 0$  and the higher positive equilibrium  $W_2^*$  are locally asymptotically stable, whereas  $W_1^*$  is unstable.*
- (iii) *When  $\Lambda = \Lambda^{crit}$ ,  $W_0 = 0$  maintains local asymptotic stability, while the only positive equilibrium  $W_c^*$  is semi-stable.*

*Proof.* The linearized model of (3.3) at equilibrium  $\tilde{W}$  is given by

$$\frac{dW}{dt} \Big|_{\tilde{W}} = \left[ \rho_1 \left( \frac{2\tilde{W}}{\tilde{W} + \gamma_1 M_s^*} - \frac{\beta \tilde{W}^2}{\tilde{W} + \gamma_1 M_s^*} - \frac{\tilde{W}^2}{(\tilde{W} + \gamma_1 M_s^*)^2} \right) e^{-\beta \tilde{W}} - \delta_W \right] W. \quad (3.6)$$

(i) For  $\tilde{W} = W_0$ , model (3.6) simplifies to  $\frac{dW}{dt} = -\delta_W W$ . Clearly,  $W_0 = 0$  is always locally asymptotically stable. To demonstrate its global attractiveness, consider the following deduction from (3.3):

$$\frac{dW}{dt} = \rho_1 \frac{W^2}{W + \gamma_1 M_s^*} e^{-\beta W} - \delta_W W = \frac{\delta_W W^2}{W + \gamma_1 M_s^*} \left( \mathcal{N}_W e^{-\beta W} - 1 - \frac{\gamma_1 M_s^*}{W} \right).$$

Observe that

$$\mathcal{N}_W e^{-\beta W} - 1 - \frac{\gamma_1 M_s^*}{W} = \frac{W + \gamma_1 M_s^*}{W} e^{-\beta W} (f_1(W, \Lambda) - f_2(W, \Lambda)).$$

Based on Lemma 2, if  $\Lambda > \Lambda^{crit}$ , then  $f_1(W, \Lambda) - f_2(W, \Lambda) < 0$  for  $W \geq 0$ . Thus,  $1 + \frac{\gamma_1 M_s^*}{W} - \mathcal{N}_W e^{-\beta W} > 0$  for  $W \geq 0$ . Let  $\eta_1 = \min_{W \geq 0} \left( 1 + \frac{\gamma_1 M_s^*}{W} - \mathcal{N}_W e^{-\beta W} \right) > 0$ , then  $\frac{dW}{dt} \leq -\frac{\eta_1 \delta_W W^2}{W + \gamma_1 M_s^*}$ . Consider the comparison model

$$\frac{dX}{dt} = -\frac{\eta_1 \delta_W X^2}{X + \gamma_1 M_s^*}, \quad X(0) = W(0), \quad (3.7)$$

leading to the inequality  $0 \leq W(t) \leq X(t)$ . Solving Eq (3.7) yields

$$\ln X(t) - \frac{\gamma_1 M_s^*}{X(t)} = \ln X(0) + \frac{\gamma_1 M_s^*}{X(0)} - \eta_1 \delta_W t. \quad (3.8)$$

Define  $g(X) \doteq \ln X - \frac{\gamma_1 M_s^*}{X}$ . It is straightforward to show that  $g'(X) = \frac{1}{X} + \frac{\gamma_1 M_s^*}{X^2} > 0$  for all  $X > 0$ , and  $\lim_{X \rightarrow 0} g(X) = -\infty$ ,  $\lim_{X \rightarrow +\infty} g(X) = +\infty$ . Additionally, (3.8) implies  $\lim_{t \rightarrow \infty} g(X(t)) = -\infty$ , suggesting  $\lim_{t \rightarrow \infty} X(t) = 0$ . Since  $0 \leq W(t) \leq X(t)$ , it follows that  $\lim_{t \rightarrow \infty} W(t) = 0$ , confirming that  $W_0 = 0$  is globally attractive.

(ii) When  $\tilde{W} = W_i^*$ ,  $i = 1, 2$ , the coefficient of the linear term in (3.6) is

$$J(W_i^*) = \rho_1 \left( \frac{2W_i^*}{W_i^* + \gamma_1 M_s^*} - \frac{\beta(W_i^*)^2}{W_i^* + \gamma_1 M_s^*} - \frac{(W_i^*)^2}{(W_i^* + \gamma_1 M_s^*)^2} \right) e^{-\beta W_i^*} - \delta_W. \quad (3.9)$$

Given that  $\rho_1 \frac{W_i^*}{W_i^* + \gamma_1 M_s^*} e^{-\beta W_i^*} = \delta_W$ , it follows that

$$\begin{aligned} J(W_i^*) &= \delta_W \left( 1 - \beta W_i^* - \frac{W_i^*}{W_i^* + \gamma_1 M_s^*} \right) \\ &= \delta_W (W_i^* + \gamma_1 M_s^*) \left( \frac{\gamma_1 M_s^*}{(W_i^* + \gamma_1 M_s^*)^2} - \frac{\beta W_i^*}{W_i^* + \gamma_1 M_s^*} \right) \\ &= \frac{\delta_W}{\mathcal{N}_W} (W_i^* + \gamma_1 M_s^*) \left( \mathcal{N}_W \frac{\gamma_1 M_s^*}{(W_i^* + \gamma_1 M_s^*)^2} - \beta e^{-\beta W_i^*} \right) \\ &= \frac{\delta_W}{\mathcal{N}_W} (W_i^* + \gamma_1 M_s^*) (f'_{1W}(W_i^*, \Lambda) - f'_{2W}(W_i^*, \Lambda)). \end{aligned}$$

Referring to Lemma 2, it is evident that  $f'_{1W}(W_1^*, \Lambda) - f'_{2W}(W_1^*, \Lambda) > 0$  and  $f'_{1W}(W_2^*, \Lambda) - f'_{2W}(W_2^*, \Lambda) < 0$  when  $\Lambda < \Lambda^{crit}$ . Consequently,  $J(W_1^*) > 0$  and  $J(W_2^*) < 0$ , indicating that  $W_1^*$  is unstable, whereas  $W_2^*$  exhibits local asymptotic stability.

(iii) When  $\Lambda = \Lambda^{crit}$ , it is deduced from the discussion in (ii) that the two positive equilibria  $W_1^*$  and  $W_2^*$  converge to establish a singular positive equilibrium  $W_c^*$ . From the stability of  $W_1^*$  and  $W_2^*$ , we have  $W_c^*$  is semi-stable. This completes the proof.  $\square$



### 3.2. Open-loop control

In scenarios where the mosquito population remains relatively stable and environmental conditions do not vary significantly, or in cases where data collection capabilities are constrained, thus hindering real-time monitoring of population densities, the adoption of an open-loop control strategy is advisable. To address this, we consider the following periodic pulse release model:

$$\begin{cases} \frac{dW(t)}{dt} = \rho_1 \frac{W^2(t)}{W(t) + \gamma_1 M_s(t)} e^{-\beta W(t)} - \delta_W W(t), \\ \frac{dM_s(t)}{dt} = -\delta_s M_s(t), \quad t \neq n\tau, \quad n = 0, 1, 2, \dots, \\ W(n\tau^+) = W(n\tau), M_s(n\tau^+) = M_s(n\tau) + \Lambda_n, \quad t = n\tau, \end{cases} \quad (3.10)$$

where  $\tau$  is the release period and  $\Lambda_n$  is the release amount for the  $n$ th release.

