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Research article

On the Cauchy problem of 3D nonhomogeneous micropolar fluids with density-dependent viscosity

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Abstract: In this paper, we considered the global well-posedness of strong solutions to the Cauchy problem of three-dimensional (3D) nonhomogeneous incompressible micropolar fluids with density-dependent viscosity and vacuum. Based on the energy method, some key a priori exponential decay-in-time rates of strong solutions are obtained. As a result, the existence and large-time asymptotic behavior of strong solutions in the whole space $\mathbb{R}^3$ are established, provided that the initial mass is sufficiently small. Note that this result is proven without any compatibility conditions.

Keywords: nonhomogeneous micropolar fluids; density-dependent viscosity; global well-posedness
Mathematics Subject Classification: 35Q35, 76D03

1. Introduction

The nonhomogeneous incompressible micropolar fluids with density-dependent viscosity ([12,20]) in $\mathbb{R}^3$ read as follows:

$$\begin{cases}\rho_t + \mathrm{div}(\rho\mathbf{u}) = 0,\\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P(\rho) = \mathrm{div}\Big((\mu(\rho)+\zeta)\nabla\mathbf{u}\Big) + 2\zeta\mathrm{rot}\mathbf{w},\\ (\rho\mathbf{w})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{w}) + 4\zeta\mathbf{w} = \mu'\Delta\mathbf{w} + (\mu'+\lambda')\nabla\mathrm{div}\mathbf{w} + 2\zeta\mathrm{rot}\mathbf{u},\\ \mathrm{div}\mathbf{u} = 0,\end{cases} \tag{1.1}$$

where $\rho, \mathbf{u} = (u^1, u^2, u^3)$, $\mathbf{w} = (w^1, w^2, w^3)$, and P denote the fluid density, velocity, micro-rotational velocity, and pressure, respectively. The viscosity coefficient $\mu(\rho)$ satisfies

$$\mu \in C^1[0,\infty), \quad \mu(\rho) \geqslant \alpha > 0, \tag{1.2}$$

for some positive constant α, while the constants μ' and λ' are the angular viscosities satisfying $\mu' > 0$ and $\mu' + \lambda' \geqslant 0$, and the constant $\zeta > 0$ denotes the dynamic micro-rotation viscosity.

In this paper, we consider the Cauchy problem of (1.1)–(1.2) with the far-field behavior

$$(\rho, \mathbf{u}, \mathbf{w})(x,t) \to (0,0,0) \quad \text{as} \quad |x| \to \infty \tag{1.3}$$

and the initial conditions

$$(\rho, \rho\mathbf{u}, \rho\mathbf{w})(x,0) = (\rho_0, \rho_0\mathbf{u}_0, \rho_0\mathbf{w}_0)(x) \quad \text{with} \quad x \in \mathbb{R}^3. \tag{1.4}$$

A micropolar fluid system is the study of fluids that exhibit micro-rotational effects and micro-rotational inertia and can be viewed as non-Newtonian fluids. It can be used to describe many phenomena that appear in a large number of complex fluids, such as suspensions, animal blood, and liquid crystals. The micropolar fluid system reduces to the Navier-Stokes equations when there is no microstructure ($\zeta = 0$ and $\mathbf{w} = 0$) and has been discussed by many mathematicians (see [1, 2, 5–10, 13, 17, 19], and references therein).

When it comes to the case that $\zeta \neq 0$ and $\mathbf{w} \neq 0$, there have been substantial developments on the global regularity problem concerning nonhomogeneous micropolar fluids (1.1) with constant viscosity μ. When the initial density is strictly away from vacuum, Braz e Silva and his cooperators [4] investigated the global existence and uniqueness of solutions for 3D nonhomogeneous asymmetric fluids by using an approach and Lagrangian coordinates under suitable initial conditions. Qian-Chen-Zhang [15] studied the global existence of weak and strong solutions to 3D nonhomogeneous incompressible asymmetric fluid equations. For initial velocities sufficiently small in the critical Besov space, global Fujita-Kato type solutions with initial density in the bounded function space and that have a positive lower bound are obtained, and this result extends the classical one on the life-span by Leray. Subsequently, Qian-He-Zhang [16] investigated the global existence and uniqueness of the solutions for the 2D inhomogeneous incompressible asymmetric fluids, with the initial (angular) velocity being located in sub-critical Sobolev spaces $H^s(\mathbb{R}^2)(0 < s < 1)$ and the initial density being bounded from above and below by some positive constants. In particular, the uniqueness of the solution in [16] is also obtained without any more regularity assumptions on the initial density. When the initial density contains a vacuum state, Braz e Silva and Santos [3] established the existence of global in-time weak solutions for the equations of asymmetric incompressible fluids with variable density. Zhang-Zhu [18] proved the global existence of strong solutions under the condition of the following compatibility:

$$\begin{cases} -(\mu+\zeta)\Delta\mathbf{u}_0 + \nabla P(\rho_0) - 2\zeta\mathrm{rot}\mathbf{w}_0 = \rho_0^{1/2} g_1, \\ -\mu'\Delta\mathbf{w}_0 - (\mu'+\lambda')\nabla\mathrm{div}\mathbf{w}_0 + 4\zeta\mathbf{w}_0 - 2\zeta\mathrm{rot}\mathbf{u}_0 = \rho_0^{1/2} g_2, \end{cases}$$

for some $(P(\rho_0), g_1, g_2) \in H^1 \times L^2 \times L^2$.

When we consider the case of $\mu = \mu(\rho)$, Liu-Zhong [12] showed that the initial boundary value problem of 2D nonhomogeneous micropolar fluids with density-dependent viscosity has a global and unique strong solution under the assumption of the smallness of $\|\nabla\mu(\rho_0)\|_{L^q}$. Qian-Qu [14] investigated the 3D inhomogeneous incompressible asymmetric fluids system and proved local well-posedness for initial velocity in the critical Besov space $\dot{B}^{3/p}_{p,1}$ for $1 < p < 6$ and initial density ρ_0 satisfying that $\rho_0 - 1$ is in the critical Besov space and that ρ_0 is bounded away from zero. Zhong [20] also considered the same model in 3D cases and established the global existence and uniqueness of strong solutions, provided that the initial energy is sufficiently small. It is worth noting that there is no need to impose some compatibility condition on the initial data.

It should be noted that although the large initial velocity is allowed in [12], it excludes large oscillations of the initial density. A natural question arises: where can we establish the global strong solutions to the 3D Cauchy problem of (1.1)–(1.4) not only with large initial velocity but also allowing large oscillations of the initial density? In fact, this is the main aim of this paper.

Before stating the main results, we set

$$\int f\,dx \triangleq \int_{\mathbb{R}^3} f\,dx, \qquad \|(f,g)\|_{L^p} \triangleq \|f\|_{L^p} + \|g\|_{L^p}.$$

For $1 \leqslant r \leqslant \infty$ and $\beta > 0$, we denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$\begin{cases} L^r = L^r(\mathbb{R}^3), \quad W^{k,r} = W^{k,r}(\mathbb{R}^3), \quad H^k = W^{k,2}, \\ D^{k,r} = D^{k,r}(\mathbb{R}^3) = \{v \in L^1_{\mathrm{loc}}(\mathbb{R}^3) | \nabla^k v \in L^r(\mathbb{R}^3)\}, \quad D^k = D^{k,2}, \\ D^1 = \{v \in L^6(\mathbb{R}^3) | \nabla v \in L^2(\mathbb{R}^3)\}, \\ C^\infty_{0,\sigma} = \{f \in C^\infty_0 | \mathrm{div} f = 0\}, \quad D^1_{0,\sigma} = \overline{C^\infty_{0,\sigma}} \text{ closure in the norm of } D^1. \end{cases}$$

The main result can be stated as follows:

