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Research article

On gradient normalized Ricci-harmonic solitons in sequential warped products

Noura Alhouiti[1], **Fatemah Mofarreh**[2], **Akram Ali**[3,*] **and Fatemah Abdullah Alghamdi**[4]

[1] Department of Basic Sciences, University College of Haqel, University of Tabuk, 71491 Tabuk, Saudi Arabia

[2] Mathematical Science Department, Faculty of Science, Princess Nourah bint Abdulrahman University. Riyadh 11546, Saudi Arabia

[3] Department of Mathematics, College of Science, King Khalid University, Abha, Saudi Arabia

[4] Financial Sciences Department, Applied College, Imam Abdulrahman Bin Faisal University, Dammam, Saudi Arabia

* **Correspondence:** Email: akali@kku.edu.sa.

Abstract: Our investigation involved sequentially warped product manifolds that contained gradient-normalized Ricci-harmonic solitons. We presented the primary connections for a gradient-normalized Ricci-harmonic soliton on sequential warped product manifolds. In practical applications, our research investigated gradient-normalized Ricci-harmonic solitons for sequential generalized Robertson-Walker spacetimes and sequential standard static space-times. Our finding generalized all results proven in [26].

Keywords: gradient normalized Ricci-harmonic soliton; sequential warped product; sequentially generalized Robertson-Walker space-time; sequential standard static space-time
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1. Introduction and motivations

The normalized Ricci flow on a compact Riemann surface of an arbitrary genus g was introduced by Hamilton [23, 24]. Under the action of the normalized Ricci flow, the smooth metric g_{ij} evolves according to the following differential equation. A one-parameter family of Riemannian metric $g(t, x)$ is called normalized Ricci flow if it satisfies the following equation:

$$\frac{\partial}{\partial t} g(t, x) = -2\mathcal{R}ic(t, x) + \frac{2r}{n} g_{ij},$$

$$g(0, x) = g_0,$$

where $r = \frac{\int_{\mathbb{B}} \mathcal{R} dv}{\int_{\mathbb{B}} dV}$ is the average of scalar curvature. If $r = 0$, then the above equation reduces to Ricci flow. After initiating these concepts, several authors studied them. For example, Abolrinwa et al. [4] constructed some results for the solitons of the normalized Ricci flow and generalized corresponding results for Ricci solitons. A complete closed Riemannian manifold evolved by a normalized Ricci flow was studied to examine the spectrum of the p-biharmonic operator by them. A flow is used to derive evolution formulas, monotonicity properties, and differentiability for the least nonzero eigenvalue. Under these flows, several monotone quantities involving the first eigenvalue are obtained. In the case $n = 2$, monotone quantities depend on compact surfaces' Euler characteristics. Additionally, the spectrum diverges in the direction of the presence of some geometric condition on which the curvature is derived. For similar studies, see [6, 7, 12–16].

In the next study, the concept of harmonic-Ricci solitons was introduced and provided some characterizations of rigidity, generalizing known results for Ricci solitons. In the complete case, the restriction to the steady and shrinking gradient soliton was imposed, and some rigidity results can be traced back to the vanishing of certain modified curvature tensors that take into account the geometry of a Riemannian manifold equipped with a smooth map φ, called φ-curvature, which is a natural generalization in the setting of harmonic-Ricci solitons of the standard curvature tensor [5]. Furthermore, almost all Ricci-harmonic solitons were defined as generalizations of Ricci-harmonic solitons and harmonic-Einstein metrics [2, 3]. It has been shown that a gradient shrinking almost Ricci-harmonic soliton on a compact domain can be almost harmonic Einstein under some necessary and sufficient conditions. Following the previous concept, the Ricci-Bourguinon harmonic solitons are introduced in [31] and use the idea of the V-harmonic map to study for geometric properties for gradient Ricci-Bourguinon harmonic solitons. As a result, the relationship between gradient Ricci-harmonic solitons and sequential warped product manifolds was established in [25, 26] by considering sequential warped product manifolds consisting of gradient Ricci-harmonic solitons. They also gave the physical applications of sequential standard static space-time and sequential generalized Robertson-Walker space-time.

In the present paper, our main focus is on studying gradient normalized Ricci-harmonic solitons inspired by [31] in sequential warped product manifolds in a similar manner with [26]. Taking motivation from Ricci-harmonic solitons in sequential warped product manifold, we then introduce the notion of normalized Ricci-harmonic solitons in sequential warped product manifold and prove some results about them which generalize previous results for Ricci-harmonic solitons in sequential warped product manifold. We also derive some significant applications for gradient normalized Ricci-harmonic solitons in sequential standard static space-time and sequential generalized Robertson-Walker space-time.

2. Basic formulas and notations

Now, we define the normalized Ricci-Harmonic soliton which is defined as follows: For a closed manifold $\mathbb{B}$, given a map φ from $\mathbb{B}$ to some closed target manifold $\mathbb{N}$;

$$\frac{\partial}{\partial t} g = -2\mathcal{R}ic + 2\alpha \nabla\varphi \otimes \nabla\varphi, \quad \frac{\partial}{\partial t}\varphi = \rho_g \varphi,$$

where $g(t)$ is a time-dependent metric on $\mathbb{B}$, Rc is the corresponding Ricci curvature, $\rho_g\varphi$ is the tension field of φ with respect to g and α is a positive constant (possibly time-dependent). Moreover, $\nabla\varphi$ stands for the gradient of the function φ. We developed normalized Ricci-Harmonic flow, which is

$$\begin{cases} \frac{\partial}{\partial t}g = -2\mathcal{R}ic - 2\frac{r}{n}g + 2\alpha\nabla\varphi\otimes\nabla\varphi, \\ \frac{\partial}{\partial t}\varphi = \rho_g\varphi. \end{cases}$$

