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Research article

Soliton solutions and a bi-Hamiltonian structure of the fifth-order nonlocal reverse-spacetime Sasa-Satsuma-type hierarchy via the Riemann-Hilbert approach

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Abstract: Our objective is to explore the intricacies of a nonlinear nonlocal fifth-order scalar Sasa-Satsuma equation in reverse spacetime which is rooted in a nonlocal 5×5 matrix AKNS spectral problem. Starting with this spectral problem, we derive both local and nonlocal symmetry relations through rotations within a defined group. We then formulate a specific type of Riemann-Hilbert problem, facilitating the generation of soliton solutions. These solutions are generated by utilizing vectors that reside in the kernel of the matrix Jost solutions. Under the condition where reflection coefficients are null, the jump matrix reduces to the identity, leading to soliton solutions via the corresponding Riemann-Hilbert problem. The explicit formulas of these soliton solutions enable a comprehensive exploration of their dynamics.

Keywords: Riemann-Hilbert problem; nonlocal reverse-spacetime; mKdV equation; Sasa-Satsuma hierarchy; soliton solutions; soliton dynamics
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1. Introduction

The exploration of integrable systems remains a captivating realm within mathematics, fascinating the interest of both mathematicians and physicists. These systems and their inherent properties serve as powerful tools for predicting a wide array of natural phenomena. They are prevalent in various fields such as nonlinear optics, plasma physics, the dynamics of ocean and water waves, gravitational fields, and fluid dynamics [1, 2].

The Korteweg-de Vries (KdV) equation, the nonlinear Schrödinger (NLS) equation, and the Kadomtsev-Petviashvili (KP) equation stand as quintessential instances of integrable systems. Within integrable systems, a diverse range of soliton solutions emerges, including breathers, lumps, and rogue waves.

Soliton solutions are a kind of special solutions that are stable localized waves. Nonlocal PT symmetric reverse-spacetime, reverse-time, and reverse-space have been studied for the NLS and KdV equations under the inverse scattering transformation and the Riemann-Hilbert problem.

Soliton solutions are unique, stable waveforms that remain localized. Researchers have explored nonlocal PT symmetric phenomena, including reverse-spacetime, reverse-time, and reverse-space, particularly for the NLS and KdV equations through the inverse scattering transformation and Riemann-Hilbert problem [3–8]. The Riemann-Hilbert approach offers a compelling method for examining and discovering new instances of nonlocal integrable equations and their soliton solutions [9–11]. In this paper, we introduce the following novel nonlocal fifth-order Sasa-Satsuma equation characterized by reverse-spacetime symmetry [12, 13]:

$$\begin{aligned} u_t =& u_{xxxxx} - 10(|u|^2 + |u(-x,-t)|^2)u_{xxx} - 15(|u|^2 + |u(-x,-t)|^2)_x u_{xx} \\ &+ [-15(|u|^2 + |u(-x,-t)|^2)_{xx} + 10(|u_x|^2 + |u_x(-x,-t)|^2) + 40(|u|^2 + |u(-x,-t)|^2)^2]u_x \\ &+ [-5(|u|^2 + |u(-x,-t)|^2)_{xxx} + 5(|u_x|^2 + |u_x(-x,-t)|^2)_x + 20((|u|^2 + |u(-x,-t)|^2)^2)_x]u. \end{aligned} \tag{1.1}$$

We establish a type of Riemann-Hilbert problem for the aforementioned integrable nonlocal Sasa-Satsuma equation, using the real line as the contour. By solving these Riemann-Hilbert problems with an identity jump matrix, we obtain soliton solutions to the integrable nonlocal Sasa-Satsuma equation (see, e.g., [14–21]).

The paper is organized as follows: In Section 2, we develop a nonlocal Sasa-Satsuma hierarchy associated with a nonlocal 5×5 matrix AKNS spectral problem and its Hamiltonian structure. In Section 3, we address the formulation of Riemann-Hilbert problems based on the corresponding matrix spectral problems. In Section 4, we derive soliton solutions by setting the reflection coefficients to zero [22–24]. In Section 5, we provide explicit and exact single soliton solutions, classify the various cases for explicit two soliton solutions, and examine their dynamic behaviors. We offer a brief conclusion and some remarks in the final section.

2. AKNS hierarchy

We consider the nonlocal 5×5 matrix AKNS spatial spectral problem [15]

$$\psi_x = \mathrm{i}U\psi, \tag{2.1}$$

where ψ is the eigenfunction and the spectral matrix $U(u;\lambda)$ is given by

$$U(u;\lambda) = U(x,t;\lambda) = \begin{pmatrix} \alpha_1\lambda & u & u(-x,-t) & \overset{*}{u} & \overset{*}{u}(-x,-t) \\ -\overset{*}{u} & \alpha_2\lambda & 0 & 0 & 0 \\ -\overset{*}{u}(-x,-t) & 0 & \alpha_2\lambda & 0 & 0 \\ -u & 0 & 0 & \alpha_2\lambda & 0 \\ -u(-x,-t) & 0 & 0 & 0 & \alpha_2\lambda \end{pmatrix} = \lambda\Lambda + P(u), \tag{2.2}$$

where $\Lambda = diag(\alpha_1, \alpha_2 I_4)$, λ is a nonzero spectral parameter, α_1, α_2 are two distinct real constants, and $u = u(x,t)$ is the potential. We assume that u and xu belong to the L^2 space and

$$P = P(x,t) = \begin{pmatrix} 0 & u & u(-x,-t) & \overset{*}{u} & \overset{*}{u}(-x,-t) \\ -\overset{*}{u} & 0 & 0 & 0 & 0 \\ -\overset{*}{u}(-x,-t) & 0 & 0 & 0 & 0 \\ -u & 0 & 0 & 0 & 0 \\ -u(-x,-t) & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Remark 2.1. *One can see that the matrix U has the following simultaneous symmetry relations:*

$$\begin{cases} U^{\dagger}(x,t;-\overset{*}{\lambda}) = -C_0 U C_0^{-1} = -U, & U(x,t;-\lambda) = -C_4 U C_4^{-1}, \\ U^{T}(x,t;-\lambda) = -C_1 U C_1^{-1}, & U^{T}(x,t;\lambda) = C_5 U C_5^{-1}, \\ U^{\dagger}(-x,-t;-\overset{*}{\lambda}) = -C_2 U C_2^{-1}, & U^{\dagger}(-x,-t;\overset{*}{\lambda}) = C_6 U C_6^{-1}, \\ \overset{*}{U}(-x,-t;\overset{*}{\lambda}) = C_3 U C_3^{-1}, & \overset{*}{U}(-x,-t;-\overset{*}{\lambda}) = -C_7 U C_7^{-1}, \end{cases} \tag{2.4}$$

where the eight 5×5 *matrices are*

$$C_i = \begin{pmatrix} 1 & \mathbf{0_{14}} \\ \mathbf{0_{41}} & \sigma_i \end{pmatrix}, \quad i \in \{0,\dots,7\}, \tag{2.5}$$

with $\mathbf{0_{14}}$, $\mathbf{0_{41}}$ *being the four-component zero row and column vectors respectively, and* σ_i *are*

$$\sigma_0 = \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix}, \tag{2.6}$$

and

$$\sigma_4 = -\sigma_0,\ \sigma_5 = -\sigma_1,\ \sigma_6 = -\sigma_2,\ \sigma_7 = -\sigma_3. \tag{2.7}$$

Note that all C_i *are symmetric and orthogonal matrices, i.e.,* $C_i = C_i^T$ *and* $C_i^2 = I_2$*, for* $i \in \{0,\dots,7\}$*. In fact, they form an orthogonal group,* $G = \{C_0, C_1, C_2, C_3, C_4, C_5, C_6, C_7\}$ *that has two connected components. The first component* G_1 *of all matrices where* $det(C_i) = 1$ *for* $i \in \{0,1,2,3\}$*, that is, the normal subgroup component* $G_1 = \{C_0, C_1, C_2, C_3\}$*. The second component is* $G_2 = \{C_4, C_5, C_6, C_7\}$*, where* $det(C_i) = -1$ *for* $i \in \{4,5,6,7\}$*.*

In this paper, we will consider all reductions generated by the matrices in the rotation group G_1*.*

In addition, since $U = \lambda\Lambda + P$, then we can easily prove that

$$\begin{cases} P^{\dagger} = -P, \\ P^{T} = -C_1 P C_1^{-1}, \\ P^{\dagger}(-x,-t) = -C_2 P C_2^{-1}, \\ \overset{*}{P}(-x,-t) = C_3 P C_3^{-1}. \end{cases} \tag{2.8}$$

We start by constructing the associated Sasa-Satsuma soliton hierarchy. To achieve this, we must solve the stationary zero curvature equation.

$$W_x = \mathrm{i}[U, W], \tag{2.9}$$

for

$$W = \begin{pmatrix} a & b_1 & b_2 & b_3 & b_4 \\ c_1 & d_{11} & d_{12} & d_{13} & d_{14} \\ c_2 & d_{21} & d_{22} & d_{23} & d_{24} \\ c_3 & d_{31} & d_{32} & d_{33} & d_{34} \\ c_4 & d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}, \tag{2.10}$$

with the scalar components a, b_i, c_i, d_{ij} for $i, j \in \{1, 2, 3, 4\}$. Solving the stationary zero curvature equation yields:

$$a_x = \mathrm{i}\Big[\overset{*}{u}b_1 + \overset{*}{u}(-x)b_2 + ub_3 + u(-x)b_4 + uc_1 + u(-x)c_2 + \overset{*}{u}c_3 + \overset{*}{u}(-x)c_4\Big], \tag{2.11}$$

$$b_{1,x} = \mathrm{i}[\alpha\lambda b_1 - ua + ud_{11} + u(-x)d_{21} + \overset{*}{u}d_{31} + \overset{*}{u}(-x)d_{41}], \tag{2.12}$$

$$b_{2,x} = \mathrm{i}[\alpha\lambda b_2 - u(-x)a + ud_{12} + u(-x)d_{22} + \overset{*}{u}d_{32} + \overset{*}{u}(-x)d_{42}], \tag{2.13}$$

$$b_{3,x} = \mathrm{i}[\alpha\lambda b_3 - \overset{*}{u}a + ud_{13} + u(-x)d_{23} + \overset{*}{u}d_{33} + \overset{*}{u}(-x)d_{43}], \tag{2.14}$$

$$b_{4,x} = \mathrm{i}[\alpha\lambda b_4 - \overset{*}{u}(-x)a + ud_{14} + u(-x)d_{24} + \overset{*}{u}d_{34} + \overset{*}{u}(-x)d_{44}], \tag{2.15}$$

$$c_{1,x} = \mathrm{i}[-\alpha\lambda c_1 - \overset{*}{u}a + \overset{*}{u}d_{11} + \overset{*}{u}(-x)d_{12} + ud_{13} + u(-x)d_{14}], \tag{2.16}$$

$$c_{2,x} = \mathrm{i}[-\alpha\lambda c_2 - \overset{*}{u}(-x)a + \overset{*}{u}d_{21} + \overset{*}{u}(-x)d_{22} + ud_{23} + u(-x)d_{24}], \tag{2.17}$$

$$c_{3,x} = \mathrm{i}[-\alpha\lambda c_3 - ua + \overset{*}{u}d_{31} + \overset{*}{u}(-x)d_{32} + ud_{33} + u(-x)d_{34}], \tag{2.18}$$

$$c_{4,x} = \mathrm{i}[-\alpha\lambda c_4 - u(-x)a + \overset{*}{u}d_{41} + \overset{*}{u}(-x)d_{42} + ud_{43} + u(-x)d_{44}], \tag{2.19}$$

$$d_{11,x} = -\mathrm{i}[\overset{*}{u}b_1 + uc_1], \qquad d_{12,x} = -\mathrm{i}[\overset{*}{u}b_2 + u(-x)c_1], \tag{2.20}$$

$$d_{21,x} = -\mathrm{i}[\overset{*}{u}(-x)b_1 + uc_2], \qquad d_{22,x} = -\mathrm{i}[\overset{*}{u}(-x)b_2 + u(-x)c_2], \tag{2.21}$$

$$d_{31,x} = -\mathrm{i}[ub_1 + uc_3], \qquad d_{32,x} = -\mathrm{i}[ub_2 + u(-x)c_3], \tag{2.22}$$

$$d_{41,x} = -\mathrm{i}[u(-x)b_1 + uc_4], \qquad d_{42,x} = -\mathrm{i}[u(-x)b_2 + u(-x)c_4], \tag{2.23}$$

$$d_{13,x} = -\mathrm{i}[\overset{*}{u}b_3 + \overset{*}{u}c_1], \qquad d_{14,x} = -\mathrm{i}[\overset{*}{u}b_4 + \overset{*}{u}(-x)c_1], \tag{2.24}$$

$$d_{23,x} = -\mathrm{i}[\overset{*}{u}(-x)b_3 + \overset{*}{u}c_2], \qquad d_{24,x} = -\mathrm{i}[\overset{*}{u}(-x)b_4 + \overset{*}{u}(-x)c_2], \tag{2.25}$$

$$d_{33,x} = -\mathrm{i}[ub_3 + \overset{*}{u}c_3], \qquad d_{34,x} = -\mathrm{i}[ub_4 + \overset{*}{u}(-x)c_3], \tag{2.26}$$

$$d_{43,x} = -\mathrm{i}[u(-x)b_3 + \overset{*}{u}c_4], \qquad d_{44,x} = -\mathrm{i}[u(-x)b_4 + \overset{*}{u}(-x)c_4], \tag{2.27}$$

where $\alpha = \alpha_1 - \alpha_2$. Now, we expand W in Laurent series, explicitly expressing the components of W as follows:

$$a = a(x, t; \lambda) = \sum_{m=0}^{\infty} a^{[m]}\lambda^{-m}, \quad d_{ii} = d_{ii}(x, t; \lambda) = \sum_{m=0}^{\infty} d_{ii}^{[m]}\lambda^{-m}, \; i \in \{1, 2, 3, 4\},$$

$$b_i = b_i(x,t;\lambda) = \sum_{m=0}^{\infty} b_i^{[m]}\lambda^{-m},\ i \in \{1,\dots,4\}, \quad c_i = c_i(x,t;\lambda) = \sum_{m=0}^{\infty} c_i^{[m]}\lambda^{-m},\ i \in \{1,\dots,4\},$$

so W can be rewritten in the following form:

$$W = W(x,t;\lambda) = \sum_{m=0}^{\infty} W_m\lambda^{-m} \quad \text{with} \quad W_m = \begin{pmatrix} a^{[m]} & b_1^{[m]} & b_2^{[m]} & b_3^{[m]} & b_4^{[m]} \\ c_1^{[m]} & d_{11}^{[m]} & d_{12}^{[m]} & d_{13}^{[m]} & d_{14}^{[m]} \\ c_2^{[m]} & d_{21}^{[m]} & d_{22}^{[m]} & d_{23}^{[m]} & d_{24}^{[m]} \\ c_3^{[m]} & d_{31}^{[m]} & d_{32}^{[m]} & d_{33}^{[m]} & d_{34}^{[m]} \\ c_4^{[m]} & d_{41}^{[m]} & d_{42}^{[m]} & d_{43}^{[m]} & d_{44}^{[m]} \end{pmatrix}, \quad m \geq 0. \tag{2.28}$$