Theorem 1.1. *For constant $\bar\rho$ and any given number $q \in (3,6)$, assume that the initial data $(\rho_0, \mathbf{u}_0, \mathbf{w}_0)$ satisfies*

$$0 \leqslant \rho_0 \leqslant \bar\rho, \quad \rho_0 \in L^1 \cap H^1, \quad \nabla\mu(\rho_0) \in L^q, \quad \mathbf{u}_0 \in D^1_{0,\sigma}, \quad \mathbf{w}_0 \in H^1_0. \tag{1.5}$$

Then there exists a positive constant ε, depending only on $\zeta, \mu', \lambda', \bar\rho, q, \alpha, \beta \triangleq \sup_{[0,\bar\rho]} \mu(\rho)$, $\|\nabla\mu(\rho_0)\|_{L^q}$, $\|\nabla \mathbf{u}_0\|_{L^2}$ and $\|\nabla \mathbf{w}_0\|_{L^2}$ such that if

$$m_0 \triangleq \|\rho_0\|_{L^1} \leqslant \varepsilon, \tag{1.6}$$

then the problem (1.1)–(1.4) possesses a unique global strong solution $(\rho, \mathbf{u}, \mathbf{w})$ in $\mathbb{R}^3 \times (0,\infty)$ satisfying that for any $0 < t < T < \infty$ and $s \in (3,q)$,

$$\begin{cases} 0 \leqslant \rho \leqslant \bar\rho, \quad \rho \in L^\infty(0,\infty; L^1 \cap H^1) \cap C([0,\infty); L^1 \cap H^1), \\ (t\nabla\mathbf{u},\ t\nabla\mathbf{w}) \in L^\infty(0,\infty; H^1) \cap L^2(0,\infty; W^{1,s}), \\ t\nabla P \in L^\infty(0,\infty; L^2) \cap L^2(0,\infty; L^s), \\ (t\nabla\mathbf{u},\ t\nabla\mathbf{w}) \in C([0,\infty); H^1), \quad (\rho\mathbf{u}, \rho\mathbf{w}) \in C([0,\infty); L^2), \\ (t\rho^{1/2}\mathbf{u}_t, t\rho^{1/2}\mathbf{w}_t) \in L^\infty(0,\infty; L^2), \quad (t\nabla\mathbf{u}_t, t\nabla\mathbf{w}_t) \in L^2(0,\infty; L^2). \end{cases} \tag{1.7}$$

Moreover,

$$\sup_{0\leqslant t\leqslant T} \|\nabla\mu(\rho)\|_{L^q} \leqslant 2\|\nabla\mu(\rho_0)\|_{L^q},$$

and for any $t \geqslant 1$,

$$\begin{aligned} &\|\nabla\mathbf{u}_t(\cdot,t)\|^2_{H^1} + \|\nabla\mathbf{w}_t(\cdot,t)\|^2_{H^1} + \|\nabla P(\cdot,t)\|^2_{L^2} \\ &\quad + \|\rho^{1/2}\mathbf{u}_t(\cdot,t)\|^2_{L^2} + \|\rho^{1/2}\mathbf{w}_t(\cdot,t)\|^2_{L^2} \leqslant Ce^{-\sigma t}, \end{aligned}$$

where $\sigma = 3\sigma_1(\frac{\pi}{2})^{4/3}\|\rho_0\|^{-1}_{L^{3/2}}$ with $\sigma_1 = \min\{\alpha, \mu'\}$.

Remark 1.1. It is worth noting that Theorem 1.1 holds for arbitrarily large initial velocity with a smallness only on the initial mass, which generalizes the result of [14], where they need the smallness assumption on $\|\mathbf{u}_0\|_{\dot{B}^{3/p}_{p,1}}$ with $1 < p < 6$.

The rest of this paper is organized as follows: In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to a priori estimates, and Theorem 1.1 is also proved in Section 3.

2. Preliminaries

In this section, we list some auxiliary lemmas that will be used later. First of all, we start with the local existence of strong solutions that can be obtained from similar arguments as used in [13,20], and we omit the details.

Lemma 2.1. *Assume that* $(\rho_0, \mathbf{u}_0, \mathbf{w}_0)$ *satisfies* (1.5). *Then, there exists a small positive time* T_0 *such that the problem* (1.1)–(1.4) *has a unique strong solution* $(\rho, \mathbf{u}, \mathbf{w})$ *on* $\mathbb{R}^3 \times (0, T_0]$.

Next, the following well-known Gagliardo-Nirenberg inequality will be used more frequently later (see [11]).

Lemma 2.2. *Let* $p \in [2, \frac{3s}{3-s}]$ *for* $s \in [2, 3)$*, or* $p \in [2, \infty]$ *for* $s = 3$*, and let* $q \in (1, \infty)$, $r \in (3, \infty)$*. There exists some generic constant* $C > 0$ *may depend on s and r such that for* $f \in L^2 \cap D_0^{1,s}$ *and* $g \in L^q \cap D_0^{1,r}$*, we have*

$$\|f\|_{L^p}^p \leqslant C\|f\|_{L^2}^{p-3s(p-2)/(5s-6)}\|\nabla f\|_{L^s}^{3s(p-2)/(5s-6)}, \tag{2.1}$$

and

$$\|g\|_{L^\infty} \leqslant C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.2}$$

The following regularity results on the Stokes equations will be used for the derivations of higher-order a priori estimates (see [10]).

Lemma 2.3. *For constants* $q \in (3, 6), \alpha > 0$ *and* $\beta > 0$*, in addition to* (1.2)*, assume that* $\mu(\rho)$ *satisfies*

$$\nabla\mu(\rho) \in L^q, \quad 0 < \alpha \leqslant \mu(\rho) \leqslant \beta < \infty.$$

Then, if $\mathbf{G} \in L^r$ *with* $r \in (2, q)$*, there exists some positive constant C depending only on* α, β, r *and q such that the unique weak solution* $(\mathbf{u}, P) \in D^1_{0,\sigma} \times L^2$ *to the following problem*

$$\begin{cases} -\mathrm{div}\Big((\mu(\rho) + \zeta)\nabla\mathbf{u}\Big) + \nabla P = \mathbf{G}, & x \in \mathbb{R}^3, \\ \mathrm{div}\mathbf{u} = 0, & x \in \mathbb{R}^3, \\ \mathbf{u}(x) \to 0, & |x| \to \infty, \end{cases}$$

satisfies

$$\|\nabla^2\mathbf{u}\|_{L^2} + \|\nabla P\|_{L^2} \leqslant C\|\mathbf{G}\|_{L^2}\Big(1 + \|\nabla\mu(\rho)\|_{L^q}^{q/(q-3)}\Big), \tag{2.3}$$

and

$$\|\nabla^2\mathbf{u}\|_{L^r} + \|\nabla P\|_{L^r} \leqslant C\|\mathbf{G}\|_{L^r}\Big(1 + \|\nabla\mu(\rho)\|_{L^q}^{q(5r-6)/2r(q-3)}\Big). \tag{2.4}$$

3. A priori estimates

In this section, we will establish some necessary a priori bounds of local strong solutions $(\rho, \mathbf{u}, \mathbf{w})$ to the Cauchy problem (1.1)–(1.4), whose existence is guaranteed by Lemma 2.1. Thus, let $T > 0$ be a fixed time and $(\rho, \mathbf{u}, \mathbf{w})$ be the smooth solution to (1.1)–(1.4) on $\mathbb{R}^3 \times (0, T]$ with smooth initial data $(\rho_0, \mathbf{u}_0, \mathbf{w}_0)$ satisfying (1.5).

We have the following key a priori estimates on $(\rho, \mathbf{u}, \mathbf{w})$.

