Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(9): 23331–23343.
DOI: 10.3934/math.20241134
Received: 13 July 2024
Revised: 22 July 2024
Accepted: 22 July 2024
Published: 02 August 2024

Research article

Advances in mathematical analysis for solving inhomogeneous scalar differential equation

Abdulrahman B. Albidah[1], **Ibraheem M. Alsulami**[2], **Essam R. El-Zahar**[3,4,*] and **Abdelhalim Ebaid**[5]

[1] Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Majmaah, 11952, Saudi Arabia

[2] Mathematics Department, Faculty of Science, Umm Al-Qura University, Makkah 21955, Saudi Arabia

[3] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam bin Abdulaziz University, P.O. Box 83, Al-Kharj 11942, Saudi Arabia

[4] Department of Basic Engineering Science, Faculty of Engineering, Menoufia University, Shebin El-Kom 32511, Egypt

[5] Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia

* **Correspondence:** Email: er.elzahar@psau.edu.sa.

Abstract: This paper considered a functional model which splits to two types of equations, mainly, advance equation and delay equation. The advance equation was solved using an analytical approach. Different types of solutions were obtained for the advance equation under specific conditions of the model's parameters. These solutions included the polynomial solutions of first and second degrees, the periodic solution and the hyperbolic solution. The periodic solution was invested to establish the analytical solution of the delay equation. The characteristics of the solution of the present model were discussed in detail. The results showed that the solution was continuous in the domain of the problem, under a restriction on the given initial condition, while the first derivative was discontinuous at a certain point and lied within the domain of the delay equation. In addition, some existing results in the literature were recovered as special cases of the current ones. The present successful analysis can be further generalized to include complex functional equations with an arbitrary function as an inhomogeneous term.

Keywords: ordinary differential equation; delay; initial value problem
Mathematics Subject Classification: 34K06, 65L03

1. Introduction

This paper focuses on a relatively new type of functional equation which can be categorized to an advance equation or a delay equation based on the relationship between the independent variable and the advanced/delay parameter. The field of delay differential equations (DDEs) has well-known applications in several areas of applied sciences such as the Pantograph model which is of practical applications in electric trains [1–8] and the Ambartsumian equation [9–13] which studies the surface brightness in the Milky Way and astronomy. Additionally, a well-known DDE is expressed as $\phi'(t) = \alpha\phi(t) + \beta\phi(t-\tau)$, where $\tau > 0$ represents the delay parameter. This differential equation is classified as a DDE because $t - \tau < t \; \forall \; \tau > 0$.

However, the differential equation $\phi'(t) = \alpha\phi(t) + \beta\phi(-t+\tau)$ isn't always a DDE. This is because $-t + \tau \nless t \; \forall \; \tau > 0$. There is a certain interval in which this differential equation represents a DDE. For declaration, if $-t + \tau < t$, i.e., $t > \tau/2$, then the differential equation $\phi'(t) = \alpha\phi(t) + \beta\phi(-t+\tau)$ belongs to the type of DDEs and τ is called the delay parameter. However, it belongs to the type of advance differential equations (ADE) if $0 \leq t \leq \tau/2$. This case simply means that $-t + \tau \geq t \; \forall \; t \in [0, \tau/2]$, where τ is called the advanced parameter in this case.

So, such types of differential equations are known as scalar differential equations (SDEs). To solve such types of SDEs, one has to find the solution in separate intervals which specify the type of the model in each of such intervals as demonstrated above. In this paper, we consider an inhomogeneous version of SDE, expressed as

$$\phi'(t) = \alpha\phi(t) + \beta\phi(-t+\tau) + \gamma, \quad \phi(t) = 0 \; \forall \; t < 0, \quad \phi(0) = \lambda, \quad \alpha, \beta, \gamma, \lambda \in \mathbb{R}, \;\; \tau \geq 0. \tag{1.1}$$

In [14], the problem (1.1) was addressed in the absence of the inhomogeneous term γ by considering the special case $\tau = 0$. Later, the authors [15] generalized the results obtained in [14] by considering $\tau \neq 0$ and $\gamma = 0$. They have divided the problem to an advance equation in the interval $t \in [0, \tau/2]$ in addition to a delay equation in the interval $t \in [\tau/2, \infty)$.

In the literature, it has been shown that the Adomian decomposition method (ADM) [16–21], the regular perturbation method [22], the homotopy perturbation method [23,24], the Laplace transform (LT) [25], and the homotopy analysis method [26] are effective methods to solve various ordinary differential equations (ODEs). However, the applications of these methods to solve delay or advance equations are very rare. On the other hand, the stability and behavior of solutions in mixed integro-differential equations with delays and advances has been discussed in [27] which was a useful study in the field of this paper.

The main motivation/objective of this paper is to generalize the results obtained in [15] by incorporating the inhomogeneous none-zero term γ, i.e., $\gamma \neq 0$, where the same initial conditions given in [15] will be assumed through this analysis. The given initial conditions in the model (1.1) assume the discontinuity at $t = 0$ for $\lambda \neq 0$. However, the solution to be derived must be continuous at $t = 0$ if $\lambda = 0$, this issue will be addressed later.