As a consquence, the system (2.11)–(2.27) generates the recursive relations:

$$b_i^{[0]} = c_i^{[0]} = 0, \quad i \in \{1,\dots,4\}, \tag{2.29}$$

$$a_x^{[m]} = \mathrm{i}\Big[\overset{*}{u}b_1^{[m]} + \overset{*}{u}(-x)b_2^{[m]} + ub_3^{[m]} + u(-x)b_4^{[m]} + uc_1^{[m]} + u(-x)c_2^{[m]} + \overset{*}{u}c_3^{[m]} + \overset{*}{u}(-x)c_4^{[m]}\Big], \tag{2.30}$$

$$b_1^{[m+1]} = \frac{1}{\alpha}[-\mathrm{i}b_{1,x}^{[m]} + ua^{[m]} - ud_{11}^{[m]} - u(-x)d_{21}^{[m]} - \overset{*}{u}d_{31}^{[m]} - \overset{*}{u}(-x)d_{41}^{[m]}], \tag{2.31}$$

$$b_2^{[m+1]} = \frac{1}{\alpha}[-\mathrm{i}b_{2,x}^{[m]} + u(-x)a^{[m]} - ud_{12}^{[m]} - u(-x)d_{22}^{[m]} - \overset{*}{u}d_{32}^{[m]} - \overset{*}{u}(-x)d_{42}^{[m]}], \tag{2.32}$$

$$b_3^{[m+1]} = \frac{1}{\alpha}[-\mathrm{i}b_{3,x}^{[m]} + \overset{*}{u}a^{[m]} - ud_{13}^{[m]} - u(-x)d_{23}^{[m]} - \overset{*}{u}d_{33}^{[m]} - \overset{*}{u}(-x)d_{43}^{[m]}], \tag{2.33}$$

$$b_4^{[m+1]} = \frac{1}{\alpha}[-\mathrm{i}b_{4,x}^{[m]} + \overset{*}{u}(-x)a^{[m]} - ud_{14}^{[m]} - u(-x)d_{24}^{[m]} - \overset{*}{u}d_{34}^{[m]} - \overset{*}{u}(-x)d_{44}^{[m]}], \tag{2.34}$$

$$c_1^{[m+1]} = \frac{1}{\alpha}[\mathrm{i}c_{1,x}^{[m]} - \overset{*}{u}a^{[m]} + \overset{*}{u}d_{11}^{[m]} + \overset{*}{u}(-x)d_{12}^{[m]} + ud_{13}^{[m]} + u(-x)d_{14}^{[m]}], \tag{2.35}$$

$$c_2^{[m+1]} = \frac{1}{\alpha}[\mathrm{i}c_{2,x}^{[m]} - \overset{*}{u}(-x)a^{[m]} + \overset{*}{u}d_{21}^{[m]} + \overset{*}{u}(-x)d_{22}^{[m]} + ud_{23}^{[m]} + u(-x)d_{24}^{[m]}], \tag{2.36}$$

$$c_3^{[m+1]} = \frac{1}{\alpha}[\mathrm{i}c_{3,x}^{[m]} - ua^{[m]} + \overset{*}{u}d_{31}^{[m]} + \overset{*}{u}(-x)d_{32}^{[m]} + ud_{33}^{[m]} + u(-x)d_{34}^{[m]}], \tag{2.37}$$

$$c_4^{[m+1]} = \frac{1}{\alpha}[\mathrm{i}c_{4,x}^{[m]} - u(-x)a^{[m]} + \overset{*}{u}d_{41}^{[m]} + \overset{*}{u}(-x)d_{42}^{[m]} + ud_{43}^{[m]} + u(-x)d_{44}^{[m]}], \tag{2.38}$$

$$d_{11,x}^{[m]} = -\mathrm{i}[\overset{*}{u}b_1^{[m]} + uc_1^{[m]}], \qquad d_{12,x}^{[m]} = -\mathrm{i}[\overset{*}{u}b_2^{[m]} + u(-x)c_1^{[m]}], \tag{2.39}$$

$$d_{21,x}^{[m]} = -\mathrm{i}[\overset{*}{u}(-x)b_1^{[m]} + uc_2^{[m]}], \qquad d_{22,x}^{[m]} = -\mathrm{i}[\overset{*}{u}(-x)b_2^{[m]} + u(-x)c_2^{[m]}], \tag{2.40}$$

$$d_{31,x}^{[m]} = -\mathrm{i}[ub_1^{[m]} + uc_3^{[m]}], \qquad d_{32,x}^{[m]} = -\mathrm{i}[ub_2^{[m]} + u(-x)c_3^{[m]}], \tag{2.41}$$

$$d_{41,x}^{[m]} = -\mathrm{i}[u(-x)b_1^{[m]} + uc_4^{[m]}], \qquad d_{42,x}^{[m]} = -\mathrm{i}[u(-x)b_2^{[m]} + u(-x)c_4^{[m]}], \tag{2.42}$$

$$d^{[m]}_{13,x} = -\mathrm{i}[\overset{*}{u} b^{[m]}_3 + \overset{*}{u} c^{[m]}_1], \qquad d^{[m]}_{14,x} = -\mathrm{i}[\overset{*}{u} b^{[m]}_4 + \overset{*}{u}(-x) c^{[m]}_1], \tag{2.43}$$
$$d^{[m]}_{23,x} = -\mathrm{i}[\overset{*}{u}(-x) b^{[m]}_3 + \overset{*}{u} c^{[m]}_2], \qquad d^{[m]}_{24,x} = -\mathrm{i}[\overset{*}{u}(-x) b^{[m]}_4 + \overset{*}{u}(-x) c^{[m]}_2], \tag{2.44}$$
$$d^{[m]}_{33,x} = -\mathrm{i}[u b^{[m]}_3 + \overset{*}{u} c^{[m]}_3], \qquad d^{[m]}_{34,x} = -\mathrm{i}[u b^{[m]}_4 + \overset{*}{u}(-x) c^{[m]}_3], \tag{2.45}$$
$$d^{[m]}_{43,x} = -\mathrm{i}[u(-x) b^{[m]}_3 + \overset{*}{u} c^{[m]}_4], \qquad d^{[m]}_{44,x} = -\mathrm{i}[u(-x) b^{[m]}_4 + \overset{*}{u}(-x) c^{[m]}_4]. \tag{2.46}$$

The first few functions involved can be explicitly determined as follows:

$$\begin{cases} a^{[0]} &= \beta_1, \\ a^{[1]} &= 0, \\ a^{[2]} &= 2\frac{\beta}{\alpha^2}(|u|^2 + |u(-x,-t)|^2), \\ a^{[3]} &= 0, \\ a^{[4]} &= \frac{\beta}{\alpha^4}\Big[12\Big(|u|^2 + |u(-x,-t)|^2\Big)^2 + 6\Big(|u_x|^2 + |u_x(-x,-t)|^2\Big) - 2\Big(|u|^2 + |u(-x,-t)|^2\Big)_{xx}\Big], \\ a^{[5]} &= 0, \\ a^{[6]} &= \frac{\beta}{\alpha^6}\Big[80(|u|^2 + |u(-x,-t)|^2)^3 + 80\Big(|u|^2 + |u(-x,-t)|^2\Big)\Big((|u_x|^2 + |u_x(-x,-t)|^2\Big) \\ & \quad - 40\Big(|u|^2 + |u(-x,-t)|^2\Big)\Big(|u|^2 + |u(-x,-t)|^2\Big)_{xx} \\ & \quad + 10\Big(|u_{xx}|^2 + |u_{xx}(-x,-t)|^2\Big) + 2\Big(|u|^2 + |u(-x,-t)|^2\Big)_{xxxx} \\ & \quad - 10\Big((|u|^2 + |u(-x,-t)|^2)_x\Big)^2 - 10\Big(|u_x|^2 + |u_x(-x,-t)|^2\Big)_{xx}\Big], \end{cases} \tag{2.47}$$

$$\begin{cases} b^{[0]}_1 &= 0, \\ b^{[1]}_1 &= \frac{\beta}{\alpha} u, \\ b^{[2]}_1 &= -\mathrm{i}\frac{\beta}{\alpha^2} u_x, \\ b^{[3]}_1 &= -\frac{\beta}{\alpha^3}\Big[u_{xx} - 4\Big(|u|^2 + |u(-x,-t)|^2\Big)u\Big], \\ b^{[4]}_1 &= \mathrm{i}\frac{\beta}{\alpha^4}\Big[u_{xxx} - 6\Big(|u|^2 + |u(-x,-t)|^2\Big)u_x - 3\Big(|u|^2 + |u(-x,-t)|^2\Big)_x u\Big], \\ b^{[5]}_1 &= \frac{\beta}{\alpha^5}\Big[u_{xxxx} - 8\Big((|u|^2 + |u(-x,-t)|^2)u_x\Big)_x \\ & \quad + \Big(24(|u|^2 + |u(-x,-t)|^2)^2 + 8(|u_x|^2 + |u_x(-x,-t)|^2) - 6(|u|^2 + |u(-x,-t)|^2)_{xx}\Big)u, \\ b^{[6]}_1 &= -\mathrm{i}\frac{\beta}{\alpha^6}\Big[u_{xxxxx} - 10(|u|^2 + |u(-x,-t)|^2)u_{xxx} - 15(|u|^2 + |u(-x,-t)|^2)_x u_{xx} \\ & \quad + [40(|u|^2 + |u(-x,-t)|^2)^2 + 10(|u_x|^2 + |u_x(-x,-t)|^2) - 15(|u|^2 + |u(-x,-t)|^2)_{xx}]u_x \\ & \quad + \Big[20\Big((|u|^2 + |u(-x,-t)|^2)^2\Big)_x + 5(|u_x|^2 + |u_x(-x,-t)|^2)_x - 5(|u|^2 + |u(-x,-t)|^2)_{xxx}\Big]u\Big], \end{cases} \tag{2.48}$$

$$\begin{cases} d_{11}^{[0]} &= \beta_2, \\ d_{11}^{[1]} &= 0, \\ d_{11}^{[2]} &= -\frac{\beta}{\alpha^2}|u|^2, \\ d_{11}^{[3]} &= 2\frac{\beta}{\alpha^3}\mathrm{Im}(u\overset{*}{u}_x), \\ d_{11}^{[4]} &= -\frac{\beta}{\alpha^4}\Big[6(|u|^2+|u(-x,-t)|^2)|u|^2+3|u_x|^2-(|u|^2)_{xx}\Big], \\ d_{11}^{[5]} &= \frac{\beta}{\alpha^5}\Big[16(|u|^2+|u(-x,-t)|^2)\mathrm{Im}(u\overset{*}{u}_x)+2\mathrm{Im}(u_{xxx}\overset{*}{u}-u_x\overset{*}{u}_{xx})\Big], \\ d_{11}^{[6]} &= \frac{\beta}{\alpha^6}\Big[\Big(10(|u|^2+|u(-x,-t)|^2)_{xx}-10(|u_x|^2+|u_x(-x,-t)|^2)-40(|u|^2+|u(-x,-t)|^2)^2\Big)|u|^2 \\ &\quad +5(|u|^2+|u(-x,-t)|^2)_x(|u|^2)_x+10(|u|^2+|u(-x,-t)|^2)((|u|^2)_{xx}-3|u_x|^2) \\ &\quad -(|u|^2)_{xxxx}+5(|u_x|^2)_{xx}-5|u_{xx}|^2\Big], \end{cases} \tag{2.49}$$

$$\begin{cases} d_{21}^{[0]} &= 0, \\ d_{21}^{[1]} &= 0, \\ d_{21}^{[2]} &= -\frac{\beta}{\alpha^2}u\overset{*}{u}(-x,-t), \\ d_{21}^{[3]} &= \mathrm{i}\frac{\beta}{\alpha^3}\Big(\overset{*}{u}(-x,-t)u_x+\overset{*}{u}_x(-x,-t)u\Big), \\ d_{21}^{[4]} &= -\frac{\beta}{\alpha^4}\Big[6(|u|^2+|u(-x,-t)|^2)u\overset{*}{u}(-x,-t)-3u_x\overset{*}{u}_x(-x,-t)-(u\overset{*}{u}(-x,-t))_{xx}\Big], \\ d_{21}^{[5]} &= \mathrm{i}\frac{\beta}{\alpha^5}\Big[8(|u|^2+|u(-x,-t)|^2)(u\overset{*}{u}_x(-x,-t)+u_x\overset{*}{u}(-x,-t)) \\ &\quad -(u\overset{*}{u}_x(-x,-t)+u_x\overset{*}{u}(-x,-t))_{xx}-2(u_x\overset{*}{u}_{xx}(-x,-t)+u_{xx}\overset{*}{u}_x(-x,-t))\Big], \\ d_{21}^{[6]} &= \frac{\beta}{\alpha^6}\Big[\Big(10(|u|^2+|u(-x,-t)|^2)_{xx}-10(|u_x|^2+|u_x(-x,-t)|^2) \\ &\quad -40(|u|^2+|u(-x,-t)|^2)^2\Big)u\overset{*}{u}(-x,-t)+5(|u|^2+|u(-x,-t)|^2)_x(u\overset{*}{u}(-x,-t))_x \\ &\quad +10(|u|^2+|u(-x,-t)|^2)((u\overset{*}{u}(-x,-t))_{xx}+3u_x\overset{*}{u}_x(-x,-t)) \\ &\quad -(u\overset{*}{u}(-x,-t))_{xxxx}-5(u_x\overset{*}{u}_x(-x,-t))_{xx}-5u_{xx}\overset{*}{u}_{xx}(-x,-t)\Big], \end{cases} \tag{2.50}$$

$$\begin{cases} d_{31}^{[0]} &= 0, \\ d_{31}^{[1]} &= 0, \\ d_{31}^{[2]} &= -\frac{\beta}{\alpha^2}u^2, \\ d_{31}^{[3]} &= 0, \\ d_{31}^{[4]} &= -\frac{\beta}{\alpha^4}\Big[6(|u|^2+|u(-x,-t)|^2)u^2+3u_x^2-(u^2)_{xx}\Big], \\ d_{31}^{[5]} &= 0, \\ d_{31}^{[6]} &= \frac{\beta}{\alpha^6}\Big[\Big(10(|u|^2+|u(-x,-t)|^2)_{xx}-10(|u_x|^2+|u_x(-x,-t)|^2)-40(|u|^2+|u(-x,-t)|^2)^2\Big)u^2 \\ &\quad +5(|u|^2+|u(-x,-t)|^2)_x(u^2)_x+10(|u|^2+|u(-x,-t)|^2)((u^2)_{xx}-3u_x^2) \\ &\quad -(u^2)_{xxxx}+5(u_x^2)_{xx}-5u_{xx}^2\Big], \end{cases} \tag{2.51}$$

$$\begin{cases} d_{41}^{[0]} &= 0, \\ d_{41}^{[1]} &= 0, \\ d_{41}^{[2]} &= -\frac{\beta}{\alpha^2} uu(-x,-t), \\ d_{41}^{[3]} &= \mathrm{i}\frac{\beta}{\alpha^3}\Big(u_x u(-x,-t) + uu_x(-x,-t)\Big), \\ d_{41}^{[4]} &= -\frac{\beta}{\alpha^4}\Big[6(|u|^2 + |u(-x,-t)|^2)uu(-x,-t) + 3u_x u_x(-x,-t) + (uu(-x,-t))_{xx}\Big], \\ d_{41}^{[5]} &= \mathrm{i}\frac{\beta}{\alpha^5}\Big[8(|u|^2 + |u(-x,-t)|^2)(uu_x(-x,-t) + u_x u(-x,-t)) \\ & - (uu_x(-x,-t) + u_x u(-x,-t))_{xx} - 2(u_x u_{xx}(-x,-t) + u_{xx} u_x(-x,-t))\Big], \\ d_{41}^{[6]} &= \frac{\beta}{\alpha^6}\Big[\Big(10(|u|^2 + |u(-x,-t)|^2)_{xx} - 10(|u_x|^2 + |u_x(-x,-t)|^2) \\ & - 40(|u|^2 + |u(-x,-t)|^2)^2\Big)uu(-x,-t) + 5(|u|^2 + |u(-x,-t)|^2)_x (uu(-x,-t))_x \\ & + 10(|u|^2 + |u(-x,-t)|^2)((uu(-x,-t))_{xx} + 3u_x u_x(-x,-t)) \\ & - (uu(-x,-t))_{xxxx} - 5(u_x u_x(-x,-t))_{xx} - 5u_{xx} u_{xx}(-x,-t)\Big], \end{cases} \tag{2.52}$$

where $\beta = \beta_1 - \beta_2$.