For simplicity, let's first consider the scenario with a fixed release amount, that is,  $\Lambda_n \equiv \Lambda$ . Then from the last two equations in (3.10), we have

$$\begin{cases} \frac{dM_s(t)}{dt} = -\delta_s M_s(t), \quad t \neq n\tau, \quad n = 0, 1, 2, \dots, \\ M_s(n\tau^+) = M_s(n\tau) + \Lambda, \quad t = n\tau. \end{cases} \quad (3.11)$$

It is evident that model (3.11) has a periodic solution  $M_s^p(t) = \frac{\Lambda e^{-\delta_s(t-n\tau)}}{1 - e^{-\delta_s\tau}}$   $t \in (n\tau, (n+1)\tau]$ ,  $n = 0, 1, 2, \dots$ , which is globally asymptotically stable, that is,  $\lim_{t \rightarrow \infty} M_s(t) = M_s^p(t)$ . Therefore, a limit form of (3.10) can be considered

$$\frac{dW}{dt} = \rho_1 \frac{W^2(t)}{W(t) + \gamma_1 M_s^p(t)} e^{-\beta W(t)} - \delta_W W(t). \quad (3.12)$$

Obviously, model (3.12) has a unique equilibrium  $W_0 = 0$ . In the following, we discuss its stability.

Denote

$$\left\langle \frac{1}{M_s^p} \right\rangle = \frac{1}{\tau} \int_0^\tau \frac{1}{M_s^p(t)} dt = \frac{e^{\delta_s\tau} + e^{-\delta_s\tau} - 2}{\delta_s\tau\Lambda} = \frac{2(\cosh(\delta_s\tau) - 1)}{\delta_s\tau\Lambda}. \quad (3.13)$$

**Theorem 2.** For any  $\tau > 0$ , if

$$\Lambda > \Lambda_p^{crit} = \frac{2\rho_1(\cosh(\delta_s\tau) - 1)}{e\beta\gamma_1\tau\delta_s\delta_W}, \quad (3.14)$$

then the unique equilibrium  $W_0 = 0$  of (3.12) is globally asymptotically stable.

*Proof.* Noting  $We^{-\beta W} \leq \frac{1}{\beta e}$  for  $W \geq 0$ , we can deduce that for  $t \in (n\tau, (n+1)\tau]$ ,

$$\frac{dW}{dt} \leq W(t) \left( \frac{\rho_1}{e\beta(W(t) + \gamma_1 M_s^p(t))} - \delta_W \right) \leq W(t) \left( \frac{\rho_1}{e\beta\gamma_1 M_s^p(t)} - \delta_W \right)$$

holds. Thus, we get

$$W(t) \leq e^{\int_{n\tau}^t \left( \frac{\rho_1}{e\beta\gamma_1 M_s^p(s)} - \delta_W \right) ds} W(n\tau), \quad t \in (n\tau, (n+1)\tau],$$

and

$$W((n+1)\tau) \leq e^{\left( \frac{\rho_1}{e\beta\gamma_1} \left\langle \frac{1}{M_s^p} \right\rangle - \delta_W \right) \tau} W(n\tau).$$

If  $\frac{\rho_1}{e\beta\gamma_1} \left\langle \frac{1}{M_s^p} \right\rangle < \delta_W$ , i.e.,  $\Lambda > \Lambda_p^{crit}$ , then  $W((n+1)\tau) < W(n\tau)$ ,  $n \in \mathbb{N}$  and  $\lim_{t \rightarrow \infty} W(t) = 0$ , which implies that the extinction equilibrium  $W_0 = 0$  is globally asymptotically stable. This completes the proof.  $\square$

Based on the conclusion in Theorem 1, we can conclude that in the scenario of periodic pulse release, if

$$\inf_{t \in [0, \tau]} M_s^p(t) = \frac{\Lambda e^{-\delta_s \tau}}{1 - e^{-\delta_s \tau}} > M_s^{*crit} = \frac{\Lambda^{crit}}{\delta_s},$$

i.e.,  $\Lambda > \Lambda^{crit} \frac{e^{\delta_s \tau} - 1}{\delta_s}$ , then  $W_0 = 0$  is also globally asymptotically stable.

#### 4. Closed-loop control

In environments where mosquito populations experience significant fluctuations, managers with adequate monitoring capabilities can implement closed-loop control strategies for population management. Depending on the challenges associated with population monitoring and data processing, varying degrees of sparsity in population estimation and data feedback may be considered. Consequently, the release function  $\Lambda(\cdot)$  changes accordingly. At this juncture, the development model of the wild mosquito population can be expressed as follows:

$$\frac{dW}{dt} = \rho_1 \frac{W^2(t)}{W(t) + \gamma_1 M_s(t)} e^{-\beta W(t)} - \delta_W W(t). \quad (4.1)$$

##### 4.1. Periodic state feedback control

Initially, it is assumed that the mosquito population in the environment is estimated periodically, every  $\tau$  time units. This estimation frequency aligns with the release schedule of sterile mosquitoes. Based on the estimated population value  $W(n\tau)$  for the  $n$ th period, the release quantity  $\Lambda_n$  for that period is determined, aiming to drive the wild mosquito population towards eventual extinction.

Considering the evolution model (3.1) of wild mosquitoes, the basic offspring number of the wild mosquito population is denoted as  $\mathcal{N}_W = \frac{\rho_1}{\delta_W}$ . The introduction of sterile mosquitoes into the environment alters this basic offspring number, which can be estimated as  $\mathcal{N}_W \frac{W(t)}{W(t) + \gamma_1 M_s(t)}$ . By choosing  $\varepsilon \in (0, \frac{1}{\mathcal{N}_W})$ , we can show that the wild mosquito population will eventually become extinct if

$$\frac{W(t)}{W(t) + \gamma_1 M_s(t)} \leq \varepsilon \quad (4.2)$$

holds for all  $t \geq 0$ .

**Lemma 3.** Assume that  $W(t)$  is any solution of model (4.1). If inequality (4.2) holds for all  $t \geq 0$ , then  $W(t)$  converges exponentially to  $W_0 = 0$ .

*Proof.* According to Eq (4.1), if inequality (4.2) is satisfied, then it follows:

$$\frac{dW(t)}{dt} \leq (\rho_1 \varepsilon e^{-\beta W(t)} - \delta_W) W(t) \leq (\rho_1 \varepsilon - \delta_W) W(t),$$

then we consider the following comparison model:

$$\frac{dW_1(t)}{dt} = (\rho_1 \varepsilon - \delta_W) W_1(t), \quad W_1(0) = W(0) \geq 0. \quad (4.3)$$

Since  $\varepsilon < \frac{1}{N_W} = \frac{\delta_W}{\rho_1}$ , so  $\rho_1\varepsilon - \delta_W < 0$ , and any solution  $W_1(t)$  of (4.3) will converge exponentially to  $W_0 = 0$ . From the comparison theorem, it follows that  $0 \leq W(t) \leq W_1(t)$ , and  $W(t)$  converges exponentially to  $W_0 = 0$ . This completes the proof.  $\square$

Next, we will focus on determining the appropriate release quantity, denoted as  $\Lambda_n$ , for each period, where  $n = 1, 2, \dots$ . This is essential to ensure the persistent satisfaction of inequality (4.2). Given the impulsive form of sterile mosquito releases, the population  $M_s(t)$  over the interval  $(n\tau, (n+1)\tau]$  is described by

$$M_s(t) = M_s(n\tau^+)e^{-\delta_s(t-n\tau)} = (M_s(n\tau) + \Lambda_n)e^{-\delta_s(t-n\tau)}. \quad (4.4)$$

To maintain inequality (4.2), it is necessary to ensure that  $\gamma_1 M_s(t) \geq \frac{1-\varepsilon}{\varepsilon} W(t)$ . From the proof of Lemma 3, we understand that  $W(t) \leq W_1(t)$  for  $t \in (n\tau, (n+1)\tau]$ , where  $W_1(t)$  is the solution to the comparison Eq (4.3) with the initial condition  $W_1(n\tau) = W(n\tau)$ . Therefore, if

$$\gamma_1 M_s(t) \geq \frac{1-\varepsilon}{\varepsilon} W_1(t) \quad (4.5)$$

is satisfied, then  $\gamma_1 M_s(t) \geq \frac{1-\varepsilon}{\varepsilon} W(t)$  naturally follows.