Definition 2.1. *Let $\varphi : (\mathbb{B}, g) \to (\mathbb{N}, h)$ be a smooth map (not necessarily harmonic map), where $(\mathbb{B}, g)$ and $(\mathbb{N}, h)$ are static Riemannian manifolds. $((\mathbb{B}, g), (\mathbb{N}, h), V, \varphi, \lambda_1, \lambda)$ is called normalized Ricci-Harmonic solitons if*

$$\begin{cases} \mathcal{R}ic - \frac{r}{n}g - \alpha\nabla\varphi\otimes\nabla\varphi - \frac{1}{2}\mathcal{L}_V g = \lambda g, \\ \rho_g\varphi + \langle\nabla\varphi, V\rangle = 0, \end{cases} \tag{2.1}$$

where $\alpha > 0$ is a positive constant depending on if m, λ_1 , and λ are real constants. On the other hand, φ is a map between $(\mathbb{B}, g)$ and $(\mathbb{N}, h)$. In particular, when $V = -\nabla f$, then $((\mathbb{B}, g), (\mathbb{N}, h), V, \varphi, \lambda_1, \lambda)$ is called a gradient normalized Ricci-harmonic soliton if it satisfies the coupled system of elliptic partial differential equations

$$\begin{cases} \mathcal{R}ic - \frac{r}{n}g - \alpha\nabla\varphi\otimes\nabla\varphi + \nabla^2 f = \lambda g, \\ \rho_g\varphi - \langle\nabla\varphi, \nabla f\rangle = 0, \end{cases} \tag{2.2}$$

where $f : \mathbb{B} \to \mathbb{R}$ is a smooth function and $\nabla^2 f = Hess(f)$. The function f is called the potential function of normalized Ricci-harmonic soliton. It is obvious that normalized Ricci-harmonic soliton $((\mathbb{B}, g), (\mathbb{N}, h), V, \varphi, \lambda_1, \lambda)$ is a Ricci-harmonic soliton if $r = 0$. Azami et al. [7] gave the condition under which the complete shrinking Ricci-harmonic Bourgainion soliton must be compact. The gradient Ricci-harmonic soliton is said to be shrinking, steady, or expanding depending on whether $\lambda > 0, \lambda = 0$, or $\lambda < 0$.

Remark 2.1. *Gradient normalized Ricci-harmonic soliton is called trivial if the potential function f is constant.*

It can be from (2.2) that when φ and f are constants, $(\mathbb{B}, g)$ must be an Einstein manifold.

2.1. Sequential warped product manifolds

Let $(\mathbb{B}_i, g)$ be a three Riemannian manifold with associated matrix g_i for $i = 1, 2, 3$, then the sequential warped product of the form $\mathbb{B} = (\mathbb{B}_1 \times_h \mathbb{B}_2) \times_f \mathbb{B}_3$ is defined as the following metric:

$$g = (g_1 \oplus h^2 g_2) \oplus f_2^2 g_3, \tag{2.3}$$

where $h : \mathbb{B}_1 \longleftarrow \mathbb{R}$ and $f : \mathbb{B}_1 \times \mathbb{B}_2 \longleftarrow \mathbb{R}$ are two smooth warping functions. Now, we denote the Levi-Civita connections on $\mathbb{B}$, $\mathbb{B}_1$, $\mathbb{B}_2$, and $\mathbb{B}_3$ are $\nabla^{\bar{g}}, \nabla_1, \nabla_2$, and ∇_3, respectively. Similarly, Ricci curvature is presented as Ric, Ric^1, Ric^2, and Ric^3, respectively. We represent the gradient of h on $\mathbb{B}_1$ by $\nabla_1 h$ and $\|\nabla_1 h\|^2 = g(\nabla_1 h, \nabla_2 h)$. Similarly, the gradient of f on $\mathbb{B}$ is by ∇h and $\|\nabla f\|^2 = g(\nabla f, \nabla f)$.

Now, we recall a lemma which will be important in the proof of our main theorems.

Lemma 2.1. *[18] Assuming that* $\mathbb{B} = (\mathbb{B}_1 \times_h \mathbb{B}_2) \times_f \mathbb{B}_3$ *is a sequential warped product manifold with metric* $g = (g_1 \oplus h^2 g_2) \oplus f_2^2 g_3$*, for any* $\mathbb{U}_i, \mathbb{V}_i, \mathbb{Z}_i \in \Gamma(\mathbb{B}_i)$*, and* $i = 1, 2, 3$*, the following holds:*

1) $\bar{\nabla}_{\mathbb{U}_1}\mathbb{V}_1 = \nabla_{1\mathbb{U}_1}\mathbb{V}_1$.
2) $\bar{\nabla}_{\mathbb{U}_1}\mathbb{U}_2 = \bar{\nabla}_{\mathbb{U}_2}\mathbb{U}_1 = \mathbb{U}_1(\ln h)\mathbb{U}_2$.
3) $\bar{\nabla}_{\mathbb{U}_2}\mathbb{V}_2 = \nabla_{2\mathbb{U}_2}\mathbb{V}_2 - hg(\mathbb{U}_2, \mathbb{V}_2)\nabla_1 h$.
4) $\bar{\nabla}_{\mathbb{U}_3}\mathbb{U}_1 = \bar{\nabla}_{\mathbb{U}_1}\mathbb{U}_3 = \mathbb{U}_1(\ln f)\mathbb{U}_3$.
5) $\bar{\nabla}_{\mathbb{U}_3}\mathbb{U}_2 = \bar{\nabla}_{\mathbb{U}_2}\mathbb{U}_3 = \mathbb{U}_2(\ln f)\mathbb{U}_3$.
6) $\bar{\nabla}_{\mathbb{U}_3}\mathbb{V}_3 = \nabla_{3\mathbb{U}_3}\mathbb{V}_3 - fg(\mathbb{U}_3, \mathbb{V}_3)\nabla_3 f_3$.
7) $\mathcal{R}(\mathbb{U}_1, \mathbb{V}_1)\mathbb{Z}_1 = \mathcal{R}_1(\mathbb{U}_1, \mathbb{V}_1)\mathbb{Z}_1$
8) $\mathcal{R}(\mathbb{U}_2, \mathbb{V}_2)\mathbb{Z}_2 = \mathcal{R}_2(\mathbb{U}_2, \mathbb{V}_2)\mathbb{Z}_2 - \|\nabla_1 h\|^2\{g_2(\mathbb{U}_2, \mathbb{Z}_2)\mathbb{V}_2 - g_2(\mathbb{V}_2, \mathbb{Z}_2)\mathbb{U}_2\}$.
9) $\mathcal{R}(\mathbb{U}_1, \mathbb{V}_2)\mathbb{Z}_1 = -\frac{1}{h}\nabla_h^2(\mathbb{U}_1, \mathbb{Z}_1)\mathbb{V}_2$.
10) $\mathcal{R}(\mathbb{U}_1, \mathbb{V}_2)\mathbb{Z}_2 = hg_2(\mathbb{V}_2, \mathbb{Z}_2)\nabla_{1\mathbb{U}_1}\nabla_1 h$.
11) $\mathcal{R}(\mathbb{U}_1, \mathbb{V}_2)\mathbb{Z}_3 = 0$.
12) $\mathcal{R}(\mathbb{U}_i, \mathbb{V}_i)\mathbb{Z}_j = 0, i \neq j$.
13) $\mathcal{R}(\mathbb{U}_i, \mathbb{V}_3)\mathbb{Z}_j = \frac{1}{f}\nabla_f^2(\mathbb{U}_i, \mathbb{Z}_j)\mathbb{V}_3, i, j = 1, 2$.
14) $\mathcal{R}(\mathbb{U}_i, \mathbb{V}_3)\mathbb{Z}_3 = fg(\mathbb{V}_3, \mathbb{Z}_3)\nabla_{\mathbb{U}_i}\nabla f, i = 1, 2$.
15) $\mathcal{R}(\mathbb{U}_3, \mathbb{V}_3)\mathbb{Z}_3 = \mathcal{R}_3(\mathbb{U}_i, \mathbb{V}_3)\mathbb{Z}_3 - \|\nabla f\|^1\{g_3(\mathbb{U}_3, \mathbb{Z}_3)\mathbb{V}_3 - g_3(\mathbb{V}_3, \mathbb{Z}_3)\mathbb{U}_3\}$.