Proposition 3.1. *There exists some positive constant ε_0 depending only on $q, \zeta, \bar{\rho}, \alpha, \beta, \mu'$, $\|\rho_0\|_{L^{3/2}}$, $\|\nabla\mathbf{u}_0\|_{L^2}, \|\nabla\mathbf{w}_0\|_{L^2}$, and $\|\nabla\mu(\rho_0)\|_{L^q}$ such that if $(\rho, \mathbf{u}, \mathbf{w})$ is a smooth solution of* (1.1)–(1.4) *on* $\mathbb{R}^3 \times (0, T]$ *satisfying*

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho)\|_{L^q} \leqslant 4\|\nabla\mu(\rho_0)\|_{L^q}, \tag{3.1}$$

the following estimates hold

$$\sup_{t\in[0,T]} \|\nabla\mu(\rho)\|_{L^q} \leqslant 2\|\nabla\mu(\rho_0)\|_{L^q}, \tag{3.2}$$

provided that

$$m_0 \leqslant \varepsilon_0. \tag{3.3}$$

The proof of Proposition 3.1 consists of Lemmas 3.1–3.4 and is to be completed by the end of this section. Throughout this section, for simplicity, we denote by C or $C_i (i = 1, 2, \cdots)$ the generic positive constants, which may depend on $q, \zeta, \bar{\rho}, \alpha, \beta, \mu', \lambda', \|\rho_0\|_{L^{3/2}}, \|\nabla\mathbf{u}_0\|_{L^2}$ and $\|\nabla\mathbf{w}_0\|_{L^2}$, but are independent of time $T > 0$ and m_0.

We begin with the following estimates:

Lemma 3.1. *Let $(\rho, \mathbf{u}, \mathbf{w})$ be a smooth solution of* (1.1)–(1.4) *on* $\mathbb{R}^3 \times (0, T]$ *satisfying* (1.5). *Then one has*

$$\sup_{t\in[0,T]} \|\rho\|_{L^p} \leqslant C\|\rho_0\|_{L^p}, \quad \text{for } p \in [1, \infty], \tag{3.4}$$

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|\rho^{1/2}\mathbf{u}\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}\|_{L^2}^2\right) + \int_0^T \left(\alpha\|\nabla\mathbf{u}\|_{L^2}^2 + \mu'\|\nabla\mathbf{w}\|_{L^2}^2\right) dt \\ &\quad + \int_0^T \zeta\|\mathrm{rot}\mathbf{u} - 2\mathbf{w}\|_{L^2}^2 dt \leqslant Cm_0^{2/3} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\sup_{t\in[0,T]} e^{\sigma t}\left(\|\rho^{1/2}\mathbf{u} + \|\rho^{1/2}\mathbf{w}\|_{L^2}^2\right) + \int_0^T e^{\sigma t} \left(\alpha\|\nabla\mathbf{u}\|_{L^2}^2 + \mu'\|\nabla\mathbf{w}\|_{L^2}^2\right) dt \\ &\quad + \int_0^T e^{\sigma t}\zeta\|\mathrm{rot}\mathbf{u} - 2\mathbf{w}\|_{L^2}^2 dt \leqslant Cm_0^{2/3}, \end{aligned} \tag{3.6}$$

where $\sigma = 3\sigma_1(\frac{\pi}{2})^{4/3}\|\rho_0\|_{L^{3/2}}^{-1}$ with $\sigma_1 = \min\{\alpha, \mu'\}$.

Proof. First, Eq (3.4) can be shown by standard arguments ([8]).

Next, in order to prove (3.5), we multiply $(1.1)_2$ and $(1.1)_3$ by $\mathbf{u}$ and $\mathbf{w}$, respectively, and in integrating the resulting equations by parts over $\mathbb{R}^3$, we get after adding them together and using $(1.1)_4$ that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\rho^{1/2}\mathbf{u},\rho^{1/2}\mathbf{w})\|_{L^2}^2+\|\mu(\rho)^{1/2}\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{w}\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2\\&\quad+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2=0.\end{aligned}\tag{3.7}$$

Integrating (3.7) over $[0,T]$ gives

$$\begin{aligned}&\sup_{t\in[0,T]}\|(\rho^{1/2}\mathbf{u},\rho^{1/2}\mathbf{w})\|_{L^2}^2+\int_0^T\Big(\alpha\|\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{w}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)\,dt\\&\leqslant C\|\rho_0\|_{L^{3/2}}\Big(\|\mathbf{u}_0\|_{L^6}^2+\|\mathbf{w}_0\|_{L^6}^2\Big),\end{aligned}\tag{3.8}$$

which, together with (2.1) and (3.4), yields (3.5).

Finally, we notice from (2.1), (3.4), and (3.8) and Hölder's inequality that

$$\begin{aligned}\|\rho^{1/2}\mathbf{u}\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}\|_{L^2}^2&\leqslant\|\rho\|_{L^{3/2}}\Big(\|\mathbf{u}\|_{L^6}^2+\|\mathbf{w}\|_{L^6}^2\Big)\\&\leqslant\frac{1}{3}\Big(\frac{2}{\pi}\Big)^{4/3}\|\rho_0\|_{L^{3/2}}\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2\Big),\end{aligned}\tag{3.9}$$

where we have used the following fact:

$$\|f\|_{L^6}^2\leqslant\frac{1}{3}\Big(\frac{2}{\pi}\Big)^{4/3}\|\nabla f\|_{L^2}^2,\quad\text{for any } f\in D^1.$$

Combining (3.7) with (3.9), one has

$$\frac{1}{2}\frac{d}{dt}\Big(\|\rho^{1/2}\mathbf{u}\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}\|_{L^2}^2\Big)+\sigma\Big(\|\rho^{1/2}\mathbf{u}\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}\|_{L^2}^2\Big)\leqslant 0,$$

where $\sigma=3\sigma_1(\frac{\pi}{2})^{4/3}\|\rho_0\|_{L^{3/2}}^{-1}$ with $\sigma_1=\min\{\alpha,\mu'\}$. By using Gronwall's inequality, one has

$$\|\rho^{1/2}\mathbf{u}\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}\|_{L^2}^2\leqslant e^{-2\sigma t}\Big(\|\rho_0^{1/2}\mathbf{u}_0\|_{L^2}^2+\|\rho_0^{1/2}\mathbf{w}_0\|_{L^2}^2\Big).\tag{3.10}$$

Multiplying (3.7) by $e^{\sigma t}$ and using (3.10) show that

$$\begin{aligned}&\frac{d}{dt}\Big[e^{\sigma t}\|(\rho^{1/2}\mathbf{u},\rho^{1/2}\mathbf{w})\|_{L^2}^2\Big]+2e^{\sigma t}\Big(\alpha\|\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{w}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)\\&\leqslant\sigma e^{\sigma t}\Big(\|\rho^{1/2}\mathbf{u}\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}\|_{L^2}^2\Big)\\&\leqslant\sigma e^{-\sigma t}\Big(\|\rho_0^{1/2}\mathbf{u}_0\|_{L^2}^2+\|\rho_0^{1/2}\mathbf{w}_0\|_{L^2}^2\Big).\end{aligned}\tag{3.11}$$

Integrating the above inequality over $[0,T]$ leads to (3.6). □

Remark 3.1. Evidently, we can infer from (1.2) and (3.4) that

$$0<\alpha\leqslant\mu(\rho)\leqslant\beta\triangleq\max_{0\leqslant\rho\leqslant\bar{\rho}}\mu(\rho)<\infty.\tag{3.12}$$