With  $W(n\tau) = W_1(n\tau)$  on  $[n\tau, (n+1)\tau]$ , we have

$$W_1(t) = W(n\tau)e^{-(\delta_W - \rho_1\varepsilon)(t-n\tau)}, \quad t \in [n\tau, (n+1)\tau],$$

which implies inequality (4.5) is equivalent to

$$\gamma_1 M_s(t) \geq \frac{1-\varepsilon}{\varepsilon} e^{-(\delta_W - \rho_1\varepsilon)(t-n\tau)} W(n\tau), \quad t \in (n\tau, (n+1)\tau].$$

From (4.4), we conclude that if

$$\Lambda_n \geq -M_s(n\tau) + \frac{1-\varepsilon}{\gamma_1\varepsilon} e^{(\delta_s - \delta_W + \rho_1\varepsilon)(t-n\tau)} W(n\tau) \quad (4.6)$$

is fulfilled over  $[n\tau, (n+1)\tau]$ , then  $\gamma_1 M_s(t) \geq \frac{1-\varepsilon}{\varepsilon} W(t)$  for all  $t \in [n\tau, (n+1)\tau]$ .

**Theorem 3.** For any given  $\varepsilon \in (0, \frac{\delta_W}{\rho_1})$ , if for every  $n \in \mathbb{N}$ , the release amount satisfies:

$$\Lambda_n \geq \max \left\{ -M_s(n\tau) + \frac{1-\varepsilon}{\gamma_1\varepsilon} e^{(\delta_s - \delta_W + \rho_1\varepsilon)\tau} W(n\tau), 0 \right\}, \quad (4.7)$$

then every solution of model (4.1) will converge to  $W_0 = 0$ . Moreover, if  $\Lambda_n$  fulfills the condition  $\Lambda_n \leq \frac{1-\varepsilon}{\gamma_1\varepsilon} e^{(\delta_s - \delta_W + \rho_1\varepsilon)\tau} W(n\tau)$ , then the series of release amounts  $\sum_{i=1}^{\infty} \Lambda_n$  converges, implying that the total number of sterile mosquitoes released throughout the control process is finite.

*Proof.* Given (4.6), its right-hand side function is monotonically increasing with respect to  $t$  on the interval  $(n\tau, (n+1)\tau]$ . Consequently, if inequality (4.6) is satisfied at  $t = (n+1)\tau$ , then it holds over the entire interval  $(n\tau, (n+1)\tau]$ . Thus, setting

$$\Lambda_n \geq -M_s(n\tau) + \frac{1-\varepsilon}{\gamma_1\varepsilon} e^{(\delta_s - \delta_W + \rho_1\varepsilon)\tau} W(n\tau)$$

ensures that

$$\gamma_1 M_s(t) \geq \frac{1-\varepsilon}{\varepsilon} W(t), \text{ for } t \in (n\tau, (n+1)\tau].$$

If inequality (4.7) is valid for all  $n \in \mathbb{N}$ , it implies that  $\gamma_1 M_s(t) \geq \frac{1-\varepsilon}{\varepsilon} W(t)$  for all  $t \geq 0$ , thereby satisfying (4.2). Based on Lemma 3, it follows that all solutions of model (4.1) converge to  $W_0 = 0$ .

Furthermore, if  $\Lambda_n \leq e^{(\delta_s - \delta_W + \rho_1 \varepsilon)\tau} \frac{1-\varepsilon}{\gamma_1 \varepsilon} W(n\tau)$  for  $n \in \mathbb{N}$ , then

$$\frac{dW}{dt} \leq (\rho_1 \varepsilon - \delta_W) W(t), \text{ for } t \geq 0,$$

and

$$W(n\tau) \leq W(0) e^{(\rho_1 \varepsilon - \delta_W)n\tau}, \text{ for } n \in \mathbb{N}.$$

Given that  $\frac{\rho_1}{\delta_W} > 1$  and  $0 < \varepsilon < \frac{\delta_W}{\rho_1}$ , it follows that  $\rho_1 \varepsilon - \delta_W < 0$ . Hence,

$$\sum_{i=1}^{\infty} \Lambda_n \leq \sum_{i=1}^{\infty} e^{(\delta_s - \delta_W + \rho_1 \varepsilon)\tau} \frac{1-\varepsilon}{\gamma_1 \varepsilon} W(0) e^{(\rho_1 \varepsilon - \delta_W)n\tau} = \frac{(1-\varepsilon)W(0)}{\gamma_1 \varepsilon} \frac{e^{(\delta_s - \delta_W + \rho_1 \varepsilon)\tau}}{1 - e^{(\rho_1 \varepsilon - \delta_W)\tau}}. \quad (4.8)$$

This demonstrates that the series of release amounts  $\sum_{i=1}^{\infty} \Lambda_n$  converges. The proof is completed.  $\square$

#### 4.2. Sparse state feedback control

Given the costs associated with estimating the mosquito population in the environment, reducing the frequency of these estimates becomes a practical consideration. Instead of conducting them in every release cycle of sterile mosquitoes, it's proposed to carry out an estimation once every several release cycles, for instance, once every  $m$  ( $m \in \mathbb{N}, m \geq 1$ ) cycles. Based on the results of the  $n$ th estimation,  $W(nm\tau)$ ,  $M_s(nm\tau)$ , the quantity of sterile mosquitoes to be released in the  $m$  cycles preceding the next estimation,  $\Lambda_{nm+k}$ ,  $k = 1, 2, \dots, m$ , is determined. This approach aims to ensure that the wild mosquito population ultimately heads towards extinction.

Regarding this, we have the following conclusion about the sparse state feedback control:

**Theorem 4.** For any given  $m \in \mathbb{N}^+$  and  $\varepsilon \in (0, \frac{\delta_W}{\rho_1})$ , suppose that for all  $n = 0, 1, 2, \dots$  and  $k = 1, 2, \dots, m$ , the amounts of mosquitoes released satisfy

$$\Lambda_{nm+k} \geq \max \left\{ \frac{1-\varepsilon}{\gamma_1 \varepsilon} e^{[(k+1)(\rho_1 \varepsilon - \delta_W) + \delta_s]\tau} W(nm\tau) - M_s(nm\tau) e^{-k\delta_s \tau} - \sum_{l=0}^{k-1} \Lambda_{nm+l} e^{-(k-l)\delta_s \tau}, 0 \right\}. \quad (4.9)$$

Then every solution of model (4.1) will converge to the extinction equilibrium  $W_0 = 0$ . Furthermore, if the release amounts fulfill  $\Lambda_{nm+k} \leq \frac{1-\varepsilon}{\gamma_1 \varepsilon} e^{[(k+1)(\rho_1 \varepsilon - \delta_W) + \delta_s]\tau} W(nm\tau)$ , then the series of released amounts  $\sum_{i=1}^{\infty} \Lambda_n$  converges.

*Proof.* First, we seek conditions under which  $\frac{W(t)}{W(t) + \gamma_1 M_s(t)} \leq \varepsilon \leq \frac{\delta_W}{\rho_1}$  is maintained for  $t \in (nm\tau, (n+1)m\tau]$ . Specifically, this requires

$$\gamma_1 M_s(t) \geq \frac{1-\varepsilon}{\varepsilon} W(t), \text{ for } t \in (nm\tau, (n+1)m\tau].$$

From model (3.11), it is evident that for any  $k = 1, 2, \dots, m$  and  $t \in ((nm + k)\tau, (nm + k + 1)\tau]$ , we have

$$M_s(t) = [(M_s(nm\tau) + \Lambda_{nm})e^{-k\delta_s\tau} + \Lambda_{nm+1}e^{-(k-1)\delta_s\tau} + \dots + \Lambda_{nm+k-1}e^{-\delta_s\tau} + \Lambda_{nm+k}]e^{-\delta_s[t-(nm+k)\tau]}. \quad (4.10)$$

To ensure  $\frac{W(t)}{W(t)+\gamma_1 M_s(t)} \leq \varepsilon$  for  $t \in (nm\tau, (n+1)m\tau]$ , it suffices to guarantee

$$\gamma_1 M_s(t) \geq \frac{1-\varepsilon}{\varepsilon} W(t), \text{ for } t \in ((nm+k)\tau, (nm+k+1)\tau], \text{ with } k = 1, 2, \dots, m. \quad (4.11)$$