Lemma 2.2. *[18] Assuming that* $\mathbb{B} = (\mathbb{B}_1 \times_h \mathbb{B}_2) \times_f \mathbb{B}_3$ *is a sequential warped product manifold with metric* $g = (g_1 \oplus h^2 g_2) \oplus f_2^2 g_3$*, for any* $\mathbb{U}_i, \mathbb{V}_i, \mathbb{Z}_i \in \Gamma(\mathbb{B}_i)$*, and* $i = 1, 2, 3$*, the following holds:*

1) $\bar{\mathcal{R}}ic(\mathbb{U}_1, \mathbb{V}_1) = \mathcal{R}ic_1(\mathbb{U}_1, \mathbb{V}_1) - \frac{n_2}{h}\nabla_h^2(\mathbb{U}_1, \mathbb{V}_1) - \frac{n_3}{f}\nabla_f^2(\mathbb{U}_1, \mathbb{V}_1)$.
2) $\bar{\mathcal{R}}ic(\mathbb{U}_2, \mathbb{V}_2) = \mathcal{R}ic_2(\mathbb{U}_2, \mathbb{V}_2) - f_1^\star g_2(\mathbb{U}_2, \mathbb{V}_2) - \frac{n_3}{f}\nabla_f^2(\mathbb{U}_2, \mathbb{V}_2)$.
3) $\bar{\mathcal{R}}ic(\mathbb{U}_3, \mathbb{V}_3) = \mathcal{R}ic_3(\mathbb{U}_3, \mathbb{V}_3) - f^\star g_3(\mathbb{U}_3, \mathbb{U}_3)$.
4) $\bar{\mathcal{R}}ic(\mathbb{U}_i, \mathbb{V}_j) = 0,\ i \neq j$.

where $h^\star = h\Delta h + (n_2 - 1)\|\nabla_1\|^2 h$ *and* $f^\star = f\Delta f + (n_3 - 1)\|\nabla_2\|^2 f$.

Now, we proof the key lemma as:

Lemma 2.3. *Assuming that* $\mathbb{B} = ((\mathbb{B}_1 \times_h \mathbb{B}_2) \times_f \mathbb{B}_3, \varphi_1, \varphi, \lambda_1, \lambda)$ *is a gradient normalized Ricci-harmonic soliton on a sequential wrapped product manifold including a nonconstant harmonic map* φ*, then the harmonic map* φ *can be expressed in the form* $\varphi = \varphi_{\mathbb{B}_1} \circ \pi_1$; $\varphi = \varphi_{\mathbb{B}_2} \circ \pi_2$; *or* $\varphi = \varphi_{\mathbb{B}_3} \circ \pi_3$ *if, and only if,* $\varphi_1 = \varphi_{1\mathbb{B}_1} \circ \pi_1$ *for a neighborhood* v *of a point* $(p_1, p_2, p_3) \in \Gamma(\bar{\mathbb{B}})$*, where* $\varphi_1 \in C^\infty(\mathbb{B}_1)$ *is a another potential function and* $\pi_i : \mathbb{B}_i \longrightarrow \mathbb{R}$ *as projection maps for* $i = 1, 2, 3$.

Proof. Operating Eq (2.2) for $\mathbb{U}_i$ and $\mathbb{U}_j$, we have

$$\bar{\mathcal{R}}ic(\mathbb{U}_i, \mathbb{U}_j) + \nabla_{\bar{g}}^2(\mathbb{U}_i, \mathbb{U}_j) - \alpha\bar{\nabla}\varphi(\mathbb{U}_i)\bar{\nabla}\varphi(\mathbb{U}_j) = (\frac{r}{n} + \lambda)\bar{g}(\mathbb{U}_i, \mathbb{U}_j), \tag{2.4}$$

$$\rho_{\bar{g}}\varphi(\mathbb{U}_i, \mathbb{U}_j) - \bar{g}(\bar{\nabla}\varphi(\mathbb{U}_i), \bar{\nabla}\varphi_1(\mathbb{U}_j)) = 0, \tag{2.5}$$

for $i \neq j$ and $i \leq i, j \leq 3$. It is implied that $\bar{g}(\mathbb{U}_i, \mathbb{U}_j) = 0$. Now, from Lemma 2.2, we have $\bar{\mathcal{R}}ic(\mathbb{U}_i, \mathbb{U}_j) = 0$. Following from [27], we get $\nabla_{\bar{g}}^2(\mathbb{U}_i, \mathbb{U}_j) = 0$. Rearranging (2.4) and (2.5), we get

$$\bar{g}(\nabla_{\mathbb{U}_i}^{\bar{g}}(\bar{\nabla}\varphi_1), \mathbb{U}_j) = 0. \tag{2.6}$$

Finally, implementing Lemma 2.1 in the above equation, it is easy to find that $\varphi_1 = \varphi_{1\mathbb{B}_1} \circ \pi_1$. Conversely, we assume that φ_1 can be written in the form $\varphi_1 = \varphi_{1\mathbb{B}_1} \circ \pi_1 \in C^\infty(\mathbb{B}_1$, then using Eqs (2.1) and (2.3), we constructed

$$\alpha\bar{\nabla}\varphi(\mathbb{U}_i)\bar{\nabla}\varphi(\mathbb{U}_j) = 0. \tag{2.7}$$

The above equation can be expressed because φ is a nonconstant map

$$\bar{\nabla}\varphi(\mathbb{U}_1 + \mathbb{U}_2 + \mathbb{U}_3)\bar{\nabla}\varphi(\mathbb{U}_1 + \mathbb{U}_2 + \mathbb{U}_3) \neq 0. \tag{2.8}$$