Lemma 3.2. *Let the condition of* (3.1) *be in force, then there exists some positive constant* ε_1, *depending only on* $q, \zeta, \bar{\rho}, \alpha, \beta, \mu', \lambda', \|\nabla\mu(\rho_0)\|_{L^q}, \|\nabla\mathbf{u}_0\|_{L^2}$ *and* $\|\nabla\mathbf{w}_0\|_{L^2}$ *such that if*

$$\begin{aligned}\sup_{t\in[0,T]}&\Big(\alpha\|\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{w}\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)\\&+\int_0^T\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)dt\\\leqslant 4&\Big(\beta\|\nabla\mathbf{u}_0\|_{L^2}^2+\mu'\|\nabla\mathbf{w}_0\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}_0\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}_0-2\mathbf{w}_0\|_{L^2}^2\Big),\end{aligned}\tag{3.13}$$

then

$$\begin{aligned}\sup_{t\in[0,T]}&\Big(\alpha\|\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{w}\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)\\&+\int_0^T\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)dt\\\leqslant 2&\Big(\beta\|\nabla\mathbf{u}_0\|_{L^2}^2+\mu'\|\nabla\mathbf{w}_0\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}_0\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}_0-2\mathbf{w}_0\|_{L^2}^2\Big),\end{aligned}\tag{3.14}$$

provided

$$m_0 \leqslant \varepsilon_1.$$

Moreover, for $i = 1, 2, 3$ *and* σ, *as in Lemma 3.1, one has*

$$\sup_{t\in[0,T]}\Big[t^i\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2\Big)\Big]+\int_0^T t^i\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)dt\leqslant Cm_0^{2/3},\tag{3.15}$$

and

$$\sup_{t\in[0,T]}\Big[e^{\sigma t}\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2\Big)\Big]+\int_0^T e^{\sigma t}\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)dt\leqslant Cm_0^{2/3}.\tag{3.16}$$

Proof. The Eq $(1.1)_1$ can be written as

$$\Big[\mu(\rho)\Big]_t+\mathbf{u}\cdot\nabla\mu(\rho)=0.\tag{3.17}$$

Multiplying $(1.1)_2$ by $\mathbf{u}_t$ and integrating the resulting equation over $\mathbb{R}^3$, one can deduce from (3.17) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\Big(\|\mu^{1/2}(\rho)\nabla\mathbf{u}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}\|_{L^2}^2\Big)+\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2=2\zeta\Big(\int\mathrm{rot}\mathbf{u}\cdot\mathbf{w}dx\Big)_t\\&-2\zeta\int\mathrm{rot}\mathbf{u}\cdot\mathbf{w}_t dx-\int\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t dx-\frac{1}{2}\int\mathbf{u}\cdot\nabla\mu(\rho)|\nabla\mathbf{u}|^2dx,\end{aligned}\tag{3.18}$$

where we have used the fact that $\Delta\mathbf{u}+\mathrm{rot}(\mathrm{rot}\mathbf{u})=\nabla\mathrm{div}\mathbf{u}=0$.

Multiplying $(1.1)_3$ by $\mathbf{w}_t$ and integrating by parts over $\mathbb{R}^3$ show that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\Big(\mu'\|\nabla\mathbf{w}\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2+4\zeta\|\mathbf{w}\|_{L^2}^2\Big)+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\\&=2\zeta\int\mathrm{rot}\mathbf{u}\cdot\mathbf{w}_t dx-\int\rho\mathbf{u}\cdot\nabla\mathbf{w}\cdot\mathbf{w}_t dx,\end{aligned}\tag{3.19}$$

Combining (3.18) with (3.19), yields

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\mu^{1/2}(\rho)\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{w}\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)\\
&\qquad+2\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+2\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\\
&=-2\int\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}_t dx-2\int\rho\mathbf{u}\cdot\nabla\mathbf{w}\cdot\mathbf{w}_t dx-\int\mathbf{u}\cdot\nabla\mu(\rho)|\nabla\mathbf{u}|^2dx=\sum_{i=1}^{3}I_i.
\end{aligned}\tag{3.20}$$

Now, we estimate $I_i(i=1,2,3.)$ as follows: Hölder's inequality, together with (2.1), (2.2), and (3.4), gives

$$\begin{aligned}
I_1&\leqslant\frac{1}{2}\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+2\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^3}^2\\
&\leqslant\frac{1}{2}\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+C(\bar\rho)\|\nabla\mathbf{u}\|_{L^2}^3\|\nabla^2\mathbf{u}\|_{L^2}.
\end{aligned}\tag{3.21}$$

Similarly,

$$\begin{aligned}
I_2&\leqslant\frac{1}{2}\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2+2\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{w}\|_{L^2}^2\\
&\leqslant\frac{1}{2}\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2+C(\bar\rho)\|\nabla\mathbf{w}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2},
\end{aligned}\tag{3.22}$$

and

$$I_3\leqslant C(\beta)\|\mathbf{u}\|_{L^6}\|\nabla^2\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^3}\leqslant C(\beta)\|\nabla\mathbf{u}\|_{L^2}^{3/2}\|\nabla^2\mathbf{u}\|_{L^2}^{3/2}.\tag{3.23}$$

Putting (3.21)–(3.23) into (3.20), one has

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\mu^{1/2}(\rho)\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{w}\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)\\
&\qquad+\frac{3}{2}\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\\
&\leqslant C\Big(\|\nabla\mathbf{u}\|_{L^2}^3+\|\nabla\mathbf{w}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}\Big)\|\nabla^2\mathbf{u}\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^2}^{3/2}\|\nabla^2\mathbf{u}\|_{L^2}^{3/2}.
\end{aligned}\tag{3.24}$$

We know from $(1.1)_2$ that $(\mathbf{u},P)$ satisfies the following system:

$$\begin{cases}
-\mathrm{div}\Big[\big(\mu(\rho)+\zeta\big)\nabla\mathbf{u}\Big]+\nabla P=-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}+2\zeta\mathrm{rot}\mathbf{w}, & x\in\mathbb{R}^3,\\
\mathrm{div}\mathbf{u}=0, & x\in\mathbb{R}^3,\\
\mathbf{u}(x)\to 0, & |x|\to\infty,
\end{cases}$$

Taking $\mathbf{G}=-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}+2\zeta\mathrm{rot}\mathbf{w}$ and $r=2$ in (2.3), we deduce from (3.1), (3.4), and Cauchy-Schwarz's inequality that

$$\begin{aligned}
&\|\nabla^2\mathbf{u}\|_{L^2}+\|\nabla P\|_{L^2}\\
&\leqslant C\|-\rho\mathbf{u}_t-\rho\mathbf{u}\cdot\nabla\mathbf{u}+2\zeta\mathrm{rot}\mathbf{w}\|_{L^2}\Big(1+\|\nabla\mu(\rho)\|_{L^q}^{q/(q-3)}\Big)\\
&\leqslant C\|\rho\|_{L^\infty}^{1/2}\|\rho^{1/2}\mathbf{u}_t\|_{L^2}+C\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}+C\|\nabla\mathbf{w}\|_{L^2}\\
&\leqslant C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^2}^{3/2}\|\nabla^2\mathbf{u}\|_{L^2}^{1/2}+C\|\nabla\mathbf{w}\|_{L^2}\\
&\leqslant\frac{1}{2}\|\nabla^2\mathbf{u}\|_{L^2}+C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^2}^3+C\|\nabla\mathbf{w}\|_{L^2},
\end{aligned}$$

which gives

$$\|\nabla^2\mathbf{u}\|_{L^2}+\|\nabla P\|_{L^2}\leqslant C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^2}^3+C\|\nabla\mathbf{w}\|_{L^2}. \tag{3.25}$$

Taking (3.25) into (3.24), one deduces from (3.13) that

$$\begin{aligned}&\frac{d}{dt}\Big(\|\mu^{1/2}(\rho)\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{w}\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)\\&\quad+\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\leqslant C\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2\Big).\end{aligned} \tag{3.26}$$

Integrating (3.26) over $[0,T]$, then using (3.5) and (3.12), shows that

$$\begin{aligned}&\sup_{t\in[0,T]}\Big(\alpha\|\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{w}\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)\\&\quad+\int_0^T\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)dt\\&\leqslant C_1m_0^{2/3}+\Big(\beta\|\nabla\mathbf{u}_0\|_{L^2}^2+\mu'\|\nabla\mathbf{w}_0\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}_0\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}_0-2\mathbf{w}_0\|_{L^2}^2\Big),\end{aligned} \tag{3.27}$$

where the positive constant C_1 depends only on $q,\zeta,\bar{\rho},\alpha,\beta,\mu',\lambda',\|\nabla\mu(\rho_0)\|_{L^q},\|\nabla\mathbf{u}_0\|_{L^2}$, and $\|\nabla\mathbf{w}_0\|_{L^2}$, but is independent of time $T>0$ and m_0. Taking

$$m_0\leqslant\varepsilon_1\triangleq\Big(\frac{\beta\|\nabla\mathbf{u}_0\|_{L^2}^2+\mu'\|\nabla\mathbf{w}_0\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}_0\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}_0-2\mathbf{w}_0\|_{L^2}^2}{C_1}\Big)^{3/2}.$$

Thus, we obtain (3.14).