Letting  $s = t - (nm + k)\tau$ , such that  $s \in (0, \tau]$ , and referring to (4.10), condition (4.11) can be rewritten as

$$\begin{aligned} & \gamma_1 [(M_s(nm\tau) + \Lambda_{nm})e^{-k\delta_s\tau} + \Lambda_{nm+1}e^{-(k-1)\delta_s\tau} + \dots + \Lambda_{nm+k-1}e^{-\delta_s\tau} + \Lambda_{nm+k}]e^{-\delta_s s} \\ & \geq \frac{1-\varepsilon}{\varepsilon} W(s + (nm+k)\tau). \end{aligned} \quad (4.12)$$

By Lemma 3, if  $W(nm\tau) = W_1(nm\tau)$ , then  $W_1(t) \geq W(t)$  for all  $t \geq nm\tau$ . This implies that condition

$$\begin{aligned} & \gamma_1 [(M_s(nm\tau) + \Lambda_{nm})e^{-k\delta_s\tau} + \Lambda_{nm+1}e^{-(k-1)\delta_s\tau} + \dots + \Lambda_{nm+k-1}e^{-\delta_s\tau} + \Lambda_{nm+k}]e^{-\delta_s s} \\ & \geq \frac{1-\varepsilon}{\varepsilon} W_1(s + (nm+k)\tau) = \frac{1-\varepsilon}{\varepsilon} e^{(\rho_1\varepsilon - \delta_w)(k\tau+s)} W(nm\tau) \end{aligned} \quad (4.13)$$

ensures (4.12) is satisfied.

From (4.13), it follows that

$$\begin{aligned} & M_s(nm\tau)e^{-k\delta_s\tau} + \Lambda_{nm}e^{-k\delta_s\tau} + \Lambda_{nm+1}e^{-(k-1)\delta_s\tau} + \dots + \Lambda_{nm+k} \\ & \geq \frac{1-\varepsilon}{\gamma_1\varepsilon} e^{(\rho_1\varepsilon - \delta_w)k\tau} e^{(\delta_s - \delta_w + \rho_1\varepsilon)s} W(nm\tau). \end{aligned}$$

Consequently, the necessary release amount for  $\Lambda_{nm+k}$  can be determined by

$$\Lambda_{nm+k} \geq \frac{1-\varepsilon}{\gamma_1\varepsilon} e^{(\rho_1\varepsilon - \delta_w)k\tau} e^{(\delta_s - \delta_w + \rho_1\varepsilon)s} W(nm\tau) - M_s(nm\tau)e^{-k\delta_s\tau} - \sum_{l=0}^{k-1} \Lambda_{nm+l}e^{-(k-l)\delta_s\tau}. \quad (4.14)$$

By selecting  $0 < \varepsilon < 1$  such that  $\delta_s - \delta_w + \rho_1\varepsilon > 0$ , the right-hand side of (4.14) increases monotonically with respect to  $s$  on the interval  $(0, \tau]$ . Therefore, if (4.14) holds for  $s = \tau$ , it holds for all  $s \in (0, \tau]$ . This implies that if the condition for (4.9) is met, then (4.14) is valid for all  $n = 0, 1, 2, \dots$  and  $k = 1, 2, \dots, m$ , ensuring

$$\gamma_1 M_s(t) \geq \frac{1-\varepsilon}{\varepsilon} W(t)$$

for all  $t \geq 0$ .

According to the result of Lemma 3, we deduce that all solutions of model (4.1) converge to  $W_0 = 0$ .

Additionally, when  $\Lambda_{nm+k}$  adheres to the upper bound  $\frac{1-\varepsilon}{\gamma_1\varepsilon} W(nm\tau)e^{[(k+1)(\rho_1\varepsilon - \delta_w) + \delta_s]\tau}$ , similar to the argument presented in Theorem 3, it follows that the series of release amounts  $\sum_{i=1}^{\infty} \Lambda_n$  converges. This completes the proof.  $\square$

### 4.3. Composite control

Following our previous discussion, it has been established that strategies for managing wild mosquito populations, either with or without state feedback, have their unique benefits. The approach lacking state feedback ensures the universal allure of the extinction equilibrium while maintaining a steady amount released across periods. Nonetheless, this method might continue releasing sterile mosquitoes even when wild populations are minimal or have been eliminated, due to the absence of state feedback. On the other hand, the strategy incorporating state feedback, though necessitating regular evaluations of the wild mosquito numbers, tailors the amount of sterile mosquitoes released in alignment with the estimated mosquito quantity. This adaptation decreases the quantity released as the wild population dwindles, eventually ceasing the release. Reflecting on the distinct advantages of these approaches, we propose a composite control strategy that merges the benefits of both.

Within this composite control framework, the two aforementioned methods of control will alternate infinitely. Our task is to pinpoint the minimal release amount needed to ensure the wild population's ultimate extinction under this integrated strategy. In this context, we arrive at the following conclusion.

**Theorem 5.** Consider any  $m \in \mathbb{N}^+$  and  $\varepsilon \in (0, \frac{\delta_W}{\rho_1})$ . If, for every  $n = 0, 1, 2, \dots$ , and  $k = 1, 2, \dots, m$ , the release quantities adhere to

$$\Lambda_{nm+k} = \min\{\Lambda_p^{crit}, \Lambda_s^{crit}\}, \quad (4.15)$$

where  $\Lambda_p^{crit}$  is defined in Theorem 2 and

$$\Lambda_s^{crit} = \max \left\{ \frac{1 - \varepsilon}{\gamma_1 \varepsilon} e^{[(k+1)(\rho_1 \varepsilon - \delta_W) + \delta_s] \tau} W(nm\tau) - M_s(nm\tau) e^{-k\delta_s \tau} - \sum_{l=0}^{k-1} \Lambda_{nm+l} e^{-(k-l)\delta_s \tau}, 0 \right\}, \quad (4.16)$$

then every solution to the model (4.1) will inevitably converge to the extinction equilibrium  $W_0 = 0$ .

*Proof.* Let's illustrate the theorem by initially considering the case where  $m = 1$ . The argument for  $m > 1$  follows a similar rationale.

Assume  $\Lambda_n = \Lambda_p^{crit}$ . Drawing from the proof of Theorem 2, it is established that any solution  $W(t)$  of model (4.1) will fulfill

$$W((n+1)\tau) \leq W(n\tau) e^{\left(\frac{\rho_1}{e\beta\gamma_1} \left\langle \frac{1}{M_s^p} \right\rangle - \delta_W\right)\tau},$$

with  $\frac{\rho_1}{e\beta\gamma_1} \left\langle \frac{1}{M_s^p} \right\rangle < \delta_W$ .

Furthermore, according to Theorem 3 (or Theorem 4 for  $m > 1$ ), it's noted that for any  $\varepsilon \in (0, \frac{\delta_W}{\rho_1})$ , and if  $\Lambda_n = \Lambda_s^{crit}$ , then any solution  $W(t)$  of model (4.1) will satisfy

$$W((n+1)\tau) \leq W(n\tau) e^{(\rho_1 \varepsilon - \delta_W)\tau},$$

where  $\rho_1 \varepsilon < \delta_W$ .

Thus, if the condition (4.15) is met, for any  $n \in \mathbb{N}$ , we have

$$W((n+1)\tau) \leq W(n\tau) e^{\max\left\{\frac{\rho_1}{e\beta\gamma_1} \left\langle \frac{1}{M_s^p} \right\rangle - \delta_W, \rho_1 \varepsilon - \delta_W\right\}\tau} = W(n\tau) e^{-\sigma\tau},$$

where  $\sigma = \min\{\delta_W - \frac{\rho_1}{e\beta\gamma_1} \left\langle \frac{1}{M_s^p} \right\rangle, \delta_W - \rho_1 \varepsilon\} > 0$ .