For a neighborhood v, applying and summing up to 3 in (2.8), we get

$$\sum_{i=1}^{3}(\bar{\nabla}\varphi(\mathbb{U}_i))^2 + \sum_{i=1}^{3}\sum_{j=1,i\neq j}^{3}\bar{\nabla}\varphi(\mathbb{U}_i)\bar{\nabla}\varphi(\mathbb{U}_j) \neq 0. \tag{2.9}$$

Now, from (2.7) and (2.9), we reached that $\bar{\nabla}\varphi(\mathbb{U}_i) \neq 0$ for $i = 1, 2, 3$. It is a complete proof of the lemma. □

2.2. Main theorems

Theorem 2.1. *Assume that a sequential warped product manifold of the type* $\mathbb{B} = ((\mathbb{B}_1 \times_h \mathbb{B}_2) \times_f \mathbb{B}_3, \bar{g}, \varphi_1, \varphi, \lambda)$ *is a gradient normalized Ricci-harmonic soliton if, and only if, the functions* f, φ_1, φ, *and* λ *satisfy one of the following conditions:*

(a) If $\varphi = \varphi_{\mathbb{B}_1} \circ \pi_1$, *then*

$$\begin{cases} \mathcal{R}ic_1 - \dfrac{n_2}{h}\nabla_1^2(h) - \dfrac{n_3}{f}\nabla^2(f) + \nabla^2(\varphi_1) - \alpha\nabla_1\varphi_{\mathbb{B}_1} \otimes \nabla_1\varphi_{\mathbb{B}_1} = \Big(\lambda + \dfrac{r}{n}\Big)g_1, \\ \Delta_1 - g_1(\nabla_1, \nabla_1(\varphi_1 - n_2\log(h))\Big\}\varphi_{\mathbb{B}_1} + n_3\nabla_1\varphi_1(\log)(f)) = 0, \end{cases} \tag{2.10}$$

$$\mathcal{R}ic_2 - \frac{n_3}{f}\nabla^2(f) = \Big\{\Big(\lambda + \frac{r}{n}\Big)h + h(\Delta_1 h) + (n_2 - 1)\|\nabla_1 h\|^2 - h(\nabla_1\varphi_1(h)\Big\}g_2, \tag{2.11}$$

and
together $\mathbb{B}_3$ *is Einstein with* $\mathcal{R}ic_3 = \lambda_3 g_3$ *such that*

$$\lambda_3 = (\lambda + \frac{r}{n})f^2 + f\Delta f + (n_3 - 1)\|\nabla f\|^2 - f(\nabla_1\varphi_1(f)). \tag{2.12}$$

(b) If $\varphi = \varphi_{\mathbb{B}_2} \circ \pi_2$, *then*

$$\mathcal{R}ic_1 - \frac{n_2}{h}\nabla_1^2(h) - \frac{n_3}{f}\nabla^2(f) + \nabla^2(\varphi_1) = \Big(\lambda + \frac{r}{n}\Big)g_1, \tag{2.13}$$

$$\begin{cases} \mathcal{R}ic_2 - \dfrac{n_3}{f}\nabla^2(f) - \dfrac{\alpha}{h^4}\nabla_2\varphi_{\mathbb{B}_2} \otimes \nabla_2\varphi_{\mathbb{B}_2} = \Big\{(\lambda + \dfrac{r}{n})h^2 + h\Delta_1 h \\ \qquad + (n_2 - 1)\|\nabla_1 h\|^2 - h(\nabla_1\varphi_1(h))\Big\}g_2, \\ \Delta_2\varphi_{\mathbb{B}_2} + n_3\nabla_2\varphi_{\mathbb{B}_2}(f) = 0, \end{cases} \tag{2.14}$$

and
together $\mathbb{B}_3$ *is Eintein with* $\mathcal{R}ic_3 - \frac{\alpha}{f}\frac{\nabla}{2}\varphi_{\mathbb{B}_2} \otimes \nabla_2\varphi_{\mathbb{B}_2} = \lambda_3 g_3$ *such that*

$$\lambda_3 = (\lambda + \frac{r}{n})f^2 + f\Delta f + (n_3 - 1)\|\nabla f\|^2 - f(\nabla_1\varphi_1(f)). \tag{2.15}$$

(c) If $\varphi = \varphi_{\mathbb{B}_3} \circ \pi_3$*, then*

$$\mathcal{R}ic_1 - \frac{n_2}{h}\nabla_1^2(h) - \frac{n_3}{f}\nabla^2(f) + \nabla^2(\varphi_1) = \Big(\lambda + \frac{r}{n}\Big)g_1, \tag{2.16}$$

$$\mathcal{R}ic_2 - \frac{n_3}{f}\nabla^2(f) = \Big\{(\lambda + \frac{r}{n})h^2 + h\Delta_1 h + (n_2 - 1)\|\nabla_1 h\|^2 - h(\nabla_1\varphi_1(h))\Big\}g_2. \tag{2.17}$$

$$\begin{cases} \mathcal{R}ic_3 - \frac{\alpha}{f^4}\nabla_3\varphi_{\mathbb{B}_3} \otimes \nabla_3\varphi_{\mathbb{B}_3} = \lambda_3 g_3, \\ \Delta_3\varphi_{\mathbb{B}_3} = 0, \quad in \quad \mathbb{B}_3, \end{cases} \tag{2.18}$$

and
together with the following

$$\lambda_3 = (\lambda + \frac{r}{n})f^2 + f\Delta f + (n_3 - 1)\|\nabla f\|^2 - f(\nabla_1\varphi_1(f)). \tag{2.19}$$

where $\nabla^2 f = Hess(f)$ *and* ∇f *is the gradient of the function* f*.*

Proof. Let $\mathbb{B} = ((\mathbb{B}_1 \times_h \mathbb{B}_2) \times_f \mathbb{B}_3, \bar{g}, \varphi_1, \varphi, \lambda_1, \lambda)$ be a gradient normalized Ricci-harmonic soliton with the assumptions $\varphi = \varphi_{\mathbb{B}_1} \circ \pi_1$. By applying Lemma 2.2 and Hessian equations from [21] in the main Eq (2.1), we arrive at (2.10). With similar procedures, again using Lemma 2.2 and putting $\varphi = \varphi_{\mathbb{B}_1} \circ \pi_1$ into the Eq (2.1), we derive that

$$\begin{aligned} \mathcal{R}ic_2(\mathbb{U}_2, \mathbb{V}_2) - \Big(h\Delta_1 h + (n_2 - 1)\|\nabla_1 h\|^2\Big)g_2(\mathbb{U}_2, \mathbb{V}_2) \\ - \frac{n_3}{f}\nabla^2(\mathbb{U}_2, \mathbb{V}_2) + \nabla^2\varphi_1(\mathbb{U}_2, \mathbb{V}_2) = \Big(\lambda + \frac{r}{n}\Big)h^2 g_2(\mathbb{U}_2, \mathbb{V}_2) \end{aligned} \tag{2.20}$$

for any $\mathbb{U}_2, \mathbb{V}_2 \in \Gamma(\mathbb{B}_2)$. Including the results from Lemma 2.1 and the relation of Hessian for any function gives the following:

$$\nabla^2\varphi_1(\mathbb{U}_2, \mathbb{V}_2) = h\nabla_1\varphi_1(h)g_2(\mathbb{U}_2, \mathbb{V}_2). \tag{2.21}$$