Multiplying (3.26) by t gives that

$$\begin{aligned}&\frac{d}{dt}\Big[t\Big(\|\mu^{1/2}(\rho)\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{u}\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)\Big]\\&\quad+t\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\\&\leqslant C\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2+\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)+Ct\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2\Big).\end{aligned} \tag{3.28}$$

Integrating (3.28) over $[0,T]$, we obtain from (3.5) and (3.6) that

$$\begin{aligned}&\sup_{t\in[0,T]}\Big[t\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2\Big)\Big]+\int_0^T t\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)dt\\&\leqslant C\int_0^T\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2+\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)dt+C\int_0^T t\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2\Big)dt\\&\leqslant Cm_0^{2/3}+C\sup_{t\in[0,T]}\Big(te^{-\sigma t}\Big)\int_0^T e^{\sigma t}\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2\Big)dt\leqslant Cm_0^{2/3},\end{aligned}$$

which implies that (3.15) holds for $i=1$. For $i=2,3$, we can take a similar approach to obtain the results of (3.15).

Next, multiplying (3.26) by $e^{\sigma t}$, one has

$$\begin{aligned}&\frac{d}{dt}\Big[e^{\sigma t}\Big(\|\mu^{1/2}(\rho)\nabla\mathbf{u}\|_{L^2}^2+\mu'\|\nabla\mathbf{u}\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2+\zeta\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)\Big]\\&\quad+e^{\sigma t}\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\\&\leqslant Ce^{\sigma t}\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2+\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big).\end{aligned} \tag{3.29}$$

Integrating (3.29) over $[0,T]$, one can deduce from (3.6) that

$$\begin{aligned}&\sup_{t\in[0,T]}\Big[e^{\sigma t}\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2\Big)\Big]+\int_0^T e^{\sigma t}\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)dt\\&\leqslant C\int_0^T e^{\sigma t}\Big(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{w}\|_{L^2}^2+\|\mathrm{rot}\mathbf{u}-2\mathbf{w}\|_{L^2}^2\Big)dt\leqslant Cm_0^{2/3}.\end{aligned}$$

Thus, we obtain (3.16). □

Lemma 3.3. *Under the condition of* (3.1)*, then for* $i=1,2,3$*,*

$$\sup_{t\in[0,T]}\Big[t^i\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\Big]+\int_0^T t^i\Big(\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\nabla\mathbf{w}_t\|_{L^2}^2\Big)dt\leqslant Cm_0^{2/3}. \tag{3.30}$$

Moreover, for σ *as in Lemma 3.1 and* $\delta(t)\triangleq\min\{1,t\}$*,*

$$\sup_{t\in[\delta(T),T]}\Big[e^{\sigma t}\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\Big]+\int_{\delta(T)}^T e^{\sigma t}\Big(\|\nabla\mathbf{u}_t\|_{L^2}^2+\|\nabla\mathbf{w}_t\|_{L^2}^2\Big)dt\leqslant C. \tag{3.31}$$

Proof. Operating ∂_t to $(1.1)_2$ and $(1.1)_3$, respectively, we infer from $(1.1)_1$ that

$$\begin{aligned}&\rho\mathbf{u}_{tt}+\rho\mathbf{u}\cdot\nabla\mathbf{u}_t-\mathrm{div}\Big((\mu(\rho)+\zeta)\nabla\mathbf{u}\Big)_t+\nabla P_t\\&=-\rho\mathbf{u}_t\cdot\nabla\mathbf{u}+(\mathbf{u}\cdot\nabla\rho)(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})+2\zeta\mathrm{rot}\mathbf{w}_t,\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}&\rho\mathbf{w}_{tt}+\rho\mathbf{u}\cdot\nabla\mathbf{w}_t-\mu'\Delta\mathbf{w}_t-(\mu'+\lambda')\nabla\mathrm{div}\mathbf{w}_t+4\zeta\mathbf{w}_t\\&=-\rho\mathbf{u}_t\cdot\nabla\mathbf{w}+(\mathbf{u}\cdot\nabla\rho)(\mathbf{w}_t+\mathbf{u}\cdot\nabla\mathbf{w})+2\zeta\mathrm{rot}\mathbf{u}_t.\end{aligned} \tag{3.33}$$

Multiplying (3.32) and (3.33) by $\mathbf{u}_t$ and $\mathbf{w}_t$, respectively, and then integrating by parts on $\mathbb{R}^3$, we deduce from (3.17) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)+\|(\mu(\rho)+\zeta)^{1/2}\nabla\mathbf{u}_t\|_{L^2}^2+\mu'\|\nabla\mathbf{w}_t\|_{L^2}^2\\&\quad+4\zeta\|\mathbf{w}_t\|_{L^2}^2+(\mu'+\lambda')\|\mathrm{div}\mathbf{w}_t\|_{L^2}^2\\&=\int\Big[-\rho\mathbf{u}_t\cdot\nabla\mathbf{u}+(\mathbf{u}\cdot\nabla\rho)(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})\Big]\cdot\mathbf{u}_t dx\\&\quad+\int\Big[-\rho\mathbf{u}_t\cdot\nabla\mathbf{w}+(\mathbf{u}\cdot\nabla\rho)(\mathbf{w}_t+\mathbf{u}\cdot\nabla\mathbf{w})\Big]\cdot\mathbf{w}_t dx\\&\quad+2\zeta\int\Big(\mathrm{rot}\mathbf{w}_t\cdot\mathbf{u}_t+\mathrm{rot}\mathbf{u}_t\cdot\mathbf{w}_t\Big)dx+\int\mathbf{u}\cdot\nabla\mu(\rho)\nabla\mathbf{u}\cdot\nabla\mathbf{u}_t dx=\sum_{i=1}^4 N_i.\end{aligned} \tag{3.34}$$

Now, we estimate $N_i (i=1,2,3,4)$ as follows: Thanks to $\mathrm{div}\mathbf{u}=0$, we deduce from (2.1), (2.2), (3.4), (3.14) and Cauchy-Schwarz's inequality that

$$\begin{aligned}N_1&\leqslant C\|\rho\|_{L^\infty}^{1/2}\|\rho^{1/2}\mathbf{u}_t\|_{L^3}\|\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{u}\|_{L^6}+C\|\rho\|_{L^\infty}\|\nabla^2\mathbf{u}\|_{L^2}\|\mathbf{u}_t\|_{L^6}\|\mathbf{u}\|_{L^6}^2\\&\quad+C\|\rho\|_{L^\infty}\|\nabla\mathbf{u}_t\|_{L^2}\|\nabla\mathbf{u}\|_{L^6}\|\mathbf{u}\|_{L^6}^2+C\|\rho\|_{L^\infty}\|\mathbf{u}_t\|_{L^6}\|\nabla\mathbf{u}\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^6}\\&\quad+\|\rho^{1/2}\mathbf{u}_t\|_{L^4}^2\|\nabla\mathbf{u}\|_{L^2}\\&\leqslant C\|\rho\|_{L^\infty}^{3/4}\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}_t\|_{L^2}^{3/2}+C\|\rho\|_{L^\infty}\|\nabla\mathbf{u}_t\|_{L^2}\|\nabla^2\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2\\&\leqslant\frac{\alpha}{6}\|\nabla\mathbf{u}_t\|_{L^2}^2+C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2+C\|\nabla^2\mathbf{u}\|_{L^2}^2,\end{aligned} \tag{3.35}$$