This establishes that  $W((n+1)\tau) < W(n\tau)$  for any  $n \in \mathbb{N}$ , and  $\lim_{t \rightarrow \infty} W(t) = 0$ . Thus, regardless of how the control modes alternate, the solution  $W(t)$  of model (4.1) is guaranteed to exponentially converge to the extinction equilibrium  $W_0 = 0$ , given  $\Lambda_n = \min\{\Lambda_p^{crit}, \Lambda_s^{crit}\}$ . The proof is completed.  $\square$

## 5. Numerical simulation

First, we present a series of numerical simulations to illustrate and compare the release strategies discussed earlier. For the model parameters, we refer to the parameter values in [2] and adjust them according to our model.

Assuming that an egg has an equal chance of hatching into either a female or male mosquito, i.e.,  $r = 0.5$  in model (2.2), it's important to note the significant difference in the quantities of female and male mosquitoes in the environment. Since they have different death rates,  $\delta_F$  and  $\delta_M$ , a simple calculation based on average lifespan reveals that the ratio of female to male mosquitoes in the environment is approximately  $1/\delta_M : 1/\delta_F$ . Let's denote this ratio as  $\alpha$  and  $W(t) = M(t) + F(t)$ . Then, from model (2.2), we have

$$M(t) = \frac{1}{1 + \alpha} W(t), \quad F(t) = \frac{\alpha}{1 + \alpha} W(t),$$

and

$$\frac{dW(t)}{dt} = \frac{\rho \frac{\alpha}{(1+\alpha)^2} W^2(t)}{\frac{1}{1+\alpha} W(t) + \gamma M_s(t)} e^{-\beta W(t)} - \frac{\delta_M + \alpha \delta_F}{1 + \alpha} W(t) = \frac{\rho_1 W^2(t)}{W(t) + \gamma_1 M_s(t)} e^{-\beta W(t)} - \delta_W W(t),$$

where

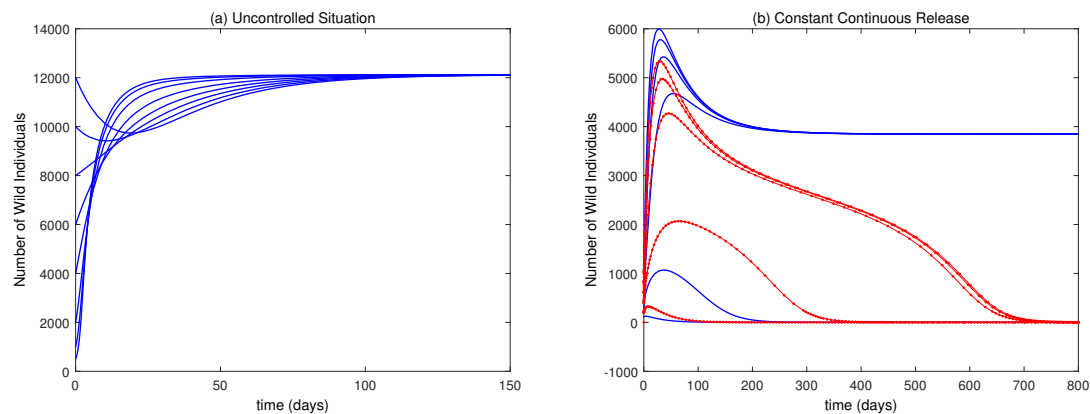
$$\rho_1 = \frac{\alpha \rho}{1 + \alpha}, \quad \gamma_1 = (1 + \alpha) \gamma, \quad \delta_W = \frac{\delta_M + \alpha \delta_F}{1 + \alpha}.$$

Using the parameter values in [2], we obtain that

$$\rho_1 = 2.6, \quad \beta = 3.57 \times 10^{-4}, \quad \gamma_1 = 2.3333, \quad \delta_W = 0.0343, \quad \delta_s = 0.04.$$

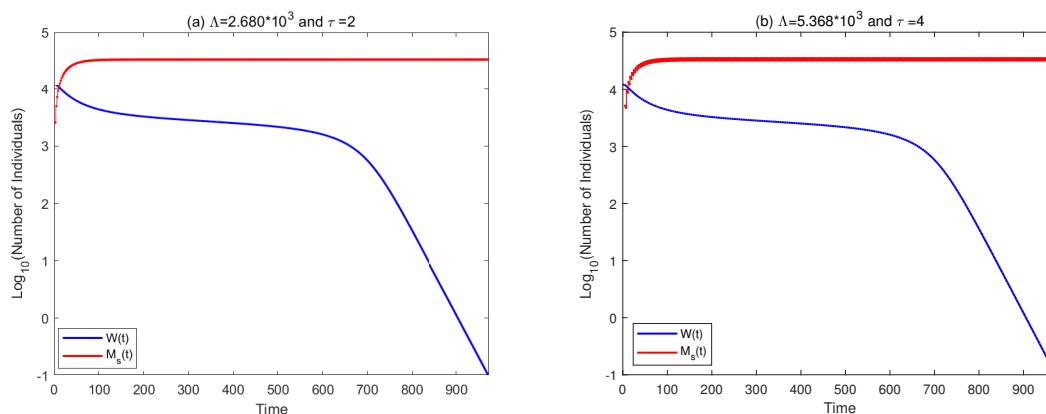
After calculation, it is easy to obtain  $N_W = \frac{\rho_1}{\delta_W} = 75.8017 > 1$  and the release threshold for constant continuous release  $\Lambda^{crit} \approx 1294$ . According to Lemma 1, if there are no sterile ones present in the field, model (3.1) exhibits a globally asymptotically stable positive equilibrium, as depicted in Figure 1(a). Furthermore, when sterile mosquitoes are continuously introduced at a fixed rate, with the release rate  $\Lambda = 1200$  being less than the critical threshold  $\Lambda^{crit}$ , model (3.2) exhibits bistability. This means it has both an extinction equilibrium and a positive equilibrium, each locally stable (see the blue curves in Figure 1(b)). When the release rate increases to  $\Lambda = 1350 > \Lambda^{crit}$ , model (3.2) is left with only the extinction equilibrium, which is globally asymptotically stable (see the red curves in Figure 1(b)).

Drawing on field experiments where sterile mosquitoes are deployed two to three times weekly, we have selected release periods of  $T = 2$  and  $T = 4$  for analytical comparison in the context of periodic impulsive releases. For simplicity in comparing outcomes, we consider the wild mosquito population eradicated when  $\log_{10}(W(t))$  reaches a value of -1.



**Figure 1.** (a) The positive equilibrium of model (3.1) is globally stable; (b) Bistability with  $\Lambda < \Lambda^{crit}$  and global stability of the extinction equilibrium with  $\Lambda > \Lambda^{crit}$  for model (3.2).

Within the open-loop control framework, employing the threshold calculation formula from Theorem 2 allows us to ascertain the critical release amounts for release periods of  $T = 2$  and  $T = 4$ , calculated as  $\Lambda_{p(2)}^{crit} = 2.680 \times 10^3$  and  $\Lambda_{p(4)}^{crit} = 5.368 \times 10^3$ , respectively. As demonstrated in Figure 2, the time required to eliminate the wild mosquito population is nearly identical for both strategies 970 and 976 days, with a cumulative release of  $1.2998 \times 10^6$  and  $1.309792 \times 10^6$  sterile mosquitoes, respectively. In terms of control efficacy, the two methods are virtually indistinguishable. However, when considering the costs associated with releasing sterile mosquitoes, the latter approach proves to be more economical, necessitating only 244 releases compared to the 485 required by the former, thereby significantly reducing the frequency of releases. This efficiency makes the second method the preferred choice in practical applications.