Following the authors [15], the advance equation of the present model is to be analytically solved in the interval $t \in [0, \tau/2]$ while the delay equation will be analyzed in the interval $t \in [\tau/2, \infty)$. It will be shown that the solution of the advance equation can be found in different forms such as polynomials of first and second degrees, the periodic solution, and the hyperbolic solution under specific relationships between the model's parameters. It will also be shown that the solution of the delay equation depends mainly on the solution of the advance equation. The solution of the delay equation will be derived in analytic forms via applying the method of steps in the intervals $[\tau/2, \tau]$ and (τ, ∞). Furthermore, the properties of the obtained solution are to be addressed theoretically and graphically. Furthermore, the existing results in the literature will be recovered as special cases of the present ones.

2. The advance equation: $0 \leq t \leq \tau/2$

The method of steps is a basic approach to solve scalar equations. However, it isn't actually valid to treat the current model in the interval $0 \leq t \leq \tau/2$. This interval implies $\tau/2 \leq -t + \tau \leq \tau$, hence, the value of $\phi(-t + \tau)$ can't be assigned in this case. The present approach suggests to convert the advance equation to a corresponding boundary value problem (BVP). This target can be achieved by using the characteristics of the model itself in such prescribed interval. For declaration, one can differentiate Eq (1.1), which yields the second-order ODE:

$$\phi''(t) + \left(\beta^2 - \alpha^2\right) \phi(t) = \gamma(\alpha - \beta). \tag{2.1}$$

Equation (2.1) is subject to the boundary conditions (BCs):

$$\phi(0) = \lambda, \quad \phi'\left(\frac{\tau}{2}\right) = (\alpha + \beta)\phi\left(\frac{\tau}{2}\right) + \gamma. \tag{2.2}$$

The second-order ODE (2.1) and the BCS (2.2) constitute the corresponding BVP of the advance equation in the interval $0 \leq t \leq \tau/2$. To solve the BVP (2.1) and (2.2), we consider four distinct cases, mainly, $\alpha = \beta$, $\alpha = -\beta$, $\left|\frac{\alpha}{\beta}\right| < 1$, and $\left|\frac{\beta}{\alpha}\right| < 1$. The solutions for these four cases are addressed in detail in the following subsections.

2.1. *First-degree polynomial solution: $\alpha = \beta$*

Suppose that $\alpha = \beta = \nu$, then Eq (2.1) reduces to

$$\phi''(t) = 0, \tag{2.3}$$

while the BCs (2.2) take the form:

$$\phi(0) = \lambda, \quad \phi'\left(\frac{\tau}{2}\right) = 2\nu\phi\left(\frac{\tau}{2}\right) + \gamma. \tag{2.4}$$

The solution of the BVP (2.3) and (2.4) can be easily obtained as

$$\phi(t) = \left(\frac{2\lambda\nu + \gamma}{1 - \nu\tau}\right) t + \lambda, \quad \nu\tau \neq 1, \tag{2.5}$$

which is the solution of the corresponding advance equation:

$$\phi'(t) = \nu\left[\phi(t) + \phi(-t + \tau)\right] + \gamma, \quad \phi(t) = 0 \; \forall \; t < 0, \quad \phi(0) = \lambda, \quad \nu, \gamma, \lambda \in \mathbb{R}. \tag{2.6}$$

in the interval $0 \leq t \leq \tau/2$.

Remark 1. *It can be seen in Eq (2.5) that the solution is discontinuous at $t = 0$ if $\lambda \neq 0$, where $\phi(0) = \lambda \neq 0$. By this, the solution is continuous at $t = 0$ if, and only if, $\lambda = 0$, and in this case the solution of the advance equation becomes $\phi(t) = \frac{\gamma t}{1-\nu\tau}$ such that $\nu\tau \neq 1$.*

2.2. *Second-degree polynomial solution:* $\alpha = -\beta$

Let $\alpha = -\beta = \rho$, then Eq (2.1) becomes

$$\phi''(t) = 2\gamma\rho. \tag{2.7}$$

The BCs (2.2) give

$$\phi(0) = \lambda, \quad \phi'\left(\frac{\tau}{2}\right) = \gamma. \tag{2.8}$$

Solving the BVP (2.7) and (2.8), we obtain

$$\phi(t) = \lambda + \gamma(1 - \rho\tau)t + \gamma\rho t^2. \tag{2.9}$$

This is the solution of the corresponding advance equation:

$$\phi'(t) = \rho\left[\phi(t) - \phi(-t+\tau)\right] + \gamma, \quad \phi(t) = 0 \;\forall\; t < 0, \quad \phi(0) = \lambda, \quad \rho, \gamma, \lambda \in \mathbb{R}, \tag{2.10}$$

in the interval $0 \le t \le \tau/2$.

2.3. *Periodic solution:* $\left|\frac{\alpha}{\beta}\right| < 1$

For $\alpha^2 < \beta^2$ or $\left|\frac{\alpha}{\beta}\right| < 1$, the solution of Eq (2.1) is periodic and given by

$$\phi(t) = c_1 \sin(\Omega t) + c_2 \cos(\Omega t) - \frac{\gamma}{\alpha+\beta}, \quad \Omega = \sqrt{\beta^2 - \alpha^2}, \qquad \left|\frac{\alpha}{\beta}\right| < 1, \tag{2.11}$$

where c_1 and c_2 are unknown constants. Applying the BCs (2.2) yields

$$c_1 = \left(\lambda + \frac{\gamma}{\alpha+\beta}\right)\left[\frac{\Omega\sin(\Omega\tau/2) + (\alpha+\beta)\cos(\Omega\tau/2)}{\Omega\cos(\Omega\tau/2) - (\alpha+\beta)\sin(\Omega\tau/2)}\right], \tag{2.12}$$