Remark 2.2. *Under the symmetry relations (2.8), one can show that W satisfies the equations:*

$$\begin{cases} W^{\dagger}(x,t;-\overset{*}{\lambda}) = -W, \\ W^{T}(x,t;-\lambda) = C_1 W C_1^{-1}, \\ W^{\dagger}(-x,-t;-\overset{*}{\lambda}) = C_2 W C_2^{-1}, \\ \overset{*}{W}(-x,-t;\overset{*}{\lambda}) = C_3 W C_3^{-1}, \end{cases} \tag{2.53}$$

for a solution W to the stationary zero curvature equation. Using the Laurent expansion (2.28) of W, we get the relations:

$$a^{[m]} = (-1)^{m+1}\overset{*}{a}^{[m]}, \tag{2.54}$$

$$b_1^{[m]} = (-1)^{m+1}\overset{*}{c}_1^{[m]}, \quad b_2^{[m]} = (-1)^{m+1}\overset{*}{c}_2^{[m]}, \tag{2.55}$$

$$b_3^{[m]} = (-1)^{m+1}\overset{*}{c}_3^{[m]}, \quad b_4^{[m]} = (-1)^{m+1}\overset{*}{c}_4^{[m]}, \tag{2.56}$$

$$\overset{*}{d}_{11}^{[m]} = (-1)^{m+1} d_{11}^{[m]}, \quad \overset{*}{d}_{21}^{[m]} = (-1)^{m+1} d_{12}^{[m]}, \tag{2.57}$$

$$\overset{*}{d}_{31}^{[m]} = (-1)^{m+1} d_{13}^{[m]}, \quad \overset{*}{d}_{41}^{[m]} = (-1)^{m+1} d_{14}^{[m]}, \tag{2.58}$$

$$\overset{*}{d}_{22}^{[m]} = (-1)^{m+1} d_{22}^{[m]}, \quad \overset{*}{d}_{32}^{[m]} = (-1)^{m+1} d_{23}^{[m]}, \tag{2.59}$$

$$\overset{*}{d}_{42}^{[m]} = (-1)^{m+1} d_{24}^{[m]}, \quad \overset{*}{d}_{33}^{[m]} = (-1)^{m+1} d_{33}^{[m]}, \tag{2.60}$$

$$\overset{*}{d}_{43}^{[m]} = (-1)^{m+1} d_{34}^{[m]}, \quad \overset{*}{d}_{44}^{[m]} = (-1)^{m+1} d_{44}^{[m]}, \tag{2.61}$$

$$a^{[m]} = (-1)^m a^{[m]}, \tag{2.62}$$
$$b_1^{[m]} = (-1)^m c_3^{[m]}, \quad b_2^{[m]} = (-1)^m c_4^{[m]}, \tag{2.63}$$
$$b_3^{[m]} = (-1)^m c_1^{[m]}, \quad b_4^{[m]} = (-1)^m c_2^{[m]}, \tag{2.64}$$
$$d_{11}^{[m]} = (-1)^m d_{33}^{[m]}, \quad d_{21}^{[m]} = (-1)^m d_{34}^{[m]}, \tag{2.65}$$
$$d_{31}^{[m]} = (-1)^m d_{31}^{[m]}, \quad d_{41}^{[m]} = (-1)^m d_{32}^{[m]}, \tag{2.66}$$
$$d_{12}^{[m]} = (-1)^m d_{43}^{[m]}, \quad d_{22}^{[m]} = (-1)^m d_{44}^{[m]}, \tag{2.67}$$
$$d_{42}^{[m]} = (-1)^m d_{42}^{[m]}, \quad d_{13}^{[m]} = (-1)^m d_{13}^{[m]}, \tag{2.68}$$
$$d_{23}^{[m]} = (-1)^m d_{14}^{[m]}, \quad d_{24}^{[m]} = (-1)^m d_{24}^{[m]}, \tag{2.69}$$

and

$$\overset{*}{a}^{[m]}(-x,-t) = (-1)^m a^{[m]}, \tag{2.70}$$
$$b_1^{[m]}(-x,-t) = (-1)^m \overset{*}{c}_2^{[m]}, \quad b_2^{[m]}(-x,-t) = (-1)^m \overset{*}{c}_1^{[m]}, \tag{2.71}$$
$$b_3^{[m]}(-x,-t) = (-1)^m \overset{*}{c}_4^{[m]}, \quad b_4^{[m]}(-x,-t) = (-1)^m \overset{*}{c}_3^{[m]}, \tag{2.72}$$
$$\overset{*}{d}_{11}^{[m]}(-x,-t) = (-1)^m d_{22}^{[m]}, \quad \overset{*}{d}_{21}^{[m]}(-x,-t) = (-1)^m d_{21}^{[m]}, \tag{2.73}$$
$$\overset{*}{d}_{31}^{[m]}(-x,-t) = (-1)^m d_{24}^{[m]}, \quad \overset{*}{d}_{41}^{[m]}(-x,-t) = (-1)^m d_{23}^{[m]}, \tag{2.74}$$
$$\overset{*}{d}_{12}^{[m]}(-x,-t) = (-1)^m d_{12}^{[m]}, \quad \overset{*}{d}_{32}^{[m]}(-x,-t) = (-1)^m d_{14}^{[m]}, \tag{2.75}$$
$$\overset{*}{d}_{42}^{[m]}(-x,-t) = (-1)^m d_{13}^{[m]}, \quad \overset{*}{d}_{33}^{[m]}(-x,-t) = (-1)^m d_{44}^{[m]}, \tag{2.76}$$
$$\overset{*}{d}_{43}^{[m]}(-x,-t) = (-1)^m d_{43}^{[m]}, \quad \overset{*}{d}_{34}^{[m]}(-x,-t) = (-1)^m d_{34}^{[m]}, \tag{2.77}$$

and finally

$$a^{[m]} = \overset{*}{a}^{[m]}(-x,-t), \tag{2.78}$$
$$b_1^{[m]} = \overset{*}{b}_4^{[m]}(-x,-t), \quad b_2^{[m]} = \overset{*}{b}_3^{[m]}(-x,-t), \tag{2.79}$$
$$c_1^{[m]} = \overset{*}{c}_4^{[m]}(-x,-t), \quad c_2^{[m]} = \overset{*}{c}_3^{[m]}(-x,-t), \tag{2.80}$$
$$d_{11}^{[m]} = \overset{*}{d}_{44}^{[m]}(-x,-t), \quad d_{12}^{[m]} = \overset{*}{d}_{43}^{[m]}(-x,-t), \tag{2.81}$$
$$d_{13}^{[m]} = \overset{*}{d}_{42}^{[m]}(-x,-t), \quad d_{14}^{[m]} = \overset{*}{d}_{41}^{[m]}(-x,-t), \tag{2.82}$$
$$d_{21}^{[m]} = \overset{*}{d}_{34}^{[m]}(-x,-t), \quad d_{22}^{[m]} = \overset{*}{d}_{33}^{[m]}(-x,-t), \tag{2.83}$$
$$d_{23}^{[m]} = \overset{*}{d}_{32}^{[m]}(-x,-t), \quad d_{24}^{[m]} = \overset{*}{d}_{31}^{[m]}(-x,-t). \tag{2.84}$$

One can easily relate all the members of the set $\mathcal{S}_1 = \{b_1^{[m]}, b_2^{[m]}, b_3^{[m]}, b_4^{[m]}, c_1^{[m]}, c_2^{[m]}, c_3^{[m]}, c_4^{[m]}\}$ directly from the above four sets of relations. Similarly, the members within each of the following sets: $\mathcal{S}_2 = \{d_{11}^{[m]}, d_{22}^{[m]}, d_{33}^{[m]}, d_{44}^{[m]}\}$, $\mathcal{S}_3 = \{d_{21}^{[m]}, d_{34}^{[m]}, d_{12}^{[m]}, d_{43}^{[m]}\}$, $\mathcal{S}_4 = \{d_{31}^{[m]}, d_{24}^{[m]}, d_{13}^{[m]}, d_{42}^{[m]}\}$ and $\mathcal{S}_5 =$

$\{d_{41}^{[m]}, d_{23}^{[m]}, d_{32}^{[m]}, d_{14}^{[m]}\}$ are related to each other. As a consequence, any member in the set $\mathcal{S}_1$ can be expressed in terms of the independents $\{a^{[m]}, d_{11}^{[m]}, d_{21}^{[m]}, d_{31}^{[m]}, d_{41}^{[m]}\}$.

We present the following Lax matrix:

$$V^{[m]}(u;\lambda) = (\lambda^m W)_+ = \sum_{i=0}^{m} W_i \lambda^{m-i} = \sum_{i=0}^{m} \begin{pmatrix} a^{[i]}\lambda^{m-i} & b_1^{[i]}\lambda^{m-i} & b_2^{[i]}\lambda^{m-i} & b_3^{[i]}\lambda^{m-i} & b_4^{[i]}\lambda^{m-i} \\ c_1^{[i]}\lambda^{m-i} & d_{11}^{[i]}\lambda^{m-i} & d_{12}^{[i]}\lambda^{m-i} & d_{13}^{[i]}\lambda^{m-i} & d_{14}^{[i]}\lambda^{m-i} \\ c_2^{[i]}\lambda^{m-i} & d_{21}^{[i]}\lambda^{m-i} & d_{22}^{[i]}\lambda^{m-i} & d_{23}^{[i]}\lambda^{m-i} & d_{24}^{[i]}\lambda^{m-i} \\ c_3^{[i]}\lambda^{m-i} & d_{31}^{[i]}\lambda^{m-i} & d_{32}^{[i]}\lambda^{m-i} & d_{33}^{[i]}\lambda^{m-i} & d_{34}^{[i]}\lambda^{m-i} \\ c_4^{[i]}\lambda^{m-i} & d_{41}^{[i]}\lambda^{m-i} & d_{42}^{[i]}\lambda^{m-i} & d_{43}^{[i]}\lambda^{m-i} & d_{44}^{[i]}\lambda^{m-i} \end{pmatrix},$$

where the modification terms are set to zero. Consequently, we obtain the spatial and temporal equations of the spectral problems [15], along with the corresponding Lax pair $\{U, V^{[m]}\}$:

$$\psi_x = \mathrm{i}U\psi, \tag{2.85}$$

$$\psi_t = \mathrm{i}V^{[m]}\psi. \tag{2.86}$$

The compatibility conditions arising from Eqs (2.85) and (2.86) lead to the subsequent zero curvature equations:

$$ZCE := U_{t_m} - V_x^{[m]} + \mathrm{i}[U, V^{[m]}] = 0. \tag{2.87}$$

The comptabiltiy of the second component in the first row and the fourth component of the first column of ZCE, namely $(ZCE)_{12}$ and $(ZCE)_{41}$ lead to the scalar Sasa-Satsuma integrable hierarchy:

$$u_{t_m} = \begin{cases} \mathrm{i}\alpha b_1^{[m+1]}, & m = \text{odd}, \\ 0, & m = \text{even}, \end{cases} \qquad m \geq 0. \tag{2.88}$$

Obtaining a hierarchy that only generates mKdV-type equations, but not NLS-type equations. More specifically, the hierarchy here gives Sasa-Satsuma-type equations due to the initial choice of the matrix $U(u;\lambda)$. Thus,

$$u_{t_m} = \mathrm{i}\alpha b_1^{[m+1]}, \quad m = \text{odd}. \tag{2.89}$$

For example, the case of $m = 3$ leads to the nonlocal reverse-spacetime Sasa-Satsuma equation [20]:

$$u_t + \frac{\beta}{\alpha^3}\Big[u_{xxx} - 6\Big(|u|^2 + |u(-x,-t)|^2\Big)u_x - 3\Big(|u|^2 + |u(-x,-t)|^2\Big)_x u\Big] = 0. \tag{2.90}$$

This soliton hierarchy possesses a bi-Hamiltonian structure

$$u_{t_m} = \mathrm{i}\alpha b_1^{[2m]} = J_1 \frac{\delta \mathcal{H}_{2m-1}}{\delta u} = J_2 \frac{\delta \mathcal{H}_{2m-3}}{\delta u}, \quad m \in \{2, 3, \ldots\}, \tag{2.91}$$

where $\mathcal{H}_{2m-3}$ are the Hamiltonian functionals and J_1 and J_2 are a Hamiltonian pair.

We derive from the recursive relations (2.30)–(2.46) and the relations (2.54)–(2.84), the following recursive formula between $b_1^{[m+1]}$ and $b_1^{[m]}$:

$$b_1^{[m+1]} = \Psi b_1^{[m]}, \tag{2.92}$$

where the recursion operator Ψ reads:

$$\begin{aligned}\Psi =&\frac{\mathrm{i}}{\alpha}\Big[-\partial+\Big((2+(-1)^m)u\partial^{-1}\overset{*}{u}+u(-x)\partial^{-1}\overset{*}{u}(-x)+(1+(-1)^m)\overset{*}{u}\partial^{-1}u+\overset{*}{u}(-x)\partial^{-1}u(-x)\Big)\Gamma_+\\&+\Big((2(-1)^{m+1}-1)u\partial^{-1}u\Big)\overset{*}{\Gamma}_+ +\Big(((-1)^{m+1}-1)u\partial^{-1}\overset{*}{u}(-x)+(-1)^{m+1}\overset{*}{u}(-x)\partial^{-1}u\Big)\Gamma_-\\&+\Big(((-1)^m+1)u\partial^{-1}u(-x)+(-1)^m u(-x)\partial^{-1}u\Big)\overset{*}{\Gamma}_-\Big],\end{aligned} \tag{2.93}$$

and where the operators $\Gamma_\pm, \overset{*}{\Gamma}_\pm$ are defined by

$$\Gamma_- f = f(-x,-t), \tag{2.94}$$

$$\overset{*}{\Gamma}_+ f = \overset{*}{f}, \tag{2.95}$$

$$\overset{*}{\Gamma}_- f = \overset{*}{f}(-x,-t), \tag{2.96}$$

for $f = f(x,t)$, and Γ_+ being the identity operator, i.e., $\Gamma_+ f = \mathrm{Id}(f) = f$.

2.1. Nonlocal reverse-spacetime Sasa-Satsuma equation

To derive the one-component nonlocal Sasa-Satsuma equation, we consider the Lax matrix

$$V^{[5]} = V^{[5]}(u;\lambda) = (\lambda^5 W)_+. \tag{2.97}$$

The spatial and temporal equations of the spectral problems defined by Eqs (2.85) and (2.86), along with the associated Lax pair $\{U, V^{[5]}\}$, are as follows:

$$\psi_x = \mathrm{i}U\psi, \tag{2.98}$$

$$\psi_t = \mathrm{i}V^{[5]}\psi, \tag{2.99}$$

with the zero curvature equation

$$U_t - V^{[5]}_x + \mathrm{i}[U, V^{[5]}] = 0, \tag{2.100}$$

that gives the scalar Sasa-Satsuma equation

$$u_t = \mathrm{i}\alpha b_1^{[6]}. \tag{2.101}$$

Explicitly,

$$\begin{aligned}u_t =&u_{xxxxx} - 10(|u|^2+|u(-x,-t)|^2)u_{xxx} - 15(|u|^2+|u(-x,-t)|^2)_x u_{xx}\\&+\big[-15(|u|^2+|u(-x,-t)|^2)_{xx}+10(|u_x|^2+|u_x(-x,-t)|^2)+40(|u|^2+|u(-x,-t)|^2)^2\big]u_x\\&+\big[-5(|u|^2+|u(-x,-t)|^2)_{xxx}+5(|u_x|^2+|u_x(-x,-t)|^2)_x+20((|u|^2+|u(-x,-t)|^2)^2)_x\big]u,\end{aligned} \tag{2.102}$$

and

$$V^{[5]} = V^{[5]}(x,t;\lambda) = (V_{ij})_{5\times 5}, \quad i,j \in \{1,\dots,5\}, \tag{2.103}$$

where the components are explicitly

$$V_{11}=\sum_{i=0}^{2}\lambda^{5-2i}a^{[2i]},\qquad V_{12}=\sum_{i=0}^{5}\lambda^{[5-i]}b_1^{[i]},\qquad V_{13}=-\sum_{i=0}^{5}\lambda^{5-i}\Gamma_-b_1^{[i]},$$
$$V_{21}=\sum_{i=0}^{5}(-1)^{i+1}\lambda^{5-i}\overset{*}{\Gamma}_+b_1^{[i]},\qquad V_{22}=\sum_{i=0}^{5}\lambda^{5-i}d_{11}^{[i]},\qquad V_{23}=\sum_{i=0}^{5}(-1)^{i+1}\lambda^{5-i}\overset{*}{\Gamma}_+d_{21}^{[i]},$$
$$V_{31}=\sum_{i=0}^{5}(-1)^{i}\lambda^{5-i}\overset{*}{\Gamma}_-b_1^{[i]},\qquad V_{32}=\sum_{i=0}^{5}\lambda^{5-i}d_{21}^{[i]},\qquad V_{33}=\sum_{i=0}^{5}(-1)^{i}\lambda^{5-i}\overset{*}{\Gamma}_-d_{11}^{[i]},$$
$$V_{41}=\sum_{i=0}^{5}(-1)^{i}\lambda^{5-i}b_1^{[i]},\qquad V_{42}=\sum_{i=0}^{2}\lambda^{5-2i}d_{31}^{[2i]},\qquad V_{43}=-\sum_{i=0}^{5}\lambda^{5-i}\Gamma_-d_{41}^{[i]},$$
$$V_{51}=\sum_{i=0}^{5}(-1)^{i+1}\lambda^{5-i}\Gamma_-b_1^{[i]},\qquad V_{52}=\sum_{i=0}^{5}\lambda^{5-i}d_{41}^{[i]},\qquad V_{53}=-\sum_{i=0}^{2}\lambda^{5-2i}\Gamma_-d_{31}^{[2i]},$$

$$V_{14}=-\sum_{i=0}^{5}\lambda^{5-i}\overset{*}{\Gamma}_+b_1^{[i]},\qquad V_{15}=\sum_{i=0}^{5}\lambda^{5-i}\overset{*}{\Gamma}_-b_1^{[i]},$$
$$V_{24}=\sum_{i=0}^{2}(-1)^{2i+1}\lambda^{5-2i}\overset{*}{\Gamma}_+d_{31}^{[2i]},\qquad V_{25}=\sum_{i=0}^{5}(-1)^{i+1}\lambda^{5-i}\overset{*}{\Gamma}_+d_{41}^{[i]},$$
$$V_{34}=-\sum_{i=0}^{5}\lambda^{5-i}\overset{*}{\Gamma}_+d_{41}^{[i]},\qquad V_{35}=\sum_{i=0}^{2}(-1)^{2i}\lambda^{5-2i}\overset{*}{\Gamma}_-d_{31}^{[2i]},$$
$$V_{44}=\sum_{i=0}^{5}(-1)^{i}\lambda^{5-i}d_{11}^{[i]},\qquad V_{45}=\sum_{i=0}^{5}\lambda^{5-i}\overset{*}{\Gamma}_-d_{21}^{[i]},$$
$$V_{54}=-\sum_{i=0}^{5}\lambda^{5-i}\overset{*}{\Gamma}_+d_{21}^{[i]},\qquad V_{55}=\sum_{i=0}^{5}\lambda^{5-i}\overset{*}{\Gamma}_-d_{11}^{[i]}.$$