and

$$
\begin{aligned}
N_2 &\leqslant C\|\rho\|_{L^\infty}^{1/2}\|\rho^{1/2}\mathbf{w}_t\|_{L^3}\|\nabla\mathbf{w}_t\|_{L^2}\|\mathbf{u}\|_{L^6} + C\|\rho\|_{L^\infty}\|\nabla^2\mathbf{w}\|_{L^2}\|\mathbf{w}_t\|_{L^6}\|\mathbf{u}\|_{L^6}^2 \\
&\quad + C\|\rho\|_{L^\infty}\|\nabla\mathbf{w}_t\|_{L^2}\|\nabla\mathbf{w}\|_{L^6}\|\mathbf{u}\|_{L^6}^2 + C\|\rho\|_{L^\infty}\|\mathbf{w}_t\|_{L^6}\|\nabla\mathbf{w}\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^6} \\
&\quad + \|\rho^{1/2}\mathbf{w}_t\|_{L^3}\|\nabla\mathbf{w}\|_{L^2}\|\mathbf{u}_t\|_{L^6}\|\rho\|_{L^\infty}^{1/2} \\
&\leqslant C\|\rho\|_{L^\infty}^{3/4}\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^{1/2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{w}_t\|_{L^2}^{3/2} + C\|\rho\|_{L^\infty}\|\nabla\mathbf{w}_t\|_{L^2}\|\nabla^2\mathbf{w}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}^2 \\
&\quad + C\|\rho\|_{L^\infty}^{3/4}\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^{1/2}\|\nabla\mathbf{w}\|_{L^2}\|\nabla\mathbf{w}_t\|_{L^2}^{1/2}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\leqslant \frac{\alpha}{6}\|\nabla\mathbf{u}_t\|_{L^2}^2 + \frac{\mu'}{2}\|\nabla\mathbf{w}_t\|_{L^2}^2 + C\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2 + C\|\nabla^2\mathbf{w}\|_{L^2}^2.
\end{aligned} \tag{3.36}
$$

Cauchy-Schwarz's inequality gives

$$
N_3 \leqslant 4\zeta\|\mathbf{w}_t\|_{L^2}^2 + \zeta\|\nabla\mathbf{u}_t\|_{L^2}^2. \tag{3.37}
$$

The inequalities (2.2) and (3.1), together with Cauchy-Schwarz's inequality, show that

$$
\begin{aligned}
N_4 &\leqslant C\|\mathbf{u}\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q}\|\nabla\mathbf{u}\|_{L^{2q/(q-2)}}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\leqslant C\|\nabla\mathbf{u}\|_{L^2}^{1-\frac{3}{q}+\frac{1}{2}}\|\nabla^2\mathbf{u}\|_{L^2}^{\frac{3}{q}+\frac{1}{2}}\|\nabla\mathbf{u}_t\|_{L^2} \\
&\leqslant \frac{\alpha}{6}\|\nabla\mathbf{u}_t\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^2}^2 + C\|\nabla^2\mathbf{u}\|_{L^2}^4.
\end{aligned} \tag{3.38}
$$

Putting (3.35)–(3.38) into (3.34) and using (3.25) yields

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big) + \|\nabla\mathbf{u}_t\|_{L^2}^2 + \|\nabla\mathbf{w}_t\|_{L^2}^2 \\
&\leqslant C\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big) + C\Big(\|\nabla^2\mathbf{u}\|_{L^2}^2 + \|\nabla^2\mathbf{w}\|_{L^2}^2\Big) + C\|\nabla^2\mathbf{u}\|_{L^2}^4 \\
&\leqslant C\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big) + C\Big(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{w}\|_{L^2}^2\Big) \\
&\quad + C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + C\|\nabla^2\mathbf{w}\|_{L^2}^2.
\end{aligned} \tag{3.39}
$$

To deal with the last term on the right side of (3.39), we first multiply $(1.1)_3$ by $\mathbf{w}$ and then integrate the resulting equation by parts on $\mathbb{R}^3$ to get that

$$
\begin{aligned}
&\mu'\|\nabla\mathbf{w}\|_{L^2}^2 + (\mu' + \lambda')\|\mathrm{div}\mathbf{w}\|_{L^2}^2 + 4\zeta\|\mathbf{w}\|_{L^2}^2 \\
&\leqslant \|\mathbf{w}\|_{L^2}\|\rho\mathbf{w}_t + \rho\mathbf{u}\cdot\nabla\mathbf{w} - 2\zeta\mathrm{rot}\mathbf{u}\|_{L^2} \\
&\leqslant \zeta\|\mathbf{w}\|_{L^2}^2 + C\Big(\|\rho\mathbf{w}_t\|_{L^2}^2 + \|\rho\mathbf{u}\cdot\nabla\mathbf{w}\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2\Big),
\end{aligned}
$$

which implies

$$
\|\nabla\mathbf{w}\|_{L^2} + \|\mathbf{w}\|_{L^2} \leqslant C\Big(\|\rho\mathbf{w}_t\|_{L^2} + \|\rho\mathbf{u}\cdot\nabla\mathbf{w}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}\Big). \tag{3.40}
$$

On the other hand, one can deduce from $(1.1)_3$ that

$$
\begin{aligned}
\|\nabla^2\mathbf{w}\|_{L^2} &\leqslant C\Big(\|\rho\mathbf{w}_t + \rho\mathbf{u}\cdot\nabla\mathbf{w} - 2\zeta\mathrm{rot}\mathbf{u} + 4\zeta\mathbf{w}\|_{L^2}\Big) \\
&\leqslant C\|\mathbf{w}\|_{L^2} + C\Big(\|\rho\mathbf{w}_t\|_{L^2} + \|\rho\mathbf{u}\cdot\nabla\mathbf{w}\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}\Big),
\end{aligned}
$$

which, together with (3.40), (2.1), and (3.4), yields

$$
\begin{aligned}
\|\mathbf{w}\|_{H^2} &\leqslant C\Big(\|\rho \mathbf{w}_t\|_{L^2} + \|\rho \mathbf{u}\cdot\nabla \mathbf{w}\|_{L^2} + \|\nabla \mathbf{u}\|_{L^2}\Big)\\
&\leqslant C\|\rho^{1/2}\mathbf{w}_t\|_{L^2}\|\rho\|_{L^\infty}^{1/2} + C\|\rho\|_{L^\infty}\|\mathbf{u}\|_{L^6}\|\nabla \mathbf{w}\|_{L^3} + C\|\nabla \mathbf{u}\|_{L^2}\\
&\leqslant \frac{1}{2}\|\mathbf{w}\|_{H^2} + C\|\rho^{1/2}\mathbf{w}_t\|_{L^2} + C\|\nabla \mathbf{u}\|_{L^2}^2\|\nabla \mathbf{w}\|_{L^2} + C\|\nabla \mathbf{u}\|_{L^2},
\end{aligned}
$$

thus

$$
\|\mathbf{w}\|_{H^2} \leqslant C\|\rho^{1/2}\mathbf{w}_t\|_{L^2} + C\|\nabla \mathbf{u}\|_{L^2}^2\|\nabla \mathbf{w}\|_{L^2} + C\|\nabla \mathbf{u}\|_{L^2}. \tag{3.41}
$$

Taking (3.41) into (3.39), we infer from (3.14) that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big) + \|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{w}_t\|_{L^2}^2\\
&\leqslant C\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big) + C\Big(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2\Big)\\
&\quad + C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2.
\end{aligned} \tag{3.42}
$$

Multiplying (3.42) by t, one has

$$
\begin{aligned}
&\frac{d}{dt}\Big[t\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\Big] + t\Big(\|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{w}_t\|_{L^2}^2\Big)\\
&\leqslant C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2\Big[t\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\Big]\\
&\quad + Ct\Big(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 + \|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\\
&\quad + C\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big),
\end{aligned}
$$

which, together with (3.14), (3.15) and Gronwall's inequality, gives that (3.30) holds for $i = 1$. Using the same methods, we can show that (3.30) for $i = 2, 3$.