$$c_2 = \lambda + \frac{\gamma}{\alpha+\beta}. \tag{2.13}$$

Inserting these constants into Eq (2.11) and simplifying, we obtain,

$$\phi(t) = \left(\lambda + \frac{\gamma}{\alpha+\beta}\right)\left[\frac{\Omega\cos(\Omega(t-\tau/2)) + (\alpha+\beta)\sin(\Omega(t-\tau/2))}{\Omega\cos(\Omega\tau/2) - (\alpha+\beta)\sin(\Omega\tau/2)}\right] - \frac{\gamma}{\alpha+\beta}, \quad \left|\frac{\alpha}{\beta}\right| < 1. \tag{2.14}$$

It is to be noted that the solution (2.14) reduces to the corresponding one in [15] as $\gamma \to 0$. Moreover, if both parameters γ and τ vanish, the solution (2.14) is in full agreement with the solution obtained in [14]. In addition, the solutions of the previous two special cases can be recovered from Eq (2.14) by taking the limit as $(\alpha - \beta) \to 0$ and $(\alpha + \beta) \to 0$, respectively.

2.4. *Hyperbolic solution:* $\left|\frac{\beta}{\alpha}\right| < 1$

If $\beta^2 < \alpha^2$ or $\left|\frac{\beta}{\alpha}\right| < 1$, the solution of Eq (2.1) takes the form:

$$\phi(t) = c_3 \sinh(\omega t) + c_4 \cosh(\omega t) - \frac{\gamma}{\alpha+\beta}, \quad \omega = \sqrt{\alpha^2 - \beta^2}, \qquad \left|\frac{\beta}{\alpha}\right| < 1. \tag{2.15}$$

Similarly, the constants c_3 and c_4 can be evaluated by applying the BCs (2.2), and this procedure leads to

$$c_1 = \left(\lambda + \frac{\gamma}{\alpha+\beta}\right)\left[\frac{-\omega\sinh(\omega\tau/2) + (\alpha+\beta)\cosh(\omega\tau/2)}{\omega\cosh(\omega\tau/2) - (\alpha+\beta)\sinh(\omega\tau/2)}\right], \tag{2.16}$$

$$c_2 = \lambda + \frac{\gamma}{\alpha+\beta}. \tag{2.17}$$

Employing the constants (2.16) and (2.17) in Eq (2.15), one can get

$$\phi(t) = \left(\lambda + \frac{\gamma}{\alpha+\beta}\right)\left[\frac{\omega\cosh(\omega(t-\tau/2)) + (\alpha+\beta)\sinh(\omega(t-\tau/2))}{\omega\cosh(\omega\tau/2) - (\alpha+\beta)\sinh(\omega\tau/2)}\right] - \frac{\gamma}{\alpha+\beta}, \quad \left|\frac{\beta}{\alpha}\right| < 1. \tag{2.18}$$

One can notice that the relation between the parameters Ω and ω is simply given by $\Omega = i\omega$, where $i = \sqrt{-1}$. So, the result in Eq (2.18) can be also obtained via replacing Ω by $i\omega$ directly in (2.14).

Remark 2. *Existence of the periodic and the hyperbolic solutions require certain constrains on the parameters α, β, and τ. The periodic solution* (2.14) *put a restriction on the advance parameter τ in terms of α and β given as $\tau \neq \frac{2}{\Omega}\tan^{-1}\left(\frac{\Omega}{\alpha+\beta}\right)$ while the hyperbolic solution* (2.18) *requires $\tau \neq \frac{2}{\omega}\tanh^{-1}\left(\frac{\omega}{\alpha+\beta}\right)$.*

3. Compact periodic solution

Theorem 1. *A compact form of the periodic solution of the advance equation is*

$$\phi(t) = \left(\lambda + \frac{\gamma}{\alpha+\beta}\right)\sqrt{1 + \left(\frac{\Omega\sin(\Omega\tau/2) + (\alpha+\beta)\cos(\Omega\tau/2)}{\Omega\cos(\Omega\tau/2) - (\alpha+\beta)\sin(\Omega\tau/2)}\right)^2}$$
$$\sin\left(\Omega t + \tan^{-1}\left(\frac{\Omega\cos(\Omega\tau/2) - (\alpha+\beta)\sin(\Omega\tau/2)}{\Omega\sin(\Omega\tau/2) + (\alpha+\beta)\cos(\Omega\tau/2)}\right)\right) - \frac{\gamma}{\alpha+\beta}, \tag{3.1}$$

provided $\left|\frac{\alpha}{\beta}\right| < 1$.