The matrix $V^{[5]}$ exhibits the properties of symmetry:

$$\begin{cases} V^{[5]\dagger}(x,t;-\overset{*}{\lambda})=V^{[5]},\\ V^{[5]T}(x,t;-\lambda)=-C_1V^{[5]}C_1^{-1},\\ V^{[5]\dagger}(-x,-t;-\overset{*}{\lambda})=-C_2V^{[5]}C_2^{-1},\\ \overset{*}{V}^{[5]}(-x,-t;\overset{*}{\lambda})=C_3V^{[5]}C_3^{-1}. \end{cases}\tag{2.104}$$

2.2. Bi-Hamiltonian structures

We start to find a bi-Hamiltonian structures of the soliton hierarchy (2.89). To do so, we are going to use the trace identity

$$\frac{\delta}{\delta u}\int tr\Big[W\frac{\partial U}{\partial\lambda}\Big]dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\Big[\lambda^{\gamma}tr(W\frac{\partial U}{\partial u})\Big],\tag{2.105}$$

where

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|tr(W^2)|. \tag{2.106}$$

Thus, from the matrix U, one can easily compute $\frac{\partial U}{\partial u}$ to obtain

$$\begin{aligned} tr\Big[W\frac{\partial U}{\partial\lambda}\Big] &= \sum_{m=0}^{\infty}\Big(\alpha_1 a^{[m]} + \alpha_2((-1)^m + 1)(\mathit{\Gamma}_+ + \overset{*}{\mathit{\Gamma}}_-)d_{11}^{[m]}\Big)\lambda^{-m}, \\ tr\Big[W\frac{\partial U}{\partial u}\Big] &= \sum_{m=0}^{\infty}((-1)^{m+1} + 1)\overset{*}{\mathit{\Gamma}}_+ b_1^{[m]}\lambda^{-m}. \end{aligned}$$

By substituting these in the trace identity formula (2.105), and matching the powers of λ^{-m-1}, we get

$$\frac{\delta}{\delta u}\int\Big(\alpha_1 a^{[m+1]} + \alpha_2((-1)^{m+1} + 1)(\mathit{\Gamma}_+ + \overset{*}{\mathit{\Gamma}}_-)d_{11}^{[m+1]}\Big)dx = (\gamma - m)((-1)^{m+1} + 1)\overset{*}{\mathit{\Gamma}}_+ b_1^{[m]},\ m \geq 1. \tag{2.107}$$

Observing when $m = 1$ and $m = 3$, we see that $\gamma = 2m$. Hence, the Hamiltonians are given by

$$\mathcal{H}_m = \frac{1}{m}\int\Big(\alpha_1 a^{[m+1]} + \alpha_2((-1)^{m+1} + 1)(\mathit{\Gamma}_+ + \overset{*}{\mathit{\Gamma}}_-)d_{11}^{[m+1]}\Big)dx, \quad m \geq 1, \tag{2.108}$$

$$= \begin{cases} \frac{1}{m}\int\Big(\alpha_1 a^{[m+1]} + 2\alpha_2(\mathit{\Gamma}_+ + \overset{*}{\mathit{\Gamma}}_-)d_{11}^{[m+1]}\Big)dx, & m = \text{odd}, \\ 0, & m = \text{even}, \end{cases} \tag{2.109}$$

$$= \begin{cases} \frac{\alpha}{m}\int a^{[m+1]}dx, & m = \text{odd}, \\ 0, & m = \text{even}, \end{cases} \tag{2.110}$$

since $a^{[m+1]} = -((-1)^{m+1} + 1)(\mathit{\Gamma}_+ + \overset{*}{\mathit{\Gamma}}_-)d_{11}^{[m+1]}$ from (2.30), (2.39), (2.40) and finally (2.45) and (2.46). Also, we have

$$\frac{\delta\mathcal{H}_{2m-1}}{\delta u} = 2\overset{*}{\mathit{\Gamma}}_+ b_1^{[2m-1]}, \quad m \in \{1, 3, \ldots\}. \tag{2.111}$$

Now since

$$u_{t_m} = \mathrm{i}\alpha b_1^{[2m]} = J_1\frac{\delta\mathcal{H}_{2m-1}}{\delta u} = J_1 2\overset{*}{\mathit{\Gamma}}_+ b_1^{[2m-1]} \tag{2.112}$$

$$= J_2\frac{\delta\mathcal{H}_{2m-3}}{\delta u} = J_2 2\overset{*}{\mathit{\Gamma}}_+ b_1^{[2m-3]}, \quad \text{for} \quad m \in \{2, 3, \ldots\}, \tag{2.113}$$

where $\overset{*}{\mathit{\Gamma}}_\pm(c\overset{*}{\mathit{\Gamma}}_\pm f) = \overset{*}{c}f$, and c is any complex number, therefore, we deduce the Hamiltonian pair J_1 and J_2 as follows:

$$J_1 = \mathrm{i}\frac{\alpha}{2}\mathit{\Psi}_o\overset{*}{\mathit{\Gamma}}_+ \tag{2.114}$$

and

$$J_2 = J_1\overset{*}{\mathit{\Gamma}}_+\mathit{\Psi}_e\mathit{\Psi}_o\overset{*}{\mathit{\Gamma}}_+, \tag{2.115}$$

where $\Psi_o = \Psi\big|_{m=odd}$ and $\Psi_e = \Psi\big|_{m=even}$. Using the matrix U, we can find the first three Hamiltonian functionals:

$$\mathcal{H}_1 = 2\frac{\beta}{\alpha}\int\Big(|u|^2 + |u(-x,-t)|^2\Big)dx, \tag{2.116}$$

$$\mathcal{H}_3 = \frac{2\beta}{3\alpha^3}\int\Big[6\Big(|u|^2 + |u(-x,-t)|^2\Big)^2 + 3\Big(|u_x|^2 + |u_x(-x,-t)|^2\Big) - \Big(|u|^2 + |u(-x,-t)|^2\Big)_{xx}\Big]dx, \tag{2.117}$$

$$\begin{aligned}\mathcal{H}_5 = \frac{\beta}{5\alpha^5}\int\Big[&80(|u|^2 + |u(-x,-t)|^2)^3 + 80\Big(|u|^2 + |u(-x,-t)|^2\Big)\Big((|u_x|^2 + |u_x(-x,-t)|^2\Big)\\ &- 40\Big(|u|^2 + |u(-x,-t)|^2\Big)\Big(|u|^2 + |u(-x,-t)|^2\Big)_{xx}\\ &+ 10\Big(|u_{xx}|^2 + |u_{xx}(-x,-t)|^2\Big) + 2\Big(|u|^2 + |u(-x,-t)|^2\Big)_{xxxx}\\ &- 10\Big((|u|^2 + |u(-x,-t)|^2)_x\Big)^2 - 10\Big(|u_x|^2 + |u_x(-x,-t)|^2\Big)_{xx}\Big]dx.\end{aligned} \tag{2.118}$$

3. Riemann-Hilbert problems

The spatial and temporal spectral problems associated with the two-component nonlocal Sasa-Satsuma equation are expressed as

$$\psi_x = iU\psi = i(\lambda\Lambda + P)\psi, \tag{3.1}$$

$$\psi_t = iV^{[5]}\psi = i(\lambda^5\Omega + Q)\psi, \tag{3.2}$$

where $\Lambda = diag(\alpha_1, \alpha_2 I_4)$, $\Omega = diag(\beta_1, \beta_2 I_4)$, and $Q = V^{[5]} - \lambda^5\Omega$. In this paper, we consistently assume that $\alpha < 0$ and $\beta < 0$, where $\beta_1 + 4\beta_2 = 0$.

To obtain soliton solutions, we start with an initial condition $u(x,0)$ and evolve it over time to reach $u(x,t)$. Assume that u decays exponentially, i.e., $u \to 0$ as $x,t \to \pm\infty$. Hence, based on spectral problems (3.1) and (3.2), the asymptotic behavior of the fundamental matrix ψ can be described as follows:

$$\psi(x,t) \rightsquigarrow e^{i\lambda\Lambda x + i\lambda^5\Omega t}. \tag{3.3}$$

Thus, the solution to the spectral problems can be expressed as

$$\psi(x,t) = \phi(x,t)e^{i\lambda\Lambda x + i\lambda^5\Omega t}. \tag{3.4}$$

The Jost solution of the eigenfunction (3.4) requires that [22, 24]

$$\phi(x,t) \to I_5, \quad \text{as} \quad x,t \to \pm\infty, \tag{3.5}$$

where I_5 is the 5×5 identity matrix. We denote

$$\phi^\pm \to I_5, \quad \text{when} \quad x \to \pm\infty. \tag{3.6}$$

Using Eq (3.4), the Lax pair (3.1) and (3.2) can be rewritten in terms of ϕ so that the spectral problems can be written equivalently as

$$\phi_x = i\lambda[\Lambda, \phi] + iP\phi, \tag{3.7}$$

$$\phi_t = i\lambda^5[\Omega, \phi] + iQ\phi. \tag{3.8}$$

To formulate the Riemann-Hilbert problems and find their solutions in the reflectionless situation, we will utilize the adjoint scattering equations corresponding to the spectral problems $\psi_x = iU\psi$ and $\psi_t = iV^{[5]}\psi$. Their adjoints are given by

$$\tilde{\psi}_x = -i\tilde{\psi}U, \tag{3.9}$$

$$\tilde{\psi}_t = -i\tilde{\psi}V^{[5]}, \tag{3.10}$$

and the equivalent spectral adjoint equations read

$$\tilde{\phi}_x = -i\lambda[\tilde{\phi}, \Lambda] - i\tilde{\phi}P, \tag{3.11}$$

$$\tilde{\phi}_t = -i\lambda^5[\tilde{\phi}, \Omega] - i\tilde{\phi}Q. \tag{3.12}$$

Due to $tr(iP) = 0$ and $tr(iQ) = 0$, and by applying Liouville's formula [22], it is evident $(det(\phi))_x = 0$, that $det(\phi)$ is a constant. Utilizing the boundary condition (3.5), we ascertain that

$$det(\phi) = 1. \tag{3.13}$$

It follows that the Jost matrix ϕ is invertible.

Furthermore, as $\phi_x^{-1} = -\phi^{-1}\phi_x\phi^{-1}$, we can derive from (3.7)

$$\phi_x^{-1} = -i\lambda[\phi^{-1}, \Lambda] - i\phi^{-1}P. \tag{3.14}$$

The spatial adjoint equation (3.11) can thus be demonstrated to be satisfied by both $(\phi^+)^{-1}$ and $(\phi^-)^{-1}$. Furthermore, both satisfy the temporal adjoint equation (3.12).

Accordingly, if the eigenfunction ϕ solves the spectral problem (3.7), then ϕ^{-1} solves the adjoint spectral problem (3.11).

Because of $\phi_x^{-1} = -\phi^{-1}\phi_x\phi^{-1}$, $C_1\phi^{-1}$ is also a solution of (3.11) with the same eigenvalue. As well as the nonlocal $\phi^T(x, t; -\lambda)C_1$, solves the spectral adjoint problem (3.11). This ensures the uniqueness of the solution since both solutions have the same boundary condition as $x \to \pm\infty$, therefore,

$$\phi^T(x, t; -\lambda) = C_1\phi^{-1}C_1^{-1}. \tag{3.15}$$

Consequently, if λ is an eigenvalue of either Eq (3.7) or (3.11), then $-\lambda$ would also be an eigenvalue, and the relation (3.15) holds.

In the same way, one can prove that $\phi^\dagger(-x, -t; -\overset{*}{\lambda})C_2$ and $C_2\phi^{-1}$ satisfy (3.11), while $\overset{*}{\phi}(-x, -t; \overset{*}{\lambda})C_3$ and $C_3\phi$ satisfy (3.7). Thus, using the boundary condition and by uniqueness of the solution, we can also derive

$$\phi^\dagger(-x, -t; -\overset{*}{\lambda}) = C_2\phi^{-1}C_2^{-1}, \tag{3.16}$$

$$\overset{*}{\phi}(-x, -t; \overset{*}{\lambda}) = C_3\phi C_3^{-1}. \tag{3.17}$$

We will now address the spatial spectral problem (3.7), assuming that the time is $t = 0$. In order to simplify notation, we will use Y^+ and Y^- to represent the boundary conditions as $x \to \infty$ and $x \to -\infty$, respectively.

Knowing that

$$\phi^{\pm} \to I_5 \quad \text{when} \quad x \to \pm\infty, \tag{3.18}$$

and from (3.4), we can write

$$\psi^{\pm} = \phi^{\pm} e^{i\lambda\Lambda x}. \tag{3.19}$$

ψ^+ and ψ^- both meet the requirements of the spectral spatial differential equation (3.1), making them both solutions to the equation. Consequently, they are linearly dependent, implying the existence of a scattering matrix $S(\lambda)$, such that

$$\psi^- = \psi^+ S(\lambda), \tag{3.20}$$

and substituting (3.19) into (3.20), leads to

$$\phi^- = \phi^+ e^{i\lambda\Lambda x} S(\lambda) e^{-i\lambda\Lambda x}, \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{3.21}$$

where

$$S(\lambda) = (s_{ij})_{5\times 5}, \quad i, j \in \{1, \ldots, 5\}. \tag{3.22}$$

With $det(\phi^{\pm}) = 1$ being considered, we can derive

$$det(S(\lambda)) = 1. \tag{3.23}$$

Furthermore, it can be demonstrated through (3.15)–(3.17) and (3.21) that $S(\lambda)$ exhibits the involution relations

$$S^T(-\lambda) = C_1 S^{-1}(\lambda) C_1^{-1}, \tag{3.24}$$

$$S^{\dagger}(-\overset{*}{\lambda}) = C_2 S^{-1}(\lambda) C_2^{-1}, \tag{3.25}$$

$$\overset{*}{S}(\overset{*}{\lambda}) = C_3 S(\lambda) C_3^{-1}. \tag{3.26}$$

We deduce from (3.24)–(3.26) that

$$\hat{s}_{11}(\lambda) = s_{11}(-\lambda), \tag{3.27}$$

$$\hat{s}_{11}(\lambda) = \overset{*}{s}_{11}(-\overset{*}{\lambda}), \tag{3.28}$$

$$\overset{*}{s}_{11}(\overset{*}{\lambda}) = s_{11}(\lambda), \tag{3.29}$$

where the scattering matrix has an inverse denoted by $S^{-1} = (\hat{s}_{ij})_{5\times 5}$ for $i, j \in \{1, \ldots, 5\}$. To formulate Riemann-Hilbert problems, it is necessary to examine the analytic properties of the Jost matrix $\phi^{\pm}$. Our solutions $\phi^{\pm}$ to this problem can be uniquely written using the Volterra integral equations in conjunction with the spatial spectral problem (3.1):

$$\phi^-(x;\lambda) = I_5 + i \int_{-\infty}^{x} e^{i\lambda(x-y)\Lambda} P(y) \phi^-(y;\lambda) e^{i\lambda(y-x)\Lambda} dy, \tag{3.30}$$

$$\phi^+(x;\lambda) = I_5 - i \int_{x}^{+\infty} e^{i\lambda(x-y)\Lambda} P(y) \phi^+(y;\lambda) e^{i\lambda(y-x)\Lambda} dy. \tag{3.31}$$

We denote the matrix ϕ^- to be

$$\phi^- = (\phi^-_{ij})_{5\times 5}, \quad i, j \in \{1, \ldots, 5\}, \tag{3.32}$$

and ϕ^+ is denoted similarly. Keep in mind that $\alpha < 0$, in the case where $Im(\lambda) > 0$ and $y < x$, it can be observed that $Re(e^{-i\lambda\alpha(x-y)})$ exhibits exponential decay. Consequently, each integral of the first column of ϕ^- converges. This implies that the elements of the first column of ϕ^- are analytic in the upper half complex plane for $\lambda \in \mathbb{C}_+$ and continuous for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

Similarly, when $y > x$, the elements of the last four columns of ϕ^+ exhibit analytic properties in the upper half plane for $\lambda \in \mathbb{C}_+$ and maintain continuity for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

It is important to note the scenario in which $Im(\lambda) < 0$. In this case, the first column ϕ^+ is analytic in the lower half plane for $\lambda \in \mathbb{C}_-$ and remains continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$. Additionally, the components of the last four columns of ϕ^- are analytic in the lower half plane for $\lambda \in \mathbb{C}_-$ and also continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Let us proceed with the construction of the Riemann-Hilbert problems. In order to build the Jost matrix within the upper-half plane, it is important to observe that

$$\phi^\pm = \psi^\pm e^{-i\lambda\Lambda x}. \tag{3.33}$$

Consider the jth columns of $\phi^\pm$ to be denoted as $\phi^\pm_j$, where j belongs to the set $\{1, 2, 3, 4, 5\}$. Consequently, the initial solution for the Jost matrix can be expressed as

$$P^{(+)}(x; \lambda) = (\phi^-_1, \phi^+_2, \phi^+_3, \phi^+_4, \phi^+_5) = \phi^- H_1 + \phi^+ H_2, \tag{3.34}$$

where $H_1 = diag(1, 0, 0, 0, 0)$ and $H_2 = diag(0, 1, 1, 1, 1)$.