Next, in order to prove (3.31), we first multiply (3.42) by $e^{\sigma t}$, then

$$
\begin{aligned}
&\frac{d}{dt}\Big[e^{\sigma t}\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\Big] + e^{\sigma t}\Big(\|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{w}_t\|_{L^2}^2\Big)\\
&\leqslant C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2\Big[e^{\sigma t}\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big)\Big]\\
&\quad + Ce^{\sigma t}\Big(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 + \|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big),
\end{aligned}
$$

which, combining (3.6), (3.14), and (3.16) with Gronwall's inequality, gives that (3.31) holds. □

Lemma 3.4. *Under the condition of* (3.1)*, then*

$$
\int_0^T \|\nabla \mathbf{u}\|_{L^\infty} dt \leqslant C m_0^{1/2r}, \tag{3.43}
$$

where $3 < r < q$ *with* $q \in (3, 6)$.

Proof. It follows from $(1.1)_2$, (2.1), (2.2), (2.4), and (3.1) that for any $r \in (3, \min\{6, q\})$

$$\begin{aligned}
&\|\nabla^2 \mathbf{u}\|_{L^r} + \|\nabla P\|_{L^r} \\
&\leqslant C\| -\rho \mathbf{u}_t - \rho \mathbf{u}\cdot\nabla\mathbf{u} + 2\zeta \mathrm{rot}\mathbf{w}\|_{L^r}\Big(1 + \|\nabla\mu(\rho)\|_{L^q}^{q(5r-6)/2r(q-3)}\Big) \\
&\leqslant C\|\rho\|_{L^\infty}^{1/2}\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{(6-r)/2r}\|\rho^{1/2}\mathbf{u}_t\|_{L^6}^{(3r-6)/2r} \\
&\quad + C\|\rho\|_{L^{6r/(6-r)}}\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^6} + C\|\nabla\mathbf{w}\|_{L^2}^{(6-r)/2r}\|\nabla\mathbf{w}\|_{L^6}^{(3r-6)/2r} \\
&\leqslant C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{(6-r)/2r}\|\nabla\mathbf{u}_t\|_{L^2}^{(3r-6)/2r} + Cm_0^{(6-r)/6r}\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla^2\mathbf{u}\|_{L^2}^{3/2} \\
&\quad + C\|\nabla\mathbf{w}\|_{L^2}^{(6-r)/2r}\|\nabla^2\mathbf{w}\|_{L^2}^{(3r-6)/2r},
\end{aligned} \tag{3.44}$$

thus

$$\begin{aligned}
\|\nabla\mathbf{u}\|_{L^\infty} &\leqslant C\|\nabla\mathbf{u}\|_{L^2}^{(2r-6)/(5r-6)}\|\nabla^2\mathbf{u}\|_{L^r}^{3r/(5r-6)} \leqslant C\|\nabla\mathbf{u}\|_{L^2} + C\|\nabla^2\mathbf{u}\|_{L^r} \\
&\leqslant C\|\nabla\mathbf{u}\|_{L^2} + C\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{(6-r)/2r}\|\nabla\mathbf{u}_t\|_{L^2}^{(3r-6)/2r} \\
&\quad + Cm_0^{(6-r)/6r}\|\nabla\mathbf{u}\|_{L^2}^{1/2}\|\nabla^2\mathbf{u}\|_{L^2}^{3/2} + C\|\nabla\mathbf{w}\|_{L^2}^{(6-r)/2r}\|\nabla^2\mathbf{w}\|_{L^2}^{(3r-6)/2r},
\end{aligned} \tag{3.45}$$

The inequality (3.6), together with Hölder's inequality, shows that

$$\begin{aligned}
\int_0^T \|\nabla\mathbf{u}\|_{L^2} dt &= \int_0^T e^{\sigma t/2}\|\nabla\mathbf{u}\|_{L^2} e^{-\sigma t/2} dt \\
&\leqslant C\Big(\int_0^T e^{\sigma t}\|\nabla\mathbf{u}\|_{L^2}^2 dt\Big)^{1/2}\Big(\int_0^T e^{-\sigma t} dt\Big)^{1/2} \\
&\leqslant Cm_0^{1/3}.
\end{aligned} \tag{3.46}$$

For any $T \in (0, 1]$, we can show from (3.30) and Hölder's inequality that

$$\begin{aligned}
&\int_0^T \|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{(6-r)/2r}\|\nabla\mathbf{u}_t\|_{L^2}^{(3r-6)/2r} dt \\
&\leqslant C\sup_{t\in[0,T]}\Big(t\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2\Big)^{(6-r)/4r}\Big(\int_0^T t\|\nabla\mathbf{u}_t\|_{L^2}^2 dt\Big)^{(3r-6)/4r} \\
&\quad \times\Big(\int_0^T t^{-2r/(r+6)} dt\Big)^{(r+6)/4r} \\
&\leqslant Cm_0^{1/3}.
\end{aligned} \tag{3.47}$$

For $T > 1$, one can deduce from (3.30), (3.31), and Hölder's inequality that

$$\begin{aligned}
&\int_1^T \|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{(6-r)/2r}\|\nabla\mathbf{u}_t\|_{L^2}^{(3r-6)/2r} dt \\
&\leqslant C\sup_{t\in[0,T]}\Big(e^{\sigma t}\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2\Big)^{(6-r)/4r}\Big(\int_0^T t^3\|\nabla\mathbf{u}_t\|_{L^2}^2 dt\Big)^{(3r-6)/4r} \\
&\quad \times\Big(\int_1^T t^{-9(r-2)/(r+6)} dt\Big)^{(r+6)/4r} \\
&\leqslant Cm_0^{(3r-6)/6r}.
\end{aligned} \tag{3.48}$$

Similarly,

$$\int_0^T \|\nabla \mathbf{w}\|_{L^2}^{(6-r)/2r}\|\nabla^2 \mathbf{w}\|_{L^2}^{(3r-6)/2r} dt \leqslant Cm_0^{(3r-6)/6r} + Cm_0^{1/3}. \tag{3.49}$$

It follows from (3.5), (3.13), and (3.25) that

$$\begin{aligned}
&\int_0^T m_0^{(6-r)/6r}\|\nabla \mathbf{u}\|_{L^2}^{1/2}\|\nabla^2 \mathbf{u}\|_{L^2}^{3/2} dt \\
&\leqslant Cm_0^{(6-r)/6r}\Big(\int_0^T \|\nabla \mathbf{u}\|_{L^2}^2 dt\Big)^{1/4}\Big(\int_0^T (\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^6 + \|\nabla \mathbf{w}\|_{L^2}^2) dt\Big)^{3/4} \\
&\leqslant Cm_0^{1/r} + Cm_0^{1/2+1/r}.
\end{aligned} \tag{3.50}$$

Due to $3 < r < q,\ q \in (3,6)$, one has

$$\frac{1}{2r} < \frac{3r-6}{6r} < \frac{1}{3}.$$

Integrating (3.45) over $t \in [0,T]$ and using (3.46)–(3.50), we can obtain (3.43). □

With Lemmas 3.1–3.4 at hand, we are in a position to prove Proposition 3.1.