Proof. Let us rewrite the solution (2.14) in the form:

$$\phi(t) = \mu\sin(\Omega t + \theta) - \frac{\gamma}{\alpha+\beta}, \tag{3.2}$$

or,

$$\phi(t) = \mu\cos\theta\sin(\Omega t) + \mu\sin\theta\cos(\Omega t) - \frac{\gamma}{\alpha+\beta}. \tag{3.3}$$

Comparing Eq (3.3) with Eq (2.11) and implementing the constants c_1 and c_2 in Eqs (2.12) and (2.13), then

$$\mu\cos\theta = \left(\lambda + \frac{\gamma}{\alpha+\beta}\right)\left[\frac{\Omega\sin(\Omega\tau/2) + (\alpha+\beta)\cos(\Omega\tau/2)}{\Omega\cos(\Omega\tau/2) - (\alpha+\beta)\sin(\Omega\tau/2)}\right], \tag{3.4}$$

$$\mu\sin\theta = \lambda + \frac{\gamma}{\alpha+\beta}. \tag{3.5}$$

Solving Eqs (3.4) and (3.5) for μ and θ leads to

$$\mu = \left(\lambda + \frac{\gamma}{\alpha+\beta}\right)\sqrt{1+\left(\frac{\Omega\sin(\Omega\tau/2)+(\alpha+\beta)\cos(\Omega\tau/2)}{\Omega\cos(\Omega\tau/2)-(\alpha+\beta)\sin(\Omega\tau/2)}\right)^2}, \tag{3.6}$$

and

$$\theta = \tan^{-1}\left(\frac{\Omega\cos(\Omega\tau/2)-(\alpha+\beta)\sin(\Omega\tau/2)}{\Omega\sin(\Omega\tau/2)+(\alpha+\beta)\cos(\Omega\tau/2)}\right). \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.3) completes the proof. □

4. The delay equation: $t \geq \tau/2$

It will be shown that the properties of $\phi(t)$ in the interval $\tau/2 \leq t \leq \tau$ are different than those properties in the interval $t > \tau$. This suggests to solve the delay model in two separate intervals as will be discussed in the following subsections. In the previous section, four different types of solutions were obtained in the interval $0 \leq t \leq \tau/2$. However, only the periodic solution is invested in this section to construct the solution of the delay equation. Before launching to the main objective of this section, let us denote to the periodic solution in the interval $0 \leq t \leq \tau/2$ by $\phi_1(t)$, i.e.,

$$\phi(t) = \phi_1(t) = c_1\sin(\Omega t) + c_2\cos(\Omega t) - \frac{\gamma}{\alpha+\beta}, \quad \Omega = \sqrt{\beta^2-\alpha^2}, \quad \left|\frac{\alpha}{\beta}\right| < 1, \quad 0 \leq t \leq \tau/2, \tag{4.1}$$

where c_1 and c_2 are already defined by Eqs (2.12) and (2.13), respectively.

4.1. *The solution in the interval* $\tau/2 \leq t \leq \tau$

For $\tau/2 \leq t \leq \tau$ we have $0 \leq -t+\tau \leq \tau/2$, thus

$$\phi(-t+\tau) = \phi_1(-t+\tau) = -c_1\sin(\Omega(t-\tau)) + c_2\cos(\Omega(t-\tau)) - \frac{\gamma}{\alpha+\beta}. \tag{4.2}$$

Hence, we have the delay model:

$$\phi'(t) = \alpha\phi(t) + \beta\phi_1(-t+\tau) + \gamma, \quad \phi(t) = \phi_1(t)\ \forall\, t \in [0,\tau/2], \ \ \tau/2 \leq t \leq \tau, \tag{4.3}$$

under the condition:

$$\phi(\tau/2) = \phi_1(\tau/2) = \delta_1, \tag{4.4}$$

where δ_1 is calculated from Eq (4.1) as

$$\delta_1 = c_1\sin(\Omega\tau/2) + c_2\cos(\Omega\tau/2) - \frac{\gamma}{\alpha+\beta}. \tag{4.5}$$

Inserting (4.2) into (4.3) leads to the ODE:

$$\begin{gathered}\phi'(t) - \alpha\phi(t) = -\beta c_1\sin(\Omega(t-\tau)) + \beta c_2\cos(\Omega(t-\tau)) + \frac{\gamma\alpha}{\alpha+\beta}, \\ \phi(t) = \phi_1(t)\ \forall\, t \in [0,\tau/2], \ \ \tau/2 \leq t \leq \tau.\end{gathered} \tag{4.6}$$

The solution of Eq (4.6) under the condition (4.4) reads

$$\phi(t) = -\frac{\gamma}{\alpha+\beta} + \left(\delta_1 + \frac{\gamma}{\alpha+\beta}\right) e^{\alpha\left(t-\frac{\tau}{2}\right)} + \beta e^{\alpha t}\left(c_2 I_1(t) - c_1 I_2(t)\right), \tag{4.7}$$

where

$$I_1(t) = \int_{\frac{\tau}{2}}^{t} e^{-\alpha t}\cos\left(\Omega\left(t-\tau\right)\right) dt, \qquad I_2(t) = \int_{\frac{\tau}{2}}^{t} e^{-\alpha t}\sin\left(\Omega\left(t-\tau\right)\right) dt. \tag{4.8}$$

The integrals $I_1(t)$ and $I_2(t)$ can be evaluated analytically and, therefore, the solution (4.8) for the delay equation (4.3) is obtained in an exact form in the interval $\tau/2 \le t \le \tau$.