Hence, $P^{(+)}$ exhibits analyticity for $\lambda \in \mathbb{C}_+$ and it demonstrates continuity for $\lambda \in \mathbb{C}_+ \cup \mathbb{R}$.

For the lower-half plane, we can construct $P^{(-)} \in \mathbb{C}_-$ which is the analytic counterpart of $P^{(+)} \in \mathbb{C}_+$. We do this by utilizing the equivalent spectral adjoint equation (3.14). In the lower-half of the plane, we can create $P^{(-)} \in \mathbb{C}_-$ as the analytic version of $P^{(+)} \in \mathbb{C}_+$. This is achieved using the corresponding spectral adjoint equation (3.14). Because $\tilde{\phi}^\pm = (\phi^\pm)^{-1}$ and $\psi^\pm = \phi^\pm e^{i\lambda\Lambda x}$, we have

$$(\phi^\pm)^{-1} = e^{i\lambda\Lambda x}(\psi^\pm)^{-1}. \tag{3.35}$$

Let $\tilde{\phi}^\pm_j$ represent the jth row of $\tilde{\phi}^\pm$, where j belongs to the set $\{1, 2, 3, 4, 5\}$. Similar to what was mentioned earlier, we can obtain the following result:

$$P^{(-)}(x; \lambda) = \left(\tilde{\phi}^-_1, \tilde{\phi}^+_2, \tilde{\phi}^+_3, \tilde{\phi}^+_4, \tilde{\phi}^+_5\right)^T = H_1(\phi^-)^{-1} + H_2(\phi^+)^{-1}. \tag{3.36}$$

Hence, $P^{(-)}$ is analytic for $\lambda \in \mathbb{C}_-$ and continuous for $\lambda \in \mathbb{C}_- \cup \mathbb{R}$.

Given that both ϕ^- and ϕ^+ fullfill

$$\phi^T(x, t; -\lambda) = C_1 \phi^{-1} C_1^{-1}. \tag{3.37}$$

Using (3.34), we have

$$P^{(+)}(x, t; -\lambda) = \phi^-(x, t; -\lambda)H_1 + \phi^+(x, t; -\lambda)H_2 \tag{3.38}$$

or equivalently

$$(P^{(+)})^T(x, t; -\lambda) = H_1(\phi^-)^T(x, t; -\lambda) + H_2(\phi^+)^T(x, t; -\lambda). \tag{3.39}$$

By substituting Eq (3.37) into Eq (3.39), we obtain the following nonlocal symmetry property:

$$(P^{(+)})^T(x,t;-\lambda) = C_1 P^{(-)} C_1^{-1}. \tag{3.40}$$

One can prove as well that

$$(P^{(+)})^\dagger(-x,-t;-\overset{*}{\lambda}) = C_2 P^{(-)} C_2^{-1}, \tag{3.41}$$

$$\overset{*}{P}{}^{(+)}(-x,-t;\overset{*}{\lambda}) = C_3 P^{(+)} C_3^{-1}. \tag{3.42}$$

Thus, the Riemann-Hilbert problems can be formulated by leveraging the analytic properties of both $P^{(+)}$ and $P^{(-)}$ as follows:

$$P^{(-)}P^{(+)} = J, \tag{3.43}$$

where $J = e^{i\lambda\Lambda x}(H_1 + H_2 S)(H_1 + S^{-1}H_2)e^{-i\lambda\Lambda x}$ for $\lambda \in \mathbb{R}$.

Replacing (3.21) in (3.34), we have

$$P^{(+)}(x;\lambda) = \phi^+(e^{i\lambda\Lambda x} S\, e^{-i\lambda\Lambda x} H_1 + H_2). \tag{3.44}$$

As x approaches positive infinity, the function $\phi^+(x;\lambda)$ converges to the identity matrix I_5. Hence, we obtain

$$\lim_{x\to+\infty} P^{(+)} = diag(s_{11}(\lambda), I_4), \quad \text{for} \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R}. \tag{3.45}$$

Similarly, we can show that

$$\lim_{x\to-\infty} P^{(-)} = diag(\hat{s}_{11}(\lambda), I_4), \quad \text{for} \quad \lambda \in \mathbb{C}_- \cup \mathbb{R}. \tag{3.46}$$

Thus, if we choose

$$G^{(+)}(x;\lambda) = P^{(+)}(x;\lambda) diag(s_{11}^{-1}(\lambda), I_4), \tag{3.47}$$

$$(G^{(-)})^{-1}(x;\lambda) = diag(\hat{s}_{11}^{-1}(\lambda), I_4) P^{(-)}(x;\lambda), \tag{3.48}$$

the two generalized matrices $G^{(+)}(x;\lambda)$ and $G^{(-)}(x;\lambda)$ form the basis for constructing the matrix Riemann-Hilbert problems on the real line for the nonlocal Sasa-Satsuma equation, they are presented as

$$G^{(+)}(x;\lambda) = G^{(-)}(x;\lambda) G_0(x;\lambda), \quad \text{for} \quad \lambda \in \mathbb{R}, \tag{3.49}$$

where the jump matrix $G_0(x;\lambda)$ can be cast as

$$G_0(x;\lambda) = diag(\hat{s}_{11}^{-1}(\lambda), I_4)\, J\, diag(s_{11}^{-1}(\lambda), I_4), \tag{3.50}$$

which reads

$$G_0(x;\lambda) = \begin{pmatrix} s_{11}^{-1}\hat{s}_{11}^{-1} & \hat{s}_{12}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} & \hat{s}_{13}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} & \hat{s}_{14}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} & \hat{s}_{15}\hat{s}_{11}^{-1}e^{i\lambda\alpha x} \\ s_{21}s_{11}^{-1}e^{-i\lambda\alpha x} & 1 & 0 & 0 & 0 \\ s_{31}s_{11}^{-1}e^{-i\lambda\alpha x} & 0 & 1 & 0 & 0 \\ s_{41}s_{11}^{-1}e^{-i\lambda\alpha x} & 0 & 0 & 1 & 0 \\ s_{51}s_{11}^{-1}e^{-i\lambda\alpha x} & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.51}$$

and its canonical normalization conditions:

$$G^{(+)}(x;\lambda) \to I_5 \quad \text{as} \quad \lambda \in \mathbb{C}_+ \cup \mathbb{R} \to \infty, \tag{3.52}$$

$$G^{(-)}(x;\lambda) \to I_5 \quad \text{as} \quad \lambda \in \mathbb{C}_- \cup \mathbb{R} \to \infty. \tag{3.53}$$

From (3.40) along with (3.47) and (3.48) and using (3.27)–(3.29), we deduce the nonlocal involution properties

$$\begin{cases} (G^{(+)})^T(x,t;-\lambda) = C_1(G^{(-)})^{-1}C_1^{-1}, \\ (G^{(+)})^\dagger(-x,-t;-\overset{*}{\lambda}) = C_2(G^{(-)})^{-1}C_2^{-1}, \\ \overset{*}{G}{}^{(+)}(-x,-t;\overset{*}{\lambda}) = C_3G^{(+)}C_3^{-1}. \end{cases} \tag{3.54}$$

Furthermore, from (3.49), (3.50), (3.54), and (3.27)–(3.29), the subsequent nonlocal involution properties are deduced for G_0

$$\begin{cases} G_0^T(x,t;-\lambda) = C_1G_0C_1^{-1}, \\ G_0^\dagger(-x,-t;-\overset{*}{\lambda}) = C_2G_0C_2^{-1}, \quad \lambda \in \mathbb{R}. \\ \overset{*}{G}_0(-x,-t;\overset{*}{\lambda}) = C_3G_0C_3^{-1}, \end{cases} \tag{3.55}$$

3.1. Time evolution of the scattering data

To accomplish this, we proceed by taking the derivative of Eq (3.21) with respect to time t and by utilizing Eq (3.8), we obtain

$$S_t = i\lambda^5[\Omega, S], \tag{3.56}$$

and thus

$$S_t = \begin{pmatrix} 0 & i\beta\lambda^5 s_{12} & i\beta\lambda^5 s_{13} & i\beta\lambda^5 s_{14} & \beta\lambda^5 s_{15} \\ -i\beta\lambda^5 s_{21} & 0 & 0 & 0 & 0 \\ -i\beta\lambda^5 s_{31} & 0 & 0 & 0 & 0 \\ -i\beta\lambda^5 s_{41} & 0 & 0 & 0 & 0 \\ -i\beta\lambda^5 s_{51} & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.57}$$

As a result, we have

$$\begin{cases} s_{12}(t;\lambda) = s_{12}(0;\lambda)e^{i\beta\lambda^3 t}, \quad s_{21}(t;\lambda) = s_{21}(0;\lambda)e^{-i\beta\lambda^3 t}, \\ s_{13}(t;\lambda) = s_{13}(0;\lambda)e^{i\beta\lambda^3 t}, \quad s_{31}(t;\lambda) = s_{31}(0;\lambda)e^{-i\beta\lambda^3 t}, \\ s_{14}(t;\lambda) = s_{14}(0;\lambda)e^{i\beta\lambda^3 t}, \quad s_{41}(t;\lambda) = s_{41}(0;\lambda)e^{-i\beta\lambda^3 t}, \\ s_{15}(t;\lambda) = s_{15}(0;\lambda)e^{i\beta\lambda^3 t}, \quad s_{51}(t;\lambda) = s_{51}(0;\lambda)e^{-i\beta\lambda^3 t}, \end{cases} \tag{3.58}$$

and $s_{11}, s_{2i}, s_{3i}, s_{4i}, s_{5i}$ are constants for $i \in \{2, ..., 5\}$.

4. Soliton solutions

4.1. General case

The classification of soliton solutions produced by the Riemann-Hilbert problems is determined by the determinant of the matrix $G^{(\pm)}$. In the case where $det(G^{(\pm)}) \neq 0$, a single unique solution is obtained. However, in the non-regular scenario where $det(G^{(\pm)}) = 0$, discrete eigenvalues may arise in the spectral plane. To find soliton solutions in this non-regular case, it is possible to convert it into the regular case [22].

From (3.44) and $det(\phi^{\pm}) = 1$, we can show that

$$det(P^{(+)}(x;\lambda)) = s_{11}(\lambda), \tag{4.1}$$
$$det(P^{(-)}(x;\lambda)) = \hat{s}_{11}(\lambda). \tag{4.2}$$

Given that $det(S(\lambda)) = 1$, it can be deduced that $S^{-1}(\lambda) = \left(cof(S(\lambda))\right)^T$. Additionally, the expression for $\hat{s}_{11}$ is represented by the determinant of the matrix:

$$\hat{s}_{11} = \begin{vmatrix} s_{22} & s_{23} & s_{24} & s_{25} \\ s_{32} & s_{33} & s_{34} & s_{35} \\ s_{42} & s_{43} & s_{44} & s_{45} \\ s_{52} & s_{53} & s_{54} & s_{55} \end{vmatrix}. \tag{4.3}$$

In the case where the matrix $S(\lambda)$ is non-regular, $\hat{s}_{11}$ should evaluate to zero.

To generate soliton solutions, it is necessary for the determinants of $P^{(+)}(x;\lambda)$ and $P^{(-)}(x;\lambda)$ to be equal to zero. Specifically, we require $det(P^{(+)}(x;\lambda)) = det(P^{(-)}(x;\lambda)) = 0$ for this purpose. When $det(P^{(+)}(x;\lambda)) = s_{11}(\lambda) = 0$, we make the assumption that $s_{11}(\lambda)$ possesses simple zeros. These zeros correspond to discrete eigenvalues $\lambda_k \in \mathbb{C}_+$, where $k \in \{1, 2, ..., 2N_1 = N\}$. On the other hand, when $det(P^{(-)}(x;\lambda)) = \hat{s}_{11}(\lambda) = 0$, we assume that $\hat{s}_{11}(\lambda)$ also has simple zeros. These zeros are associated with discrete eigenvalues $\hat{\lambda}_k \in \mathbb{C}_-$, where $k \in \{1, 2, ..., 2N_1 = N\}$. These eigenvalues represent the poles of the transmission coefficients [24].

From Eqs (3.27)–(3.29) and $det(P^{(\pm)}(x;\lambda)) = 0$, one can deduce that

$$\text{if} \quad \lambda \in \mathbb{C}_+, \quad \text{then} \quad \begin{cases} -\lambda \in \mathbb{C}_-, \\ -\overset{*}{\lambda} \in \mathbb{C}_+, \quad \lambda \notin \mathrm{i}\mathbb{R}. \\ \overset{*}{\lambda} \in \mathbb{C}_-, \end{cases} \tag{4.4}$$

If $\lambda = \mathrm{i}m \in \mathrm{i}\mathbb{R}$, for $m > 0$, the couple $(\lambda, -\overset{*}{\lambda}) \in \mathbb{C}_+^2$ coincide, forcing $\hat{\lambda} = -\lambda = -im \in \mathbb{C}_-$.

To make this more clear, we can view the choices of the eigenvalues in a more systematic way. Recall that the Riemann-Hilbert problem requires the same number of eigenvalues in the upper-half plane and in the lower-half plane. Assume $\lambda_k \in \mathbb{C}_+$ for all $k = 1, 2, \ldots, 2N_1$. Fix n for $1 \leq n \leq N_1$ and let λ_n lies off the imaginary axis. The eigenvalues in the upper-half plane are given by the N_1-couples $(\lambda_n, \lambda_{N_1+n}) = (\lambda, -\overset{*}{\lambda}) \in \mathbb{C}_+^2$, which are assumed to be the zeros of $det(P^{(+)}(x;\lambda)) = 0$. For any λ_n, the choice of λ_{N_1+n} depends on λ_n, that is, $\lambda_n = -\overset{*}{\lambda}_{N_1+n}$, where λ_n is freely chosen. If λ_n lies on the imaginary axis, then the pair of eigenvalues coincide.

In the lower-half plane, the eigenvalues are determined by the choice of the eigenvalue λ_n in the upper-half plane. We have $\hat{\lambda}_k \in \mathbb{C}_-$ for all $k = 1, 2, \ldots, 2N_1$ and similarly the eigenvalues are given by the N_1-couples $(\hat{\lambda}_n, \hat{\lambda}_{N_1+n}) = (-\lambda, \overset{*}{\lambda}) \in \mathbb{C}_-^2$, which are assumed to be the zeros of $det(P^{(-)}(x;\lambda)) = 0$, and $\hat{\lambda}_n = -\overset{*}{\hat{\lambda}}_{N_1+n}$.

In other words, if λ_n is not pure imaginary, then the scheme of the eigenvalues take the form

$$(\lambda_n, \lambda_{N_1+n}, \hat{\lambda}_n, \hat{\lambda}_{N_1+n}) = (\lambda, -\overset{*}{\lambda}, -\lambda, \overset{*}{\lambda}). \tag{4.5}$$

Every $Ker(P^{(+)}(x; \lambda_k))$ consists of just one column vector $\mathrm{v}_k = \mathrm{v}_k(x, t)$, while each $Ker(P^{(-)}(x; \hat{\lambda}_k))$ comprises only one row vector $\hat{\mathrm{v}}_k = \hat{\mathrm{v}}_k(x, t)$, such that

$$P^{(+)}(x; \lambda_k)\mathrm{v}_k = 0, \quad \text{for} \quad k \in \{1, 2, ..., 2N_1\}, \tag{4.6}$$

and

$$\hat{\mathrm{v}}_k P^{(-)}(x; \hat{\lambda}_k) = 0, \quad \text{for} \quad k \in \{1, 2, ..., 2N_1\}. \tag{4.7}$$

In order to acquire explicit soliton solutions, we select $G_0 = I_5$ within the framework of the Riemann-Hilbert problems. This choice will result in the reflection coefficients $s_{21} = s_{31} = s_{41} = s_{51} = 0$ and $\hat{s}_{12} = \hat{s}_{13} = \hat{s}_{14} = \hat{s}_{15} = 0$. Consequently, the Riemann-Hilbert problems can be formulated in the following manner [25]:

$$G^{(+)}(x; \lambda) = I_5 - \sum_{k,j=1}^{N} \frac{\mathrm{v}_k (M^{-1})_{kj} \hat{\mathrm{v}}_j}{\lambda - \hat{\lambda}_j}, \tag{4.8}$$

and

$$(G^{(-)})^{-1}(x; \lambda) = I_5 + \sum_{k,j=1}^{N} \frac{\mathrm{v}_k (M^{-1})_{kj} \hat{\mathrm{v}}_j}{\lambda - \lambda_k}, \tag{4.9}$$

where $M = (m_{kj})_{N\times N}$ is a matrix defined by [25]

$$m_{kj} = \begin{cases} \frac{\hat{\mathrm{v}}_k \mathrm{v}_j}{\lambda_j - \hat{\lambda}_k}, & \text{if} \quad \lambda_j \neq \hat{\lambda}_k, \\ 0, & \text{if} \quad \lambda_j = \hat{\lambda}_k, \end{cases} \quad k, j \in \{1, 2, ..., N\}. \tag{4.10}$$

It is possible to analyze the spatial and temporal development of the scattering vectors v_k and $\hat{\mathrm{v}}_k$, since the zeros λ_k and $\hat{\lambda}_k$ remain constant values and are not affected by changes in space and time.