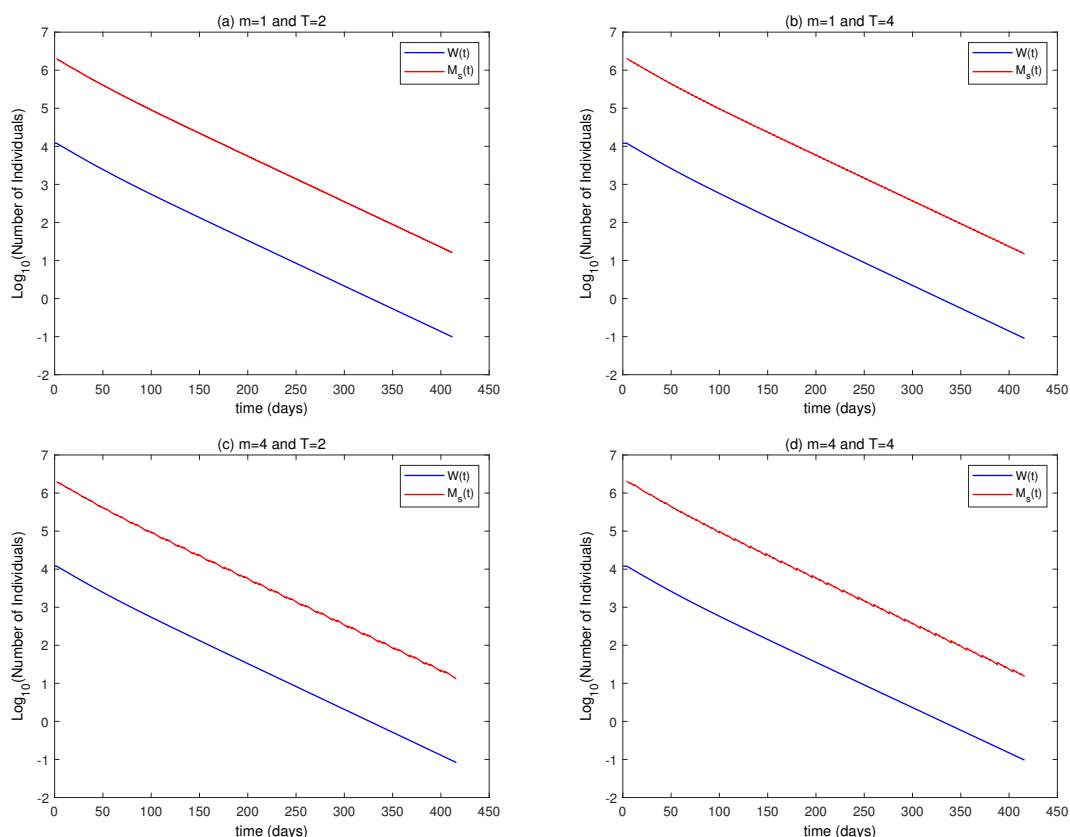
**Figure 2.** Open-loop control with release periods of  $T = 2$  and  $T = 4$ .

Within the framework of closed-loop control, as discussed in the previous section, the basic offspring number of the wild mosquito population after the release of sterile mosquitoes is represented by  $\varepsilon N_W$ . This parameter acts as an indicator of the level of control exerted on the wild mosquito population, where lower values indicate more effective control, and higher values suggest less effective control. For the purpose of comparison, we choose two distinct values of  $\varepsilon N_W$ , specifically 0.2 and 0.8, to represent scenarios of high and low control effectiveness, respectively. Additionally, to

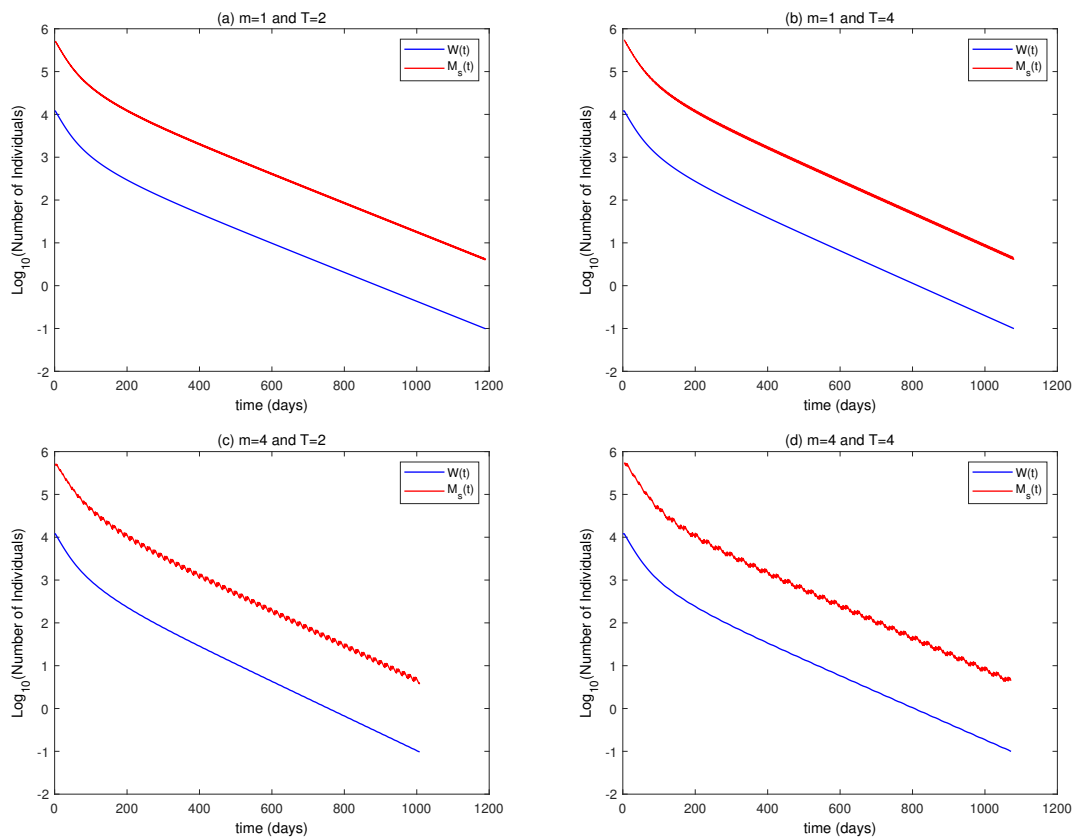


account for different frequencies of population estimation, we employ  $m = 1$  and  $m = 4$  as markers. These represent scenarios where estimations are conducted every release period and every four release periods, respectively. This setup allows us to examine the impact of both the intensity of control (as varied by  $\varepsilon$ ) and the frequency of population assessments on the overall effectiveness of the mosquito population management strategy.

Upon observing Figures 3 and 4, it is noted that within the closed-loop control process, the intensity of population control significantly influences the timeliness of the entire control operation. Compared to control strategies with lower intensity ( $\varepsilon N_W = 0.8$ ), higher intensity control ( $\varepsilon N_W = 0.2$ ) can markedly reduce the duration of control. In Tables 1 and 2, the cumulative quantity of sterile mosquitoes released, the duration, and the number of effective release events for each combination of release period and estimation frequency are documented. Through comparison, it is also observed that shorter release periods ( $T = 2$ ) and more frequent population estimations ( $m = 1$ ) can reduce the total number of sterile mosquitoes required to achieve population extinction, whereas larger release periods ( $T = 4$ ) can significantly decrease the number of effective releases. Furthermore, stronger control intensity ( $\varepsilon N_W = 0.2$ ) necessitates the release of a larger number of sterile mosquitoes. Specifically, at  $m = 4$  (estimation conducted once every four periods), it significantly increases the total number of sterile mosquitoes released throughout the control process but can greatly reduce the number of effective release events in the case of low-intensity control.