4.2. The solution in the interval $t > \tau$

Let us define $\phi(t) = \phi_2(t)$ for the solution in the previous interval $\tau/2 \le t \le \tau$. At $t = \tau$, we obtain $\phi(\tau) = \phi_2(\tau) = \delta_2$, where

$$\delta_2 = -\frac{\gamma}{\alpha+\beta} + \left(\delta_1 + \frac{\gamma}{\alpha+\beta}\right) e^{\frac{\alpha\tau}{2}} + \beta e^{\alpha\tau}\left(c_2 I_1(\tau) - c_1 I_2(\tau)\right), \tag{4.9}$$

and

$$I_1(\tau) = \int_{\frac{\tau}{2}}^{\tau} e^{-\alpha t}\cos\left(\Omega\left(t-\tau\right)\right) dt, \qquad I_2(\tau) = \int_{\frac{\tau}{2}}^{\tau} e^{-\alpha t}\sin\left(\Omega\left(t-\tau\right)\right) dt. \tag{4.10}$$

In the interval $t > \tau$, we have $-t + \tau < 0$ and thus $\phi(-t + \tau) = 0$. Therefore, the delay equation reduces to:

$$\phi'(t) - \alpha\phi(t) = \gamma, \quad \phi(t) = \phi_2(t)\ \forall\ t \in [\tau/2, \tau], \quad t > \tau. \tag{4.11}$$

The solution of the ODE (4.11) under the condition $\phi(\tau) = \delta_2$ is

$$\phi(t) = -\frac{\gamma}{\alpha} + \left(\delta_2 + \frac{\gamma}{\alpha}\right) e^{\alpha(t-\tau)}, \quad t > \tau. \tag{4.12}$$

5. Characteristics of the solution

In the previous sections, the solutions $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$ were obtained in the interval $[0, \tau/2]$ (advance equation) and in the intervals $[\tau/2, \tau]$ and (τ, ∞) for the delay equation. As mentioned in Remark 1, the continuity of the solution at $t = 0$ requires $\lambda = 0$. In this case, the solution $\phi(t)$ of the problem (1.1) is continuous for $t \ge 0$. For $\lambda \ne 0$, the solution is discontinuous at $t = 0$ but continuous at the points $t = \tau/2$ and $t = \tau$. The MATHEMATICA software is used to generate the results of the present paper. In Figures 1–4, the solutions $\phi_1(t)$ (blue curve), $\phi_2(t)$ (red curve), and $\phi_3(t)$ (green curve) are displayed.

The first black dot is the end point of the first interval $[0, \tau/2]$ and it represents the value of $\phi(t)$ at $t = \tau/2$, i.e., $\phi(\tau/2)$. Similarly, the second black dot is the end point of the second interval $[\tau/2, \tau]$ and it represents the value of $\phi(t)$ at $t = \tau$, i.e., $\phi(\tau)$. It is obvious from these figures that $\phi(t)$ is a continuous function in the full domain, where $\phi_1(\tau/2) = \phi_2(\tau/2)$ and $\phi_2(\tau) = \phi_3(\tau)$. The periodicity of the solution is clear in the intervals $[0, \tau]$ and $[\tau/2, \tau]$, while exponential decay/growth is observed in the rest of the domain (τ, ∞).

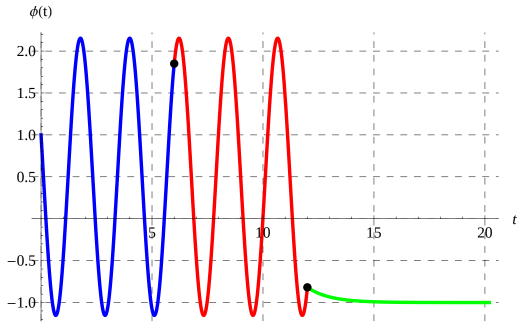

Figure 1. Plot of the solution in the interval $[0, \tau/2]$ for the advanced equation (blue curve) and in the intervals $[\tau/2, \tau]$ (red curve), (τ, ∞) (green curve) for the delay equation. The black dots represent the connection points between the solutions at $\lambda = 1$, $\alpha = -1$, $\beta = 3$ and $\gamma = -1$, when $\tau = 12$.

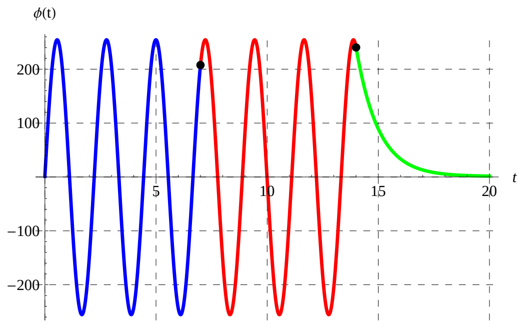

Figure 2. Plot of the solution in the interval $[0, \tau/2]$ for the advanced equation (blue curve) and in the intervals $[\tau/2, \tau]$ (red curve), (τ, ∞) (green curve) for the delay equation. The black dots represent the connection points between the solutions at $\lambda = 1$, $\alpha = -1$, $\beta = 3$ and $\gamma = 1$, when $\tau = 14$.

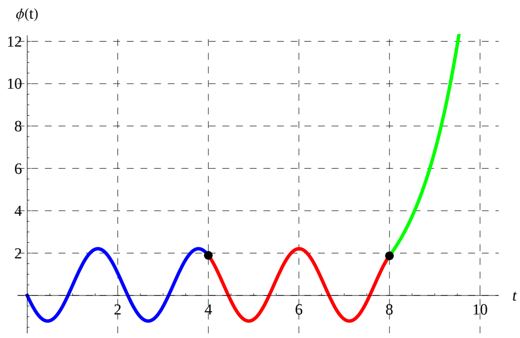

Figure 3. Plot of the solution in the interval $[0, \tau/2]$ for the advanced equation (blue curve) and in the intervals $[\tau/2, \tau]$ (red curve), (τ, ∞) (green curve) for the delay equation. The black dots represent the connection points between the solutions at $\lambda = 0$, $\alpha = 1$, $\beta = -3$ and $\gamma = 1$, when $\tau = 8$.