Taking the x-derivative of both sides of the equation

$$P^{(+)}(x; \lambda_k)\mathrm{v}_k = 0, \quad 1 \leq k \leq N. \tag{4.11}$$

Upon recognizing that $P^{(+)}$ satisfies the spectral spatial equivalent equation (3.7) in conjunction with (4.6), we deduce that

$$P^{(+)}(x; \lambda_k)\left(\frac{d\mathrm{v}_k}{dx} - i\lambda_k \Lambda \mathrm{v}_k\right) = 0, \quad \text{for} \quad k, j \in \{1, 2, ..., N\}. \tag{4.12}$$

Similarly, by differentiating with respect to t and utilizing the temporal equation (3.8) along with (4.6), we obtain

$$P^{(+)}(x; \lambda_k)\left(\frac{d\mathrm{v}_k}{dt} - i\lambda_k^5 \Omega \mathrm{v}_k\right) = 0, \quad \text{for} \quad k, j \in \{1, 2, ..., N\}. \tag{4.13}$$

The adjoint spectral equations (3.11) and (3.12) yield similar results

$$\left(\frac{d\hat{\mathrm{v}}_k}{dx} + i\hat{\lambda}_k \hat{\mathrm{v}}_k \Lambda\right) P^{(-)}(x; \hat{\lambda}_k) = 0, \tag{4.14}$$

and

$$\left(\frac{d\hat{\mathrm{v}}_k}{dt}+i\hat{\lambda}_k^5\hat{\mathrm{v}}_k\Omega\right)P^{(-)}(x;\hat{\lambda}_k)=0. \tag{4.15}$$

Given that v_k is a unique vector in the kernel of $P^{(+)}$, it follows that both $\frac{d\mathrm{v}_k}{dx}-\mathrm{i}\lambda_k\Lambda\mathrm{v}_k$ and $\frac{d\mathrm{v}_k}{dt}-\mathrm{i}\lambda_k^5\Omega\mathrm{v}_k$ are proportional to v_k.

Therefore, it is possible to assume, without loss of generality, that the space dependence of v_k is represented by

$$\frac{d\mathrm{v}_k}{dx}=i\lambda_k\Lambda\mathrm{v}_k,\quad 1\leq k\leq N, \tag{4.16}$$

while the time dependence of v_k is represented by

$$\frac{d\mathrm{v}_k}{dt}=i\lambda_k^5\Omega\mathrm{v}_k,\quad 1\leq k\leq N. \tag{4.17}$$

So, we can conclude that

$$\mathrm{v}_k=\mathrm{v}_k(x,t;\lambda_k)=e^{i\lambda_k\Lambda x+i\lambda_k^5\Omega t}\mathrm{w}_k,\quad \text{for}\quad k\in\{1,2,...,N\}, \tag{4.18}$$

by solving Eqs (4.16) and (4.17). Likewise, we get

$$\hat{\mathrm{v}}_k=\hat{\mathrm{v}}_k(x,t;\hat{\lambda}_k)=\hat{\mathrm{w}}_k e^{-i\hat{\lambda}_k\Lambda x-i\hat{\lambda}_k^5\Omega t},\quad \text{for}\quad k\in\{1,2,...,N\}, \tag{4.19}$$

where w_k and $\hat{\mathrm{w}}_k$ are constant column and row vectors in $\mathbb{C}^5$, respectively. In addition, they need to satisfy the orthogonality condition:

$$\hat{\mathrm{w}}_k\mathrm{w}_l=0,\quad \text{when}\quad \lambda_l=\hat{\lambda}_k,\quad 1\leq k,l\leq N. \tag{4.20}$$

From (4.6) and using the formula (3.40), it is easy to see

$$\mathrm{v}_k^T(x,t;-\lambda_k)(P^{(+)})^T(x,t;-\lambda_k)=\mathrm{v}_k^T(x,t;-\lambda_k)C_1P^{(-)}(x,t;\lambda_k)C_1^{-1}=0. \tag{4.21}$$

Because $\mathrm{v}_k^T(x,t;-\lambda_k)C_1P^{-}(x,t;\lambda_k)$ can be zero and using (4.7) this leads to

$$\mathrm{v}_k^T(x,t;-\lambda_k)C_1P^{(-)}(x,t;\lambda_k)=\mathrm{v}_k^T(x,t;\lambda_k)C_1P^{(-)}(x,t;-\lambda_k)=0 \tag{4.22}$$

$$=\hat{\mathrm{v}}_k(x,t;\hat{\lambda}_k)P^{(-)}(x,t;\hat{\lambda}_k), \tag{4.23}$$

thus, we can take

$$\hat{\mathrm{v}}_k(x,t;\hat{\lambda}_k)=\mathrm{v}_k^T(x,t;\lambda_k)C_1. \tag{4.24}$$

Therefore, the involution relations (4.18) and (4.19) give

$$\mathrm{v}_k(x,t)=e^{i\lambda_k\Lambda x+i\lambda_k^5\Omega t}\mathrm{w}_k, \tag{4.25}$$

$$\hat{\mathrm{v}}_k(x,t)=\mathrm{w}_k^T e^{-i\hat{\lambda}_k\Lambda x-i\hat{\lambda}_k^5\Omega t}C_1. \tag{4.26}$$

Now, in order to satisfy the orthogonality condition (4.20), one can notice that we require

$$\mathrm{w}_k^\dagger C_2\mathrm{w}_l=0,\quad \text{as}\quad \lambda_l=\hat{\lambda}_k,\quad 1\leq k,l\leq N. \tag{4.27}$$

Due to the fact that, $\hat{\lambda}_k = -\overset{*}{\lambda}_k = \lambda_k$ still occurs for $\lambda_k \in \mathrm{i}\mathbb{R}$, while $\hat{\lambda}_k = -\overset{*}{\lambda}_k$ holds, when $\lambda_k \neq \hat{\lambda}_k$.

The Riemann-Hilbert problem can be accurately resolved, leading to the determination of potentials through the calculation of the matrix $P^{(+)}$. Since $P^{(+)}$ exhibits analytic properties, the expansion of $G^{(+)}$ can be carried out as follows:

$$G^{(+)}(x;\lambda) = I_5 + \frac{1}{\lambda}G_1^{(+)}(x) + O\Big(\frac{1}{\lambda^2}\Big), \quad \text{when} \quad \lambda \to \infty. \tag{4.28}$$

Having established that $G^{(+)}$ satisfies the spectral problem, by inserting it into (3.7) and equating the coefficients with the corresponding power of $\frac{1}{\lambda}$ at order $O(1)$, we obtain

$$P = -[\Lambda, G_1^{(+)}]. \tag{4.29}$$

If we denote

$$G_1^{(+)} = (\mathsf{G}_{ij})_{5\times 5}, \quad i, j \in \{1, \ldots, 5\}, \tag{4.30}$$

then, since P satisfies the symmetry relations (2.8) simultaneously, therefore from (4.29), $G_1^{(+)}$ satisfies the following symmetry relations:

$$(G_1^{(+)})^\dagger = G_1^{(+)}, \tag{4.31}$$

$$(G_1^{(+)})^T(-x,-t) = C_1 G_1^{(+)} C_1^{-1}, \tag{4.32}$$

$$(G_1^{(+)})^\dagger(-x,-t) = C_2 G_1^{(+)} C_2^{-1}, \tag{4.33}$$

$$\overset{*}{G}{}_1^{(+)}(-x,-t) = C_3 G_1^{(+)} C_3^{-1}. \tag{4.34}$$

Accordingly, it can be reduced to the form

$$G_1^{(+)} = \begin{pmatrix} \mathsf{G}_{11} & \mathsf{G}_{12} & \mathsf{G}_{12}(-x,-t) & \overset{*}{\mathsf{G}}_{12} & \overset{*}{\mathsf{G}}_{12}(-x,-t) \\ \overset{*}{\mathsf{G}}_{12} & \mathsf{G}_{22} & \overset{*}{\mathsf{G}}_{32} & \overset{*}{\mathsf{G}}_{42} & \overset{*}{\mathsf{G}}_{52}(-x,-t) \\ \overset{*}{\mathsf{G}}_{12}(-x,-t) & \mathsf{G}_{32} & \overset{*}{\mathsf{G}}_{22}(-x,-t) & \overset{*}{\mathsf{G}}_{52}(-x,-t) & \overset{*}{\mathsf{G}}_{42}(-x,-t) \\ \mathsf{G}_{12} & \mathsf{G}_{42} & \mathsf{G}_{52}(-x,-t) & \mathsf{G}_{22} & \overset{*}{\mathsf{G}}_{32}(-x,-t) \\ \mathsf{G}_{12}(-x,-t) & \mathsf{G}_{52} & \mathsf{G}_{42}(-x,-t) & \mathsf{G}_{32}(-x,-t) & \overset{*}{\mathsf{G}}_{22}(-x,-t) \end{pmatrix}. \tag{4.35}$$

Thus,

$$P = -[\Lambda, G_1^{(+)}] = \begin{pmatrix} 0 & -\alpha\mathsf{G}_{12} & -\alpha\mathsf{G}_{12}(-x,-t) & -\alpha\overset{*}{\mathsf{G}}_{12} & -\alpha\overset{*}{\mathsf{G}}_{12}(-x,-t) \\ \alpha\overset{*}{\mathsf{G}}_{12} & 0 & 0 & 0 & 0 \\ \alpha\overset{*}{\mathsf{G}}_{12}(-x,-t) & 0 & 0 & 0 & 0 \\ \alpha\mathsf{G}_{12} & 0 & 0 & 0 & 0 \\ \alpha\mathsf{G}_{12}(-x,-t) & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.36}$$

Matching the components of (4.36) to the components of the P matrix, We are able to recover the potential u:

$$u = -\alpha\mathsf{G}_{12}. \tag{4.37}$$

Note that all components in (4.36) are equivalent and compatible to the components in (2.3). It can be seen from (4.28) that

$$G_1^{(+)} = \lambda \lim_{\lambda\to\infty}(G^{(+)}(x;\lambda) - I_5), \tag{4.38}$$

then using Eq (4.8), we deduce

$$G_1^{(+)} = -\sum_{k,j=1}^{N} \mathrm{v}_k (M^{-1})_{k,j} \hat{\mathrm{v}}_j, \tag{4.39}$$

where

$$\mathrm{v}_k = (\mathrm{v}_{k,1}, \mathrm{v}_{k,2}, \mathrm{v}_{k,3}, \mathrm{v}_{k,4}, \mathrm{v}_{k,5})^T,\ \hat{\mathrm{v}}_k = (\hat{\mathrm{v}}_{k,1}, \hat{\mathrm{v}}_{k,2}, \hat{\mathrm{v}}_{k,3}, \hat{\mathrm{v}}_{k,4}, \hat{\mathrm{v}}_{k,5}). \tag{4.40}$$

We deduce that the specific Riemann-Hilbert problem solutions determined by (4.8)–(4.10), satisfy (3.54). Hence the matrix $G_1^{(+)}$ posses the symmetry relations (4.31)–(4.34), which are generated from the non-local symmetry (2.4).

Now, by substituting (4.39) into (4.37) and using (4.25) and (4.26), we generate the N-soliton solution to the nonlocal fifth-order Sasa-Satsuma equation

$$u = \alpha \sum_{k,j=1}^{N} \mathrm{v}_{k,1} (M^{-1})_{kj} \hat{\mathrm{v}}_{j,2}. \tag{4.41}$$

5. Exact soliton solutions and dynamics

5.1. Explicit one-soliton solution

For a general explicit formula for the one-soliton solution of the Sasa-Satsuma equation (1.1), i.e., when $N = 1$, the eigenvalues configuration gives $\lambda_1 = \mathrm{i}m$ and $\hat{\lambda}_1 = -\mathrm{i}m$, where $m > 0$, in order to fulfill condition (4.4). Taking w_1 to be the vector $\mathrm{w}_1 = (\mathrm{w}_{11}, \mathrm{w}_{12}, -\mathrm{w}_{12}, \mathrm{w}_{12}, -\mathrm{w}_{12})^T$, for $\mathrm{w}_{11}, \mathrm{w}_{12} \in \mathbb{R}\backslash\{0\}$, the explicit solution yields

$$u(x,t) = \frac{\mathrm{i}2\alpha m \mathrm{w}_{11}\mathrm{w}_{12}}{\mathrm{w}_{11}^2 e^{-\alpha m x - \beta m^5 t} + 4\mathrm{w}_{12}^2 e^{\alpha m x + \beta m^5 t}}. \tag{5.1}$$

Since this Sasa-Satsuma equation require the orthogonality condition

$$\mathrm{w}_1^\dagger C_2 \mathrm{w}_1 = 0, \tag{5.2}$$

resulting in $\mathrm{w}_{11}^2 = 4\mathrm{w}_{12}^2$.

As a consequence, the solution to the scalar nonlocal reverse-spacetime Sasa-Satsuma equation (2.90) in the one-soliton case simplifies to

$$u(x,t) = \pm \frac{\mathrm{i}\alpha m}{e^{\alpha m x + \beta m^5 t} + e^{-\alpha m x - \beta m^5 t}} = \pm \mathrm{i}\frac{\alpha}{2} m\, sech(\alpha m x + \beta m^5 t). \tag{5.3}$$

5.1.1. Dynamics of the one-soliton

For the one-soliton, the soliton moves with constant speed $V = \frac{\beta}{\alpha}m^4$ along the line $x = -\frac{\beta}{\alpha}m^4 t$. In that case, the amplitude is given by $|u| = \frac{1}{2}\alpha m$. The amplitude of the moving soliton stays constant as seen in Figure 1.

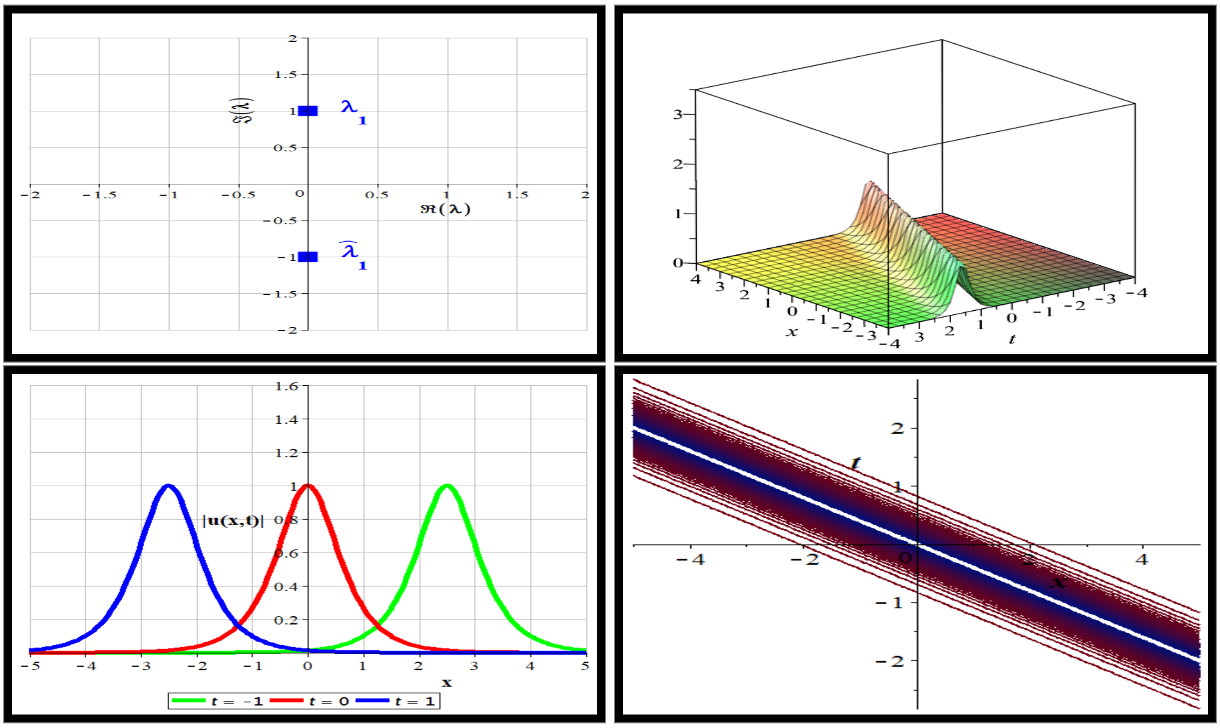


Figure 1. Spectral plane alongside three-dimensional (3D) and two-dimensional (2D), and the contours plots of $|u|$ of the travelling one-soliton with parameters $(\alpha, \beta) = (-2, -5)$, $(\lambda_1, \hat{\lambda}_1) = (i, -i)$, $\mathrm{w}_1 = (2, 1, -1, 1, -1)$.