Combing the Eqs (2.20) and (2.21), we get our supposed result (2.11). Now for any $\mathbb{U}_3, \mathbb{V}_3 \in \Gamma(\mathbb{B}_3)$ and using Lemma 2.2 with $\varphi = \varphi_{\mathbb{B}_1} \circ \pi_1$, we get

$$\begin{aligned} \mathcal{R}ic_3(\mathbb{U}_3, \mathbb{V}_3) - \Big(f\Delta_2 f + (n_3 - 1)\|\nabla f\|^2\Big)g_3(\mathbb{U}_3, \mathbb{V}_3) + \nabla^2\varphi_1(\mathbb{U}_3, \mathbb{V}_3) \\ = \Big(\lambda + \frac{r}{n}\Big)f^2 g_2(\mathbb{U}_3, \mathbb{V}_3). \end{aligned} \tag{2.22}$$

Again with same property as in (2.21), we have

$$\nabla^2\varphi_1(\mathbb{U}_3, \mathbb{V}_3) = f\nabla_2\varphi_1(f)g_2(\mathbb{U}_3, \mathbb{V}_3). \tag{2.23}$$

Inserting (2.23) into (2.22), we derive

$$\mathcal{R}ic_3(\mathbb{U}_3,\mathbb{V}_3) - \{f\Delta_2 f + (n_3-1)\|\nabla f\|^2\}g_3(\mathbb{U}_3,\mathbb{V}_3) + f\nabla_2\varphi_1(f)g_3(\mathbb{U}_3,\mathbb{V}_3) = \Big(\lambda+\frac{r}{n}\Big)f^2 g_2(\mathbb{U}_3,\mathbb{V}_3). \tag{2.24}$$

From the above equation, it is concluded that $\mathbb{B}_3$ is an Einstein manifold. The same procedures will apply to another case, and then we complete the proof of the theorem. □

Theorem 2.2. *Let a sequential warped product manifold of the type* $\mathbb{B} = ((\mathbb{B}_1 \times_h \mathbb{B}_2) \times_f \mathbb{B}_3, \bar{g}, \varphi_1, \varphi, \lambda)$ *is a gradient normalized Ricci-harmonic soliton with noncosntant harmonic map* φ*. If* $(\lambda + \frac{r}{n}) \geq 0$, φ_1 *tends to maximum or minimum in* $\mathbb{B}_1$ *with the following inequality*

$$\frac{n_1}{f} tr_{g_1}\nabla^2(f) + \frac{n_2}{h}\Delta_1(h) \geq \mathcal{R}_1, \tag{2.25}$$

then $\varphi_1 = \varphi_{1\mathbb{B}_1} \circ \pi_1$ *are constant functions, where* $\mathcal{R}_1$ *represents the scalar curvature on* $\mathcal{R}_1$.

Proof. From the first statement of the theorem and taking trace in (2.10) for any $\mathbb{U}_1, \mathbb{V}_1 \in \Gamma(\mathbb{B}_1)$,

$$\Delta_1\varphi_{1\mathbb{B}_1} = n_1\Big(\lambda+\frac{r}{n}\Big) + \alpha\|d\pi_1(\varphi)\|^2 - \mathcal{R}_1 + \frac{n_3}{f}tr_{g_1}\nabla^2(f) + \frac{n_2}{h}\Delta_1(h). \tag{2.26}$$

Now from (2.25) and $\big(\lambda + \frac{r}{n}\big) \geq 0$ together with φ_1 tending to the maximum or minimum in $\mathbb{B}_1$, it easily concludes from (2.26) that the map $\varphi_1 = \varphi_{1\mathbb{B}_1} \circ \pi_1$ is a constant function. □

Theorem 2.3. *Let a sequential warped product manifold of the type* $\mathbb{B} = ((\mathbb{B}_1 \times_h \mathbb{B}_2) \times_f \mathbb{B}_3, \bar{g}, \varphi_1, \varphi, \lambda)$ *be a gradient normalized Ricci-harmonic soliton with nonconstant harmonic map* φ *such that* f *tends to the maximum or minimum and the following inequalities hold:*

$$\left\{\Big(\lambda+\frac{r}{n}\Big) \leq \frac{\mu}{f^2} \quad or \quad \Big(\lambda+\frac{r}{n}\Big) \geq \frac{\mu}{f^2}\right\} \in \mathbb{B}_1 \times \mathbb{B}_2, \tag{2.27}$$

then f *is a constant function.*

Proof. One of the most useful elliptic operators of 2*nd* order is defined by

$$\omega(\cdot) = \Delta(\cdot) - \nabla\varphi_1(\cdot) + \frac{n_1-1}{f}\nabla f(\cdot). \tag{2.28}$$

Implementing (2.12), (2.15), (2.20), and (2.28), we get the following:

$$\omega(\cdot) = \frac{\mu - \big(\lambda+\frac{r}{n}\big)f^2}{f}. \tag{2.29}$$

Applying our assumption (2.27) together with Eq (2.29), if f tends to a maximum or minimum, then f is a constant function. It completes the proof of the theorem. □

3. Applications in sequential standard static space-time

We consider $\mathbb{B}_3 = \mathcal{I}$ to be an open interval associated with a subinterval of $\mathbb{R}$. In this case, dt^2 is the Euclidean metric tensor on $\mathcal{I}$, then a sequential warped product manifold of the form $\mathbb{B} = ((\mathbb{B}_1 \times_h \mathbb{B}_2) \times_f \mathcal{I}, \bar{g})$ turns into *sequential standard static space-time* with metric tensor $\bar{g} = (g_1 \oplus h^2 g_2) \oplus f^2(-dt^2)$. This type of space-time is defined in [19, 20]. If $\varphi : \mathbb{B} \longrightarrow \mathbb{R}$ is a harmonic map, then we have the following result:

Theorem 3.1. *Assume that a sequential warped product manifold of the type $\mathbb{B} = ((\mathbb{B}_1 \times_h \mathbb{B}_2) \times_f \mathcal{I}, \bar{g}, \varphi_1, \varphi, \lambda)$ is a gradient normalized Ricci harmonic soliton if, and only if, the functions f, φ_1, φ and λ satisfy one of the following conditions:*

(a) If $\varphi = \varphi_{\mathbb{B}_1} \circ \pi_1$, then

$$\begin{cases} \mathcal{R}ic_1 - \frac{n_2}{h}\nabla_1^2(h) - \frac{n_3}{f}\nabla^2(f) + \nabla^2(\varphi_1) - \alpha\nabla_1\varphi_{\mathbb{B}_1} \otimes \nabla_1\varphi_{\mathbb{B}_1} = \Big(\lambda + \frac{r}{n}\Big)g_1, \\ \qquad \Delta_1 - g_1(\nabla_1, \nabla_1(\varphi_1 - n_2\log(h))\Big\}\varphi_{\mathbb{B}_1} + n_3\nabla_1\varphi_1(\log)(f)) = 0, \end{cases} \tag{3.1}$$

$$\mathcal{R}ic_2 - \frac{n_3}{f}\nabla^2(f) = \Big\{\Big(\lambda + \frac{r}{n}\Big)h + h(\Delta_1 h) + (n_2 - 1)\|\nabla_1 h\|^2 - h(\nabla_1\varphi_1(h)\Big\}g_2, \tag{3.2}$$

and together with the following

$$(\lambda + \frac{r}{n})f^2 + f\Delta f - f(\nabla_1\varphi_1(f)) = 0. \tag{3.3}$$

(b) If $\varphi = \varphi_{\mathbb{B}_2} \circ \pi_2$, then

$$\mathcal{R}ic_1 - \frac{n_2}{h}\nabla_1^2(h) - \frac{n_3}{f}\nabla^2(f) + \nabla^2(\varphi_1) = \Big(\lambda + \frac{r}{n}\Big)g_1, \tag{3.4}$$

$$\begin{cases} \mathcal{R}ic_2 - \frac{n_3}{f}\nabla^2(f) - \frac{\alpha}{h^4}\nabla_2\varphi_{\mathbb{B}_2} \otimes \nabla_2\varphi_{\mathbb{B}_2} = \Big\{(\lambda + \frac{r}{n})h^2 + h\Delta_1 h + (n_2 - 1)\|\nabla_1 h\|^2 \\ \qquad\qquad - h(\nabla_1\varphi_1(h))\Big\}g_2, \\ \Delta_2\varphi_{\mathbb{B}_2} + n_3\nabla_2\varphi_{\mathbb{B}_2}(f) = 0, \end{cases} \tag{3.5}$$

and together with the following

$$(\lambda + \frac{r}{n})f^2 + f\Delta f - f(\nabla_1\varphi_1(f)) = 0. \tag{3.6}$$

(c) If $\varphi = \varphi_{\mathcal{I}} \circ \pi_{\mathcal{I}}$, then

$$\mathcal{R}ic_1 - \frac{n_2}{h}\nabla_1^2(h) - \frac{n_3}{f}\nabla^2(f) + \nabla^2(\varphi_1) = \Big(\lambda + \frac{r}{n}\Big)g_1, \tag{3.7}$$

$$\mathcal{R}ic_2 - \frac{n_3}{f}\nabla^2(f) = \Big\{(\lambda + \frac{r}{n})h^2 + h\Delta_1 h + (n_2 - 1)\|\nabla_1 h\|^2 - h(\nabla_1\varphi_1(h))\Big\}g_2, \tag{3.8}$$

$$\begin{cases} \alpha\nabla_{\mathcal{I}}\varphi_{\mathcal{I}} \otimes \nabla_{\mathcal{I}}\varphi_{\mathcal{I}} + f^4\Big\{big(\lambda + \frac{r}{n})f^2 + f\Delta f - f(\nabla_1\varphi_1(f))\}\Big\} = 0. \\ \Delta_{\mathcal{I}}\varphi_{\mathcal{I}} = 0, \quad in \quad \mathcal{I}. \end{cases} \tag{3.9}$$

Proof. For the interval $\mathcal{I}$, the metric tensor is defined as $g_{\mathcal{I}}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -1$ and the Ricci curvature is given as $\mathcal{R}ic(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = 0$ in Theorem 2.1, the desire result of theorem. The proof is completed. □

4. Applications in generalized Robertson-Walker space-time

If we consider $\varphi : \mathbb{B} \longrightarrow \mathbb{R}$ is a harmonic map through the sequential generalized Robertson-Walker space-time $\mathbb{B} = ((\mathcal{I} \times_h \mathbb{B}_2) \times_f \mathbb{B}_3, \bar{g}, \varphi_1, \varphi, \lambda)$, then we have the following results.

Theorem 4.1. *A sequential generalized Robertson-Walker space-time $\mathbb{B} = ((\mathcal{I} \times_h \mathbb{B}_2) \times_f \mathbb{B}_3, \bar{g}, \varphi_1, \varphi, \lambda)$ is a gradient normalized Ricci harmonic soliton if, and only if, the following differential equations satisfy*

(a) If $\varphi = \varphi_{\mathbb{B}_1} \circ \pi_1$, then

$$\begin{cases} \dfrac{n_2 f_1''}{f} + \dfrac{n_3 \nabla^2(f)}{f} - \varphi_1'' + \alpha\varphi_{\mathcal{I}}'' = \lambda + \dfrac{r}{n}, \\ \varphi_{\mathcal{I}}'' - \varphi_{\mathcal{I}}'\varphi_1' + \dfrac{n_2 h'}{h}\varphi_{\mathcal{I}}' + \dfrac{n_3 \nabla f}{f}\varphi_{\mathcal{I}}' = 0, \end{cases}$$

$$\mathcal{R}ic_2 - \frac{n_3}{f}\nabla^2(f) = \Big\{\Big(\lambda + \frac{r}{n}\Big)h^2 + hh'' + (n_2 - 1)(h')^2 - hh'\varphi_1'\Big\}g_2,$$

and together $\mathbb{B}_3$ is Einstein with $\mathcal{R}ic_3 = \lambda_3 g_3$ such that

$$\lambda_3 = (\lambda + \frac{r}{n})f^2 + f\Delta f + (n_3 - 1)\|\nabla f\|^2 - (\nabla f)f\varphi_1.$$