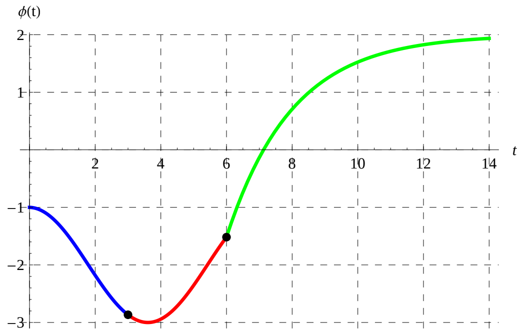

Figure 4. Plot of the solution in the interval $[0, \tau/2]$ for the advanced equation (blue curve) and in the intervals $[\tau/2, \tau]$ (red curve), (τ, ∞) (green curve) for the delay equation. The black dots represent the connection points between the solutions at $\lambda = -1$, $\alpha = -1/2$, $\beta = 1$ and $\gamma = 1$, when $\tau = 6$.

Regarding the continuity of the derivative $\phi'(t)$, Figures 5 and 6 indicate that $\phi'(t)$ is continuous at $t = \tau/2$ but discontinuous at $t = \tau$. The discontinuity of $\phi'(t)$ at $t = \tau$ returns to the fact that the initial condition is selected as a none-zero value, i.e., $\lambda \neq 0$. However, the derivative $\phi'(t)$ is still continuous at $t = \tau$ if $\lambda = 0$, where Figures 7 and 8 confirm this point. Moreover, $\phi'(t)$ is continuous at $t = \tau/2$, whatever the value of λ (zero or not).

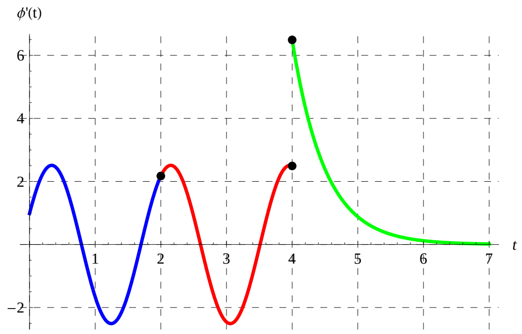

Figure 5. Plot of $\phi'(t)$ in the intervals $[0, \tau/2]$ (blue curve), $[\tau/2, \tau]$ (red curve), and (τ, ∞) (green curve) at $\lambda = 1$, $\alpha = -2$, $\beta = -4$ and $\gamma = 10$, when $\tau = 4$.

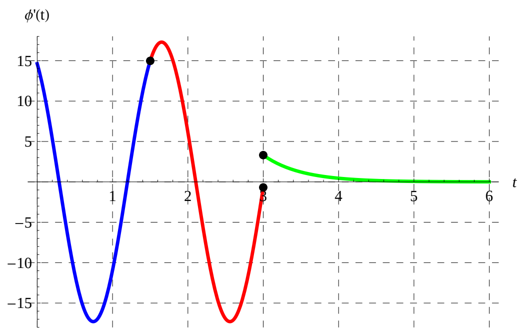

Figure 6. Plot of $\phi'(t)$ in the intervals $[0, \tau/2]$ (blue curve), $[\tau/2, \tau]$ (red curve), and (τ, ∞) (green curve) at $\lambda = 1$, $\alpha = -2$, $\beta = -4$ and $\gamma = -10$, when $\tau = 3$.

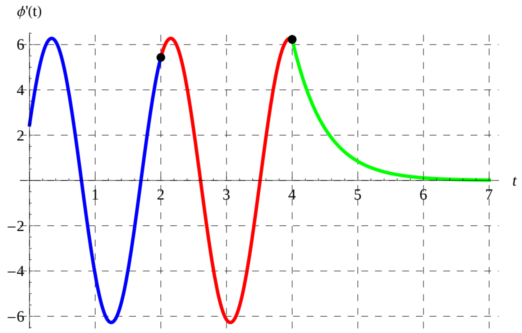

Figure 7. Plot of $\phi'(t)$ in the intervals $[0, \tau/2]$ (blue curve), $[\tau/2, \tau]$ (red curve), and (τ, ∞) (green curve) at $\lambda = 0$, $\alpha = -2$, $\beta = -4$ and $\gamma = 10$, when $\tau = 4$.

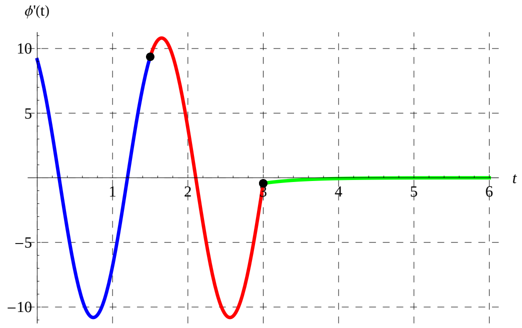

Figure 8. Plot of $\phi'(t)$ in the intervals $[0, \tau/2]$ (blue curve), $[\tau/2, \tau]$ (red curve), and (τ, ∞) (green curve) at $\lambda = 0$, $\alpha = -2$, $\beta = -4$ and $\gamma = -10$, when $\tau = 3$.