Remark 5.1. ***Breather case:*** *When the parameter λ_1 equals m and is a real number, a breather with a period of $\frac{\pi}{|\beta m^5|}$ is obtained, as illustrated in Figure 2. The solution for this breather can be expressed as*

$$u(x,t) = \pm \frac{\alpha m}{e^{\mathrm{i}(\alpha m x + \beta m^5 t)} + e^{-\mathrm{i}(\alpha m x + \beta m^5 t)}} = \pm \frac{\alpha}{2} m\, sec(\alpha m x + \beta m^5 t). \tag{5.4}$$

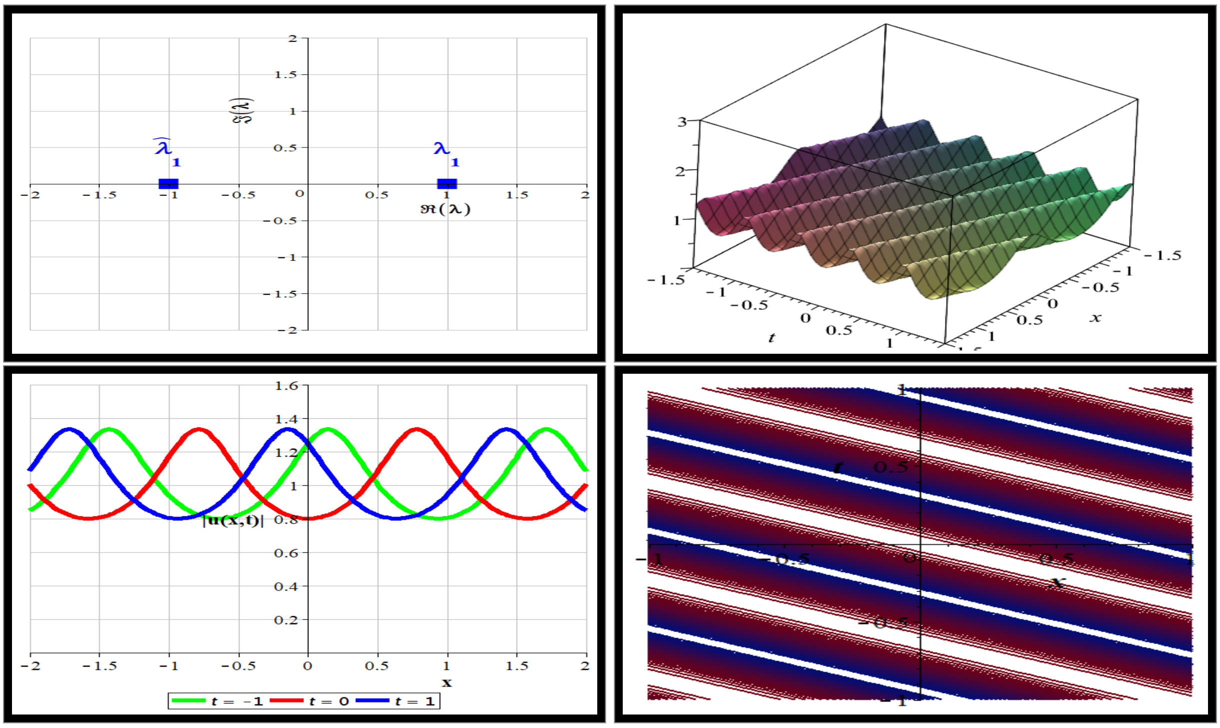
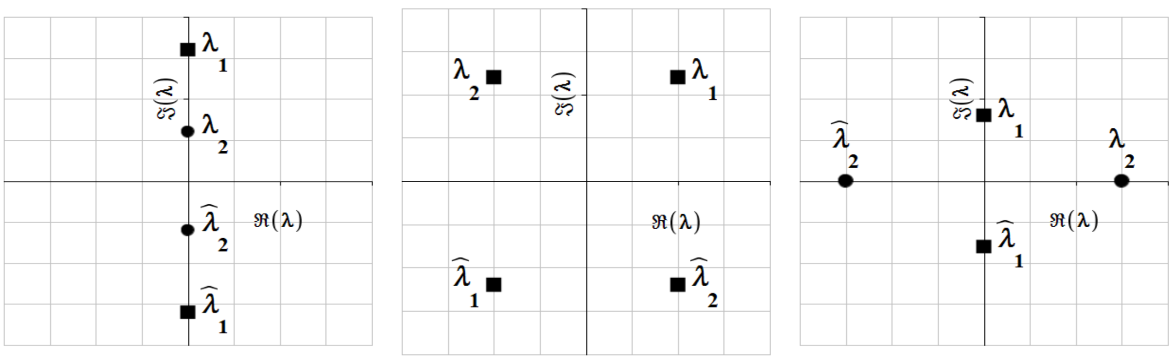


Figure 2. Spectral plane alongside three-dimensional (3D) and two-dimensional (2D), and the contours plots of $|u|$ of the one-soliton breather with parameters $(\alpha, \beta) = (-2, -5)$, $(\lambda_1, \hat{\lambda}_1) = (-2, -5)$, $\mathrm{w}_1 = (1, 1, 1, 1, 1)$.

5.2. *Explicit two-soliton solutions*

For a general explicit formula for two-soliton solutions of the Sasa-Satsuma equation (1.1), i.e., when $N = 2$, the configuration of the eigenvalues for this equation is given by $(\lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2) = (\lambda, -\overset{*}{\lambda}, -\lambda, \overset{*}{\lambda})$. As a result, we have three distinct cases as shown in Figure 3. In all cases, the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}_+ \cup \mathbb{R}$ and $\hat{\lambda}_1, \hat{\lambda}_2 \in \mathbb{C}_- \cup \mathbb{R}$ are all taken to be distinct, i.e., $\lambda_1 \neq \lambda_2$ and $\hat{\lambda}_1 \neq \hat{\lambda}_2$.



(a) Case I **(b)** Case II **(c)** Case III

Figure 3. Spectral planes of two-soliton eigenvalues cases.

Case I: If all eigenvalues in the complex plane are pure imaginary, that is $\lambda_1 = im_1$, $\lambda_2 = im_2$, $\hat{\lambda}_1 = -im_1$, $\hat{\lambda}_2 = -im_2$, for $m_1, m_2 > 0$ and $\mathrm{w}_1 = (\mathrm{w}_{11}, \mathrm{w}_{12}, -\mathrm{w}_{12}, \mathrm{w}_{12}, -\mathrm{w}_{12})^T$, then for simplicity of the solution, we take $\mathrm{w}_2 = \mathrm{w}_1$. Hence, the solution in this nonlocal reverse-spacetime case is given by

$$u(x,t) = \pm \mathrm{i}\alpha(m_1^2 - m_2^2)\frac{\mathsf{N}_1}{\mathsf{D}_1}, \tag{5.5}$$

where

$$\begin{aligned}
\mathsf{N}_1(x,t) =& m_2 e^{(\alpha_1+\alpha_2)m_2 x-(\beta_1+\beta_2)m_2^5 t}\Big(e^{-2(\alpha_1 m_1 x+\beta_1 m_1^5 t)} + e^{-2(\alpha_2 m_1 x+\beta_2 m_1^5 t)}\Big)\\
&- m_1 e^{(\alpha_1+\alpha_2)m_1 x-(\beta_1+\beta_2)m_1^5 t}\Big(e^{-2(\alpha_1 m_2 x+\beta_1 m_2^5 t)} + e^{-2(\alpha_2 m_2 x+\beta_2 m_2^5 t)}\Big),
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\mathsf{D}_1(x,t) =& (m_1 - m_2)^2\Big(e^{-2\alpha_1(m_1+m_2)x-2\beta_1(m_1^5+m_2^5)t} + e^{-2\alpha_2(m_1+m_2)x-2\beta_2(m_1^5+m_2^5)t}\Big)\\
&+ (m_1+m_2)^2\Big(e^{-2(\alpha_1 m_1+\alpha_2 m_2)x-2(\beta_1 m_1^5+\beta_2 m_2^5)t} + e^{-2(\alpha_1 m_2+\alpha_2 m_1)x-2(\beta_1 m_2^5+\beta_2 m_1^5)t}\Big)\\
&- 8m_1 m_2 e^{-(\alpha_1+\alpha_2)(m_1+m_2)x-(\beta_1+\beta_2)(m_1^5+m_2^5)t}.
\end{aligned} \tag{5.7}$$

5.2.1. Dynamics of the two-soliton solution: Case I

If the eigenvalues $\lambda_1 = -\hat{\lambda}_1$ and $\lambda_2 = -\hat{\lambda}_2$, then the two solitons move in the same direction before and after collision, where the faster soliton overtakes the slower one. An overlay of two traveling waves is shown in Figure 4, where the faster soliton overtakes the slower one from the right. Furthermore, in the pre and post collision, the amplitude remains unchanged.

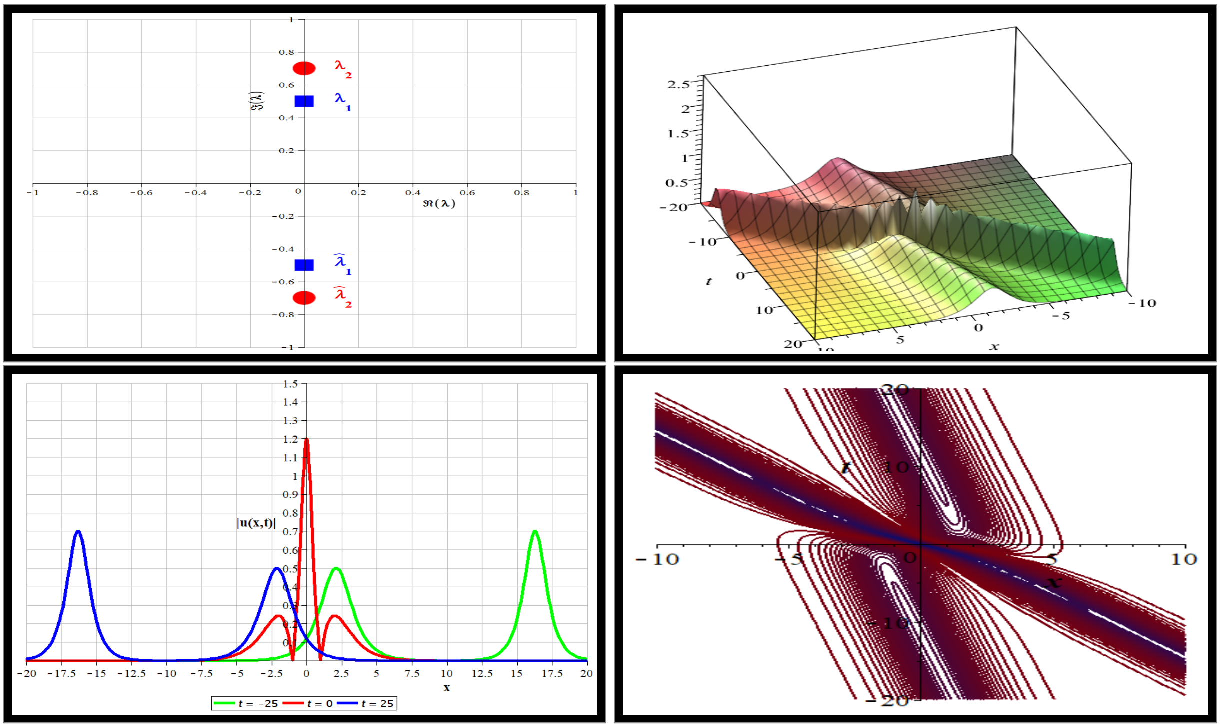


Figure 4. Spectral plane alongside three-dimensional (3D) and two-dimensional (2D), and the contours plots of $|u|$ of the two solitons interaction with parameters $(\alpha, \beta) = (-2, -5)$, $(\lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2) = (0.5\mathrm{i}, 0.7\mathrm{i}, -0.5\mathrm{i}, -0.7\mathrm{i})$, $\mathrm{w}_1 = \mathrm{w}_2 = (2, 1, -1, 1, -1)$.

Case II: In that case, if $\lambda_1, \lambda_2 \in \mathbb{C}_+$ are not pure imaginary, then the involution property (4.4) requires that $\lambda_2 = -\overset{*}{\lambda}_1$, while in the lower half-plane $\hat{\lambda}_1 = -\lambda_1$ and $\hat{\lambda}_2 = \overset{*}{\lambda}_1$.

Let $w_1 = (w_{11}, w_{12}, w_{12}, w_{12}, w_{12})^T$ and $w_2 = w_1$. the solution in this nonlocal reverse-spacetime case is given by

$$u(x,t) = i4\alpha \mathrm{Im}(\lambda^2) w_{11} w_{12} \frac{N_2}{D_2}, \tag{5.8}$$

where

$$N_2(x,t) = 2w_{11}^2 \mathrm{Re}\left(\overset{*}{\lambda} e^{i\left(2\alpha_1\lambda-(\alpha_1+\alpha_2)\overset{*}{\lambda}\right)x+i\left(2\beta_1\lambda^5-(\beta_1+\beta_2)\overset{*}{\lambda}^5\right)t}\right) + 8w_{12}^2 \mathrm{Re}\left(\overset{*}{\lambda} e^{i\left(2\alpha_2\lambda-(\alpha_1+\alpha_2)\overset{*}{\lambda}\right)x+i\left(2\beta_2\lambda^5-(\beta_1+\beta_2)\overset{*}{\lambda}^5\right)t}\right). \tag{5.9}$$

$$D_2(x,t) = 4(\mathrm{Re}(\lambda))^2 w_{11}^4 e^{-4\alpha_1 \mathrm{Im}(\lambda)x-4\beta_1 \mathrm{Im}(\lambda^5)t} + 64(\mathrm{Re}(\lambda))^2 w_{12}^4 e^{-4\alpha_2 \mathrm{Im}(\lambda)x-4\beta_2 \mathrm{Im}(\lambda^5)t} \tag{5.10}$$

$$- 32(\mathrm{Im}(\lambda))^2 w_{11}^2 w_{12}^2 \mathrm{Re}\left(e^{i2(\alpha_2\lambda-\alpha_1\overset{*}{\lambda})x+i2(\beta_2\lambda^5-\beta_1\overset{*}{\lambda}^5)t}\right)$$

$$+ 32|\lambda|^2 w_{11}^2 w_{12}^2 e^{-2(\alpha_1+\alpha_2)\mathrm{Im}(\lambda)x-2(\beta_1+\beta_2)\mathrm{Im}(\lambda^5)t}. \tag{5.11}$$

5.2.2. Dynamics of the two-soliton solution: Case II

In this configuration of the eigenvalues, the two solitons S_1 and S_2 move in the same direction as shown in Figure 5. The soliton wave S_2 with the higher speed overtakes the wave S_1 and after the collision, the wave S_1 gains speed and overtakes S_2. Therefore, we have a continuously occurring phenomenon of periodic collisions or an oscillatory-breather. While in Figure 6, we have a two-soliton double-humped keeping the shape as it moves.

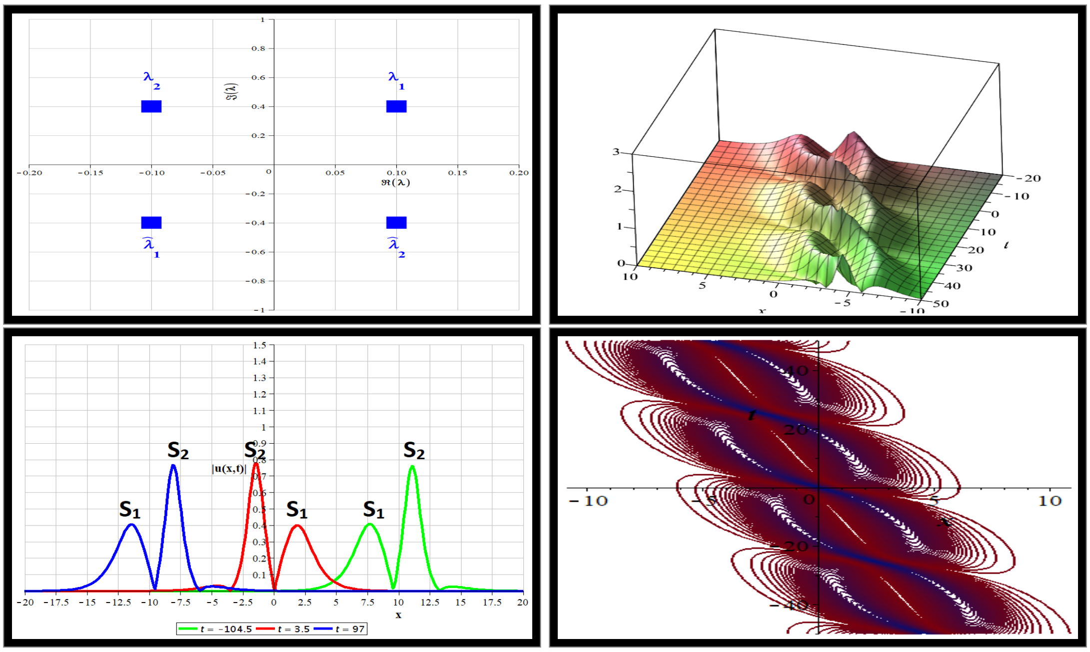


Figure 5. Spectral plane alongside three-dimensional (3D) and two-dimensional (2D), and the contours plots of $|u|$ of the two solitons interaction with parameters $(\alpha, \beta) = (-2, -5)$, $(\lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2) = (0.1 + 0.4i, -0.1 + 0.4i, -0.1 - 0.4i, 0.1 - 0.4i)$, $w_1 = w_2 = (2, 1, 1, 1, 1)$.