(b) If $\varphi = \varphi_{\mathbb{B}_2} \circ \pi_2$, then

$$\frac{n_2 f_1''}{f} + \frac{n_3 \nabla^2(f)}{f} - \varphi_1'' = \lambda + \frac{r}{n},$$

$$\begin{cases} \mathcal{R}ic_2 - \dfrac{n_3}{f}\nabla^2(f) - \dfrac{\alpha}{h^4}\nabla_2\varphi_{\mathbb{B}_2} \otimes \nabla_2\varphi_{\mathbb{B}_2} = \Big\{\Big(\lambda + \dfrac{r}{n}\Big)h^2 + hh'' + (n_2 - 1)(h')^2 \\ \qquad\qquad - hh'\varphi_1'\Big\}g_2, \\ \Delta_2\varphi_{\mathbb{B}_2} + n_3\nabla_2\varphi_{\mathbb{B}_2}(f) = 0, \end{cases}$$

and together $\mathbb{B}_3$ is Eintein with $\mathcal{R}ic_3 - \frac{\alpha}{f}\nabla_2\varphi_{\mathbb{B}_2} \otimes \nabla_2\varphi_{\mathbb{B}_2} = \lambda_3 g_3$ such that

$$\lambda_3 = (\lambda + \frac{r}{n})f^2 + f\Delta f + (n_3 - 1)\|\nabla f\|^2 - (\nabla f)f\varphi_1.$$

(c) If $\varphi = \varphi_{\mathbb{B}_3} \circ \pi_3$, then

$$\frac{n_2 f_1''}{f} + \frac{n_3 \nabla^2(f)}{f} - \varphi_1'' = \lambda + \frac{r}{n},$$

$$\mathcal{R}ic_2 - \frac{n_3}{f}\nabla^2(f) = \Big\{\Big(\lambda + \frac{r}{n}\Big)h^2 + hh'' + (n_2 - 1)(h')^2 - hh'\varphi_1'\Big\}g_2,$$

$$\begin{cases} \mathcal{R}ic_3 - \dfrac{\alpha}{f^4}\nabla_3\varphi_{\mathbb{B}_3} \otimes \nabla_3\varphi_{\mathbb{B}_3} = \lambda_3 g_3, \\ \Delta_3\varphi_{\mathbb{B}_3} = 0, \quad in \quad \mathbb{B}_3, \end{cases}$$

and together with the following

$$\lambda_3 = (\lambda + \frac{r}{n})f^2 + f\Delta f + (n_3 - 1)\|\nabla f\|^2 - (\nabla f)f\varphi_1.$$

Proof. Now, we define the following for the first factor $\mathcal{I}$:

$$\begin{aligned} \nabla_1 h &= -h', \\ \nabla_1^2 h(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) &= h'', \\ \Delta_1 h &= -h'', \\ g_{\mathcal{I}}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) &= -1, \\ g_{\mathcal{I}}(\nabla_1 h, \nabla_1 h) &= -(h')^2. \end{aligned}$$

All the above equations substitute in Theorem 2.1, and we get our desired results. It completes the proof of our theorem. □

Remark 4.1. *As we know, if $r = 0$ in* (2.2)*, then a gradient normalized Ricci-harmonic soliton is generalized to a gradient Ricci-Harmonic soliton which is given in [2, 6]. Now substitute $r = 0$ in Theorems 2.1, 2.2, 2.3, 3.1, and 4.1. Then Theorems 2.1, 2.2, 2.3, 3.1, and 4.1 coincide with Theorems 2.1, 2.2, 3.1, and 3.2 in [26]. As a result, our results are the natural generalization of gradient Ricci-Harmonic solitons on sequentially warped product manifolds.*

5. Conclusions

The geometry of warped product manifolds is rich and varied, and their properties depend crucially on the choice of the warping function. Understanding the behavior of this function is therefore of fundamental importance in the study of these objects. In recent years, there has been a surge of interest in the study of warped product manifolds, driven in part by their wide-ranging applications and connections to other mathematics areas. Therefore, the study of warped product manifolds has many important applications in geometry and physics. For example, in general relativity, warped product manifolds are used to model certain black hole space-times. In algebraic geometry, they arise in studying moduli spaces of vector bundles on algebraic varieties. In topology, they have been used to construct examples of exotic manifolds that do not admit a smooth structure [11].

Normalized Ricci solitons are solutions to the Ricci flow equation in Riemannian geometry, and they have found applications in various areas of mathematics and physics. In physics, particularly in the study of general relativity and the behavior of space-time, normalized Ricci solitons have been of interest. Here are some potential physical applications: In the context of gravitational collapse: Normalized Ricci solitons can be used to model the behavior of space-time in the context of gravitational collapse. In the study of black holes and other astrophysical phenomena, these solitons can provide insights into the dynamics of space-time near singularities. About cosmology, normalized

Ricci solitons may have implications for cosmological models, particularly in understanding the behavior of the universe at large scales. They can potentially shed light on the evolution of the universe and the behavior of space-time in the early universe. Quantum gravity: In the quest to develop a consistent theory of quantum gravity that unifies general relativity and quantum mechanics, space-time behavior at small scales is crucial. Normalized Ricci solitons could play a role in understanding the quantum nature of space-time and its dynamics in a quantum gravity framework.

In singularities and space-time geometry: Normalized Ricci solitons can be used to study the behavior of space-time near singularities, such as those found in black holes or cosmological models. Understanding the geometric properties of space-time near singularities is important for understanding the fundamental nature of space-time. The study of geometric flows, including the Ricci flow, has applications in understanding the evolution of manifolds and geometric structures. Normalized Ricci solitons are important solutions in this context and can provide insights into the long-term behavior of geometric evolution. These are just a few potential physical applications of normalized Ricci solitons. Their study can contribute to our understanding of the fundamental nature of space-time, gravitational phenomena, and the behavior of geometric structures in physics [1–4, 8–10, 22, 28–30].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Author contributions