Such a conclusion can be easily proved theoretically as follows. At $t = \tau/2$, one can find from the second condition in Eq (2.2) that the left derivative at $t = \tau/2$ is $\phi'_-\left(\frac{\tau}{2}\right) = (\alpha+\beta)\phi_1\left(\frac{\tau}{2}\right) + \gamma$, where $\phi_1(t)$ is the solution in the interval $[0, \tau/2]$. The right derivative at $t = \tau/2$ can be obtained from Eq (4.3) as $\phi'_+\left(\frac{\tau}{2}\right) = (\alpha+\beta)\phi_2\left(\frac{\tau}{2}\right) + \gamma$, where $\phi_2(t)$ is the solution in the interval $[\tau/2, \tau]$. Since $\phi_1\left(\frac{\tau}{2}\right) = \phi_2\left(\frac{\tau}{2}\right)$, then $\phi'_-\left(\frac{\tau}{2}\right) = \phi'_+\left(\frac{\tau}{2}\right)$, thus $\phi'(t)$ is always continuous at $t = \tau/2$.

For $t = \tau$, Eq (4.3) gives the left derivative at $t = \tau$ by $\phi'_-(\tau) = \alpha\phi_2(\tau) + \beta\phi_1(0) + \gamma$, i.e., $\phi'_-(\tau) = \alpha\phi_2(\tau) + \beta\lambda + \gamma$. From Eq (4.11), the right derivative at $t = \tau$ can be obtained as $\phi'_+(\tau) = \alpha\phi_3(\tau) + \gamma$. Since $\phi_2(\tau) = \phi_3(\tau)$, then $\phi'_-(\tau) - \phi'_+(\tau) = \beta\lambda$, which implies that $\phi'_-(\tau) = \phi'_+(\tau)$ if $\beta\lambda = 0$. Hence, $\lambda = 0$ (assuming that $\beta \neq 0$) must be satisfied to ensure the continuity of the derivative at $t = \tau$.

6. Conclusions

In this paper, the SDE: $\phi'(t) = \alpha\phi(t) + \beta\phi(-t+\tau) + \gamma$ was investigated under the conditions $\phi(t) = 0\ \forall\ t < 0$, $\phi(0) = \lambda$, $\tau \geq 0$. It was shown that the present SDE belongs to the type of ADEs in the interval $[0, \tau/2]$ while it belongs to the type of DDEs in the interval $[\tau/2, \infty)$. Four different types of solutions were obtained for the advance equation in terms of the advance parameter τ under specific conditions such as the polynomial of first degree ($\alpha = \beta$), polynomial of second degree ($\alpha = -\beta$), the periodic solution ($\left|\frac{\alpha}{\beta}\right| < 1$), and the hyperbolic solution ($\left|\frac{\beta}{\alpha}\right| < 1$). The periodic solution was implemented to construct the solution of the delay equation. The characteristics of the solution were discussed theoretically and graphically in detail. The results indicated that the solution $\phi(t)$ is continuous in the full domain of the problem while the first derivative $\phi'(t)$ is discontinuous at a certain point and lies within the domain of the delay equation, mainly, at $t = \tau$ if $\lambda \neq 0$. However, it was demonstrated theoretically that $\phi'(t)$ is still continuous at $t = \tau$ if $\lambda = 0$, this conclusion was confirmed through some plots.

Furthermore, the existing results in the literature [15] were determined as special cases of the present ones. The current successful analysis suggests to generalize the results in the future by addressing the SDE: $\phi'(t) = \alpha\phi(t) + \beta\phi(-t + \tau) + f(t)$ as a future work, where $f(t)$ is an arbitrary real function. Generalization of the current results can also be extended to include fractional delay differential equations, which may be analyzed numerically [28] or analytically.

Author contributions

Abdulrahman B. Albidah: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing-review and editing, Visualization; Ibraheem M. Alsulami: Methodology, Validation, Formal analysis, Investigation, Writing-original draft preparation; Essam R. El-Zahar: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing-review and editing; Abdelhalim Ebaid: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data curation, Writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors extend the appreciation to the Deanship of Postgraduate Studies and Scientific Research at Majmaah University for funding this research work through the project number (R-2024-1211). This study is also supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2024/R/1445).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. H. I. Andrews, Third paper: Calculating the behaviour of an overhead catenary system for railway electrification, *Proc. Inst. Mech. Eng.*, **179** (1964), 809–846. https://doi.org/10.1243/PIME_PROC_1964_179_050_02