**Figure 3.** Closed-loop control for model (4.1) with  $\varepsilon N_W = 0.2$ .



**Figure 4.** Closed-loop control for model (4.1) with  $\varepsilon\mathcal{N}_W = 0.8$ .

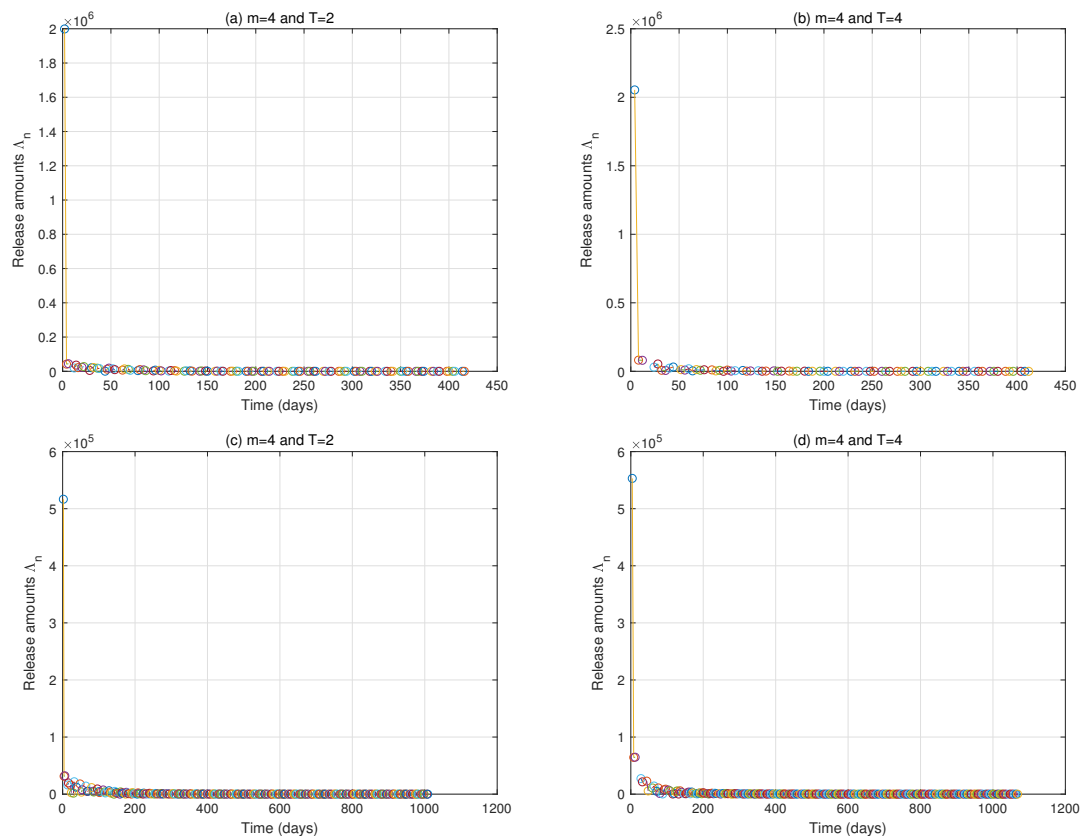
**Table 1.** Closed-loop control with  $\varepsilon\mathcal{N}_W = 0.2$ .

	Total release amount		Total duration of control		Effective release counts	
	$m = 1$	$m = 4$	$m = 1$	$m = 4$	$m = 1$	$m = 4$
$T = 2$	2430877	2465778	410	416	205	208
$T = 4$	2477977	2507602	412	412	103	103

**Table 2.** Closed-loop control with  $\varepsilon\mathcal{N}_W = 0.8$ .

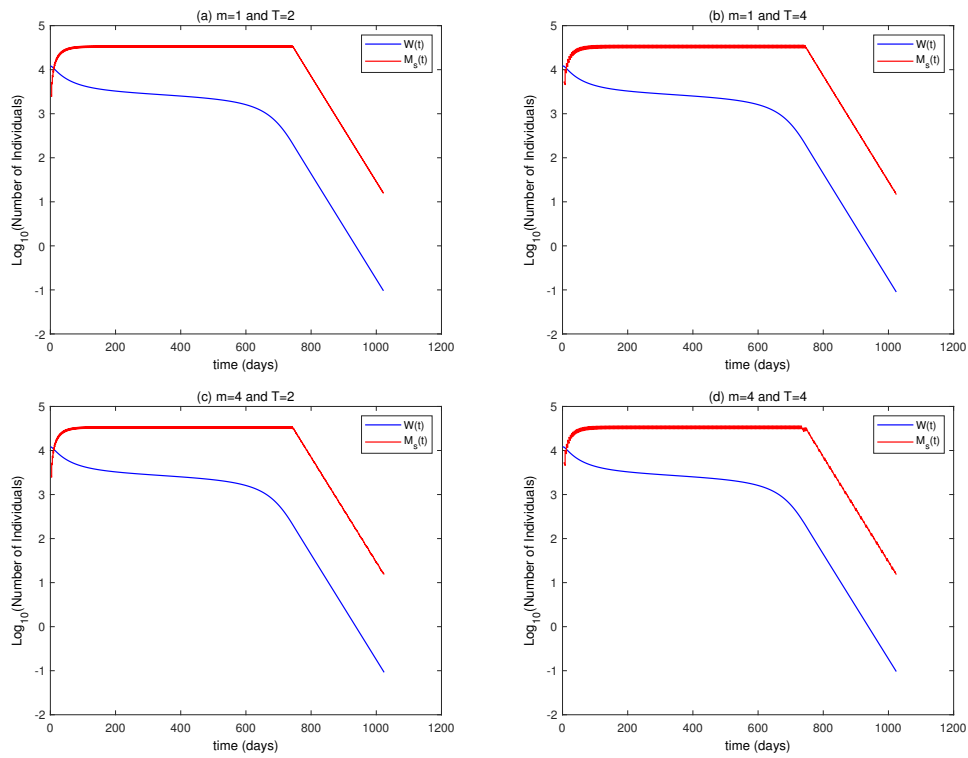
	Total release amount		Total duration of control		Effective release counts	
	$m = 1$	$m = 4$	$m = 1$	$m = 4$	$m = 1$	$m = 4$
$T = 2$	794517	839233	1188	1008	594	246
$T = 4$	820622	904891	1076	1068	269	228

Additionally, with  $m = 4$  (performing estimations once every four periods), releasing sterile mosquitoes in each cycle becomes unnecessary. If the environment’s accumulated sterile mosquito count meets or exceeds the threshold outlined by inequality (4.9), then the release amount for that cycle could be reduced to zero. Consequently, the actual count of release events might fall below the total number of periods. Figure 5 illustrates how the release volumes vary with different settings of release intervals, estimation frequencies, and control strengths.

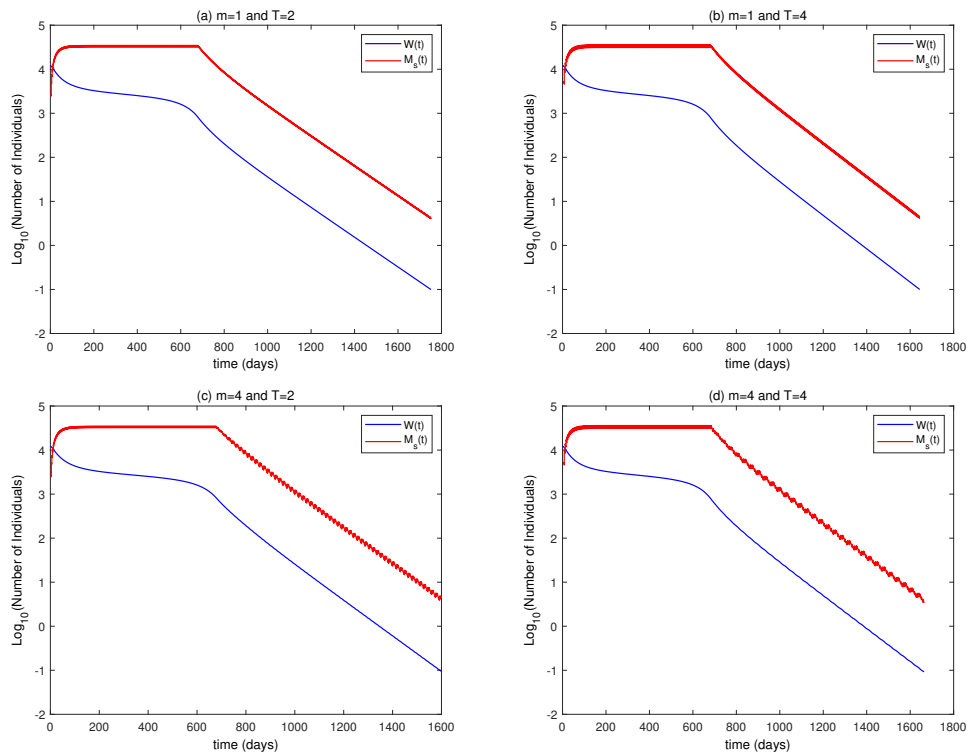


**Figure 5.** Distribution of the release amount for closed-loop control; (a) and (b)  $\varepsilon N_W = 0.2$ ; (c) and (d)  $\varepsilon N_W = 0.8$ .