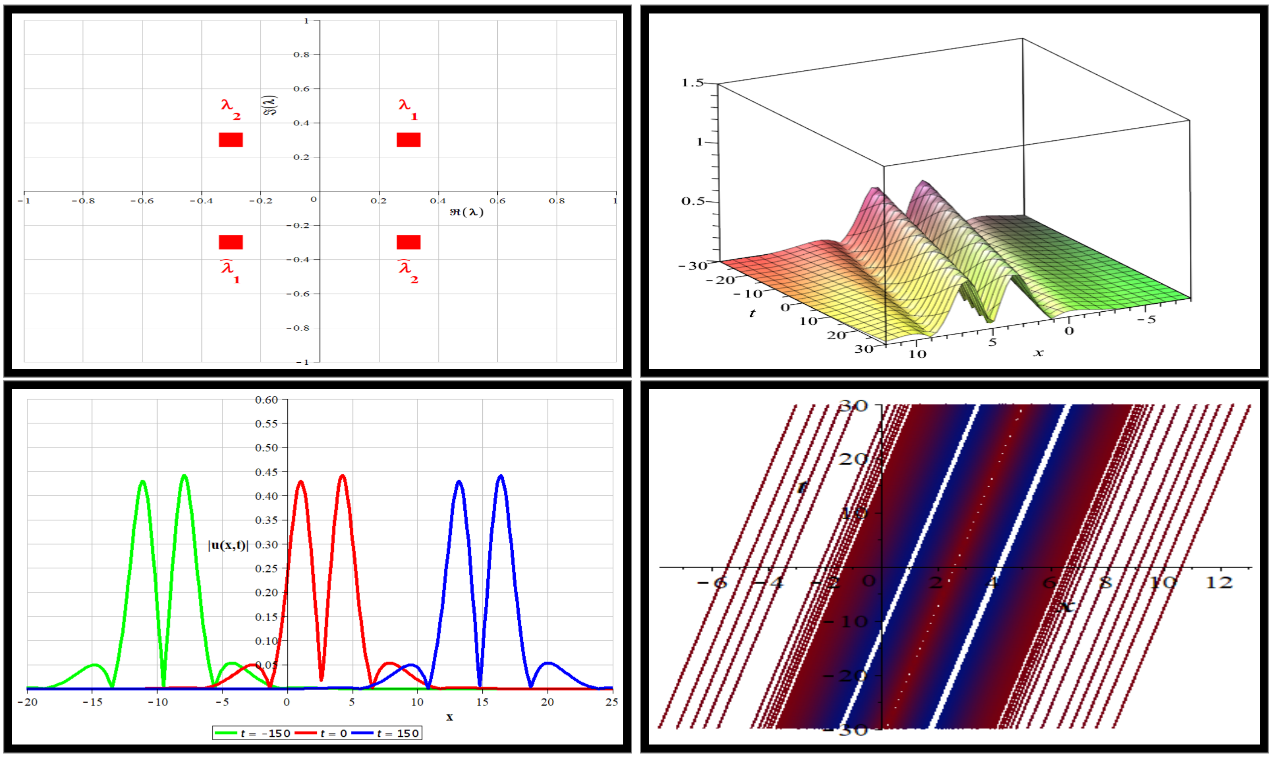


Figure 6. Spectral plane alongside three-dimensional (3D) and two-dimensional (2D), and the contours plots of $|u|$ of the two solitons interaction with parameters $(\alpha, \beta) = (-2, -5)$, $(\lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2) = (0.3 + 0.3\mathrm{i}, -0.3 + 0.3\mathrm{i}, -0.3 - 0.3\mathrm{i}, 0.3 - 0.3\mathrm{i})$, $\mathrm{w}_1 = \mathrm{w}_2 = (2, 1, 1, 1, 1)$.

Case III: In that case, if $\lambda_1 = \mathrm{i}m \in \mathrm{i}\mathbb{R}_+$ is pure imaginary and $\lambda_2 = n \in \mathbb{R}_+$, then the involution property (4.4) requires that $\hat{\lambda}_1 = -\mathrm{i}m$ and $\hat{\lambda}_2 = -n$.

Let $\mathrm{w}_1 = \mathrm{w}_2 = (\mathrm{w}_{11}, \mathrm{w}_{12}, -\mathrm{w}_{12}, \mathrm{w}_{12}, -\mathrm{w}_{12})^T$. The solution for this nonlocal reverse-spacetime case reads

$$u(x,t) = -2\alpha(m^2 + n^2)\mathrm{w}_{11}\mathrm{w}_{12}\frac{\mathsf{N}_3}{\mathsf{D}_3}, \tag{5.12}$$

where

$$\begin{aligned}\mathsf{N}_3(x,t) =& \mathrm{i}me^{-(\alpha_1+\alpha_2)mx-(\beta_1+\beta_2)m^5t}\left(\mathrm{w}_{11}^2e^{\mathrm{i}2\alpha_1nx+\mathrm{i}2\beta_1n^5t} + 4\mathrm{w}_{12}^2e^{\mathrm{i}2\alpha_2nx+\mathrm{i}2\beta_2n^5t}\right)\\ &- ne^{\mathrm{i}(\alpha_1+\alpha_2)nx+i(\beta_1+\beta_2)n^5t}\left(\mathrm{w}_{11}^2e^{-2\alpha_1mx-2\beta_1m^5t} + 4\mathrm{w}_{12}^2e^{-2\alpha_2mx-2\beta_2m^5t}\right),\end{aligned} \tag{5.13}$$

$$\begin{aligned}\mathsf{D}_3(x,t) =& 4(\mathrm{i}m+n)^2\mathrm{w}_{11}^2\mathrm{w}_{12}^2\left(e^{2(\mathrm{i}\alpha_1n-\alpha_2m)x+2(\mathrm{i}\beta_1n^5-\beta_2m^5)t} + e^{2(\mathrm{i}\alpha_2n-\alpha_1m)x+2(\mathrm{i}\beta_2n^5-\beta_1m^5)t}\right)\\ &- (\mathrm{i}m+n)^2\mathrm{w}_{11}^4e^{2\alpha_1(\mathrm{i}n-m)x+2\beta_1(\mathrm{i}n^5-m^5)t} - 16(\mathrm{i}m+n)^2\mathrm{w}_{12}^4e^{2\alpha_2(\mathrm{i}n-m)x+2\beta_2(\mathrm{i}n^5-m^5)t}\\ &- \mathrm{i}32mn\mathrm{w}_{11}^2\mathrm{w}_{12}^2e^{(\alpha_1+\alpha_2)(\mathrm{i}n-m)x+(\beta_1+\beta_2)(\mathrm{i}n^5-m^5)t}.\end{aligned} \tag{5.14}$$

5.2.3. Dynamics of the two-soliton solution: Case III

Taking a look at this dynamics, we can observe a soliton and a breather moving in the same direction. They interact continuously while the soliton travels through the breather. This is shown in Figure 7. Another possible different dynamics is when a soliton and a breather travel together at the same speed without interacting, as seen in Figure 8, where the soliton in the latter situation keeps its shape at all times while moving with the breather as a packet.

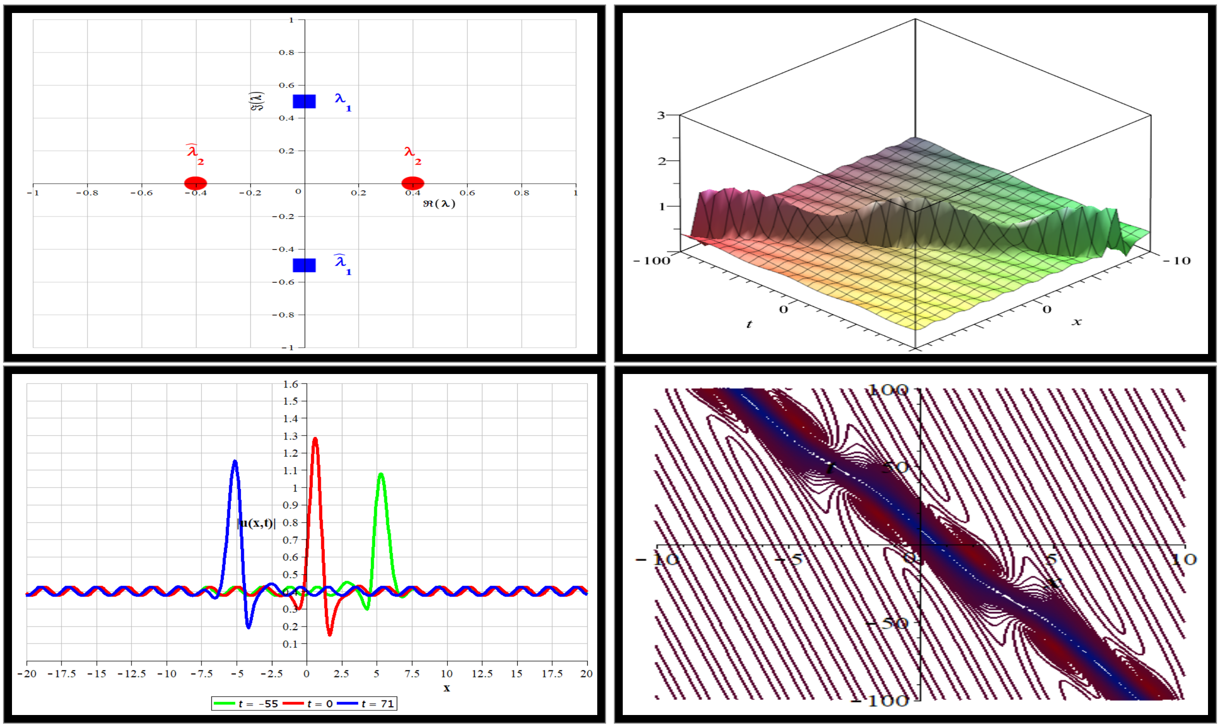


Figure 7. Spectral plane alongside three-dimensional (3D) and two-dimensional (2D), and the contours plots of $|u|$ of the continuous interaction between the soliton wave and the breather. The parameters are $(\alpha,\beta) = (-4,-5)$, $(\lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2) = (0.5i, 0.4, -0.5i, -0.4)$, $w_1 = w_2 = (2, 4, -4, 4, -4)$.

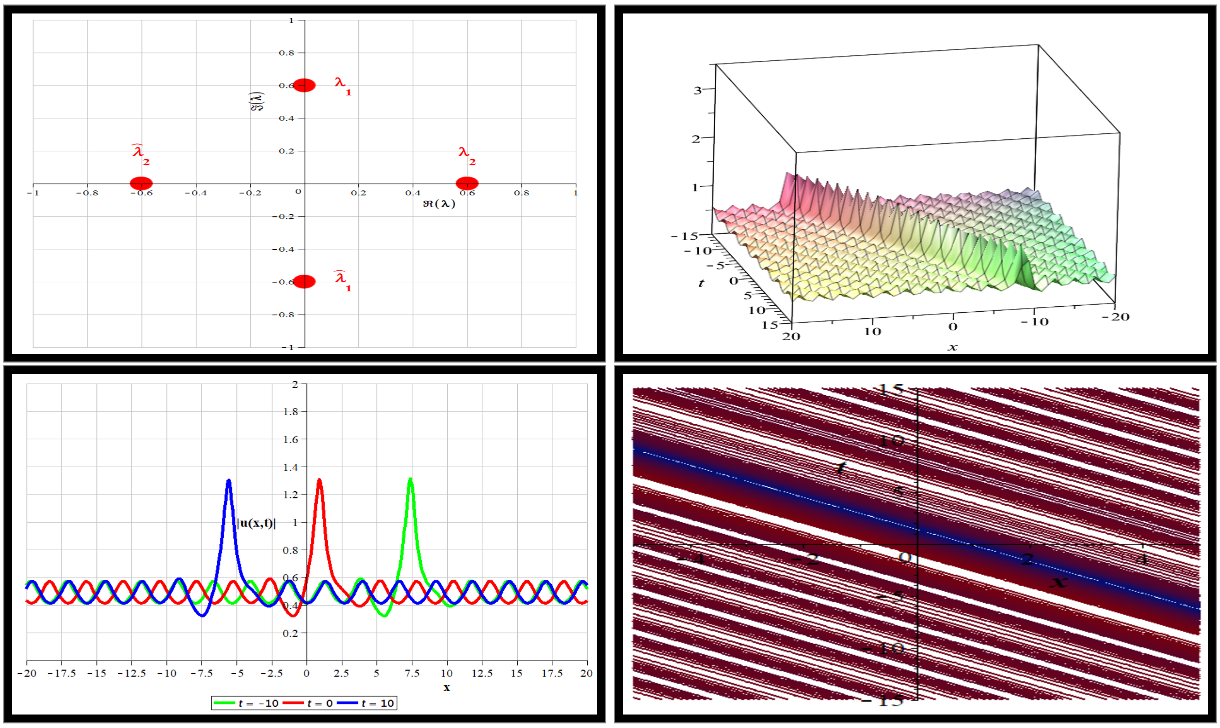


Figure 8. Spectral plane alongside three-dimensional (3D) and two-dimensional (2D), and the contours plots of $|u|$ of the soliton wave and the breather. The parameters are $(\alpha, \beta) = (-2, -10)$, $(\lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2) = (0.6i, 0.6, -0.6i, -0.6)$, $w_1 = w_2 = (2, 2.5, -2.5, 2.5, -2.5)$.

Two soliton breather: In this particular case, the configuration (4.5) compels a two-soliton breather to behave as a one-soliton breather if all eigenvalues are real. That is, since $\lambda_1 \neq \lambda_2$ and $\hat{\lambda}_1 \neq \hat{\lambda}_2$, then λ_2 and $\hat{\lambda}_2$ are redundant, as seen in Figure 9, and we take $\lambda_2 = \hat{\lambda}_2 = 0$. Consequently, it reduces to the breather solution, previously illustrated in Figure 2.

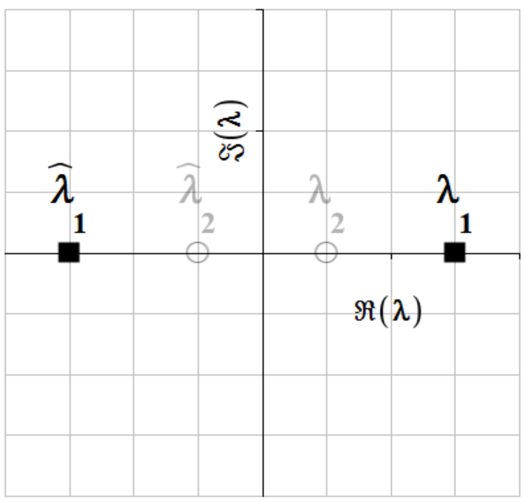


Figure 9. Spectral plane of the two-soliton breather.

6. Conclusions

To summarize, in this paper, we investigated a nonlocal reverse-spacetime scalar Sasa-Satsuma equation. This equation is derived from a nonlocal integrable hierarchy, where the nonlocal nature is embodied within the hierarchy's structure. The latter construction allows nonlocal systems to be constructed without using reductions and guarantees integrability. Moreover, the hierarchy provides mKdV-type nonlocal integrable equations and eliminate NLS-type ones. Further, a kind of soliton solutions was generated, and the Hamiltonian structure was derived for the resulting nonlocal Sasa-Satsuma equation.

For fundamental soliton solutions, Solitons in local equations exhibit elastic interactions in a superposition fashion, whereas in nonlocal equations, this behavior may not always hold true. Additionally, soliton solutions in nonlocal equations may develop singularities at a specific time, a phenomenon that does not occur in the presented Sasa-Satsuma equation.

Moreover, reverse-spacetime equations exhibit very different dynamical behaviors than reverse-time and reverse-space equations [26]. For instance, it can be seen from the plotted figures that the one-soliton to the reverse-spacetime Sasa-Satsuma equation is a moving soliton, while there is a stationary one-soliton in the reverse-time and reverse-space NLS equation [27].

Finally, we remark that it remains intriguing to solve nonlocal integrable equations in the cases of reverse-spacetime, reversespace, and reverse-time by different techniques such as Darboux transformations and the Hirota bilinear method [28–30].

Author contributions

Ahmed M. G. Ahmed, Alle Adjiri and Solomon Manukure: Conceptualization, Formal analysis, Investigation, Methodology, Resources, Software, Validation, Visualization, Writing-original draft, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

Dr. Solomon Manukure is a Guest Editor of special issue "Emerging Trends in Algebra, Geometry, and Topology of Soliton Systems" for AIMS Mathematics. Dr. Solomon Manukure was not involved in the editorial review and the decision to publish this article. The authors declare that they have no conflicts of interest.

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