Conceptualization, A. A. and F. A. A.; methodology, A. A. and N. A; software, F. A. A.; validation, A.A., F.A.A., and F.M.; formal analysis, A. A.; investigation, A. A.; resources, N. A.; data curation, A. A., F. M.; writing—original draft preparation, A. A.; writing—review and editing, F. M.; visualization, N. A; supervision, N. A.; project administration, F.A.A., and N.A; funding acquisition, N.A. All authors have read and agreed to the published version of the manuscript.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. A. Abolarinwa, Evolution and monotonicity of the first eigenvalue of p-Laplacian under the Ricci-harmonic flow, *J. Appl. Anal.*, **21** (2015), 147–160. https://doi.org/10.1515/jaa-2015-0013
2. A. Abolarinwa, Gap theorems for compact almost Ricci-harmonic solitons, *Int. J. Math.*, **30** (2019), 1950040. https://doi.org/10.1142/S0129167X1950040X
3. A. Abolarinwa, N. K. Oladejo, S. O. Salawu, On the entropy formulas and solitons for the Ricci-harmonic flow, *Bull. Iran. Math. Soc.*, **45** (2019), 1177–1192. https://doi.org/10.1007/s41980-018-00192-1
4. A. Abolarinwa, Y. Chong, D. Zhang, On the spectrum of the p-biharmonic operator under the Ricci flow, *Results Math.*, **75** (2020), 54. https://doi.org/10.1007/s00025-020-1182-9
5. A. Anselli, On the rigidity of harmonic-Ricci solitons, *Adv. Geom.*, **22** (2022), 171–198. https://doi.org/10.1515/advgeom-2022-0003
6. S. Azami, Ricci-Bourguignon flow coupled with harmonic map flow, *Int. J. Math.*, **30** (2019), 1950049. https://doi.org/10.1142/S0129167X19500496
7. S. Azami, V. Pirhadi, G. Fasihi-Ramandi, Complete shrinking Ricci-Bourguignon harmonic solitons, *Int. J. Math.*, **33** (2022), 2250046. https://doi.org/10.1142/S0129167X2250046X
8. H. Al-Sodais, H. Alodan, S. Deshmukh, Hypersurfaces of Euclidean space as gradient Ricci solitons, *An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (NS)*, **61** (2015), 437–444.
9. A. Ali, F. Mofarreh, D. S. Patra, Geometry of almost Ricci solitons on paracontact metric manifolds, *Quaest. Math.*, **45** (2022), 1167–1180. https://doi.org/10.2989/16073606.2021.1929539
10. A. Ali, N. Alshehri, F. Mofarreh, Y. Li, Geometry of gradient Einstein harmonic solitons in sequential warped products manifolds, *Eur. Phys. J. Plus*, **139** (2024), 339. https://doi.org/10.1140/epjp/s13360-024-05120-3
11. R. L. Bishop, B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.*, **145** (1969), 1–49.
12. A. M. Blaga, H. M. Tastan, Gradient solitons on doubly warped product manifolds, *Rep. Math. Phys.*, **89** (2022), 319–333. https://doi.org/10.1016/S0034-4877(22)00036-2
13. U. C. De, M. Turan, A. Yildiz, A. De, Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds, *Publ. Math. Debrecen*, **80** (2012), 127–142.
14. U. C. De, K. Mandal, Ricci solitons and gradient Ricci solitons on N(k)-paracontact manifold, *Zh. Mat. Fiz. Anal. Geom.*, **15** (2019), 307–320. https://doi.org/10.15407/mag15.03.307
15. S. Dwivedi, Some results on Ricci-Bourguignon solitons and almost solitons, *Can. Math. Bull.*, **64** (2021), 591–604. https://doi.org/10.4153/S0008439520000673
16. S. Dwivedi, D. S. Patra, Some results on almost $*$-Ricci-Bourguignon solitons, *J. Geom. Phys.*, **178** (2022), 104519. https://doi.org/10.1016/j.geomphys.2022.104519
17. U. C. De, S. Shenawy, *Generalized quasi-Einstein GRW space-times*, *Int. J. Geom. Methods M.*, **16** (2019), 1950124. https://doi.org/10.1142/S021988781950124X

18. U. C. De, S. Shenawy, B. Unal, Sequential warped products: Curvature and conformal vector fields, *Filomat*, **33** (2019), 4071–4083. https://doi.org/10.2298/FIL1913071D
19. F. Dobarro, B. Unal, Special standard static space–times, *Nonlinear Anal.-Theor.*, **59** (2004), 759–770. https://doi.org/10.1016/j.na.2004.07.035
20. H. X. Guo, R. Philipowski, A. Thalmaier, On gradient solitons of the Ricci-harmonic flow, *Acta Math. Sin. (Engl. Ser.)*, **31** (2015), 1798–1804. https://doi.org/10.1007/s10114-015-4446-7
21. S. Guler, Sequential warped products and their applications, *Int. Electron. J. Geom.*, **14** (2021), 277–291. https://doi.org/10.36890/iejg.937419
22. D. Ganguly, S. Dey, A. Ali, A. Bhattacharyya, Conformal Ricci soliton and quasi-Yamabe soliton on generalized Sasakian space form, *J. Geom. Phys.*, **169** (2021), 104339. https://doi.org/10.1016/j.geomphys.2021.104339
23. R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, *J. Differential Geometry*, **17** (1982), 255–306. https://doi.org/10.4310/jdg/1214436922
24. R. S. Hamilton, The Ricci flow on surfaces, In: *Mathematics and general relativity*, Contemporary Mathematics, 1988, 237–262. https://doi.org/10.1090/conm/071
25. F. Karaca, C. Ozgur, Gradient Ricci-harmonic solitons on doubly warped product manifolds, *Filomat*, **37** (2023), 5969–5977. https://doi.org/10.2298/FIL2318969K
26. F. Karaca, C. Ozgur, On sequential warped product manifolds admitting gradient Ricci-harmonic solitons, *Phys. Scr.*, **98** (2023), 085213. https://doi.org/10.1088/1402-4896/ace1b4
27. M. L. de Sousa, R. Pina, Gradient Ricci solitons with the structure of warped products, *Results Math.*, **71** (2017), 825–840. https://doi.org/10.1007/s00025-016-0583-2
28. Y. Li, D. S. Patra, N. Alluhaibi, F. Mofarreh, A. Ali, *Geometric classifications of k-almost Ricci solitons admitting paracontact metrices*, *Open Math.*, **21** (2023), 20220610. https://doi.org/10.1515/math-2022-0610
29. R. Muller, *Ricci flow coupled with harmonic map flow*, 2012, arXiv: 0912.2907. https://doi.org/10.48550/arXiv.0912.2907
30. D. S. Patra, A. Ali, F. Mofarreh, Characterizations of Ricci-Bourguignon almost solitons on pseudo-Riemannian manifolds, *Mediterr. J. Math.*, **19** (2022), 176. https://doi.org/10.1007/s00009-022-02085-4
31. X. Cao, *A remark of Ricci-Bourguignon harmonic soliton*, 2023, arXiv: 2309.16485. https://doi.org/10.48550/arXiv.2309.16485

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