Proof of Proposition 3.1. It follows from (3.17) that

$$\Big[\nabla \mu(\rho)\Big]_t + \mathbf{u}\cdot\nabla^2\mu(\rho) + \nabla \mathbf{u}\cdot\nabla\mu(\rho) = 0.$$

Multiplying the above equation by $q|\nabla\mu(\rho)|^{q-2}\nabla\mu(\rho)$ and integrating the resulting equations on $\mathbb{R}^3$, we can obtain

$$\begin{aligned}
\frac{d}{dt}\|\nabla\mu(\rho)\|_{L^q}^q = &-q\int \mathbf{u}\cdot\nabla^2\mu(\rho)\cdot|\nabla\mu(\rho)|^{q-2}\nabla\mu(\rho)dx \\
&- q\int \nabla\mathbf{u}\cdot\nabla\mu(\rho)\cdot|\nabla\mu(\rho)|^{q-2}\nabla\mu(\rho)dx.
\end{aligned}$$

Due to $\mathrm{div}\mathbf{u} = 0$, then

$$q\int \mathbf{u}\cdot\nabla^2\mu(\rho)\cdot|\nabla\mu(\rho)|^{q-2}\nabla\mu(\rho)dx = -\int |\nabla\mu(\rho)|^q \mathrm{div}\mathbf{u} dx = 0.$$

Thus

$$\Big(\|\nabla\mu(\rho)\|_{L^q}^q\Big)_t \leqslant q\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q}^q,$$

which implies that

$$\Big(\|\nabla\mu(\rho)\|_{L^q}\Big)_t \leqslant \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q},$$

which, together with (3.43) and Gronwall's inequality, shows that

$$\begin{aligned}
\sup_{t\in[0,T]}\|\nabla\mu(\rho)\|_{L^q} &\leqslant \exp\Big\{\int_0^T \|\nabla\mathbf{u}\|_{L^\infty} dt\Big\}\|\nabla\mu(\rho_0)\|_{L^q} \\
&\leqslant \exp\Big\{C_2 m_0^{1/2r}\Big\}\|\nabla\mu(\rho_0)\|_{L^q} \\
&\leqslant 2\|\nabla\mu(\rho_0)\|_{L^q},
\end{aligned} \tag{3.51}$$

provided

$$m_0 \leqslant \varepsilon_0 \triangleq \min\Big\{\varepsilon_1, \Big(\frac{\ln 2}{C_2}\Big)^{2r}\Big\}.$$

Thus, we complete the proof of Proposition 3.1. □

Lemma 3.5. *Under the condition of* (3.1)*, then*

$$\sup_{t\in[0,T]} \Big(\|\nabla\rho\|_{L^2} + \|\rho_t\|_{L^{3/2}}\Big) \leqslant C. \tag{3.52}$$

Proof. Similar to the method of (3.51), we can deduce from $(1.1)_1$ that

$$\sup_{t\in[0,T]} \|\nabla\rho\|_{L^2} \leqslant C. \tag{3.53}$$

Hölder's inequality, together with $(1.1)_1$, (3.14), and (3.53), yields

$$\|\rho_t\|_{L^{3/2}} = \|\mathbf{u}\cdot\nabla\rho\|_{L^{3/2}} \leqslant \|\nabla\rho\|_{L^2}\|\mathbf{u}\|_{L^6} \leqslant C\|\nabla\rho\|_{L^2}\|\nabla\mathbf{u}\|_{L^2} \leqslant C.$$

Thus, we complete the proof of Lemma 3.5. □

Lemma 3.6. *Under the condition of* (3.1)*, then for* $3 < r < q$ *with* $q \in (3,6)$*, the following estimates hold:*

$$\sup_{t\in[0,T]} t\Big(\|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{H^1}^2 + \|\nabla P\|_{L^2}^2\Big) + \int_0^T t\Big(\|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{W^{1,r}}^2 + \|\nabla P\|_{L^r}^2\Big)dt \leqslant C, \tag{3.54}$$

and

$$\sup_{t\in[\delta(T),T]} e^{\sigma t}\Big(\|(\nabla\mathbf{u},\nabla\mathbf{w})\|_{H^1}^2 + \|\nabla P\|_{L^2}^2\Big) \leqslant C, \tag{3.55}$$

for σ *as in Lemma 3.1 and* $\delta(t)$ *as in Lemma 3.3.*

Proof. It follows from (3.14), (3.15), (3.25), (3.30), and (3.41) that

$$\begin{aligned}
&\|\nabla\mathbf{u}\|_{H^1}^2 + \|\nabla\mathbf{w}\|_{H^1}^2 + \|\nabla P\|_{L^2}^2\\
&\leqslant C\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big) + C\Big(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{w}\|_{L^2}^2\Big),
\end{aligned}$$

thus

$$\begin{aligned}
&\sup_{t\in[0,T]} \Big(t\|\nabla\mathbf{u}\|_{H^1}^2 + \|\nabla\mathbf{w}\|_{H^1}^2 + \|\nabla P\|_{L^2}^2\Big)\\
&\leqslant Ct\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big) + Ct\Big(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{w}\|_{L^2}^2\Big)\\
&\leqslant C.
\end{aligned} \tag{3.56}$$

Hence, we can use the same methods to obtain (3.55).

By virtue of $r \in (3, q)$ with $q \in (3, 6)$, one can obtain from (3.44) that

$$
\begin{aligned}
&\|\nabla \mathbf{u}\|_{W^{1,r}}^2 + \|\nabla P\|_{L^r}^2 + \|\nabla \mathbf{w}\|_{W^{1,r}}^2 \\
&\leqslant C\Big(\|\nabla \mathbf{u}\|_{L^r}^2 + \|\nabla \mathbf{w}\|_{L^r}^2\Big) + C\Big(\|\nabla^2 \mathbf{u}\|_{L^r}^2 + \|\nabla P\|_{L^r}^2\Big) + C\|\nabla^2 \mathbf{w}\|_{L^r}^2 \\
&\leqslant C\Big(\|\nabla^2 \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{w}\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2\Big) \\
&\quad + C\Big(\|\nabla^2 \mathbf{u}\|_{L^r}^2 + \|\nabla P\|_{L^r}^2\Big) + C\|\nabla^2 \mathbf{w}\|_{L^r}^2 \\
&\leqslant C\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 + \|\mathbf{w}\|_{L^2}^2\Big) \\
&\quad + C\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^{(6-r)/r}\|\nabla \mathbf{u}_t\|_{L^2}^{(3r-6)/r} + \|\nabla \mathbf{u}\|_{L^2}\|\nabla^2 \mathbf{u}\|_{L^2}^3 + \|\nabla \mathbf{w}\|_{L^2}\|\nabla^2 \mathbf{w}\|_{L^2}^3\Big) \\
&\quad + C\Big(\|\rho^{1/2}\mathbf{w}_t\|_{L^2}^{(6-r)/r}\|\nabla \mathbf{w}_t\|_{L^2}^{(3r-6)/r} + \|\nabla \mathbf{u}\|_{L^2}\|\nabla^2 \mathbf{u}\|_{L^2}\|\nabla^2 \mathbf{w}\|_{L^2}^2\Big) \\
&\leqslant C\Big(\|\rho^{1/2}\mathbf{u}_t\|_{L^2}^2 + \|\rho^{1/2}\mathbf{w}_t\|_{L^2}^2\Big) + C\Big(\|\nabla \mathbf{u}_t\|_{L^2}^2 + \|\nabla \mathbf{w}_t\|_{L^2}^2\Big) \\
&\quad + C\Big(\|\nabla \mathbf{u}\|_{L^2}\|\nabla^2 \mathbf{u}\|_{L^2}^3 + \|\nabla \mathbf{w}\|_{L^2}\|\nabla^2 \mathbf{w}\|_{L^2}^3\Big) \\
&\quad + C\Big(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{w}\|_{L^2}^2 + \|\mathbf{w}\|_{L^2}^2\Big),
\end{aligned}
$$

which, together with (3.5), (3.15), (3.25), (3.30), (3.41), and (3.56), yields

$$
\int_0^T t\Big(\|(\nabla \mathbf{u}, \nabla \mathbf{w})\|_{W^{1,r}}^2 + \|\nabla P\|_{L^r}^2\Big)dt \leqslant C.
$$

Thus, we complete the proof of Lemma 3.6. □

Proof of Theorem 1.1. Similar to the standard arguments in [10, 18, 20], with all the a priori estimates established in Sections 3 at hand, we can immediately obtain our main results. □

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that she has no conflict of interest.

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