2. M. R. Abbott, Numerical method for calculating the dynamic behaviour of a trolley wire overhead contact system for electric railways, *Comput. J.*, **13** (1970), 363–368. https://doi.org/10.1093/comjnl/13.4.363
3. G. Gilbert, H. E. H. Davtcs, Pantograph motion on a nearly uniform railway overhead line, *Proc. Inst. Electr. Eng.*, **113** (1966), 485–492. https://doi.org/10.1049/piee.1966.0078
4. P. M. Caine, P. R. Scott, Single-wire railway overhead system, *Proc. Inst. Electr. Eng.*, **116** (1969), 1217–1221. https://doi.org/10.1049/piee.1969.0226
5. J. R. Ockendon, A. B. Tayler, The dynamics of a current collection system for an electric locomotive, *Proc. R. Soc. Lond. A Math. Phys. Eng. Sci.*, **322** (1971), 447–468. https://doi.org/10.1098/rspa.1971.0078
6. L. Fox, D. F. Mayers, J. R. Ockendon, A. B. Tayler, On a functional differential equation, *IMA J. Appl. Math.*, **8** (1971), 271–307. https://doi.org/10.1093/imamat/8.3.271
7. T. Kato, J. B. McLeod, The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$, *Bull. Am. Math. Soc.*, **77** (1971), 891–935.
8. A. Iserles, On the generalized pantograph functional-differential equation, *Eur. J. Appl. Math.*, **4** (1993), 1–38. https://doi.org/10.1017/S0956792500000966
9. V. A. Ambartsumian, On the fluctuation of the brightness of the milky way, *Doklady Akad Nauk USSR*, **44** (1994), 244–247.
10. J. Patade, S. Bhalekar, On analytical solution of Ambartsumian equation, *Natl. Acad. Sci. Lett.*, **40** (2017), 291–293. https://doi.org/10.1007/s40009-017-0565-2
11. H. O. Bakodah, A. Ebaid, Exact solution of Ambartsumian delay differential equation and comparison with Daftardar-Gejji and Jafari approximate method, *Mathematics*, **6** (2018), 331. https://doi.org/10.3390/math6120331
12. S. M. Khaled, E. R. El-Zahar, A. Ebaid, Solution of Ambartsumian delay differential equation with conformable derivative, *Mathematics*, **7** (2019), 425. https://doi.org/10.3390/math7050425
13. D. Kumar, J. Singh, D. Baleanu, S. Rathore, Analysis of a fractional model of the Ambartsumian equation, *Eur. Phys. J. Plus*, **133** (2018), 1330–259. https://doi.org/10.1140/epjp/i2018-12081-3
14. A. Ebaid, H. K. Al-Jeaid, On the exact solution of the functional differential equation $y'(t) = ay(t) + by(-t)$, *Adv. Differ. Equ. and Contr.*, **26** (2022), 39–49.
15. N. A. M. Alshomrani, A. Ebaid, F. Aldosari, M. D. Aljoufi, On the exact solution of a scalar differential equation via a simple analytical approach, *Axioms*, **13** (2024), 129. https://doi.org/10.3390/axioms13020129
16. G. Adomian, *Solving frontier problems of physics: The decomposition method*, Boston: Springer Science & Business Media, 2013.
17. A. M. Wazwaz, Adomian decomposition method for a reliable treatment of the Bratu-type equations, *Appl. Math. Comput.*, **166** (2005), 652–663. https://doi.org/10.1016/j.amc.2004.06.059
18. J. S. Duan, R. Rach, A new modification of the Adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations, *Appl. Math. Comput.*, **218** (2011), 4090–4118. https://doi.org/10.1016/j.amc.2011.09.037

19. J. Diblík, M. Kúdelcíková, Two classes of positive solutions of first order functional differential equations of delayed type, *Nonlinear Anal.*, **75** (2012), 4807–4820. https://doi.org/10.1016/j.na.2012.03.030
20. A. Alshaery, A. Ebaid, Accurate analytical periodic solution of the elliptical Kepler equation using the Adomian decomposition method, *Acta Astronaut.*, **140** (2017), 27–33. https://doi.org/10.1016/j.actaastro.2017.07.034
21. W. J. Li, Y. N. Pang, Application of Adomian decomposition method to nonlinear systems, *Adv. Differ. Equ.*, **67** (2020). https://doi.org/10.1186/s13662-020-2529-y
22. A. Ebaid, H. K. Al-Jeaid, H. Al-Aly, Notes on the perturbation solutions of the boundary layer flow of nanofluids past a stretching sheet, *Appl. Math. Sci.*, **122** (2013), 6077–6085. http://dx.doi.org/10.12988/ams.2013.36277
23. A. Ebaid, Remarks on the homotopy perturbation method for the peristaltic flow of Jeffrey fluid with nano-particles in an asymmetric channel, *Comput. Math. Appl.*, **68** (2014), 77–85. https://doi.org/10.1016/j.camwa.2014.05.008
24. Z. Ayati, J. Biazar, On the convergence of Homotopy perturbation method, *J. Egyptian Math. Soc.*, **23** (2015), 424–428. http://dx.doi.org/10.1016/j.joems.2014.06.015
25. S. M. Khaled, The exact effects of radiation and joule heating on Magnetohydrodynamic Marangoni convection over a flat surface, *Therm. Sci.*, **22** (2018), 63–72. https://doi.org/10.2298/TSCI151005050K
26. O. Nave, Modification of semi-analytical method applied system of ODE, *Mod. Appl. Sci.*, **14** (2020), 75–81. https://doi.org/10.5539/mas.v14n6p75
27. A. F. Yeniçerioğlu, C. Yazıcı, S. Pinelas, On the stability and behavior of solutions in mixed differential equations with delays and advances, *Math. Method. Appl. Sci.*, **45** (2022), 4468–4496. https://doi.org/10.1002/mma.8049
28. W. M. Abd-Elhameed, Y. H. Youssri, A. G. Atta, Tau algorithm for fractional delay differential equations utilizing seventh-kind Chebyshev polynomials, *J. Math. Model.*, **12** (2024), 277–299. https://doi.org/10.22124/jmm.2024.25844.2295