In exploring the composite control strategy, we investigate the dynamics of the mosquito population's trajectory towards extinction across various combinations of release periods and estimation frequencies, while also considering the level of control intensity (as illustrated in Figures 6 and 7). Tables 3 and 4 provide detailed data on the cumulative number of sterile mosquitoes released, the duration of control, and the number of effective release instances for each scenario. It is observed that under the composite control strategy, the effects of release periods and population control intensity on the convergence towards the extinction equilibrium mirror those seen with the closed-loop control strategy. Moreover, we find that although the composite control strategy can significantly reduce the amount of each individual release compared to closed-loop control, the low-intensity composite control strategy does not exhibit advantages in terms of the cumulative release quantity, control duration, and the number of effective releases. Conversely, the high-intensity composite control strategy demonstrates a significant advantage in the cumulative quantity of sterile mosquitoes released but does not show improvement in the overall duration of control and the number of effective releases. Figure 8 further details the distribution of release amounts under the composite strategy. Observations indicate that state feedback from the wild mosquito population initiates a progressive reduction in release amounts as the wild population decreases, ultimately ceasing when the number reaches zero.



**Figure 6.** Composite control strategy for model (4.1) with  $\varepsilon N_W = 0.2$ .



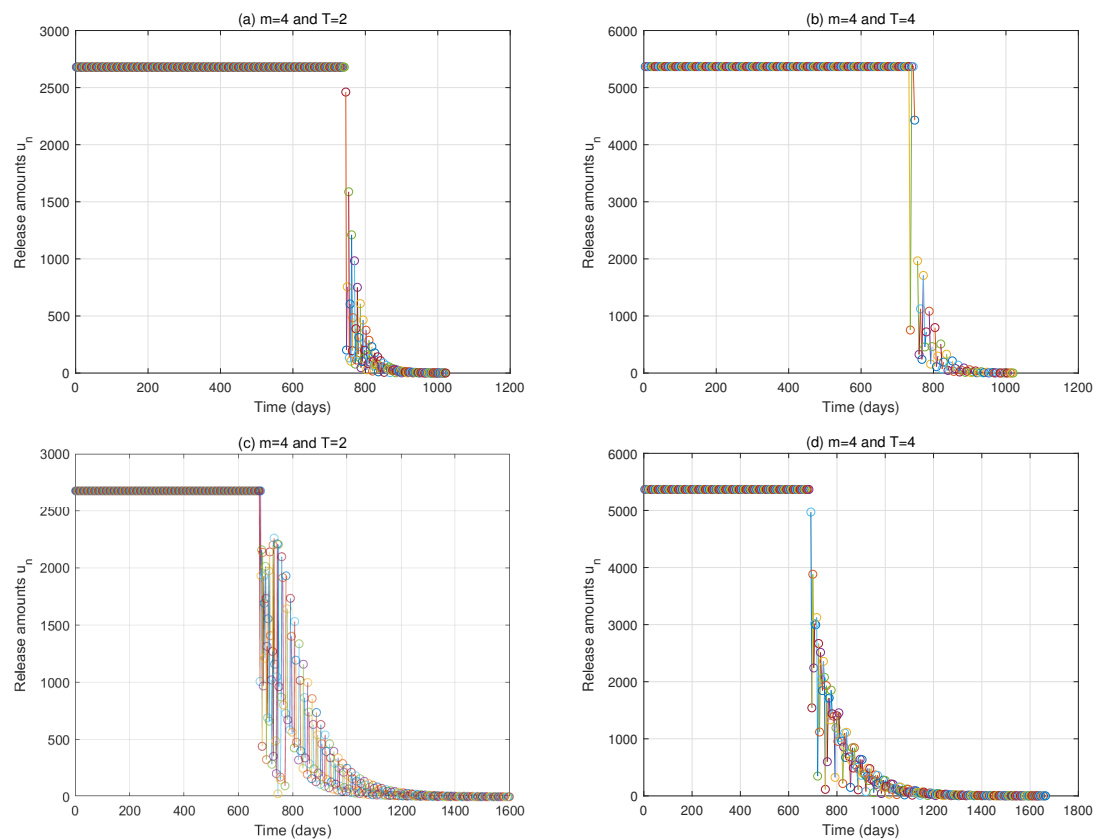
**Figure 7.** Composite control strategy for model (4.1)  $\varepsilon N_W = 0.8$ .

**Table 3.** Composite control strategy with  $\varepsilon\mathcal{N}_W = 0.2$ .

	Total release amount		Total duration of control		Effective release counts	
	$m = 1$	$m = 4$	$m = 1$	$m = 4$	$m = 1$	$m = 4$
$T = 2$	1010317	1009699	1020	1022	510	492
$T = 4$	1011775	1009844	1020	1020	255	239

**Table 4.** Composite control strategy with  $\varepsilon\mathcal{N}_W = 0.8$ .

	Total release amount		Total duration of control		Effective release counts	
	$m = 1$	$m = 4$	$m = 1$	$m = 4$	$m = 1$	$m = 4$
$T = 2$	998683	994126	1750	1598	875	588
$T = 4$	994726	995392	1640	1660	410	386

**Figure 8.** Distribution of the release amount for composite control strategy; (a) and (b)  $\varepsilon\mathcal{N}_W = 0.2$ ; (c) and (d)  $\varepsilon\mathcal{N}_W = 0.8$ .

## 6. Conclusions

In this work, we have developed a novel type of mosquito population suppression model. Utilizing this model, we explored various release strategies for sterile mosquitoes, including the constant continuous release strategy, open-loop control strategy, closed-loop control strategy, and composite control strategy. Through extensive theoretical analysis, we identified the release amount thresholds

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for these strategies that can ensure the ultimate extinction of the wild mosquito population.

Furthermore, we carried out a comprehensive suite of numerical simulations to both validate our theoretical results and assess the effectiveness of different mosquito release strategies. The implementation of closed-loop control mechanisms aims to minimize the unpredictability of releases, thus preventing the unnecessary and excessive delivery of sterile mosquitoes in scenarios where the population of wild mosquitoes is minimal or has already been eradicated. It was observed that within the closed-loop control framework, shortening the intervals between releases and increasing the frequency of population assessments contribute to reducing the overall quantity of sterile mosquitoes needed. Additionally, while the composite control strategy significantly lowers the amount of each individual release compared to closed-loop control, the low-intensity composite control strategy does not exhibit advantages in terms of cumulative release quantity, control duration, and the number of effective releases. Conversely, the high-intensity composite control strategy demonstrates a significant reduction in the cumulative quantity of sterile mosquitoes released but does not offer improvements in the overall control duration and the number of effective releases. Hence, in field applications, when there is state feedback on mosquito population numbers, the closed-loop control strategy should be preferred for low-intensity control. For high-intensity control, the composite control strategy may be selected with the aim of reducing the total quantity of sterile mosquitoes released. At this juncture, strategically choosing larger release periods and estimation intervals becomes imperative to decrease the number of effective releases and minimize the operational costs associated with field releases of sterile mosquitoes.

### **Author contributions**

Shouzong Liu: Methodology, Writing; Yang Xu: Data Curation, Visualization; Mingzhan Huang: Software.

### **Use of AI tools declaration**

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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### **Conflict of interest**

The authors declare no conflicts of interest in this paper.

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